硕士学位论文

$\mathbb{CP}^2$ 和 $\mathbb{P}^1\times\mathbb{P}^1$ 上的相对格罗莫夫-威腾不变量

RELATIVE GROMOV-WITTEN INVARIANTS OF $\mathbb{CP}^2$ AND $\mathbb{P}^1\times\mathbb{P}^1$

研　　究　　生：欧阳灏

指　导　教　师：邬龙挺助理教授

南方科技大学

二〇二四年六月

国内图书分类号：O151.24 学校代码：14325
国际图书分类号：514 密级：公开

理学硕士学位论文

$\mathbb{CP}^2$ 和 $\mathbb{P}^1 \times \mathbb{P}^1$ 上的相对格罗莫夫-威腾不变量

学 位 申 请 人：欧阳灏
指 导 教 师：邬龙挺助理教授
学 科 名 称：数学
答 辩 日 期：2024 年 5 月
培 养 单 位：数学系
学位授予单位：南方科技大学

Classified Index: O151.24
U.D.C: 514

Thesis for the Degree of Master of Science

RELATIVE GROMOV-WITTEN INVARIANTS OF $\mathbb{CP}^2$ AND $\mathbb{P}^1 \times \mathbb{P}^1$

Candidate:	Hao Ouyang
Supervisor:	Assistant Prof. Longting Wu
Discipline:	Mathematics
Date of Defence:	May, 2024
Affiliation:	Mathematics
Degree-Conferring-Institution:	Southern University of Science and Technology

学位论文公开评阅人和答辩委员会名单

公开评阅人名单

无（匿名评阅人）

答辩委员会名单

主席	马梓铭	副教授	南方科技大学
委员	向子卿	副教授	南方科技大学
	胡晓文	副教授	大湾区大学
秘书	刘孟晴	博士后	南方科技大学

DECLARATION OF ORIGINALITY AND AUTHORIZATION OF THESIS, SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Declaration of Originality of Thesis

I hereby declare that this thesis is my own original work under the guidance of my supervisor. It does not contain any research results that others have published or written. All sources I quoted in the thesis are indicated in references or have been indicated or acknowledged. I shall bear the legal liabilities of the above statement.

Signature: Date:

Declaration of Authorization of Thesis

I fully understand the regulations regarding the collection, retention and use of thesis of Southern University of Science and Technology.

1. Submit the electronic version of thesis as required by the University.
2. The University has the right to retain and send the electronic version to other institutions that allow the thesis to be read by the public.
3. The University may save all or part of the thesis in certain databases for retrieval, and may save it with digital, cloud storage or other methods for the purpose of teaching and scientific research. I agree that the full text of the thesis can be viewed online or downloaded within the campus network.

（1）I agree that once submitted, the thesis can be retrieved online and the first 16 pages can be viewed within the campus network.

（2）I agree that ☐ upon submission/ ☐ 12 months after submission, the full text of the thesis can be viewed and downloaded by the public.

4. This authorization applies to decrypted confidential thesis.

Signature of Author: Date:

Signature of Supervisor: Date:

摘　要

这篇论文中，由 Fan 与 Wu 的工作的启发，我们展示了 $(\mathbb{P}^1\times\mathbb{P}^1, diagonal)$ 的相对 Gromov-Witten 不变量的精确公式。更进一步地，由于这类相对 Gromov-Witten 不变量的生成函数的闭公式，我们可以讨论 $(\mathbb{P}^1\times\mathbb{P}^1, diagonal)$ 和 $(\mathbb{P}^2, conic)$ 的相对 Gromov-Witten 不变量的渐进性质并揭示这两类相对 Gromov-Witten 不变量的关系。

关键词：相对 Gromov-Witten 不变量；渐进性质；Hirzebruch 曲面；复射影平面

ABSTRACT

In this thesis we present the explicit formula of relative Gromov-Witten of $(\mathbb{P}^1 \times \mathbb{P}^1, diagonal)$, motivated by the work of Fan and Wu. Furthermore, due to the closed form of the generating functions of these Gromov-Witten invariants, we are able to deduce the asymptotic expansions of relative Gromov-Witten invariants of $(\mathbb{P}^1 \times \mathbb{P}^1, diagonal)$ and $(\mathbb{P}^2, conic)$ and reveal the underlying relation of these two sets of relative Gromov-Witten invariants.

Keywords: Relative Gromov-Witten invariants; Asymptotic expansion; Hirzebruch surfaces; Complex projective plane

TABLE OF CONTENTS

CHAPTER 1 INTRODUCTION

Enumerative geometry is a old area that counts geometry objects satisfy given properties. For example, problem of Apollonius is one of the earliest counting problem, that asks the number and construction of circles that are tangent to three given circles, points or lines.

In the late 20 century, enumerative geometry got a new development with motivation from physics, mainly string theory and topological field theory. While mathematicians conjectured that for a generic quintic threefold $X \subset \mathbb{P}^4$, there exists only finitely many curves with fixed degree $d > 0$ and genus g. Via mirror symmetry, string theorists predicted the numbers by a different set of numbers N_d, counting maps of degree d from a source curve of genus 0 to X, which are called Gromov-Witten invariants. Gromov-Witten invariants originated to the research of pseudoholomorphic curves by Gromov[1] and study of non-linear σ model by Witten[2]. The analysis of string theorists told that it would be natural to study maps from a source curve to the target X. In the 1990's, mathematicians established the mathematical foundations of the theory of Gromov-Witten invariants, such as Ruan and Tian[3], Li and Tian[4] [5], Behrend and Fantechi[6] [7].

Generally speaking, Gromov-Witten theory is the theory in enumerative geometry, counting the pseudoholomorphic curves in symplectic manifolds, introducing amounts of remarkable structures, such as quantum cohomology. This theory answers many old questions. For example, from the associativity of quantum cohomology, which is equivalent to WDVV equation, Kontsevich gave the famous Kontsevich's formula,

$$N_d = \sum_{\substack{d_A+d_B=d \\ d_A,d_B\geq 1}} \binom{3d-4}{3d_A-2} d_A^2 d_B^2 N_{d_A} N_{d_B} - \binom{3d-4}{3d_A-1} d_A^3 d_B N_{d_A} N_{d_B},$$

recursively giving the (virtual) count of the degree d, genus 0 rational curves in $\mathbb{P}^2$.

A subarea of Gromov-Witten theory is to determine the (virtual) numbers of 'relative' Gromov-Witten invariants. 'Relative' here is a specific tangency condition to a given divisor. The relative Gromov-Witten theory was defined by Li and Ruan[8], Ionel and Parker[9] in symplectic geometry and Li[10] in algebraic geometry.

A simple nontrivial case of relative enumerative problems is counting degree d rational, genus 0 curves passing through 2d-1 points which has a order d tangent at a given point to a given line in $\mathbb{P}^2$ which was solved by Takahashi[11] and Reineke and Weist[12] with different methods. And they both found these invariants satisfy a recursion formula, which is similar to the Kontsevich's formula. Especially, Reineke and Weist presented there is a relation between the relative invariants and moduli of representations of quiver. Inspired by the work of Reineke and Weist, Bousseau[13] constructed a correspondence of quiver DT (Donaldson-Thomas) invariants and relative Gromov-Witten invariants.

In parallel, Fan and Wu[14] enlarged a WDVV-like recursion formula for this relative invariant of genus 0 case by defining the relative Gromov-Witten invariants of negative contact order, which works on relative pair (X, D), where X is a smooth projective surface and D is a smooth ample divisor. By packaging the invariants into a generating function, the recursion formula deduces a 3rd order nonlinear PDE, and derives an explicit formula for relative pairs $(\mathbb{P}^2, line)$ and $(\mathbb{P}^2, conic)$.

In the later work of Bousseau and Wu[13], they deduced the recursion formula of relative Gromov-Witten invariants of a family of relative pairs in all genus, which reproves the the simplest form of the recursion formula of Fan and Wu, which derives a second order non-linear PDE. And this simplified formula could be derived from the previous one by proper choice of hypersurface divisor, which is called 'T_{log}'.

We would note the total space X and the divisor D as (X, D) and $N_{g,\beta}(X, D)$ (resp. $\bar{N}_{g,\beta}(X, D)$) the count of stable maps of genus g, type β with contact order $D \cdot \beta$ to D at an unspecific (resp. specific) point.

Mostly mathematicians concern about Gromov-Witten invariants of one parameter, while this thesis presents that the PDE of the case $H_0 = \mathbb{P}^1 \times \mathbb{P}^1$, of two parameters, admits an exact solution due to symmetry and degenerating to the cases as $\mathbb{P}^2$[14]. By this method, we have the following result, proved in Chapter 3,

Theorem 1.1:

$$
\begin{aligned}
&N_{0,(m,n)}(\mathbb{P}^1 \times \mathbb{P}^1, diagonal) = \\
&\quad \frac{d!}{d-1} \sum_{j=1}^{d-2} \frac{(d-1)^{j-1}(-1)^j}{j!(d-1)} \sum_{\substack{a_1+\cdots+a_j=d-2 \\ a_i \geq 1}} (-1)^{d-2} \prod_{i=1}^{j} \frac{a_i^{a_i-1}}{a_i!} \sum_{\substack{\sum_K a_k = d-1-n \\ K \subset [j]}} 1,
\end{aligned}
$$

where $d = m + n$.

Except the explicit formula of invariants, we also concern the asymptotic behavior, or say asymptotic expansion. In the viewpoint of mathematics, the asymptotic nature of the Gromov-Witten invariants may encode message of the manifold. And there are lots of works done about the Gromov-Witten invariants in $\mathbb{P}^2$ [15] [16] [17] [18]. Wei and Tian proved the following result of the asymptotic expansion of N_d,

There exist $x_0, a_k^0 \in \mathbb{R}$ such that $a_3^k > 0$ and for any fixed $N \geq 4$,

$$\frac{N_d}{(3d-1)!} = e^{-dx_0}(\sum_{k=3}^{N-1} a_k^0 d^{-k-\frac{1}{2}} + O(d^{-N-\frac{1}{2}})),$$

equivalently say, there exist $x_0, a_k^0 \in \mathbb{R}$ such that $a_3^k > 0$,

$$\frac{N_d}{(3d-1)!} \sim e^{-dx_0} \sum_{k\geq 3} a_k^0 d^{-k-\frac{1}{2}}.$$

In this thesis, we focus on the case $(\mathbb{P}^1 \times \mathbb{P}^1, diagonal)$, which could derive an exact formula of the generating function with coefficients $N_{0,(m,n)}(\mathbb{P}^1 \times \mathbb{P}^1, diagonal))$, the relative Gromov-Witten invariants on $H_0 = \mathbb{P}^1 \times \mathbb{P}^1$. The radius of convergence is determined by analysis tools, which is equivalent to say the following limit, proved in Chapter 4,

Theorem 1.2:

$$\lim_{d\to\infty} \sqrt[d]{\frac{\sum_{m+n=d} N_{0,(m,n)}(\mathbb{P}^1 \times \mathbb{P}^1, diagonal))}{(d-1)!}} = e.$$

In fact, a similar discussion and computation tell us the exponential growth rate of relative Gromov-Witten invariants of the relative pair $(\mathbb{P}^2, conic)$, while a slight modification to determine $(\mathbb{P}^2, line)$.

Theorem 1.3:

$$\lim_{d\to\infty} \sqrt[d]{\frac{N_{0,d}(\mathbb{P}^2, conic)}{d!}} = 2e.$$

Since we have the exact formula of this generating function, the sum of the anti-diagonal relative invariants, $\sum_{m+n=d} N_{0,(m,n)}(\mathbb{P}^1 \times \mathbb{P}^1, diagonal))$ admits an asymptotic expansion,

Theorem 1.4:

$$\frac{\sum_{m+n=d} N_{0,(m,n)}(\mathbb{P}^1 \times \mathbb{P}^1, diagonal))}{d!} \sim e^d(\frac{\sqrt{2}}{\sqrt{\pi}} d^{-\frac{3}{2}} + \sum_{k\geq 2} a_k d^{-k-\frac{1}{2}}),$$

where a_k are computable constants.

Especially, we have the following theorem,

Theorem 1.5:

$$\frac{\bar{N}_{0,d}(\mathbb{P}^2, conic)}{(d-1)!} \sim (2e)^d(\frac{\sqrt{2}}{\sqrt{\pi}}d^{-\frac{3}{2}} + \sum_{k\geq 2} a_k d^{-k-\frac{1}{2}}).$$

We notice the generating functions of genus 0, relative Gromov-Witten invariants of the relative pair $(\mathbb{P}^2, conic)$ and $(\mathbb{P}^1 \times \mathbb{P}^1, diagonal)$ are similar. Moreover, the recursion formulas of these different relative pairs are similar. I guess there may exist a proper morphism preserving relative pairs between $(\mathbb{P}^2, conic)$ and $(\mathbb{P}^1 \times \mathbb{P}^1, diagonal)$ to explain it geometrically.

Although we consider only genus 0 Gromov-Witten invariants in the whole thesis, via the work of Bousseau and Wu[19], we could compute higher genus relative Gromov-Witten invariants.

For general Hirzebruch surfaces, we need to solve an exact solution of a 2nd order PDE with an essential difference to the H_0 case. Nevertheless, from the information from the quiver side, we could derive inequalities for those invariants, presented in Chapter 5.

In general surfaces, that, with two non-symmetric curve classes in homology classes, the work of Fan and Wu[14] introduces two recursion formulas. We compare the recursion formulas from the different choice of hypersurface divisors by an example and illustrate they are nonequivalent in Chapter 5.

CHAPTER 2 PRELIMINARY

2.1 Gromov–Witten invariants

Definition 2.1: An n-pointed, genus g, complex, quasi-stable curve

$$(C, p_1, ..., p_n),$$

is a projective, connected, reduced, (at worst) nodal curve of arithmetic genus g with n distinct, nonsingular, marked pointed.

An isomorphism η of two n-pointed curves $(C, p_1, ..., p_n)$ and $(C', p_1', ..., p_n')$ if η is an isomorphism of curves C and C' and $\eta(p_i) = p_i'$. We call a quasi-stable curve is stable if its automorphism group is finite.

There is a graphical way to realize n-pointed, genus 0 stable curves.

Definition 2.2: A tree of projective lines is a connected curve with the following properties:

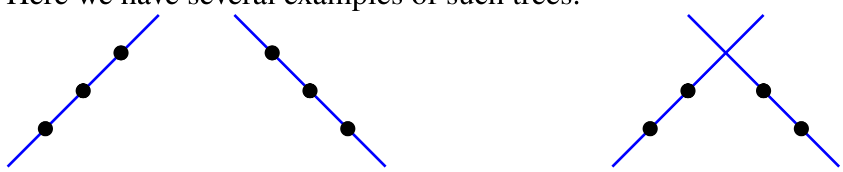
i) Each irreducible component is isomorphic to a projective line.

ii) The points of intersection of the components are ordinary double points.

iii) There are no closed circuits. That is, if a node is removed, the curve becomes disconnected.

A n-pointed stable curve C is a tree of projective lines, with n distinct marked points that are smooth and every lines contains 3 special points, that marks and nodes.

Here we have several examples of such trees.



Before we get into the moduli spaces of stable maps, we take moduli spaces of stable curves as an example. Note $M_{0,n}$ is the moduli space of smooth, genus 0, n-pointed curves.

Example 2.1: $M_{0,3}$ is a point. Simply we take three marked points as $0, 1, \infty$. Then for another stable curve with marks p_1, p_2, p_3. By choice of affine coordinate chart, we suppose $p_i \neq \infty$, $i = 1, 2, 3$ and note $p_i = [x_i : 1]$. The morphism

$$f : x \mapsto \frac{ax + b}{cx + d},$$

where $ax_1 + b = 0$, $cx_3 + d = 0$, $ax_2 + b = cx_2 + d$ is an isomorphism of stable curves.

Example 2.2: $M_{0,4}$ is slightly different from $M_{0,3}$. Assume we have a 4-tuple of distinct points (p_1, p_2, p_3, p_4) in $\mathbb{P}^1$. Simply we would assume none of them is ∞. A morphism f of $\mathbb{P}^1$ such that $f(p_1) = 0, f(p_2) = 1, f(p_3) = \infty$ is uniquely determined. We find $f(p_4)$ suitably parameterizes the equivalent classes of smooth, genus 0, 4-pointed curves.

Remark 2.1: We notice that $M_{0,4} \cong \mathbb{P}^1 \backslash \{0, 1, \infty\}$, that it is not closed. A proper compactification is $\mathbb{P}^1$ that the curves correspondence to $0, 1, \infty$ are nodal curves. Geometrically, we could say, if two marks coincide at one point, then we 'blow up' this point.

Definition 2.3: A family of maps over S from n-pointed, genus g curves to X consists of the data $(\pi : \mathcal{C} \to S, \{p_i\}_{1 \le i \le n}, \mu : \mathcal{C} \to X)$ that

i) A family of n-pointed, genus g, quasi-stable curves $\pi : \mathcal{C} \to S$ with n sections p_i and each geometric fiber is an n-pointed, genus g, quasi-stable curve;

ii) A morphism $\mu : \mathcal{C} \to X$.

Let $(C, p_1, ..., p_n, \mu)$ be a map from a n-pointed, genus g, quasi-stable curve to X. The special points of an irreducible component are the marked points and the component intersections of the other components. The map is stable if for every component $E \subset C$

i) If $E \cong \mathbb{P}^1$ and E is mapped to a point by μ, then E must contain at least three special points;

ii) If E has arithmetic genus 1 and E is mapped to a point by μ, then E must contain at least one special points.

A family is stable if the pointed map on each fiber is stable.

Let X be a scheme over $\mathbb{C}$ and C a pointed curve, $\beta \in H_2(X)$, a map $f : C \to X$ represents β if $f_*([C]) = \beta$. We could define a contravariant functor $\bar{\mathcal{M}}_{g,n}(X, \beta)$ from the category of complex algebraic schemes to sets that $\bar{\mathcal{M}}_{g,n}(X, \beta)(S)$ is the set of isomorphism classes of stable families over S of maps from n-pointed, genus g curves to X representing the class β. We have

Theorem 2.1: There exists a projective, coarse moduli space $\bar{M}_{g,n}(X, \beta)$ defined by the following natural transformation of functors

$$\phi : \bar{\mathcal{M}}_{g,n}(X, \beta) \to Hom_{sch}(*, \bar{M}_{g,n}(X, \beta)),$$

satisfying

i) $\phi(Spec(\mathbb{C}))$ is a bijection;

ii) If Z is a scheme and $\psi : \phi : \bar{\mathcal{M}}_{g,n}(X, \beta) \to Hom_{sch}(*, Z)$, then there exists a unique morphism of schemes $\gamma : \bar{M}_{g,n}(X, \beta) \to Z$ such that $\psi = \gamma\phi$.

Especially we would consider the moduli space $M_{g,n}$ parameterized the n-pointed, genus g, complex, quasi-stable curves. In general, the moduli space is not smooth.

2.1.1 Virtual cycles

If the fundamental homology class of the moduli space exists with proper dimension, then Gromov-Witten classes defined by some axioms stated in [K,M][20] works. However, since general moduli spaces are usually neither smooth nor of proper dimension, the fundamental classes are not well-defined or do not work well. Virtual classes are the analogy of fundamental classes. Such cycles were constructed by Li and Tian[4] [5], Behrend and Fantechi[7].

Theorem 2.2: For any smooth projective variety variety X, and any choice of integers n and g and $\alpha \in A_1(X)/\sim_{alg}$, there is a virtual moduli cycle $[\mathcal{M}^X_{\alpha,g,n}]^{vir} \in A_k X \otimes_{\mathbb{Z}} \mathbb{Q}$, where k is the virtual dimension of $\mathcal{M}^X_{\alpha,g,n}$.

2.1.2 Gromov-Witten invariants

The variety $\bar{M}_{0,n}(X,\beta)$ with n morphisms ρ_i to X, where ρ_i takes $[(C, p_1, ..., p_n, \mu)]$ to $\mu(p_i)$, for given classes $\gamma_1, ..., \gamma_n$ in $H^{2*}(X)$, holds the cohomology classes $\rho_1^*(\gamma_1) \cup ... \rho_n^*(\gamma_n)$. We would like to consider if the classes are homogeneous and $\sum \frac{deg\gamma_i}{2} = dimX + (-K_X \cdot \beta) + n - 3$, where K_X denote the canonical divisor of X. The number

$$I_\beta(\gamma_1, \dots, \gamma_n) = \int_{[\bar{M}_{0,n}(X,\beta)]^{vir}} \rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n),$$

is the Gromov-Witten invariant.

If we restrict to consider X is a homogeneous variety, we have the following

i) If $\beta = 0$, $\bar{M}_{0,n}(X,\beta) = \bar{M}_{0,n} \times X$ with ρ_i are all equal to the projection p onto the second factor.

$$\begin{aligned} I_\beta(\gamma_1, \dots, \gamma_n) &= \int_{\bar{M}_{0,n}(X,\beta)} \rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n) \\ &= \int_{p_*[\bar{M}_{0,n} \times X]} \gamma_1 \cup \cdots \cup \gamma_n. \end{aligned}$$

The projection has positive relative dimension, which tells $p_*[\bar{M}_{0,n} \times X] = 0$ by definition except $n = 3$.

ii) If $\gamma_1 = 1 \in A^0(X)$ and $\beta \neq 0$, $I_\beta(\gamma_1, \dots, \gamma_n)$ vanishes.

iii) If $\gamma_1 \in A^1 X$ and $\beta \neq 0$, then

$$I_\beta(\gamma_1, \dots, \gamma_n) = (\int_\beta \gamma_1) \cdot I_\beta(\gamma_2, \dots, \gamma_n).$$

2.1.3 Quantum product

Simply we would only deal with $X = \mathbb{P}^r$ with basis $\{h^0, h^1, \dots, h^r\} \in A^*(\mathbb{P}^r)$ as h^0 is the fundamental class, h^1 is the hyperplane class. And we would note

$$I(\gamma_1, \dots, \gamma_n) = \sum_{\beta} I_\beta(\gamma_1, \dots, \gamma_n).$$

Particularly, we consider the following $\gamma_1, \dots, \gamma_n$ of the form $(h^0)^{a_0}(h^1)^{a_1} \dots (h^r)a^r$, that there are a_i copies of h^i, $i = 1, \dots, r$.

With formal variables $\mathbf{x} = (x_0, \dots, x_r)$, we define the following generating function

$$\Phi(x_0, \dots, x_r) := \sum_{a_0, \dots, a_r} \frac{x_0^{a_0} \cdots x_r^{a_r}}{a_0! \cdots a_r!} I((h^0)^{a_0}(h^1)^{a_1} \dots (h^r)a^r).$$

If we introduce the notation $\mathbf{x}^{\mathbf{a}} = (x_0^{a_0} \cdots x_r^{a_r})$, $\mathbf{a}! = a_0! \cdots a_r!$ and $\mathbf{h}^{\mathbf{a}} = (h^0)^{a_0}(h^1)^{a_1} \dots (h^r)a^r$, then

$$\Phi(x) = \sum_{\mathbf{a}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} I(\mathbf{h}^{\mathbf{a}}).$$

Formally, we have

$$\Phi_i := \frac{\partial}{\partial x_i}\Phi(x) = \sum_{\mathbf{a}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} I(\mathbf{h}^{\mathbf{a}}, h^i),$$

and

$$\Phi_{ijk} := \frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\Phi(x) = \sum_{\mathbf{a}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} I(\mathbf{h}^{\mathbf{a}}, h^i, h^j, h^k).$$

Definition 2.4: The quantum product $*$ is defined by

$$h^i * h^j := \sum_{e+f=r} \Phi_{ije} h^f.$$

Now we could state the significant theorem,

Theorem 2.3: The quantum product is associative.

From the theorem above, we get

$$\sum_{e+f=r} \Phi_{ije}\Phi_{fkl} = \sum_{e+f=r} \Phi_{ike}\Phi_{fil}.$$

for every i, j, k, l, which is the famous WDVV equations.

2.1.4 Relative Gromov-Witten invariants

Relative Gromov-Witten invariants counts the stable maps with some tangency condition. Let the divisor D be rational and smooth, we associate a tuple Γ, called topological type, (g, n, β, ρ, μ), where g is the genus, n are marked points, $\beta \in H_2(X, \mathbb{Z})$, $\mu \in \mathbb{Z}^\rho$ is a partition of $D \cdot \beta$ that encodes the contact orders. The issue comes when we consider the compactification of the moduli space. The method is to construct a geometry object ζ of the relative pair.

ζ is a geometry object constructed from (X, D) via blowing up, called expanded pairs. The motivation is when we take the compactification of the moduli spaces of smooth marked curves of given topological type Γ, the points of boundary may represent the curves that an irreducible component entirely maps to D. It causes trouble to deal with contact order.

We let $X[1]$ be the blow up of $X \times \mathbb{A}^1$ along $D \times 0$ and $D[1]$ be the proper transformation of $D \times \mathbb{A}^1$, viewing $(X[1], D[1])$ as a family over $\mathbb{A}^1$. The relative pair $(X[n], D[n])$ can be contracted inductively from blowing $X[n-1] \times \mathbb{A}^1$ along $D[n-1] \times 0$. And the central fiber of $X[n]$, noted as $X[n]_0 = X \cup_D (\Delta \cup \cdots \cup \Delta)$, with n copies of a ruled variety Δ attached to X along D. We notice that if irreducible component of a map entirely lies in $(\Delta \cup \cdots \cup \Delta)$, by composition with blow down $X \cup_D (\Delta \cup \cdots \cup \Delta) \to X$, it entirely lies in D. We would take the compactification of the moduli space of smooth relative maps of type Γ as the relative maps to $(X[n], D[n])$ of some n up to an equivalence. ζ is the direct limit of the system of $(X[n], D[n])$ quotient by a group action.

Li[21] had proven the following theorem

Theorem 2.4: The moduli functor $\mathcal{M}(\zeta, \Gamma)$ of relative stable maps to ζ of topological type Γ is separated and proper Deligne-Mumford stack over $\mathbb{C}$.

This functor gives moduli spaces of relative stable maps. And in the later work[22], Li proved that the moduli spaces admits a virtual class, deriving a degeneration formula of relative Gromov-Witten invariants.

For a topological type $\Gamma = (0, n, \beta, \rho, \mu)$, we know $\mathcal{M}(\zeta, \Gamma)$ has n (resp. ρ) morphisms to X (resp. D), that if $[f]$ is a point of $\mathcal{M}(\zeta, \Gamma)$, $ev_{X,i}([f])$ (resp. $ev_{D,j}([f])$) takes the i-th marked point (resp. j-th contact point), for given classes $\alpha_1, ..., \alpha_n \in H^*(X)$ and $\epsilon_1, ..., \epsilon_\rho \in H^*(D)$, we define relative Gromov-Witten invariants as

$$I(\alpha_1, ..., \alpha_n, \epsilon_1, ..., \epsilon_\rho)_{(X,D)} = \int_{\mathcal{M}(\zeta,\Gamma)} (\cup_i ev^*_{X,i}(\alpha_i)) \cup (\cup_j ev^*_{D,j}(\epsilon_j)).$$

In this thesis, we only consider $g = 0$.

We would note the total space X and the divisor D as (X, D) and consider $N_{g,\beta}(X, D)$, the count of stable maps of genus g, type β with contact order $D \cdot \beta$ to D at an unspecific point.

2.2 Hirzebruch surfaces

Hirzebruch surfaces consist the a family of ruled surfaces. In this thesis, we need some basic properties about them.

Definition 2.5: The Hirzebruch surface H_k is the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ associated to the sheaf

$$\mathcal{O} \oplus \mathcal{O}(-k).$$

For example, H_0 is $\mathbb{P}^1 \times \mathbb{P}^1$. There is a natural map from $\mathbb{P}^1$ to H_k as the zero section.

The Picard group of H_k is $\mathbb{Z} \oplus \mathbb{Z}$, as the first term represents the divisor class of the zero section and the second one is the fiber class, with the self-intersection numbers -k and 0. The intersection matrix is

$$\begin{pmatrix} -k & 1 \\ 1 & 0 \end{pmatrix},$$

with canonical divisor $K = (-2, -2-k)$.

Remake: With these notations, $(\mathbb{P}^1 \times \mathbb{P}^1, diagonal)$ is $(H_0, (1,1))$ in this thesis.

2.3 Lambda W function

In this thesis we need some properties of the Lambda W function, mainly on the principle branch, to express the exact solution of the PDE we concern and compute the asymptotic expansion. Lambda W function (We call it W if no ambiguous) is a multi-branched function satisfies the following equation

$$W(z)e^{-W(z)} = z.$$

The principle branch of W is an analytic function on $\mathbb{C} - (\frac{1}{e}, \infty)$, with the expansion of series $W = \sum_{k>0} \frac{k^{k-1}}{k!} z^k$ around 0. Usually we would take $W(\frac{1}{e}) = 1$. Here, the branch points of W are $\frac{1}{e}$ and ∞. Especially, we list some important values of W,

$$W(0) = 0, W(\frac{1}{e}) = 1, W(e) = -1.$$

To deduce the differential of W, we directly derivative of the equation$W(z)e^{-W(z)} = z$, the differential of W is

$$\frac{dW}{dz} = \frac{e^W}{1-W} = \frac{W}{z(1-W)}.$$

In addition, the convergence of the expansion $W = \sum_{k>0} \frac{k^{k-1}}{k!} z^k$ is important. By

Stirling's formula

$$n! = \sqrt{2\pi n}(\frac{n}{e})^n e^{\lambda_n},$$

where $\frac{1}{12n+1} < \lambda_n < \frac{1}{12n}$ and by Cauchy condensation test, the convergence radius of the expansion $W = \sum_{k>0} \frac{k^{k-1}}{k!} z^k$ is $r = \frac{1}{e}$.

Remark 2.2: In some references, Lambda function is defined as the inverse function of $f(w) = we^w$, with the expansion of the principle branch around 0 $w(z) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1}}{k!} x^k$. From the expansion we directly know that $W(-z) = -w(z)$.

The n-th derivatives of $w(z)$ is

$$w^{(n)}(z) = \frac{w^{n-1}(z)}{z^n(1+w(z))^{2n-1}} \sum_{k=1}^{n} a_{k,n} w^k(z).$$

The $a_{k,n}$ is a number triangle (OEIS, A042977).

$a_{k,n}$	n=1	2	3	4	5
k=1	1				
2	-2	-1			
3	9	8	2		
4	-64	-79	-36	-6	
5	625	974	622	192	24

2.4 Exact formula of $N_{0,d}(\mathbb{P}^2, line)$ and $N_{0,d}(\mathbb{P}^2, conic)$

We would note the total space X and the rational divisor D as (X, D) and $N_{g,\beta}(X, D)$ (resp. $\bar{N}_{g,\beta}(X, D)$) the count of stable maps of genus g, type β with contact order $D \cdot \beta$ to Y at an unspecific (resp. specific) point[19]. And we have the following 2 theorems[14],

Theorem 2.5: If $T_{log} \cdot \beta > 0$, then

$$N_{g,\beta}(X, D) = (D \cdot \beta)\bar{N}_{g,\beta}(X, D).$$

Theorem 2.6: For any smooth projective surface X, smooth ample rational divisor D, and curve class β such that $T_{log} \cdot \beta \geq 3$, we have

$$\frac{N_{0,\beta}(X, D)}{D \cdot \beta} = \sum_{\beta_1+\beta_2=\beta, \beta_1, \beta_2>0} (D \cdot \beta_1)^2 \binom{T_{log} \cdot \beta - 3}{T_{log} \cdot \beta_1 - 1} \frac{N_{0,\beta_1}(X, D)}{D \cdot \beta_1} \frac{N_{0,\beta_2}(X, D)}{D \cdot \beta_2},$$

where $T_{log} = -(K_X + D)$ and $\binom{a}{b} = 0$ if $a < b$ or $b < 0$.

The recursion gives a ODE at case $(\mathbb{P}^2, line)$,

$$(q\frac{d}{dq}-2)(q\frac{d}{dq}-1)F^L = \frac{(q\frac{d}{dq})^2F}{4}\cdot(q\frac{d}{dq}-1)F^L,$$

where $F^L = \sum \frac{\bar{N}_{0,d}(\mathbb{P}^2,line)}{(2d-1)!}q^{2d}$.

The solution is implicitly given by[14]

$$A^L e^{W(\frac{A^L}{2i})+W(\frac{A^L}{-2i})} = q,$$

where $A^L = (\frac{d}{dq}-\frac{1}{q})F^L$.

And by Lagrangian Inversion Theorem,

$$\bar{N}_{0,d+1}(\mathbb{P}^2, line) = \frac{(2d)!}{2d+1}\sum_{s=1}^{d}\sum_{\sum_{i=1}^{s} a_i, a_i>0}\frac{(-1)^{d-s}(2d+1)^s}{s!}\prod_{i=1}^{s}\frac{a_i^{a_i-1}}{(2a_i)!}.$$

Similar discuss works for $(\mathbb{P}^2, conic)$ with $F^C = \sum \frac{16\bar{N}_{0,d}(\mathbb{P}^2,conic)}{(d-1)!}q^d$,

$$\bar{N}_{0,d+2}(\mathbb{P}^2, conic) = \frac{d!}{d+1}\sum_{s=1}^{d}\sum_{\sum_{i=1}^{s} a_i, a_i>0}\frac{(-1)^{d-s}2^{d+s}(d+1)^s}{s!}\prod_{i=1}^{s}\frac{a_i^{a_i-1}}{a_i!}.$$

CHAPTER 3 EXPLICIT FORMULA FOR $(\mathbb{P}^1 \times \mathbb{P}^1, diagonal)$

We have such a recursion formula
$$\frac{N_{0,\beta}(X,D)}{D\cdot\beta} = \sum_{\beta_1+\beta_2=\beta}(D\cdot\beta_1)^2\binom{T_{log}\cdot\beta-3}{T_{log}\cdot\beta_1-1}\frac{N_{0,\beta_1}(X,D)}{D\cdot\beta_1}\frac{N_{0,\beta_2}(X,D)}{D\cdot\beta_2}.$$
We would note
$$N_\beta = \frac{N_{0,\beta}(X,D)}{(D\cdot\beta)(T_{log}\cdot\beta-1)!},$$
if no ambiguous with the relative pair. There is a simplified recursion formula
$$(T_{log}\cdot\beta-2)(T_{log}\cdot\beta-1)N_\beta = \sum_{\beta_1+\beta_2=\beta}(D\cdot\beta_1)^2(T_{log}\cdot\beta_2-1)N_{\beta_1}N_{\beta_2}.$$

In this chapter, we choose $D = (1,1)$ which means $K = -2D$. By the intersection matrix of H_0, set $\beta = (m,n)$, we have the following
$$D\cdot\beta = m+n,$$
$$T_{log}\cdot\beta = m+n.$$

We define the generating function
$$F = \sum N_{m,n}x^m y^n,$$
and one operators,
$$G = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$
Deduced by the recursion formula, the following PDE holds
$$(G-2)(G-1)F = G^2F\cdot(G-1)F.$$

Notice this PDE seems like one ODE. By observation, if we take $y = sx$ for some s, and set $F_s(q) = F(q,sq)$, then $GF_s = q\frac{dF_s}{dq}$. Here we have $F_s(q) = \sum_d q^d \sum_{m+n=d} N_{m,n}s^n$. And F_s meets the following ODE
$$(q\frac{d}{dq}-2)(q\frac{d}{dq}-1)F_s = (q\frac{d}{dq})^2F_s\cdot(q\frac{d}{dq}-1)F_s,$$
here we define $A = (\frac{d}{dq}-\frac{1}{q})F_s$ and $B = q\frac{d}{dq}F_s$.

Combine

$$A\frac{dB}{dq} = q\frac{dA}{dq} - A,$$
$$B = q\frac{d}{dq}(B - qA).$$

Notice the first equation is from the ODE and the second equation is from the definition of A and B,

We get

$$A = Mq \exp B,$$
$$A\frac{dB}{dA} = A\frac{\frac{dB}{dq}}{\frac{dA}{dq}} = \frac{B + 2qA}{1 + B + qA}.$$

where M is a constant determined by the initial date.

We would note $B = f(A)$. The above system of equations concludes

$$A\frac{df}{dA} = \frac{f + \frac{2A^2}{M}e^{-f}}{1 + f + \frac{A^2}{M}e^{-f}}.$$

The differential equation appeared in the work of Fan and Wu and we have the following lemma[14]

Lemma 3.1: Let

$$f(x) = \sum_{k=0}^{\infty} a_k x^k,$$

be a formal series, and

$$x\frac{d}{dx}f = \frac{f + \frac{x^2}{2}e^{-f}}{1 + f + \frac{x^2}{4}e^{-f}}.$$

Then

$$f = -W(c_1 x) - W(c_2 x)$$

where $W(x) = \sum_{k\geq 1} \frac{k^{k-1}}{k!}x^k$, the Lambert W function, and $c_1 c_2 = \frac{1}{4}$.

We note $x = \frac{2A}{\sqrt{M}}$. The initial data comes from[19]

$$N_{0,1} = N_{1,0} = 1,\ N_{1,1} = 1,\ N_{0,2} = N_{2,0} = 0.$$

We get $M = s$. By comparing the coefficients, we get

$$c_1 + c_2 = -\frac{1}{2}(\sqrt{s} + \frac{1}{\sqrt{s}}).$$

Thus

$$f(A) = -W(-A) - W(-\frac{1}{s}A),$$

and

$$A = sqe^{-W(-A)-W(-\frac{1}{s}A)}.$$

Convention: For a power series $f = \sum_{k=0}^{\infty} a_k x^k$, we would use the notation $a_k = [x^k]f$, or say we take the k-th coefficient of the power series.

The definition of A gives $A = \sum_d (d-1)q^{d-1} \sum_{m+n=d} N_{m,n}s^n$. By Lagrange Inversion,

$$[q^n]A = \frac{1}{n}[x^{n-1}]s^n e^{n(-W(-x)-W(-\frac{1}{s}x))}.$$

We conclude

$$[q^n]A = s^n \sum_{j=1}^{n-1} \frac{n^{j-1}(-1)^j}{j!} \sum_{\substack{a_1+\cdots+a_j=n-1 \\ a_i \geq 1}} \prod_{i=1}^{j} \frac{a_i^{a_i-1}}{a_i!}((-1)^{a_i} + (-\frac{1}{s})^{a_i}). \tag{3-1}$$

We will note $A(s,k) = [q^k]A$.

By comparing the both sides as polynomials of s, it directly gives the number,

$$(d-1)N_{m,n} = \sum_{j=1}^{d-2} \frac{(d-1)^{j-1}(-1)^j}{j!} \sum_{\substack{a_1+\cdots+a_j=d-2 \\ a_i \geq 1}} (-1)^{d-2} \prod_{i=1}^{j} \frac{a_i^{a_i-1}}{a_i!} \sum_{\substack{\sum_K a_k=d-1-n \\ K\subset[j]}} 1. \tag{3-2}$$

From the view of geometry, we have $N_{m,n} = N_{n,m}$. It is sufficient to check that $A(s,d-1) = s^d A(\frac{1}{s}, d-1)$ since $(d-1)\sum_{m+n=d} N_{m,n}s^n = (d-1)\sum_{m+n=d} N_{m,n}s^{d-m}$.

Direct computation of Matlab presents the following results,

m+n=	$\sum_{m+n} N_{m,n}s^n$
3	$s + s^2$
4	$s + 6s^2 + s^3$
5	$s + 24s^2 + 24s^3 + s^4$
6	$s + 80s^2 + 270s^3 + 80s^4 + s^5$
7	$s + 240s^2 + 2160s^3 + 2160s^4 + 240s^5 + s^6$

However, if we consider the case of $k > 0$, note $F = \sum_\beta N_\beta p^{T_{log}\cdot\beta} q^{D\cdot\beta}$, the recursion formula presents,

$$(p\frac{\partial}{\partial p} - 2)(p\frac{\partial}{\partial p} - 1)F = (q\frac{\partial}{\partial q})^2 F \cdot (p\frac{\partial}{\partial p} - 1)F.$$

This kind of PDE is non-linear and the method used in H_0 does not work on these cases.

CHAPTER 4 ASYMPTOTIC

4.1 Singularity and exponential growth

4.1.1 $(\mathbb{P}^2, conic)$ and $(\mathbb{P}^1 \times \mathbb{P}^1, diagonal)$

There are lots of works done to discuss the asymptotic expansion of N_d, the Gromov-Witten invariants of $\mathbb{P}^2$. Inspired by those works, we are interested in asymptotic expansions of relative Gromov-Witten invariants. We would firstly give a rough control of the exponential growth.

As section 3.1, we are have the following recursion formula of H_0

$$[q^d]F_s = \sum_{m+n=d} \frac{1}{2}(\frac{m^2(n-1)}{(d-1)(d-2)} + \frac{n^2(m-1)}{(d-1)(d-2)})[q^m]F_s \cdot [q^n]F_s,$$

for $d \geq 3$.

By roughly enlarging and shrinking, we have

$$\frac{1}{2}(\frac{m^2(n-1)}{(d-1)(d-2)} + \frac{n^2(m-1)}{(d-1)(d-2)}) \leq \frac{mnd}{2(d-2)^2} \leq \frac{2m \cdot 2n}{2d},$$

for $d > 2$. For the sequence $\{b_k\}$ of number satisfying

$$b_d = \sum_{m+n=d} \frac{f(m)f(n)}{f(d)} b_m b_n,$$

for some $f : \mathbb{N}^+ \to \mathbb{R}$, then

$$b_d = \frac{(2d-2)!}{d!(d-1)!}(f(1)b_1)^d.$$

Here we let $b_1 = \max\{|1+s|, \sqrt{|s|}\}$, $f(d) = 2d$, then we have $b_1 \geq |[q^1]F_s|$ and $b_2 \geq |[q^2]F_s|$. By induction we get

$$|[q^d]F_s| \leq \frac{(2d-2)!}{d!(d-1)!}(2b_1)^d.$$

Here we cite the Stirling's Formula again

$$n! = \sqrt{2\pi n}(\frac{n}{e})^n e^{\lambda_n},$$

where $\frac{1}{12n+1} < \lambda_n < \frac{1}{12n}$.

Therefore we get $[q^d]F_s < C2^{3d}(b_1)^d d^{-\frac{3}{2}}$.

Similarly, for $d \geq 3$, we have the following inequality

$$\frac{1}{2}\left(\frac{m^2(n-1)}{(d-1)(d-2)} + \frac{n^2(m-1)}{(d-1)(d-2)}\right) \geq \frac{1}{2}, \tag{4-1}$$

which would give a upper bound of the convergence radius.

We record the invention that $[q^d]F_s = \frac{1}{d-1}[q^{d-1}A]$. The work on Gromov-Witten invariants of $\mathbb{P}^2$ derives us to guess that

$$\lim_{d\to+\infty} \sqrt[d]{[q^d]A} = e^{x_0(s)},$$

for some $x_0(s) \in \mathbb{R}$. Directly We could find that the convergence radius of F_s and A are the same. From the coefficient we have deduced for A,

$$[q^n]A = s^n \sum_{j=1}^{n-1} \frac{n^{j-1}(-1)^{n-1-j}}{j!} \sum_{\substack{a_1+\cdots+a_j=n-1 \\ a_i\geq 1}} \prod_{i=1}^{j} \frac{a_i^{a_i-1}}{a_i!}(1+(\frac{1}{s})^{a_i}).$$

We replace the $\frac{a_i^{a_i-1}}{a_i!}$ by $\frac{e^{a_i}}{a_i e^{\lambda_{a_i}}\sqrt{2\pi a_i}}$, we get

$$[q^n]A = s^n e^{n-1} n^{n-2} \sum_{j=1}^{n-1} \frac{n^{j-1}(-1)^{n-1-j}}{j!} \sum_{\substack{a_1+\cdots+a_j=n-1 \\ a_i\geq 1}} \prod_{i=1}^{j} \frac{1}{a_i e^{\lambda_{a_i}}\sqrt{2\pi a_i}}(1+(\frac{1}{s})^{a_i}).$$

Now we need to consider the part

$$\sum_{\substack{a_1+\cdots+a_j=n-1 \\ a_i\geq 1}} \prod_{i=1}^{j} \frac{1}{a_i e^{\lambda_{a_i}}\sqrt{2\pi a_i}}(1+(\frac{1}{s})^{a_i}).$$

Here notice if $s \geq 1$, then $\frac{1}{a_i e^{\lambda_{a_i}}\sqrt{2\pi a_i}}(1+(\frac{1}{s})^{a_i}) \leq 1$, hence we are to count the sum

$$\sum_{\substack{a_1+\cdots+a_j=n-1 \\ a_i\geq 1}} 1,$$

which counts the lattice points $(a_i)_1^j(a_i > 0)$ such that $\sum a_i = n-1$. It is sufficient to count the lattice points $(a_i)_1^{j-1}, a_i > 0$ such that $\sum a_i \leq n-2$. Directly, for a lattice point $(a_i)_1^{j-1}, a_i > 0$ we can define the cube $\{(x_i)_1^{j-1}, |x_i - a_i| \leq \frac{1}{2}\}$ and the cube is volume 1. All cubes are almost distinct and the union is contained in $\{(x_i)_1^{j-1}, \sum x_i \leq n-1\}$, which has volume $\frac{(n-1)^{j-1}}{(j-1)!}$.

Hence

$$[q^n]A \leq s^n e^{n-1} \sum_{j=1}^{n-1} \frac{n^{j-1}(n-1)^{j-1}}{j!(j-1)!}.$$

We would take $c_{j-1} = \frac{n^{j-1}(n-1)^{j-1}}{j!(j-1)!}$, it follows that $\frac{c_j}{c_{j-1}} = \frac{n(n-1)}{(j+1)j}$. Since $j = 1, ..., n-1$, $\frac{c_j}{c_{j-1}} = \frac{n(n-1)}{(j+1)j} \leq 1$.

$$s^n e^{n-1} \sum_{j=1}^{n-1} \frac{n^{j-1}(n-1)^{j-1}}{j!(j-1)!} \leq \frac{e^{2n-3}(\frac{1}{n-1}+1)^{n-1}(\frac{1}{n-2}+1)^{n-2}}{2\pi\sqrt{(n-1)(n-2)}e^{\lambda_{n-1}}e^{\lambda_{n-2}}}.$$

From this inequality, we would get the following

$$\lim_{d\to\infty} \frac{\sqrt[d]{[q^d]A}}{se^3} \leq 1.$$

We would like to treat A and W as complex functions, where W is the principal branch of $We^{-W} = z$. And the inequality above presents that A is holomorphic in a small disk around 0. Then we could use some results from complex analysis to deduce the radius of convergence of the power series of A centered at 0.

Lemma 4.1: An analytic function has at least one singularity on its circle of convergence.

It has some alternative statements, as the radius of convergence is the distance from the center to the nearest singularity. Hence we should find the singularity of A. And we need the following lemma,

Lemma 4.2 (Inverse function theorem of holomorphic function): Let f be a holomorphic function on U and $p \in U$ such that $f'(p) \neq 0$, then there exists a neighborhood $v \subset U$ that $f_{|V}$ is biholomorphic.

Theorem 4.1 (Vivanti–Pringsheim theorem): A complex function defined by a power series with non-negative real coefficients and convergence radius R, has a singularity at $z = R$.

For simpler discuss, we would take $s = 1$, and have

$$Ae^{2W(-A)} = q.$$

We know $W(z)e^{-W(z)} = z$ and

$$\frac{dW}{dz}e^{-W} - We^{-W}\frac{dW}{dz} = 1,$$

hence

$$\frac{dW}{dz} = \frac{e^W}{1-W},$$

expect $z = \frac{1}{e}$. Similarly, we have

$$\frac{dA}{dq} e^{2W(-A)} \frac{1+W(-A)}{1-W(-A)} = 1,$$

By inverse function theorem of holomorphic function, for every A such that $-A \notin [\frac{1}{e}, \infty)$ and $W(-A) \neq 1$, then there exists local inverse holomorphic $A(q)$.

Here we have if $W(-A) = 1$ then $A = -\frac{1}{e}$ and $q = -e$; if $W(-A) = -1$, then $A = e$ and $q = \frac{1}{e}$. Now we need to prove that A defined by power series could be extend to radius $\frac{1}{e}$. We are supposed to check the following assertion,

Theorem 4.2: There exists holomorphic A on the open disk with radius $\frac{1}{e}$ centred at 0 and $A(0) = 0$.

Proof: Firstly, we should check the transcendental equation $Ae^{2W(-A)} = q$ is solvable if $|q| < \frac{1}{e}$. Due to the property of W, there are the following equation,

$$W(-A)e^{-W(-A)} = -A,$$
$$W(-A)e^{W(-A)} = -q.$$

Replace $W(-A)$ as ω, we notice that if $\omega e^{\omega} = -q$ is solvable for given q, then the original equation admits a solution. Explicitly, the solution of $\omega e^{\omega} = -q$ with $|q| < \frac{1}{e}$ is uniquely $\omega = w(-q) = -W(q)$.

We notice that $W(-A) = -W(q)$ since we choose the principal branch of W function, and directly get $A = W(q)e^{W(q)}$. ■

Corollary 4.1: If $s = 1$, then $\lim_{d \to \infty} \sqrt[d]{[q^d]A} = e$.

The above case has an advantage that for $|q| < \frac{1}{e}$, $A(q) \neq -\frac{1}{e}$, which takes the branch point of the principle branch out of our discuss.

Example 4.1: By Fan and Wu[14], the relative invariants of pair $(\mathbb{P}^2, conic)$ is determined. It defines the function

$$A^c = \sum_{1}^{\infty} (d-1) \frac{16 N_{0,d}(\mathbb{P}^2, conic)}{(d-1)!2d} q^d,$$

and get the following

$$A^c e^{2W(-\frac{A^C}{8})} = 16q.$$

Here by derivative,

$$\frac{dA}{dq} e^{2W(-\frac{A^C}{8})} \frac{1+W(-\frac{A^C}{8})}{1-W(-\frac{A^C}{8})} = 16q.$$

We have if $W(-\frac{A^C}{8}) = -1$, then $A^C = 8e$ and $q = \frac{1}{2e}$. Hence $\lim_{d\to\infty} \sqrt[d]{(d-1)\frac{16N_{0,d}(\mathbb{P}^2,conic)}{(d-1)!2d}} = 2e$. As the case above, we would claim that there exists a holomorphic A^c on the open disk with radius $\frac{1}{2e}$ centred at 0 and $A(0) = 0$. Firstly, we should check that the transcendental equation $A^c e^{2W(-\frac{A^c}{8})} = 16q$ is solvable if $|q| < \frac{1}{2e}$. Due to the properties of W, we could deduce the following system,

$$W(-\frac{A^c}{8})e^{-W(-\frac{A^c}{8})} = \frac{-A^c}{8},$$
$$W(-\frac{A^c}{8})e^{W(-\frac{A^c}{8})} = -2q.$$

Notice that if $\omega e^{\omega} = -2q$ admits a solution, then the original equation admits a solution. Explicitly, the solution of $\omega e^{\omega} = -2q$ with $|q| < \frac{1}{2e}$ is $\omega = w(-2q) = -W(2q)$. Similarly, we know that $A^C = 8W(2q)e^{W(2q)}$. Therefore the proof is done.

Computation by Matlab shows that when $d = 6858$, $\sqrt[d]{[q^d]F^c} = 5.42000302$.

d	$\sqrt[d]{[q^d]A}$	d	$\sqrt[d]{[q^d]F^c}$
100	2.579275010344777	100	4.901859498847964
200	2.634377590369906	200	5.117769221538866
300	2.656748136057127	300	5.204288917049157
400	2.669157565936843	1000	5.349687842651101
600	2.682742253579137	2000	5.388280315881144
800	2.690144771152465	3000	5.40250120618875
1000	2.694853361954298	4000	5.410023951314717
1200	2.69813303807613	5000	5.414717348598933
1400	2.700558760292524	6000	5.417940694563656

4.1.2 $(\mathbb{P}^2, line)$

By the cases of H_0 and $(\mathbb{P}^2, conic)$, we would guess that there exists similar discuss working for the case $(\mathbb{P}^2, line)$.

Recall the function of $(\mathbb{P}^2, line)$ is

$$F^L = \sum \frac{\bar{N}_{0,d}((\mathbb{P}^2, line))}{(2d-1)!} q^{2d},$$

with the solution

$$A^L e^{W(-\frac{A^L}{2i}) + W(\frac{A^L}{2i})} = q,$$

where $A^L = (\frac{d}{dq} - \frac{1}{q})F^L$.

We compute the convergence radius of F^L and A^L by computing $\sqrt[2d]{\frac{\bar{N}_{0,d}((\mathbb{P}^2,line))}{(2d-1)!}}$,

d	$\sqrt[2d]{\frac{\bar{N}_{0,d}((\mathbb{P}^2,line))}{(2d-1)!}}$
600	0.967697520703326
1000	0.9722386183798555
2000	0.9760145405494061
16000	0.9799140280241565
18000	0.979984915713517

Meanwhile, we consider the following

$$\frac{d}{dq}A^L e^{W(-\frac{A^L}{2i})+W(\frac{A^L}{2i})} = \frac{1-W(-\frac{A^L}{2i})W(\frac{A^L}{2i})}{(1+W(-\frac{A^L}{2i}))(1+W(\frac{A^L}{2i}))} e^{W(-\frac{A^L}{2i})+W(\frac{A^L}{2i})}\frac{dA^L}{dq} = 1.$$

It forces that if there exists q_0 such that $1-W(-\frac{A^L(q_0)}{2i})W(\frac{A^L(q_0)}{2i}) = 0$, q_0 must be a singularity of A^L.

Combine the following

$$\begin{aligned} A^L e^{W(-\frac{A^L}{2i})+W(\frac{A^L}{2i})} &= q, \\ W(-\frac{A^L}{2i})e^{-W(-\frac{A^L}{2i})} &= -\frac{A^L}{2i}, \\ W(\frac{A^L}{2i})e^{-W(\frac{A^L}{2i})} &= \frac{A^L}{2i}, \\ 1-W(-\frac{A^L}{2i})W(\frac{A^L}{2i}) &= 0. \end{aligned} \tag{4-2}$$

We notice that the conjugate of $W(z)$ satisfies

$$\overline{W(z)}e^{-\overline{W(z)}} = \bar{z},$$

which tells, $\overline{W(z)} = W(\bar{z})$. Thus if A^L is real, then $W(-\frac{A^L}{2i})W(\frac{A^L}{2i})$ and $W(-\frac{A^L}{2i}) + W(\frac{A^L}{2i})$ are real numbers.

We get $A^L q = 4$, and note the minimal positive A^L such that $1-W(-\frac{A^L}{2i})W(\frac{A^L}{2i}) = 0$ as R. And in the closed disk $\bar{B}_R(0)$, there exists only finitely many zeros of $1 - W(-\frac{A^L}{2i})W(\frac{A^L}{2i})$.

Firstly we claim that the power series $\sum \frac{\bar{N}_{0,d}((\mathbb{P}^2,line))}{(2d-1)!}q^{2d}$ absolutely converges in a small disk since A^L is a transformation of the $F_s|_{s=-1}$ that we could apply the discussion of F_s. Here we note the convergence radius of A^L as r.

We know A^L is defined by a power series of p with all non-negative coefficients around 0. Recall theorem4.1, one of singularities closest to 0 of A^L of is $q = r$.

Let ϵ be an arbitrary positive number and small enough that A^L is well-defined in $(-2\epsilon, 2\epsilon)$. And we are able to find a positive small η such that $\{a + bi, a \in (A^L(\frac{\epsilon}{2}), R - \epsilon), |b| < \eta\} \times [\epsilon, M]$ contains no singularity of $\frac{d}{dx}f(x)$.

We consider the solution of the initial value problem below

$$\begin{cases} \frac{dx}{dq} = \frac{(1+W(-\frac{x}{2i}))(1+W(\frac{x}{2i}))}{(1-W(-\frac{x}{2i})W(\frac{x}{2i}))} e^{-W(-\frac{x}{2i})-W(\frac{x}{2i})} = f(x), \\ x(\epsilon) = A^L(\epsilon). \end{cases}$$

Here we cite a theorem from ODE,

Theorem 4.3 (Cauchy-Lipschitz theorem/Picard-Lindelof theorem): Let $U \in X$ be an open set of a Banach space X and $f : U \times [0, T] \to X$ be a continuous function satisfying the Lipschitz condition

$$|f(x_1, t) - f(x_2, t)|_X \leq M|x_1 - x_2|_X \quad \forall (x_1, t), (x_2, t) \in U \times [0, T],$$

the following initial value problem

$$\begin{cases} \frac{dx}{dt} = f(x, t), \\ x(0) = x_0. \end{cases}$$

such that $x_0 \in U$ has a unique solution in $[0, \delta]$ for some positive δ. Particularly, either δ can be taken equal to T or there is a maximal interval $[0, T_0)$ of existence of the solution, characterized by the property that $x(t)$ approaches the boundary of U as $t \to T_0$.

Here we have

$$\frac{d}{dx}\frac{(1 + W(\frac{x}{2i}))(1 + W(-\frac{x}{2i}))}{(1 - W(\frac{x}{2i})W(-\frac{x}{2i}))}e^{-W(\frac{x}{2i})-W(-\frac{x}{2i})} =$$

$$\frac{e^{-W\left(\frac{x}{2i}\right)-W\left(-\frac{x}{2i}\right)}\left(1 + W\left(-\frac{x}{2i}\right)\right)\left(\frac{-W\left(\frac{x}{2i}\right)}{x\left(-W\left(\frac{x}{2i}\right)+1\right)} + \frac{-W\left(-\frac{x}{2i}\right)}{x\left(-W\left(-\frac{x}{2i}\right)+1\right)}\right)\left(1 + W\left(\frac{x}{2i}\right)\right)}{1 - W\left(\frac{x}{2i}\right)W\left(-\frac{x}{2i}\right)}$$

$$- \frac{e^{-W\left(\frac{x}{2i}\right)-W\left(-\frac{x}{2i}\right)}\left(1 + W\left(-\frac{x}{2i}\right)\right)\left(-\frac{-W\left(\frac{x}{2i}\right)W\left(\frac{x}{2i}\right)}{x\left(-W\left(\frac{x}{2i}\right)+1\right)} - \frac{-W\left(\frac{x}{2i}\right)W\left(\frac{x}{2i}\right)}{x\left(-W\left(-\frac{x}{2i}\right)+1\right)}\right)\left(1 + W\left(\frac{x}{2i}\right)\right)}{\left(1 - W\left(\frac{x}{2i}\right)W\left(-\frac{x}{2i}\right)\right)^2}$$

$$- \frac{e^{-W\left(\frac{x}{2i}\right)}\left(1 + W\left(\frac{x}{2i}\right)\right)}{-2i\left(W\left(-\frac{x}{2i}\right) + 1\right)\left(1 - W\left(\frac{x}{2i}\right)W\left(-\frac{x}{2i}\right)\right)}$$

$$+ \frac{e^{-W\left(-\frac{x}{2i}\right)}\left(1 + W\left(-\frac{x}{2i}\right)\right)}{2i\left(-W\left(\frac{x}{2i}\right) + 1\right)\left(1 - W\left(\frac{x}{2i}\right)W\left(-\frac{x}{2i}\right)\right)}.$$

We find the non-differential points of $f(x)$ are zeros of $1-W(\frac{x}{2i}), 1-W(-\frac{x}{2i}), 1-W(-\frac{x}{2i})W(\frac{x}{2i})$. From the definition of Lambert W function, the solution of $1-W(\frac{x}{2i}) = 0$ (resp. $1-W(-\frac{x}{2i}) = 0$) is $x = \frac{2i}{e}$ (resp. $x = -2ei$).

Hence, we could say, this ODE meets Lipschitz continuity in $\{a+bi, a \in (A^L(\frac{\epsilon}{2}), R-\epsilon), |b| < \eta\} \times [\epsilon, M]$ for arbitrarily large M, which shows there is a solution $C(q)$ in maximal interval $[\epsilon, \epsilon+\delta_\epsilon)$ for some positive δ_ϵ such that

$$\begin{cases} C(\epsilon) = A^L(\epsilon), \\ C(q)e^{W(-\frac{C(q)}{2i})+W(\frac{C(q)}{2i})} - q = \int_\epsilon^q \frac{d}{dt}(C(t)e^{W(-\frac{C(t)}{2i})+W(\frac{C(y)}{2i})} - t)dt = 0. \end{cases}$$

and C is unique by Picard-Lindelöf theorem. Hence we consider C as an extension of A^L in $[\epsilon, \epsilon+\delta_\epsilon)$.

We claim that $\epsilon + \delta_\epsilon = M$ or $C(\epsilon+\delta_\epsilon) \in \partial\{a+bi, a \in (A^L(\frac{\epsilon}{2}), R-\epsilon), |b| < \eta\}$ by Picard-Lindelof theorem.

Now we show that in any $t \in [\epsilon, \epsilon+\delta_\epsilon]$, A^L is holomorphic. We take $q(A^L)$, that consider q as a holomorphic function around $A^L(t)$. Notice at this point, $\frac{dq}{dA^L} \neq 0$, we have a local inverse of $q(A^L)$. We choose a neighborhood as a disk of t with radius r_t that $A^L(q)$ is holomorphic. Glue these disks with the original disk we hold of 0, we deduce that A^L has no singularity in $[0, \epsilon+\delta_\epsilon)$, the power series of A^L converges in $|q| < \epsilon+\delta_\epsilon$ and A^L is a real function in $[0, \epsilon+\delta_\epsilon)$. It tells that $C(q)$ is a real, non-decreasing function in $[0, \epsilon+\delta_\epsilon)$ that we would say $\epsilon+\delta_\epsilon = M$ or $C(\epsilon+\delta_\epsilon) = R-\epsilon$.

Since ϵ is arbitrarily small, we take $\epsilon \to 0$ and $M \to \infty$, deriving C is defined in a maximal interval $(0, T_0)$ such that $T_0 = \infty$ or $C(T_0) = R$. Since we know there is a upper bound of the convergence radius r by (4-1). For sufficiently large M, we need to choose T_0 as the positive $r' = \frac{4}{R}$ such that $C(r') = R$.

And we know we have no way to extend $A^L(q)$ holomorphically around $q = r'$. Hence $r = r'$.

Lastly, we claim there exist exactly one positive r is a solution of (4-2). If there is another positive $u \neq r$ as a solution. If $r > u > 0$, then $A^L(r) < A^L(u)$, while in the interval $[0, r]$, A^L is non-decreasing, it derives a contradiction. If $u > r$, then $A^L(r) > A^L(u)$, which contracts to our assumption of $R = A^L(r)$.

Approximately, $\frac{1}{r} \approx 0.980654423229753$.

4.2 Asymptotic expansion

Tian and Wei[18] have proven that the classical genus 0 GW invariants in $\mathbb{P}^2$ is of the following asymptotic expansion:

$$n_d = e^{dx_0}(\sum_{k=3}^{N-1} a_k d^{-k-\frac{1}{2}} + O(d^{-N-\frac{1}{2}})),$$

or say

$$n_d \sim e^{dx_0} \sum_{k=3}^{\infty} a_k d^{-k-\frac{1}{2}},$$

for some $x_0, a_k \in \mathbb{R}$ with $a_3 > 0$ and any $N \geq 4$.

It derives the search that we may have similar asymptotic expansions for the $(\mathbb{P}^2, conic)$ and $(H_0, (1,1))$.

To simplify our discussion, we would take $(H_0, (1,1))$ as the example to illustrate the standard process[23].

Firstly we determine the singular expansion of A of the singularity $q = \frac{1}{e}$, that analytic in a slit neighborhood Ω^0, where Ω is a neighborhood of $q = \frac{1}{e}$ and $\Omega^0 = \Omega \setminus \mathbb{R}_{\geq \frac{1}{e}}$.

We get the formal expansion of $Ae^{2W(-A)}$ at $A = e$.

$$\begin{aligned} Ae^{2W(-A)} =& \frac{1}{e} - \frac{(A-e)^2}{8e^3} + \frac{13(A-e)^3}{96e^4} \\ & - \frac{187(A-e)^4}{1536e^5} + \frac{3187(A-e)^5}{30720e^6} - \frac{21299(A-e)^6}{245760e^7} + \\ & \frac{1483037(A-e)^7}{20643840e^8} - \frac{13078241(A-e)^8}{220200960e^9} + \frac{1167442979(A-e)^9}{23781703680e^{10}} + \dots \end{aligned}$$

Solving the equation $Ae^{2W(-A)} = q$ we conclude that

$$1 - eq = (A-e)^2(\frac{1}{8e^2} - \frac{13}{96 \cdot e^3}(A-e) + \cdots).$$

The formal square root of the equation above on $\mathbb{C} - [\frac{1}{e}, \infty)$ is

$$-\sqrt{1-eq} = (A-e)(\frac{1}{2\sqrt{2}e} - \frac{13}{48\sqrt{2}e^2}(A-e) + \frac{49}{288\sqrt{2}e^3}(A-e)^2 \cdots).$$

Take the formal inverse function we know that

$$\begin{aligned} A - e =& -2\sqrt{2}e(1-eq)^{\frac{1}{2}} + \frac{13e(1-eq)}{3} - \frac{71e(1-eq)^{\frac{3}{2}}}{9\sqrt{2}} \\ & + \frac{896e(1-eq)^2}{135} - \frac{2309e(1-eq)^{\frac{5}{2}}}{216\sqrt{2}} + \frac{14248e(1-eq)^3}{1701} \end{aligned}$$

$$-\frac{17534803e(1-eq)^{\frac{7}{2}}}{1360800\sqrt{2}}+\frac{7130e(1-eq)^4}{729}+O\left(\sqrt{1-eq}^9\right).$$

We would check that the above formal series is proper for us to apply Theorem B.4.

Due to the choice of the analytic branch, we know that the choice of Δ domain is arbitrary. Recall that

$$W(-A)e^{-W(-A)}=-A,$$
$$W(-A)e^{W(-A)}=-q.$$

We know $A(q)=W(q)e^{W(q)}$ is holomorphic in $\mathbb{C}-[\frac{1}{e},\infty)$. Furthermore, we take the square roots in this domain, which derives a biholomorphic function(Notice $z^{\frac{1}{2}}=e^{\frac{1}{2}\ln z}$). From above, we note that $A-e$ is a holomorphic to $(1-eq)^{\frac{1}{2}}$ in a proper domain. And we could apply theorem B.4.

Here we list

$$[z^n](1-z)^{\frac{1}{2}}\sim-\frac{1}{\sqrt{\pi n^3}}(\frac{1}{2}+\frac{3}{16n}+\frac{25}{256n^2}+O(\frac{1}{n^3})),$$
$$[z^n](1-z)^{\frac{3}{2}}\sim\frac{1}{\sqrt{\pi n^5}}(\frac{3}{4}+\frac{45}{32n}+\frac{1155}{512n^2}+O(\frac{1}{n^3})),$$
$$[z^n](1-z)^{\frac{5}{2}}\sim\frac{1}{\sqrt{\pi n^7}}(\frac{15}{8}+\frac{525}{64n}+\frac{26775}{1024n^2}+O(\frac{1}{n^3})).$$

It follows that

$$[q^d]A=e^d[(eq)^d]A=\frac{e^{d+1}}{\sqrt{\pi d^3}}(\sqrt{2}-\frac{31\sqrt{2}}{12d}-\frac{8855\sqrt{2}}{576d^2}+O(d^{-3})).$$

We could get the asymptotic expansion of

$$\sum_{m+n=d}N_{m,n}=\frac{1}{d-1}[q^{d-1}]A=\frac{e^d}{\sqrt{\pi}}(\sqrt{2}d^{-\frac{3}{2}}-\frac{13}{6\sqrt{2}}d^{-\frac{5}{2}}-\frac{11495}{288\sqrt{2}}d^{-\frac{7}{2}}+O(d^{-\frac{9}{2}})).$$

Theorem 4.4:

$$\frac{\sum_{m+n=d}N_{0,(m,n)}(H_0,(1,1))}{d!}\sim e^d(\frac{\sqrt{2}}{\sqrt{\pi}}d^{-\frac{3}{2}}+\sum_{k\geq2}a_kn^{-k-\frac{1}{2}}),$$

where a_k is computable constant.

We could apply this process to the case $(\mathbb{P}^2,conic)$ and the result is strongly related to the case of $(H_0,(1,1))$ if we take the substitution $B=\frac{A^c}{8},p=2q$, we directly from the result of $(H_0,(1,1))$, notice that $[q^d]A^C=2^{d+3}[p^d]B=(2e)^d(\frac{2^3e}{\sqrt{\pi}}d^{-\frac{3}{2}}+O(d^{-\frac{5}{2}}))$.

Lemma 4.3:

$$\frac{\bar{N}_{0,d}(\mathbb{P}^2,conic)}{(d-1)!}=(2e)^d(\frac{\sqrt{2}}{\sqrt{\pi}}d^{-\frac{3}{2}}+O(d^{-\frac{7}{2}})).$$

Similarly, we could consider the $A_{-1} = \sum_d (d-1) q^{d-1} \sum_{m+n=d} N_{m,n} (-1)^n$. Combine A_{-1} and A^L, we have

$$\begin{cases} A^L e^{W(\frac{A^L}{2i}) + W(\frac{A^L}{-2i})} = q, \\ A_{-1} e^{W(A_{-1}) + W(-A_{-1})} = q. \end{cases}$$

Let $C = \frac{A^L}{2i}$ and $p = \frac{q}{2i}$, we know that

$$\frac{\sum_{m+n=d} (-1)^n N_{0,(m,n)}(H_0, (1,1))}{d!} = \frac{1}{2i(d-1)} [p^{d-1}] (\sum \frac{\bar{N}_{0,d}(\mathbb{P}^2, line)}{(2d-2)!} (2ip)^{2d-1}).$$

If d is odd,

$$\frac{\sum_{m+n=d} (-1)^n N_{0,(m,n)}(H_0, (1,1))}{d!} = 0;$$

if d is even, we would note as $2d$, then

$$\frac{\sum_{m+n=2d} (-1)^n N_{0,(m,n)}(H_0, (1,1))}{d!} = (2i)^{2d-2} \frac{\bar{N}_{0,d}(\mathbb{P}^2, line)}{(2d-1)!}.$$

CHAPTER 5 FURTHER DIRECTION

5.1 Inequality

The quiver side presents that for $k > 0$, only $N_{0,\beta}(H_k, D)$ with $T_{log} \cdot \beta \geq 0$ are non vanishing. Suppose $\beta = (m, n)$, we have $T_{log} \cdot \beta = (1-k)m + n \geq 0$. As k=1, we would deduce $T_{\log} \cdot \beta = n \leq m + n \leq 2n$. Generally, we have $D \cdot \beta = n \geq T_{\log} \cdot \beta$. Here we would separate to 2 cases.

5.1.1 H_1

Set k=1, the recursion formula transforms to

$$N_{m,n} = \sum_{\substack{m_1+m_2=m \\ n_1+n_2=n}} \frac{(m_1+n_1)^2(n_2-1)}{(n-1)(n-2)} N_{m_1,n_1} N_{m_2,n_2}.$$

The quiver side tells $N_{m,n} = 0$ if $m > n$. Therefore for those non-vanishing $N_{m,n}$, we only need to consider the recursion

$$N_{m,n} = \sum_{\substack{m_1+m_2=m \\ n_1+n_2=n \\ m_i \leq n_i}} \frac{(m_1+n_1)^2(n_2-1)}{(n-1)(n-2)} N_{m_1,n_1} N_{m_2,n_2}.$$

The following inequality holds in this recursion formula,

$$\frac{n_1^2(n_2-1)}{(n-1)(n-2)} \leq \frac{(m_1+n_1)^2(n_2-1)}{(n-1)(n-2)} \leq 4\frac{n_1^2(n_2-1)}{(n-1)(n-2)}.$$

Define the generating series

$$f_1(q,s) = \sum N_{m,n} s^{m+n} q^n = \sum F_{1,n}(s) q^n,$$

$$g_1(q,s) = \sum G_{m,n} s^{m+n} q^n = \sum G_{1,n}(s) q^n,$$

$$e_1(q,s) = \sum E_{m,n} s^{m+n} q^n = \sum E_{1,n}(s) q^n.$$

where $G_{m,n}$ (resp. $E_{m,n}$) is generated by

$$G_{m,n} = \sum_{\substack{m_1+m_2=m \\ n_1+n_2=n}} \frac{n_1^2(n_2-1)}{(n-1)(n-2)} G_{m_1,n_1} G_{m_2,n_2},$$

resp.

$$E_{m,n} = \sum_{\substack{m_1+m_2=m\\ n_1+n_2=n}} 4\frac{n_1^2(n_2-1)}{(n-1)(n-2)} E_{m_1,n_1}E_{m_2,n_2},$$

with the same initial date as $N_{m,n}$. The initial date are $N_{0,1} = N_{1,1} = N_{1,2} = N_{2,2} = 1$. Hence we have

$$G_{m,n} \leq N_{m,n} \leq E_{m,n},$$

$$G_1 e^{W(c_1G_1)+W(c_2G1)} = (s^3+s^4)q,$$

where $G_1 = (\frac{d}{dq} - \frac{1}{q})g_1$, and $c_1, c_2 = -\frac{-(s+s^2)\pm\sqrt{s^2-2s^3-3s^4}}{2s^3+2s^4}$.

$$E_1 e^{W(c_1E_1)+W(c_2E_1)} = (s^3+s^4)q,$$

where $E_1 = (\frac{d}{dq} - \frac{1}{q})e_1$, and $c_1, c_2 = -2\frac{-(s+s^2)\pm\sqrt{s^2+s^3}}{s^3+s^4}$.

Remark Noticing the symmetry of $G_{1,n}$ and $E_{1,n}$, we guess that $F_{1,n}(-1) = 0$.

The recursion formula presents that $(n-1)(n-2)F_{1,n}(s) = \sum_{n_1+n_2=n}(n_1 - 1)F_{1,n_1}(s)\cdot(s\frac{d}{ds})^2F_{1,n_2}(s)$. Since $F_{1,1}(-1), F_{1,2}(-1) = 0$. By induction hypothesis, we get $F_{1,n}(-1) = 0$.

5.1.2 H_k

If $k > 1$, we have $D\cdot\beta = n + m \geq T_{log}\cdot\beta$. Hence we define

$$f_k(q,s) = \sum N_{m,n}s^{m+n}q^n,$$

and

$$g_k(q,s) = \sum G_{m,n}s^{m+n}q^n,$$

where $G_{m,n}$ is generated by

$$G_{m,n} = \sum_{\substack{m_1+m_2=m\\ n_1+n_2=n}} \frac{((1-k)m_1+n_1)^2((1-k)m_2+n_2-1)}{((1-k)m+n-1)((1-k)m+n-2)} G_{m_1,n_1}G_{m_2,n_2}.$$

Remark: Here the generating series e_k would give a lower bound. The upper bound e_k are not defined for $k > 1$ since the suitable inequality consist of $T_{log}\cdot\beta$ or $D\cdot\beta$ is not found now. Furthermore if we consider

$$\frac{((1-k)m_1+n_1)^2((1-k)m_2+n_2-1)}{((1-k)m+n-1)((1-k)m+n-2)} \leq$$

$$\frac{((1-k)m_1+n_1)(m_1+n_1)((1-k)m_2+n_2-1)}{((1-k)m+n-1)((1-k)m+n-2)} \leq \frac{(m_1+n_1)^2((1-k)m_2+n_2-1)}{((1-k)m+n-1)((1-k)m+n-2)}.$$

The intermediate formula gives a new generating series. But the closed form of this series is improper for counting coefficient, so we ignore it.

5.2 Recursion formula in another direction

There is a fun fact that if $k > 0$, the generating function packaging $\bar{N}_{0,(m,n)}(H_k,(1,k+1))$ admits two independent PDE. First we introduce the initial degeneration formula[24]

Theorem 5.1: For any smooth projective surface X, smooth ample rational divisor D, and curve class β such that $T_{log}\cdot\beta \geq 3$, we have

$$\begin{aligned}(H\cdot D)\frac{N_{0,\beta}(X,D)}{D\cdot\beta} = &\sum_{\beta_1+\beta_2=\beta,\beta_1,\beta_2>0}((D\cdot\beta_1)^2(H\cdot\beta_2)\binom{T_{log}\cdot\beta-3}{T_{log}\cdot\beta_1-1}\\ &+(D\cdot\beta_1)(D\cdot\beta_2)(H\cdot\beta_2)\binom{T_{log}\cdot\beta-3}{T_{log}\cdot\beta_1-2}\\ &-(D\cdot\beta_1)^2(H\cdot\beta_1)\binom{T_{log}\cdot\beta-3}{T_{log}\cdot\beta_1}\\ &-(D\cdot\beta_1)(D\cdot\beta_2)(H\cdot\beta_1)\binom{T_{log}\cdot\beta-3}{T_{log}\cdot\beta_1-1})\frac{N_{0,\beta_1}(X,D)}{D\cdot\beta_1}\frac{N_{0,\beta_2}(X,D)}{D\cdot\beta_2}.\end{aligned}$$

Here by choice of H, we could get various degeneration formulas. If we let $H = T_{log}$, it is the degeneration formula we mainly use in this thesis. However, Wu and Fang originally chose $H = D$, deriving the formula

$$\begin{aligned}(D\cdot D)\bar{N}_\beta(X,D) = &\sum_{\beta_1+\beta_2=\beta,\beta_1,\beta_2>0}((D\cdot\beta_1)(D\cdot\beta_2)^2\binom{T_{log}\cdot\beta-3}{T_{log}\cdot\beta_1-2}\\ &-(D\cdot\beta_1)^3\binom{T_{log}\cdot\beta-3}{T_{log}\cdot\beta_1})\bar{N}_{\beta_1}(X,D)\bar{N}_{\beta_2}(X,D).\end{aligned} \tag{5-1}$$

Recall the formula mainly applied in the thesis,

$$\bar{N}_\beta(X,D) = \sum_{\beta_1+\beta_2=\beta,\beta_1,\beta_2>0}(D\cdot\beta_1)^2\binom{T_{log}\cdot\beta-3}{T_{log}\cdot\beta_1-1}\bar{N}_{\beta_1}(X,D)\bar{N}_{\beta_2}(X,D). \tag{5-2}$$

One fact is that the two recursion formulas might be nonequivalent. Here we use $(H_1,(1,2))$ as an example to illustrate.

By (5-1), we get

$$3\bar{N}_{2,3}(X,D) = 4\bar{N}_{2,2}(X,D)\bar{N}_{0,1}(X,D) + 12\bar{N}_{1,1}(X,D)\bar{N}_{1,2}(X,D) - \bar{N}_{1,0}(X,D)\bar{N}_{1,3}(X,D).$$

While by (5-2)

$$\bar{N}_{2,3}(X,D) = \bar{N}_{2,2}(X,D)\bar{N}_{0,1}(X,D) + 4\bar{N}_{1,1}(X,D)\bar{N}_{1,2}(X,D).$$

Form these two equations, we realize that these two recursion formulas derived from different directions are nonequivalent.

5.3 Further direction

In this thesis, we leave several questions incomplete. For the weighted sum of Gromov-Witten invariants of $(\mathbb{P}^1 \times \mathbb{P}^1, diagonal)$, $\sum_{m+n=d} N_{m,n}s^n$ we only deal with the case $s = \pm 1$. A direct question is we could determine the maximal region of s such that the asymptotic expansion is exactly computable.

We find there are some numerical relations between the relative Gromov-Witten invariants of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. Possibly there exists an explanation to reveal the geometry.

For general Hirzebruch surfaces, we have to face a non-linear second order PDE,

$$(p\frac{\partial}{\partial p} - 2)(p\frac{\partial}{\partial p} - 1)F = (q\frac{\partial}{\partial q})^2 F \cdot (p\frac{\partial}{\partial p} - 1)F.$$

Typically, separation of variables does not present the solution we need.

APPENDIX A SINGULAR EXPANSION OF SINGLE SINGULARITY

Here we present definition and theorems[23] we need in this thesis to compute the singular expansion.

Theorem A.1 (Theorem VI.1): Let α be an complex number in $\mathbb{C}-\mathbb{Z}_{\leq 0}$, the coefficient of z^n in the power series of function around 0

$$f(z) = (1-z)^{-\alpha},$$

admits for large n an asymptotic expansion of n,

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(1+\sum_{k=1}^{\infty}\frac{e_k}{n^k}),$$

where e_k is a polynomial in α of degree 2k. Particularly:

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(1+\frac{\alpha(\alpha-1)}{2n}+\frac{\alpha(\alpha-1)(\alpha-2)(3\alpha-1)}{24n^2} + \frac{\alpha^2(\alpha-1)^2(\alpha-2)(\alpha-3)}{48n^3}+O(\frac{1}{n^4})).$$

The coefficients e_k satisfy[23]

$$e_k = \sum_{l=k}^{2k}\lambda_{k,l}(\alpha-1)(\alpha-2)\cdots(\alpha-l),$$

where $\lambda_{k,l} := [v^k t^l]e^t(1+vt)^{-1-\frac{1}{v}}$.

Theorem A.2 (Theorem VI.2): Assume α is a complex number in $\mathbb{C}-\mathbb{Z}_{\leq 0}$. The coefficients of z^n in the power series of function around 0

$$f(z) = (1-z)^{-\alpha}(\frac{1}{z}\ln\frac{1}{1-z})^{\beta},$$

admits for large n an asymptotic expansion of $\ln n$,

$$f_n = [z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(\ln n)^{\beta}(1+\frac{C_1}{\ln n}+\frac{C_2}{\ln^2 n}+\cdots),$$

where $C_k = \binom{\beta}{k}\Gamma(\alpha)\frac{d^k}{ds^k}\frac{1}{\Gamma(s)}|_{s=\alpha}$

With the asymptotic expansions stated above, we apply them combinationally to form a method computing asymptotic expansions of general functions.

Firstly, we could define $\mathcal{S}$ as the set of functions

$$S = \{(1-z)^{\alpha}\lambda(z)^{\beta} | \alpha, \beta \in \mathbb{C}\}, \lambda(z) := \frac{1}{z}\ln\frac{1}{1-z}.$$

Definition A.1 (Definition VI.1): Given two real numbers ϕ, R with $R > 1$ and $0 < \phi < \frac{\pi}{2}$, a (open) Δ-domain $\Delta(\phi, R)$ is defined as

$$\Delta(\phi, R) = \{z \big| |z| < R, z \neq 1, |arg(z-1)| > \phi\}.$$

A domain is a Δ-domain at 1 if it is a $\Delta(\phi, R)$ for some R and ϕ. For a complex number $\zeta \neq 0$, a Δ-domain at ζ is the image by the mapping $z \mapsto \zeta z$ of a Δ-domain at 1. A function is Δ-analytic in some Δ-domain if it is analytic in some Δ-domain.

Theorem A.3 (Theorem VI.3/Transfer, Big-Oh and little-oh): Let α, β be arbitrary real numbers, $\alpha, \beta \in \mathbb{R}$ and let $f(z)$ be a function that is Δ-analytic.

(i) Assume that $f(z)$ satisfies the condition

$$f(z) = O((1-z)^{\alpha}(\ln\frac{1}{1-z})^{\beta}),$$

in the intersection of a neighborhood of 1 with its Δ-domain, then one has : $[z^n]f(z) = O(n^{\alpha-1}(\ln n)^{\beta})$

(ii) Assume that $f(z)$ satisfies the condition

$$f(z) = o((1-z)^{\alpha}(\ln\frac{1}{1-z})^{\beta}),$$

in the intersection of a neighborhood of 1 with its Δ-domain, then one has : $[z^n]f(z) = o(n^{\alpha-1}(\ln n)^{\beta})$

Theorem A.4 (Theorem VI.4/Singularity analysis, single singularity): Let $f(z)$ be a function which is analytic at 0 with a singularity at ζ, such that $f(z)$ can be extended to a domain of the form $U \cap \zeta\Delta_0$ for a Δ-domain Δ_0 and a neighborhood of ζ, where $\zeta\Delta_0$ is the image of Δ_0 by the mapping $z \mapsto \zeta z$. Assume that there exist two function σ, ι, where σ is finite linear combination of functions in $\mathcal{S}$ and $\iota \in \mathcal{S}$, such that

$$f(z) = \sigma(z/\zeta) + O(\iota(z/\zeta)),$$

as $z \to \zeta$ in $\zeta\Delta_0$. Then, the coefficients of $f(z)$ satisfy the asymptotic estimate

$$f_n = \zeta^{-n}\sigma_n + O(\zeta^{-n}\iota_n^*),$$

where $\sigma_n = [z^n]\sigma(z)$ and $\iota_n^* = n^{a-1}(\ln n)^b$, if $\iota(z) = (1-z)^{-a}\lambda(z)^b$.

APPENDIX B PROGRAMS OF MATLAB

Program to compute the limit of $(P^2, conic)$ case

```
d=6000;%maximal number of degree m+n
N00=0;
N20=0;
N02=0;
N01=1;
N10=1;
N11=1/2;
N=NaN(d);
N(1)=16/2/exp(1);%/8exp(1)
N(2)=16/4/exp(2);%/64exp(2), initial date divided by a factor to control the growth rate
for i=3:d
N(i)=0;
for k=1:i-1
    N(i)=N(i)+N(k)*N(i-k)*k^2*(i-k-1)/(4*(i-1)*(i-2));
end
disp(append( "n_", num2str(i), " = ", num2str( 2*exp(1)*N(i)^(1/i),16)) )
end
```

Program to compute the limit of $(P^2, line)$ case

```
d=3000;
N00=0;
N20=0;
N02=0;
N01=1;
N10=1;
N11=1/2;
N(1)=1;
for j=2:d
N(j)=0;
for k=1:j-1
    N(j)=N(j)+N(k)*N(j-k)*k^2*(2*(j-k)-1)/(2*j-1)/(2*j-2);
```

```
end
disp(append( "n_", num2str(j), " = ", num2str(N(j)^(1/2/j),16)) )
end
```

Program to compute the limit of $(H_0, (1,1))$ case

```
d=1500;
N00=0;
N20=0;
N02=0;
N01=1;
N10=1;
N11=1/2;
N=NaN(d);
N(1,1)=0;
N(2,1)=1/2;
N(1,2)=1/2;
N(3,1)=0;
N(1,3)=0;
N(2,2)=1/4;%initial date divided by a factor to control the growth
    rate
N_i=0;
for i=3:d
    for j=0:i%计算N_j,i-j, as N(j+1,i-j+1)
comp=0;
for k=0:j
    for l=0:i-j
        if k+l==0 || k+l==i
            continue;
        end
        comp=comp+(k+l)^2*(j-k+i-j-l-1)*N(k+1,l+1)*N(j-k+1,i-j-l+1)
            ;
    end
end
N(j+1,i-j+1)=comp/(i-1)/(i-2);
N_i=N_i+N(j+1,i-j+1);
    end
disp(append( "n_", num2str(j), " = ",  num2str(2*((i-3)*N_i)^(1/(i
    -2)),16) ))
end
```

REFERENCES

[1] GROMOV M. Pseudo holomorphic curves in symplectic manifolds[J]. Invent. Math., 1985, 82 (2): 307-347.

[2] WITTEN E. Topological sigma models[J]. Comm. Math. Phys., 1988, 118(3): 411-449.

[3] RUAN Y, TIAN G. A mathematical theory of quantum cohomology[J]. J. Differential Geom., 1995, 42(2): 259-367.

[4] LI J, TIAN G. Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties[J]. J. Amer. Math. Soc., 1998, 11(1): 119-174.

[5] LI J, TIAN G. Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds[M]//First Int. Press Lect. Ser.: I Topics in symplectic 4-manifolds (Irvine, CA, 1996). Int. Press, Cambridge, MA, 1998: 47-83.

[6] BEHREND K. Gromov-Witten invariants in algebraic geometry[J]. Invent. Math., 1997, 127 (3): 601-617.

[7] BEHREND K, FANTECHI B. The intrinsic normal cone[J]. Invent. Math., 1997, 128(1): 45-88.

[8] LI A M, RUAN Y. Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds[J]. Invent. Math., 2001, 145(1): 151-218.

[9] IONEL E N, PARKER T H. Relative Gromov-Witten invariants[J]. Ann. of Math. (2), 2003, 157(1): 45-96.

[10] LI J. Stable Morphisms to Singular Schemes and Relative Stable Morphisms[J]. Journal of Differential Geometry, 2001, 57(3): 509 - 578.

[11] TAKAHASHI N. Local and relative Gromov-Witten invariants of the projective plane[J]. manuscripta mathematica, 2003, 111: 413-426.

[12] REINEKE M, WEIST T. Moduli spaces of point configurations and plane curve counts[J]. Int. Math. Res. Not. IMRN, 2021(13): 10339-10372.

[13] BOUSSEAU P. On an example of quiver Donaldson-Thomas/relative Gromov-Witten correspondence[J]. Int. Math. Res. Not. IMRN, 2021(15): 11845-11888.

[14] FAN H, WU L. Witten-Dijkgraaf-Verlinde-Verlinde equation and its application to relative Gromov-Witten theory[J]. Int. Math. Res. Not. IMRN, 2021(13): 9834-9852.

[15] DI FRANCESCO P, ITZYKSON C. Quantum intersection rings[M]//Progr. Math.: volume 129 The moduli space of curves (Texel Island, 1994). Birkhäuser Boston, Boston, MA, 1995: 81-148.

[16] ZINGER A. Some conjectures on the asymptotic behavior of Gromov-Witten invariants[M]// Adv. Lect. Math. (ALM): volume 47 Handbook for mirror symmetry of Calabi-Yau & Fano manifolds. Int. Press, Somerville, MA, [2020] ©2020: 523-550.

[17] COUSO-SANTAMARÍA R, SCHIAPPA R, VAZ R. On asymptotics and resurgent structures of enumerative Gromov-Witten invariants[J]. Commun. Number Theory Phys., 2017, 11(4): 707-790.

[18] TIAN G, WEI D. Asypmtotics of enumerative invariants in CP^2[A]. 2016. arXiv: 1609.06425.

[19] BOUSSEAU P, WU L. All-genus WDVV recursion, quivers, and BPS invariants[A]. 2023.

[20] KONTSEVICH M, MANIN Y. Gromov-Witten classes, quantum cohomology, and enumerative geometry[J]. Communications in Mathematical Physics, 1994, 164: 525-562.

[21] LI J. Stable morphisms to singular schemes and relative stable morphisms[J]. J. Differential Geom., 2001, 57(3): 509-578.

[22] LI J. A degeneration formula of GW-invariants[J]. J. Differential Geom., 2002, 60(2): 199-293.

[23] FLAJOLET P, SEDGEWICK R. SYMBOLIC METHODS[M]. Cambridge University Press, 2009: 13–14.

[24] FAN H, WU L. Witten–Dijkgraaf–Verlinde–Verlinde Equation and its Application to Relative Gromov–Witten Theory[J]. International Mathematics Research Notices, 2021, 2021(13): 9834-9852.

ACKNOWLEDGEMENTS

First of all, I would like to give my heartfelt thanks to all the people who have ever helped me in this paper.

My sincere and hearty thanks and appreciations go firstly to my supervisor, Prof. Longting Wu, whose suggestions and encouragement have given me much insight into Gromov-Witten Theory. It has been a great privilege and joy to study under his guidance and supervision. Furthermore, it is my honor to benefit from his personality and diligence, which I will treasure my whole life. My gratitude to him knows no bounds.

I am also extremely grateful to my colleagues, Zepei Liu and Yicong Wang as their help to code and introduction of Matlab.

RESUME AND ACADEMIC ACHIEVEMENTS

Resume

Ouyang Hao was born in 2000, Yizhang, Hunan, China.

In September 2018, he was admitted to Southern University of Science and Technology (SUSTech). In June 2022, he obtained a bachelor's degree in Science from the Department of Mathematics, SUSTech.

In September 2022, he began his graduate study in the Department of Mathematics, SUSTech, and expected to get a master of science degree in mathematics, in July 2024.

Academic Achievements during the Study for an Academic Degree

None