Mathematics I - Code - BCA 101

by

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Syallabus

UNIT-I
DETERMINANTS:

Definition, Minors, Cofactors, Properties of Determinants MATRICES: Definition, Types of Matrices, Addition, Subtraction, Scalar Multiplication and Multiplication of Matrices, Adjoint, Inverse, Cramers Rule, Rank of Matrix Dependence of Vectors, Eigen Vectors of a Matrix, Caley-Hamilton Theorem (without proof).

UNIT-II
LIMITS & CONTINUITY:
Limit at a Point, Properties of Limit, Computation of Limits of Various Types of Functions, Continuity at a Point, Continuity Over an Interval, Intermediate Value Theorem, Type of Discontinuities

UNIT-III
DIFFERENTIATION:
Derivative, Derivatives of Sum, Differences, Product & Quotients, Chain Rule, Derivatives of Composite Functions, Logarithmic Differentiation, Rolle's Theorem, Mean Value Theorem,
Expansion of Functions (Maclaurin's & Taylor's), Indeterminate Forms, L' Hospitals Rule,
Maxima & Minima, Curve Tracing, Successive Differentiation & Liebnitz Theorem.

UNIT-IV
INTEGRATION:
Integral as Limit of Sum, Fundamental Theorem of Calculus(without proof.), Indefinite Integrals, Methods of Integration Substitution, By Parts, Partial Fractions, Reduction Formulae for Trigonometric Functions, Gamma and Beta Functions(definition).

UNIT-V
VECTOR ALGEBRA:
Definition of a vector in 2 and 3 Dimensions; Double and Triple Scalar and Vector Product and physical interpretation of area and volume.

Text Books

1. *Elementary Engineering Mathematics* by Dr. B.S. Grewal, Khanna Publications
2. *Higher Engineering Mathematics* by B.S. Grewal, Khanna Publications

Reference Books

1. *Differential Calculus* by Shanti Narayan, Publishers S. Chand & Co.
2. *Integral Calculus* by Shanti Narayan, Publishers S. Chand & Co.
3. *Modern Abstract Algebra* by Shanti Narayan, Publishers S. Chand & Co.

CONTENTS

MATRIX THEORY

Review of the fundamentals

A rectangular array of mn elements arranged in m rows & n columns is called a **'Matrix'** of a order $m \times n$ matrices are denoted
by capital letters of The English Alphabet.

Examples

$$\text{Matrix of order } 3 \times 2 \text{ is } \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$$

$$\text{Matrix of order } 4 \times 3 \text{ is } \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix}$$

$$\text{Matrix of order } 3 \times 3 \text{ is } \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Note :- Elements of Matrices are written in rows and columns with in the bracket ( ) or [ ].

Types of Matrices

(1) Equivalent Matrices : Two matrices are said to be equivalent if the order is the same.

(2) Equal Matrices : Two matrices are said to be equal if the corresponding elements are equal.

(3) Rectangular & Square Matrices : A matrix of order $m \times n$ is said to be rectangular if $m \neq n$, square if $m = n$.

(4) Row Matrix : A matrix having only one row is called Row Matrix.

(5) Column Matrix : A matrix having only one column is called Column Matrix.

(6) Null Matrix or Zero Matrix : A matrix in which all the elements are zeros is called Null Matrix or Zero Matrix
denoted as O. [English alphabet O not zero where as elements are zeros]

(7) Diagonal Matrix : A diagonal matrix is a square matrix in which all elements except the elements in the principal
diagonal are zeros.

$$\text{Example} \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

are diagonal matrices of order 2 & 3.

(8) Scalar Matrix : A diagonal matrix in which all the elements in the principal diagonal are same.

Example $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ $\begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$

are Scalar Matrices of order 2 & 3.

(9) Unit Matrix or Identity Matrix : A diagonal matrix in which all the elements in the principal diagonal is 1 is called Unit Matrix or Identity Matrix denoted by I.

Example: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

are unit matrices of order 2 & 4.

(10) Transpose of a Matrix : If A is any matrix then the matrix obtained by interchanging the rows & columns of A is called 'Transpose of A and it is written as A' or A^T.

Example: If $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ then A' is $\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$

A is of order 3×2 but A' is of order 2×3.

Matrix addition

Two matrices can be added or subtracted if their orders are same.

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$$\text{Example: If } A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \,\&\, B = \begin{pmatrix} c_1 & d_1 & e_1 \\ c_2 & d_2 & e_2 \end{pmatrix}$$

$$A + B = \begin{pmatrix} a_1 + c_1 & b_1 + d_1 & c_1 + e_1 \\ a_2 + c_2 & b_2 + d_2 & c_2 + e_2 \end{pmatrix}$$

$$A - B = \begin{pmatrix} a_1 - c_1 & b_1 - d_1 & c_1 - e_1 \\ a_2 - c_2 & b_2 - d_2 & c_2 - e_2 \end{pmatrix}$$

Matrix Multiplication

If A is a matrix of order $m \times p$ and B is matrix of order $p \times n$, then the product AB is defined and its order is $m \times n$. (ie. for AB to be defined number of columns of A must be same as number of rows of B)

$$\text{Example: Let } A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \,\&\, B = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{pmatrix}$$

$$\text{then } AB = \begin{pmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & a_1\beta_1 + b_1\beta_2 + c_1\beta_3 \\ a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3 & a_2\beta_1 + b_2\beta_2 + c_2\beta_3 \end{pmatrix}$$

which is of order 2×2.

Note :- If A is multiplied by A then AA is denoted as $A2$, AAA.... as $A3$ etc.

Scalar Multiplication of a Matrix

If A is a matrix of any order and K is a scalar (a constant), then KA represent a matrix in which every element of A is multiplied by K.

$$\text{Example: If } A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \text{ then } KA = \begin{pmatrix} Ka_1 & Kb_1 & Kc_1 \\ Ka_2 & Kb_2 & Kc_2 \\ Ka_3 & Kb_3 & Kc_3 \end{pmatrix}$$

Symmetric and Skew Symmetric Matrices

Let A be a matrix of order $n \times n$ an element in i^{th} row and j^{th} column can be denoted as aij. Hence a matrix of order $n \times n$ can be denoted as (aij) or $[aij]$ where $i = 1, 2, \ldots\ldots n,\ \ j = 1, 2, \ldots\ldots n$

A matrix of order $n \times n$ is said to be Symmetric if $aij = aji$ and Skew Symmetric if $aij = -aji$ or A is symmetric if $A = A^T$ or $A = A'$, skew symmetric if $A = -A^T$ or $A = -A'$ also $A + A'$ is symmetric & $A - A'$ is skew symmetric.

Note :- In a skew symmetric matrix the elements in principal diagonal are all zeros.

$$\text{Example: } A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 7 & 6 \\ 5 & 6 & 8 \end{bmatrix} \text{ is symmetric where } A = A'$$

$$B = \begin{pmatrix} 0 & -2 & 7 \\ 2 & 0 & 6 \\ -7 & -6 & 0 \end{pmatrix} \text{ is skew symmetric where } B = -B'$$

Determinant

A determinant is defined as a mapping (function) from the set of square matrices to the set of real numbers.

If A is a square matrix its determinant is denoted as $|A|$.

$$\text{Example: Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ then det. } A \text{ or } |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Minors and Co-factors

Let $A = (aij) \quad i = 1, 2, 3 \quad j = 1, 2, 3$

$$\text{ie } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Consider $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ which is a determinant formed by leaning all the elements of row and column in which all lies

determinant is called Minor of a_{11}. Thus we can form nine minors. In general if A is matrix of order $n \times n$ then minor of

obtained by leaning all the elements in the row and column in which aij lies in A. The order of this minor is $n - 1$where as the order of given determinant is n if this minor is multiplied by $(-1)i + j$ then it is called Co-factors of aij.

$$\text{Example: Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\text{Minor of } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\text{Co-factor of } a_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\text{Minor of } a_{21} \text{ is } \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\text{Co-factor of } a_{21} \text{ is } (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Value of a determinant

Consider a matrix A of order $n \times n$. Consider all the elements of any row or column and multiply each element by its corresponding
co-factor. Then the algebraic sum of the product is the value of the determinant.

Example: Let $|A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Co-factor of a_1 is b_2

Co-factor of b_1 is $-a_2$

$\therefore \ |A| = a_1b_2 - b_1a_2$

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Co - factor of a_1 is $(-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

Co - factor of b_1 is $(-1)^{1+2}\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$

Co - factor of c_1 is $(-1)^{1+3}\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$$\therefore \ |A| = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$= a_1b_2c_3 - a_1b_3c_2) - a_2b_1c_3 - a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1$$

Properties of determinants

(1) If the elements of any two rows or columns are interchanged then value of the determinant changes only in sign.

(2) If the elements of two rows or columns are identical then the value of the determinant is zero.

(3) If all the elements of any row or column is multipled by a constant K, then the value of the determinant is multipled by K.

(4) If all the elements of any row or column are written as sum of two elements then the determinant can be written as sum of two determinants.

(5) If all the elements of any row or column are multiplied by a constant and added to the corresponding elements of any other row or column then the value of the determinant donot alter.

Adjoint of a Matrix

$$\text{Let } A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

Let us denoted the co-factors of $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$ as $A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3$ transpose of matrix of co-factors is called **Adjoint** of the Matrix.

$$\text{Matrix of Co-factors} = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$$

$$\text{Adjoint of } A = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

Theorem $A.\text{adj}.A = |A| I = \text{adj}.A.A$

$$A.\text{adj}.A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

$$= \begin{bmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{bmatrix}$$

Now $a_1A_1 + b_1B_1 + c_1C_1 = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = \Delta$ The value of the det. A.

Similarly $a_2A_2 + b_2B_2 + c_2C_2 = \Delta$

$a_3A_3 + b_3B_3 + c_3C_3 = \Delta$

$$a_1A_2 + b_1B_2 + c_1C_2 = -a_1\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_1\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_1\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= -a_1(b_1c_3 - b_3c_1) + b_1(a_1c_3 - a_3c_1) - c_1(a_1b_3 - a_3b_1)$$

$$= -a_1b_1c_3 + a_1b_3c_1 + a_1b_1c_3 - a_3b_1c_1 - a_1b_3c_1 + a_3b_1c_1 = 0$$

Similarly the other five elements of A adj A is zero.

$$\therefore\ A.\text{adj}.A = \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix} \text{ where } \Delta = |A|$$

$$= \Delta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \Delta.I$$

$$\therefore\ A.\text{adj}.A = |A| \cdot I = \text{adj}.A$$

Singular and Non-singular Matrices

A square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

Inverse of a Matrix

Two non-singular matrices A & B of the same order is said to be inverse of each other if $AB = I = BA$. Inverse of A is de as A^{-1}. Inverse of B is denoted as B^{-1} and further $(AB)^{-1} = B^{-1}A^{-1}$.

To find the inverse of A

$$A.\text{adj}.A = |A| \cdot I$$

multiply by A^{-1}, $AA^{-1}.\text{adj}.A = |A| A^{-1}$

ie $\text{adj}.A = |A| A^{-1} \Rightarrow A^{-1} = \dfrac{\text{adj}.A}{|A|}$

Example : Find the inverse of $\begin{pmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{pmatrix}$

$$\text{Matrix of Co-factors} = \begin{bmatrix} \begin{vmatrix} -5 & 4 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & 4 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} -2 & -5 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 4 & -2 \\ -5 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ -2 & -5 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (-5+8) & -(-2-4) & (4+5) \\ -(4-4) & (1+2) & -(-2-4) \\ (16-10) & -(4-4) & (-5+8) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & 9 \\ 0 & 3 & 6 \\ 6 & 0 & 3 \end{bmatrix}$$

$$\therefore \ \text{adj}.A = \begin{bmatrix} 3 & 0 & 6 \\ 6 & 3 & 0 \\ 9 & 6 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{vmatrix} = 1(-5+8) - 4(-2-4) - 2(4+5) = 3 + 24 - 18 = 9$$

$$A^{-1} = \frac{1}{|A|}\text{adj}.A = \frac{1}{9}\begin{pmatrix} 3 & 0 & 6 \\ 6 & 3 & 0 \\ 9 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 3/9 & 0 & 6/9 \\ 6/9 & 3/9 & 0 \\ 9/9 & 6/9 & 3/9 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \\ 1 & 2/3 & 1/3 \end{pmatrix}$$

Solutions of Linear equations

Cramer's Rule

To solve the equations

$$a_1x+b_1y+c_1z=d_1$$

$$a_2x+b_2y+c_2z=d_2$$

$$a_3x+b_3y+c_3z=d_3$$

$$\text{Consider } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

first evaluate & if it is not zero then multiply both sides of (1) by x.

$$\Delta x = x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_{3x} & b_3 & c_3 \end{vmatrix}$$

multiply the elements of columns 2 & 3 by y & z and add to elements of column 1.

$$\text{then } \Delta x = \begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \Delta_1 \text{ (say)}$$

multiply both sides of (1) by y

$$\Delta y = y\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1y & c_1 \\ a_2 & b_2y & c_2 \\ a_3 & b_3y & c_3 \end{vmatrix}$$

multiply the elements of columns 1 & 3 by x & z and add to the elements of column 2.

$$= \begin{vmatrix} a_1 & a_1x+b_1y+c_1z & c_1 \\ a_2 & a_2x+b_2y+c_2z & c_2 \\ a_3 & a_3x+b_3y+c_3z & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = \Delta_2 \text{ (say)}$$

multiply both sides of (1) by z

$$\Delta z = z\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1z \\ a_2 & b_2 & c_2z \\ a_3 & b_3 & c_3z \end{vmatrix}$$

multiply the elements of columns 1 & 2 by x & y and add to the elements of column 3.

$$= \begin{vmatrix} a_1 & b_1 & a_1x+b_1y+c_1z \\ a_2 & b_2 & a_2x+b_2y+c_2z \\ a_3 & b_3 & a_3x+b_3y+c_3z \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = \Delta_3 \text{ (say)}$$

then $x = \dfrac{\Delta_1}{\Delta}$ from (2)

$$y = \frac{\Delta_2}{\Delta} \text{ from } (3)$$

$$z = \frac{\Delta_3}{\Delta} \text{ from } (4)$$

Note :- Verification of values of x, y, z can be done by substituting in the given equations.

Example - 1

Solve $2x + y - z = 3$

$x + y + z = 1$

$x - 2y - 3y = 4$

$$\text{Let } \Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{vmatrix}$$

$$= 2(-3+2) - 1(-3-1) - 1(-2-1) = -2 + 4 + 3 = 5$$

multiply both sides of (1) by x

$$\text{then } \Delta x = x \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 2x & 1 & -1 \\ x & 1 & 1 \\ x & -2 & -3 \end{vmatrix}$$

multiply the elements of columns 2 & 3 by y and z and add to the elements of column 1.

$$\Delta x = \begin{vmatrix} 2x + y - z & 1 & -1 \\ x + y + z & 1 & 1 \\ x - 2y - 3y & -2 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & -2 & -3 \end{vmatrix} = 3(-3+2) - 1(-3-4) - 1(-2-4) = -3+7+6 = 10$$

$$\therefore \; x = \frac{10}{\Delta} = \frac{10}{5} = 2$$

multiply both sides of (1) by y

$$\Delta y = y \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{vmatrix} == \begin{vmatrix} 2 & y & -1 \\ 1 & y & 1 \\ 1 & -2y & -3 \end{vmatrix}$$

multiply the elements of column 1 by x & 3 by z and to the corresponding elements of column 2.

$$\text{then } \Delta y = \begin{vmatrix} 2 & 2x+y-z & -1 \\ 1 & x+y+z & 1 \\ 1 & x-2y-3z & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{vmatrix} = 2(-3-4) - 3(-3-1) - 1(4-1) = -14+12-3 = -5$$

$$\therefore \; y = \frac{-5}{\Delta} = \frac{-5}{5} = -1$$

multiply both sides of (1) by z

$$\Delta z = z\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -z \\ 1 & 1 & z \\ 1 & 4 & -3z \end{vmatrix}$$

multiply the elements of column 1 by x & column 2 by y and to the corresponding elements of column 3.

$$\text{then } \Delta z = \begin{vmatrix} 2 & 1 & 2x+y-z \\ 1 & 1 & x+y+z \\ 1 & -2 & x-2y-3z \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix} = 2(4+2)-1(4-1)+3(-2-1) \;= 12-3-9=0$$

$$\therefore \; z = \frac{0}{\Delta} = \frac{0}{5} = 0$$

Thus solution is $x = 2$, $y = -1$ & $z = 0$ which can be verified by substituting in the given equations.

Example - 2

Solve $4x + y = 7$

$3y + 4z = 5$

$5x + 3z = 2$

$$\Delta = \begin{vmatrix} 4 & 1 & 0 \\ 5 & 3 & 4 \\ 2 & 0 & 3 \end{vmatrix} = 4(9-0)-1(0-20)+0(0-15) \;= 36+20=56$$

$$\Delta_1 = \begin{vmatrix} 7 & 1 & 0 \\ 5 & 3 & 4 \\ 2 & 0 & 3 \end{vmatrix} = 7(9-0) - 1(15-8) + 0(0-6) \ = 63 - 7 = 56$$

$$\Delta_2 = \begin{vmatrix} 4 & 7 & 0 \\ 0 & 5 & 4 \\ 5 & 2 & 3 \end{vmatrix} = 4(15-8) - 7(0-20) + 0(0-25) = 28 + 140 = 168$$

$$\Delta_3 = \begin{vmatrix} 4 & 1 & 7 \\ 0 & 3 & 5 \\ 5 & 0 & 2 \end{vmatrix} = 4(6-0) - 1(0-25) + 7(0-15) \ = 24 + 25 - 105 = -56$$

$$\therefore \ x = \frac{\Delta_1}{\Delta} = \frac{56}{56} = 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{168}{56} = 3$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-56}{56} = -1$$

Solution of Linear equations by Matrix Method

Given $$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

Consider $A=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ & $B=\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

then given equations can be written in Matrix form as $AX=B$. If $|A|\neq 0$ solution exists multiply both sides

$$A^{-1}(AX)=A^{-1}B$$
$$A^{-1}AX=A^{-1}B$$
ie $IX=A^{-1}B$

$\therefore\ X=A^{-1}B$

Example

Solve $3x-y+2z=13$
$$2x+y-z=3$$
$$x+3y-5z=-8$$

Let $A=\begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & -1 \\ 1 & 3 & -5 \end{pmatrix}$ $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B=\begin{pmatrix} 13 \\ 3 \\ -8 \end{pmatrix}$

then given equations can be written as $AX=B$

$$\therefore\ X = A^{-1}B$$

To find A^{-1}

$$|A| = \begin{vmatrix} 3 & -1 & 2 \\ 2 & 1 & -1 \\ 1 & 3 & -5 \end{vmatrix} = 3(-5+3)+1(-10+1)+2(6-1) = -6-9+10 = -5$$

$$\text{Matrix of Co-factors} = \begin{bmatrix} \begin{vmatrix} 1 & -1 \\ 3 & -5 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & -5 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 3 & -5 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & -5 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (-5+3) & -(-10+1) & (6-1) \\ -(5-6) & (-15-2) & -(9+1) \\ 1-2 & -(-3-4) & (3+2) \end{bmatrix} = \begin{bmatrix} -2 & 9 & 5 \\ 1 & -17 & -10 \\ -1 & -7 & 5 \end{bmatrix}$$

$$\text{adj}.A = \begin{pmatrix} -2 & 1 & -1 \\ 9 & -17 & 7 \\ 5 & -10 & 5 \end{pmatrix} \qquad \therefore\ A^{-1} = \frac{1}{|A|}\text{adj}.A = -\frac{1}{5}\begin{pmatrix} -2 & 1 & -1 \\ 9 & -17 & 7 \\ 5 & -10 & 5 \end{pmatrix}$$

Using this in (1)

$$X = -\frac{1}{5}\begin{pmatrix} -2 & 1 & -1 \\ 9 & -17 & 7 \\ 5 & -10 & 5 \end{pmatrix}\begin{pmatrix} 13 \\ 3 \\ -8 \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -26 & 3 & 8 \\ 117 & -51 & -56 \\ 65 & -30 & -40 \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -15 \\ 10 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$\therefore\ x = 3, y = -2, z = 1$ is the solution

Verification : Consider the first equation

$$3x - y + 2z = 9 + 2 + 2 = 13$$

Characteristic equation, Eigen Values & Eigen Vectors

Let A & I be square matrices of same order and λ a scalar then $|A - \lambda I| = 0$ is called **Characteristic equation** and the roots of this equation ie values of λ are called **Eigen Values or Characteristic roots.** The matrix X satisfying $AX = \lambda X$ is called **Eigen Vector.**

Example - 1

Find the eigen roots and eigen vectors of the matrix $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ Characteristic equation is $\left|\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 0$

ie $\begin{vmatrix} 1-\lambda & 4 \\ 4 & 3-\lambda \end{vmatrix} = 0 \quad \Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$

$\Rightarrow 3 - 3\lambda - \lambda + \lambda^2 - 8 = 0 \quad \Rightarrow \lambda^2 - 4\lambda - 5 = 0$

$$\Rightarrow (\lambda - 5)(\lambda + 1) = 0 \Rightarrow \lambda = -1, 5$$

$\therefore$ Eigen roots are -1 & 5.

To find eigen vector X, corresponding to -1,

$$AX = -1$$

$$\text{ie } \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x + 4y \\ 2x + 3y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

$$\Rightarrow \begin{matrix} x + 4y = -x \\ 2x + 3y = -y \end{matrix} \Rightarrow \begin{matrix} ix = -4y \\ \text{ie } x = -2y \end{matrix} \Rightarrow \therefore \frac{x}{-2} = \frac{y}{1}$$

$\therefore$ Eigen vector corresponding to eigen value -1 is $(-2, 1)$

To find the eigen vector corresponding to 5.

$$\text{ie } \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 5\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 + 4x_2 \\ 2x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} 5x_1 \\ 5x_2 \end{pmatrix}$$

$\therefore$ Eigen vector is (1, 1)

$\therefore$ Eigen vector corresponding to eigen root 5 is (1, 1)

Example - 2

Find the eigen roots and eigen vectors of the matrix $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

Let $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ Characteristic equation is $\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$

ie $(6-\lambda)\left[(3-\lambda)^2 - 1\right] + 2\left[-2(3-\lambda)+2\right] + 2\left[2-2(3-\lambda)\right] = 0$

ie $(6-\lambda)\left[9+\lambda^2-6\lambda-1\right] + 2\left[-6+2\lambda+2\right] + 2\left[2-6+2\lambda)\right] = 0$

ie $(6-\lambda)(\lambda^2-6\lambda+8) + 2(2\lambda-4) + 2(2\lambda-4) = 0$

ie $6\lambda^2 - 36\lambda + 48 - \lambda^3 + 6\lambda^2 - 8\lambda + 4\lambda - 8 + 4\lambda - 8 = 0$

which is the characteristic equation, by inspection 2 is a root

$\therefore$ dividing $\lambda^3 - 12\lambda^2 + 36\lambda - 32$ by $\lambda - 2$, we have

$$(\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$$

ie $(\lambda-2)(\lambda-2\lambda)(\lambda-8) = 0 \quad \therefore \ \lambda = 2, 2, 8$

To find eigen vector or $\lambda = 2$

Consider $AX = 2X$

$$\text{ie} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{where } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} 6x_1 - 2x_2 + 2x_3 &= 2x_1 \\ -2x_1 + 3x_2 - x_3 &= 2x_2 \\ 2x_1 - x_2 + 3x_3 &= 2x_3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 4x_1 - 2x_2 + 2x_3 &= 0 \\ -2x_1 + x_2 - x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 0 \end{aligned}$$

the above three equations represent one equation $2x_1 - x_2 + x_3 = 0$.

Let $x_3 = 0$, then $2x_1 = x_2$ ie $\frac{x_1}{1} = \frac{x_2}{2}$ ie $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{0}$

$\therefore$ Eigen Vector is (1, 2, 0)

To find the eigen vector for $\lambda = 8$

Consider $AX = 8X$

$$\text{ie} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8x_1 \\ 8x_2 \\ 8x_3 \end{pmatrix}$$

ie $6x_1 - 2x_2 + 2x_3 = 8x_1$ ie $-2x_1 - 2x_2 + 2x_3 = 0$

$-2x_1 + 3x_2 - x_3 = 8x_2$ $-2x_1 - 5x_2 - x_3 = 0$

$2x_1 - x_2 + 3x_3 = 8x_3$ $2x_1 - x_2 - 5x_3 = 0$

ie $x_1 + x_2 - x_3 = 0$

$2x_1 + 5x_2 + x_3 = 0$

$2x_1 - x_2 - 5x_3 = 0$

adding (1) & (2) we get $3x_1 + 6x_2 = 0$

ie $x_1 + 2x_2 = 0$ ie $x_1 = -2x_2$ $\therefore \frac{x_1}{-2} = \frac{x_2}{1} = K$ (Say)

then $x_1 = -2K,\ x_2 = K$

subsitituting in (1)

$-2K + K - x_3 = 0 \Rightarrow x_3 = -K$

$\therefore\ x_1 = -2K,\ x_2 = K\ \&\ x_3 = -K$

$\therefore\ x_1 = -2K,\ x_2 = K\ \&\ x_3 = -K$

$\therefore$ Eigen vector is $(-2, 1, -1)$ or $(2, -1, 1)$

$\therefore$ Eigen roots are 2, 2, 8

& Eigen vectors are **(1, 2, 0) & (2, –1, 1)**

Properties of Eigen values

(1) The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal.

(2) The product of the eigen values of a matrix is equal to the value of its determinant.

(3) If λ is an eigen value of A then $\frac{1}{\lambda}$ is the eigen value of A^{-1}.

Cayley - Hamilton Theorem

Every square matrix satisfies its characteristic equation.

Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Characteristic equation is $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}=0$

which on simplification becomes a quadric equation in λ in the form $\lambda^2+a_1\lambda+a_2=0$ where a_1, a_2 are constants. Cayley Hamilton Theorem states that $A^2+a_1A+a_2I=0$ where I is a unit matrix of order 2 & 0 is a null matrix of order 2.

If $A=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

then characteristic equation is $\begin{vmatrix} a_1-\lambda & b_1 & c_1 \\ a_2 & b_2-\lambda & c_2 \\ a_3 & b_3 & c_3-\lambda \end{vmatrix}=0$

which on simplification becomes $-\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0$ which is a cubic equation.

Then as per Cayley Hamilton Theorem $-A^3+a_1A^2+a_2A+a_3I=0$ where I is a unit matrix of order 3 & 0 is a null matrix of order 3.

In general if A is a square matrix of order n then characteristic equation will be of the form

$$(-1)^n\lambda^n+a\lambda^{n-1}+a_2\lambda^{n-z}+\dots\dots+a_nI=0$$

and by Cayley Hamilton Theorem

$$(-1)^n A^n + aA^{n-1} + a_2 A^{n-z} + \dots\dots + a_n I = 0$$

where I is a unit matrix of order n & 0 is a null matrix of order n.

<u>**Note**</u> :- If we put $\lambda = 0$ in the characteristic equation then $a_n = |A|$

$\therefore$ If $a_n = 0$, matrix A is singular & $a_n \neq 0$ the matrix A is non-singular & hence inverse exists and we can find the inverse of A using Cayley Hamilton Theorem.

<u>**Example - 1**</u>

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Characteristic equation is $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

ie $\lambda^2 + a_1\lambda + a_2 = 0$ where a_1, a_2 are constants.

By Cayley Hamilton Theorem

$$A^2 + a_1 A + a_2 I = 0$$

multiply both sides by A^{-1}

then $A^2 A^{-1} + a_1 A A^{-1} + a_2 A^{-1} = 0$

ie $A + a_1 I + a_2 A^{-1} = 0$

$$\therefore\ a_2 A^{-1} = -(A + a_1 I)$$

$$\therefore\ A^{-1} = -\frac{1}{a_2}(A + a_1 I)$$

Example - 2

The characteristic equation of a matrix A of order 2 is $\lambda^2 - 5\lambda + 10 = 0$ find $|A|$.

Solution : put $\lambda = 0$ in C.E. then the constant 10 is $|A|$.

Example - 3

Find the inverse of $\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$ using Cayley Hamilton Theorem.

Solution : Let $A = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$ C.E. is $\begin{vmatrix} 2-\lambda & -1 \\ -3 & 4-\lambda \end{vmatrix} = 0$

ie $(2-\lambda)(4-\lambda) - 3 = 0$ ie $8 - 4\lambda - 2\lambda + \lambda^2 - 3 = 0$

ie $\lambda^2 - 6\lambda + 5 = 0$

by Cayley Hamilton Theorem

$$A^2 - 6A + 5I = 0$$

multiply both sides by A^{-1}

$$A - 6I + 5A^{-1} = 0$$

$$\therefore\ 5A^{-1} = -A + 6I$$

$$A-6I+5A^{-1}=0$$

$$\therefore\ 5A^{-1}=-A+6I$$

$$=-\begin{bmatrix}2 & -1\\ -3 & 4\end{bmatrix}+6\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-2+6 & 1+0\\ 3+0 & -4+6\end{bmatrix}=\begin{bmatrix}4 & 1\\ 3 & 2\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{5}\begin{bmatrix}4 & 1\\ 3 & 2\end{bmatrix}$$

Diagonalisation of Matrices

If A is a square matrix of order n where all the eigen values are linearly independent then a matrix P can be found such that $P^{-1}AP$ is a Diagonal Matrix.

Let A be a square matrix of order 3 and let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values, corresponding to these. Let X_1, X_2, X_3 be three vectors where

$$X_1=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix},\ X_2=\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix},\ X_3=\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}$$

$$\text{Let } P=\begin{bmatrix}x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{bmatrix}\ \text{ Then } P^{-1}AP=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}$$

Example - 1

$$\text{Let } A=\begin{pmatrix}1 & -2\\ -5 & 4\end{pmatrix}$$

$$\text{C.E. is}\begin{vmatrix}1-\lambda & -2\\ -5 & 4-\lambda\end{vmatrix}=0\Rightarrow(1-\lambda)(4-\lambda)-10=0$$

ie $\lambda^2-5\lambda-6=0 \Rightarrow (\lambda+1)(\lambda-6)=0 \quad \Rightarrow \lambda=-1,6$ are eigen values

For $\lambda=-1$, let $AX=-X$

$$\text{ie } \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$$

$$\text{ie } \left.\begin{array}{l} x_1-2x_2=-x_1 \\ -5x_1+4x_2=-x_2 \end{array}\right\} \Rightarrow x_1=x_2 \quad \therefore \frac{x_1}{1}=\frac{x_2}{1} \text{....eigen vector is} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda=6$, $AX=6X$

$$\text{ie } \begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 6x_1 \\ 6x_2 \end{pmatrix}$$

$$\left.\begin{array}{l} x_1-2x_2=-x_1 \\ -5x_1+4x_2=-x_2 \end{array}\right\} \Rightarrow 5x_1=-2x_2$$

$$\text{ie } \frac{x_1}{-2}=\frac{x_2}{5} \quad \therefore \text{ eigen vector is } \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

Let $P=\begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}$ Then $P^{-1}=\frac{1}{7}\begin{bmatrix} 5 & 2 \\ -1 & 1 \end{bmatrix}$

$$\therefore\ P^{-1}AP = \frac{1}{7}\begin{bmatrix} 5 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 5-10 & -10+8 \\ -1-5 & 2+4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -5 & -2 \\ -6 & 6 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} -5-2 & 10-10 \\ -6+6 & 12+30 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -7 & 0 \\ 0 & 42 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}$$

Thus $P = \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}$ diagonalize the matrix $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$

<u>Example - 2</u>

Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

Characteristic equation is $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

ie $(1-\lambda)[(5-\lambda)(1-\lambda)-1]-1[1-\lambda-3]+3[1-3(5-\lambda)]=0$

ie $(1-\lambda)[(5-\lambda)(1-\lambda)-1]-1[1-\lambda-3]+3[1-3(5-\lambda)]=0$

ie $(1-\lambda)(\lambda^2-6\lambda+4)-(-\lambda-2)+3(-14+3\lambda)=0$

ie $\lambda^2-6\lambda+4-\lambda^3+6\lambda^2-4\lambda+\lambda+2-42+9\lambda=0$

ie $-\lambda^3+7\lambda^2-36=0$ ie $\lambda^3-7\lambda^2+36=0$

by inspection -2 is a root $\therefore$ $\lambda+2$ is a factor. $\therefore$ equation becomes

$(\lambda+2)(\lambda^2-9\lambda+18)=0$

ie $(\lambda+2)(\lambda-3(\lambda-6)=0 \quad \therefore \lambda=-2,3,6$

ie characteristic roots are $-2, 3, 6$.

To find the eigen vector for $\lambda=-2$ Consider $AX_1=-2X_1$

where $X_1=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\text{ie} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} -2x_1 \\ -2x_2 \\ -2x_3 \end{pmatrix}$$

$$\begin{array}{llll} \text{ie} & x_1+x_2+3x_3=-2x_1 & \text{ie} & 3x_1+x_2+3x_3=0 \\ & x_1+5x_2+x_3=-2x_2 & & x_1+7x_2+x_3=0 \\ & 3x_1+x_2+x_3=-2x_3 & & 3x_1+x_2+3x_3=0 \end{array}$$

(1) & (3) are same.

Put $x_2=0$ in (1) or (2), then $x_1+x_3=0$

ie $x_1=-x_3 \Rightarrow \dfrac{x_1}{-1}=\dfrac{x_3}{1} \quad \therefore$ eigen vector $X_1=\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Let X_2 be the eigen vector for $\lambda = 3$.

ie $AX_2 = 3X_2$

$$\text{ie} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3y_1 \\ 3y_2 \\ 3y_3 \end{pmatrix} \text{ where } X_2 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\text{ie} \quad y_1 + y_2 + 3y_3 = 3y_1 \quad \text{ie} \quad -2y_1 + y_2 + 3y_3 = 0 \quad (1)$$

$$y_1 + 5y_2 + 3y_3 = 3y_2 \qquad y_1 + 2y_2 + y_3 = 0 \quad (2)$$

$$3y_1 + y_2 + 3y_3 = 3y_3 \qquad 3y_1 + y_2 - 2y_3 = 0 \quad (3)$$

Let us eliminate y_1 from (1) & (2)

$$(1)\times 1 \text{ is} \quad -2y_1 + y_2 + 3y_3 = 0$$

$$(2)\times 2 \text{ is} \quad \underline{2y_1 + 4y_2 + 2y_3 = 0}$$

$$\text{adding} \quad 5y_2 + 5y_3 = 0$$

$$\Rightarrow y_2 + y_3 = 0 \quad \text{ie } y_2 = -y_3 \Rightarrow \frac{y_2}{-1} = \frac{y_3}{1} \quad (4)$$

Let us eliminate y_2 from (2) & (3)

$$(2)\times 1 \text{ is} \quad y_1 + 2y_2 + y_3 = 0$$

$$(3)\times 2 \text{ is} \quad \underline{6y_1 + 2y_2 - 4y_3 = 0}$$

$$\text{subtracting} \quad -5y_2 + 5y_3 = 0$$

$$\Rightarrow 5y_1 = 5y_3 \Rightarrow y_1 = y_3 \quad \therefore \frac{y_1}{1} = \frac{y_3}{1}$$

From (4) & (5) $\frac{y_1}{1}=\frac{y_2}{-1}=\frac{y_3}{1}$ $\therefore\ X_2=\begin{bmatrix}1\\-1\\1\end{bmatrix}$

Next, let X_3 be the eigen vector for $\lambda = 6$

ie $AX_3 = 6X_3$

ie $\begin{pmatrix}1 & 1 & 3\\1 & 5 & 1\\3 & 1 & 1\end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}6z_1\\6z_2\\6z_3\end{pmatrix}$ where $X_3=\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}$

ie $z_1+z_2+3z_3=6z_1$ ie $-5z_1+z_2+3z_3=0$ (1)

$z_1+5z_2+z_3=6z_2$ $z_1-z_2+z_3=0$ (2)

$3z_1+z_2+z_3=6z_3$ $3z_1+z_2-5z_3=0$ (3)

adding (1) & (2), $-4z_1+4z_3=0$

ie $z_1=z_3$ $\therefore\ \frac{z_1}{1}=\frac{z_3}{1}$ (4)

Let us eliminate z_3 from (2) & (3)

$(2)\times 5$ is $5z_1-5z_2+5z_3=0$

$$\begin{array}{ll} (3)\times 1 \text{is} & \underline{3z_1 + z_2 + -5y_3 = 0} \\ \text{adding} & \quad 8z_1 - 4z_2 = 0 \end{array}$$

$$\text{ie } 2z_1 = z_2 \Rightarrow \frac{z_1}{1} = \frac{z_2}{2}$$

From (4) & (5), $\frac{z_1}{1} = \frac{z_2}{2} = \frac{z_3}{1} \quad \therefore \; X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Let $P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ Then $P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

$\therefore \; P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ diagonalize $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Exercise

1. Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix}$.

2. Evaluate $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$.

UNIT - 2

Limits of functions

Consider $y = f(x) = \frac{x^2 - 1}{x - 1}$ the function is defined for all values of x except for $x = 1$.

$\therefore$ for $x = 1$, $f(x) = \frac{1-1}{1-1} = \frac{0}{0}$ which is indeterminate.

Let us consider the values of $f(x)$ as x approaches 1

x	$f(x) = \frac{x^2 - 1}{x - 1}$
.9	1.9
.99	1.99
.999	1.999
1.01	2.01
1.001	2.001
1.0001	2.0001

It can be seen from the above values that as x approaches 1, $\frac{x^2 - 1}{x - 1}$ approaches 2.

This value 2 is called "Limit of $\frac{x^2-1}{x-1}$ as x approaches 1" which can be written as $\lim\limits_{x\to 1}\frac{x^2-1}{x-1}$ or $\underset{x\to 1}{Lt}\,\frac{x^2-1}{x-1}=2.$

In general the limit of a function $f(x)$ as x approaches a is denoted as l and which is written as

$$\lim_{x\to a} f(x)=l \text{ or } \underset{x\to a}{Lt}\, f(x)=l$$

Properties

(1) $\lim\limits_{x\to a}\left[f(x)\pm g(x)\right]=\lim\limits_{x\to a} f(x)\pm\lim\limits_{x\to a} g(x)$

(2) $\lim\limits_{x\to a}\left[f(x)g(x)\right]=\lim\limits_{x\to a} f(x)\lim\limits_{x\to a} g(x)$ in particular $\lim\limits_{x\to a} kf(x)=k\cdot\lim\limits_{x\to a} f(x)$ where k is a contant

(3) $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ provided $\lim\limits_{x\to a} g(x)\neq 0.$

Standard Limits

(1) $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$

(2) $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ (θ in radians) also $\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1$

(3) $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \quad 2 < e < 3$ or $\lim_{x \to 0} (1+x)^{1/x} = e$

(4) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a \quad (a > 0)$ in particular $\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$

Examples

(1) $\lim_{x \to 0} \frac{x^2 + 4x + 3}{x^2 - 5x + 4} = \frac{0 + 0 + 3}{0 - 0 + 4} = \frac{3}{4}$

(2) $\lim_{x \to \infty} \frac{2x^2 - 3x + 4}{3x^2 + 2x + 1}$ (dividing Nr & Dr by x^2) $= \lim_{x \to \infty} \frac{2 - \frac{3}{x} + \frac{4}{x^2}}{3 + \frac{2}{x} + \frac{1}{x^2}} = \frac{2 - 0 + 0}{3 + 0 + 0} = \frac{2}{3}$

(3) $\lim_{x \to 3} \frac{x^4 - 81}{x - 3} = \lim_{x \to 3} \frac{x^4 - 3^4}{x - 3} = 4(3)^3 = 108$

(4) $$\lim_{x\to -a}\frac{x^7+a^7}{x^5+a^5}=\lim_{x\to -a}\frac{\frac{x^7+a^7}{x+a}}{\frac{x^5+a^5}{x+a}}=\lim_{x\to -a}\frac{\frac{x^7-(-a)^7}{x-(-a)}}{\frac{x^5-(-a)^5}{x-(-a)}}=\frac{7(-a)^6}{5-(-a)^4}=\frac{7}{5}a^2$$

(5) $$\lim_{x\to 0}\frac{\sin 7x}{x}=\lim_{x\to 0}\frac{\sin 7x}{x}\times 7=1\times 7=7$$

(6) $$\lim_{\theta\to 0}\frac{1-\cos 2\theta}{\theta^2}=\lim_{\theta\to 0}\frac{2\sin^2\theta}{\theta^2}=2$$

(7) $$\lim_{x\to 0}\frac{\tan 3x-x}{3x-\sin x}\ \text{(dividing Nr \& Dr by } x)=\lim_{x\to 0}\frac{\frac{\tan 3x}{x}-1}{3-\frac{\sin x}{x}}=\frac{3-1}{3-1}=1$$

(8) $$\lim_{x\to 0}(1+ax)^{b/x}=\lim_{x\to 0}\left[(1+ax)^{\frac{1}{ax}}\right]^{ab}=e^{ab}$$

(9) $$\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n=\lim_{n\to\infty}\left[\left(1+\frac{3}{n}\right)^{n/3}\right]^3=e^3$$

(10) $$\lim_{x\to 0}(1-2x)^{1/x}=\lim_{x\to 0}\left[(1-2x)^{-\frac{1}{2x}}\right]^{-2}=e^{-2}$$

(11) $$\lim_{x\to 0}\frac{a^x-b^x}{x}=\lim_{x\to 0}\frac{(a^x-x)-(b^x-x)}{x}=\lim_{x\to 0}\frac{a^x-x}{x}-\lim_{x\to 0}\frac{b^x-x}{x}=\log_e a-\log_e b=\log_e\frac{a}{b}$$

(12) $\lim\limits_{x\to 0}\dfrac{2^x-1}{\sin x}=\lim\limits_{x\to 0}\dfrac{\dfrac{2^x-1}{x}}{\dfrac{\sin x}{x}}=\dfrac{\log_e 2}{1}=\log_e 2$

Continuity of a function

A function $f(x)$ is said to be continuousat $x=a$ if $\lim\limits_{x\to a} f(x)=f(a)$

$\therefore$ A function $f(x)$ is said to be continuous if $\lim\limits_{x\to 0} f(x)$ exists, $f(a)$ exists and they are equal.

If these donot happen then the function is said to be not continuous or discontinuous.

A function $f(x)$ is said to be continuous in an interval if it is continuous at all points in the interval.

Examples

(1) $f(x)=\dfrac{x^2-1}{x-1}$ is not continuous at $x=1$ $\because \lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}=2$ exists but $f(0)=\dfrac{0}{0}$ donot exists.

where as it is continuous at all other values of x.

(2) Discuss the continuity of function $f(x)=\begin{cases}4x+3 & \text{for } x\geq 4\\ 3x+7 & \text{for } <4\end{cases}$ at $x = 4$.

Solution : While finding the limit of a function $f(x)$ as x approaches a, if we consider the limit of the function as x approaches a from left hand side, the limit is called **'Left Hand Limit' (LHL)** and if x approaches a from right hand side the limit is called **'Right Hand Limit' (RHL)** and the limit of the function is said to exists if both LHL & RHL exists and are equal, for convenience LHL & RHL are denoted as $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ and further

$$\text{LHL} = \lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h) \ \ \& \ \ \text{RHL} = \lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$$

For the given problem

LHL at $x=4$ is $\lim_{x\to 4^-} f(x) = \lim_{h\to 0} f(4-h) = \lim_{h\to 0} 3(4-h)+7 = 19$

RHL at $x=4$ is $\lim_{x\to 4^+} f(x) = \lim_{h\to 0} f(4+h) = \lim_{h\to 0} 4(4+h)+3 = 19$

and $f(4)=19$

$\therefore$ The function is continuous at $x=4$

(3) Examine the continuity of $f(x)=\begin{cases}\dfrac{\sin x}{x} & \text{for } x\neq 0\\ 2 & \text{for } x=0\end{cases}$ at $x = 0$.

Solution : $\lim_{x\to 0}\dfrac{\sin x}{x}=1$, but $f(0)=2$

$\therefore$ The function is discontinuous at $x=0$.

(4) Examine the continuity of the function $f(x)=\begin{cases}5x-4 & \text{for } 0<x\leq 1\\ 4x^2-2x & \text{for } 1<x<2\\ 4x+4 & \text{for } x\geq 2\end{cases}$

at $x=1$ and $x=2$.

at $x=1$, $\text{LHL} = \lim\limits_{x\to 1^-} f(x) = \lim\limits_{h\to 0} f(1-h) = \lim\limits_{h\to 0} 5(1-h)-4 = 1$

$\text{RHL} = \lim\limits_{h\to 0} f(1+h) = \lim\limits_{h\to 0} 4(1+h)^2 - 2(1+h) = 4-2 = 2$

$\text{LHL} \neq \text{RHL}$ at $x=1$

$\therefore$ The function is discontinuous at $x=1$.

at $x=2$, $\text{LHL} = \lim\limits_{h\to 0} f(2-h) = \lim\limits_{h\to 0} 4(2-h)^2 - 2(2-h) = 16-4 = 12$

$\text{RHL} = \lim\limits_{h\to 0} f(2+h) = \lim\limits_{h\to 0} 4(2+h)+4 = 12$

and $f(2) = 4\times 2+4 = 12$

$\therefore \lim\limits_{x\to 2} f(x) = f(2)$

$\therefore$ The function is continuous at $x=2$.

UNIT – 3

Differentiation

Differentiation is all about finding rates of change of one quantity compared to another. We need differentiation when the rate of change is not constant.

First, let's take an example of a car travelling at a constant 60 km/h.

We notice that the distance from the starting point increases at a constant rate of 60 km each hour, so after 5 hours we have travelled 300 km. We notice that the slope (gradient) is always 300/5 = 60. There is a constant rate of change of the distance compared to the time. The slope is positive all the way

Rate of Change that is Not Constant

Now let's throw a ball straight up in the air. Because gravity acts on the ball it slows down, then it reverses direction and starts to fall. All the time during this motion the velocity is changing. It goes from positive (when the ball is going up), slows down to zero, then becomes negative (as the ball is coming down). During the "up" phase, the ball has negative acceleration and as it falls, the acceleration is positive.

Development of Differential Calculus

Up until the time of Newton and Leibniz, there was no reliable way to describe or predict this constantly changing velocity. There was a real need to understand how constantly varying quantities could be analysed and predicted. That's why they developed differential calculus, which we will learn about in the next few chapters.

There are many applications of differentiation in science and engineering. You can see some of these in Applications of Differentiation.

Differentiation is also used in analysis of finance and economics.

One important application of differentiation is in the area of optimisation, which means finding the condition for a maximum (or minimum) to occur. This is important in business (cost reduction, profit increase) and engineering (maximum strength, minimum cost.)

Derivative of a function

In mathematics, the derivative is a way to show rate of change: that is, the amount by which a function is changing at one given point. ... The derivative is often written using "dy over dx" (meaning the difference in y divided by the difference in x) and is given by

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The Derivative of a Constant

If f(x)=c where c is a constant, then f'(x) = 0

The Power Rule

If n is a positive integer, then for all real values of x.

$$\frac{d(x^n)}{dx} = nx^{n-1}$$

Derivatives of Sums and Differences

If f and g are two differentiable functions at x, then

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

And

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

The Product Rule

If f and g are differentiable at x, then

$$\frac{d}{dx}[f(x).g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

The Quotient Rule

If f and g are differentiable functions at x and $g(x)\neq 0$, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Some basic formulas of differentiation

$$\frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$$

$$\frac{d}{dx}(\text{sechx}) = \text{sechxtanhx}$$

$$\frac{d}{dx}(\text{cosechx}) = -\text{cosechxcothx}$$

Illustrative Examples

1. Find the derivatives of the following functions with respect to x

(i) x^{-8} (ii) 5^x (iii) $5x^{-7}$ (iv) $(x^{1/2} + 1/x^{1/2})^2$

(v) $\frac{a\cos x + b\sin x + c}{\sin x}$

Solution. We know power formula $\frac{d(x^n)}{dx} = nx^{n-1}$

(i) $\frac{d(x^{-8})}{dx} = -8x^{-8-1} = -8x^{-9}$

(ii) We know that $\frac{d}{dx}(a^x) = a^x \log a$

Therefore $\frac{d}{dx}(5^x) = 5^x \log 5$

(iii) $\frac{d(5x^{-7})}{dx} = 5\frac{d(x^{-7})}{dx} = 5(-7)x^{-8} = -35x^{-8}$

(iv) $(x^{1/2} + 1/x^{1/2})^2 = x + \frac{1}{x} + 2$

$$\frac{d}{dx}(x+\frac{1}{x}+2) = \frac{d}{dx}(x) + \frac{d}{dx}(\frac{1}{x}) + \frac{d}{dx}(2) = 1 + (-1)x^{-2} + 0 = 1 - x^{-2}$$

(v) $\frac{d}{dx}(\frac{a\cos x + b\sin x + c}{\sin x}) = \frac{d}{dx}(a\cot x) + \frac{d}{dx}(b) + \frac{d}{dx}(c\ \mathrm{cosec}x)$

$= -a\,\mathrm{cosec}^2x + 0 - c\,\mathrm{cosec}x\cot x$

2. Find the derivatives of the following functions with respect to x

(i) $X^2\log x$ (ii) $\frac{e^x}{1+\sin x}$ (iii) $\sin x\log x$ (iv) $\frac{x^2+2x+4}{x+4}$

Solution. By using product formula

$$\frac{d}{dx}(x^2\log x) = x^2\frac{d}{dx}(\log x) + \log x\,.\frac{d}{dx}(x^2)$$

$$= x^2\frac{1}{x} + \log x\,.2x = x + 2x\log x$$

By using quotient formula

$$\frac{d}{dx}\left(\frac{e^x}{1+\sin x}\right) = \frac{(1+\sin x)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2} = \frac{(1+\sin x)e^x - e^x\cos x}{(1+\sin x)^2}$$

$$= \frac{e^x(1+\sin x-\cos x)}{(1+\sin x)^2}$$

Rest of the parts may be solved accordingily

Derivative of function of function (chain rule)

3. Find the derivatives of the following functions

(i) $\tan^3 x$ (ii) $e^{\tan^{-1}x}$

Solution (i) $\frac{d}{dx}(\tan^3 x) = 3\tan^2 x\frac{d}{dx}(\tan x) = 3\tan^2 x.\sec^2 x$

(ii) $\frac{d}{dx}(e^{\tan^{-1}x}) = e^{\tan^{-1}x}\frac{d}{dx}(\tan^{-1}x) = e^{\tan^{-1}x}\frac{1}{1+x^2}$

Logarithmic differentiation – Whenever we are required to differentiate a function of x is raised to a power which itself is a function of x, neither the formula for a^x nor for x^n is applicable .In such cases we first take logarithm of the function and then differentiate.This process is called logarithmic differentiation.

Differentiate the function with respect to x

Let $y = x^{\sin x}$

Taking log of both sides

$\log y = \log(x^{\sin x}) = \sin x \log x$

Differentiating with respect to x $\frac{1}{y}\frac{dy}{dx} = \sin x.\frac{1}{x} + \log x \cos x$

$$\frac{dy}{dx} = y(\sin x.\frac{1}{x} + \log x \cos x)$$

$$= x^{\sin x}(\sin x.\frac{1}{x} + \log x \cos x)$$

Also remember $\frac{d(\sinh^{-1}x)}{dx} = \frac{1}{\sqrt{x^2+1}}$

$$\frac{d(\cosh^{-1}x)}{dx} = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d(\tanh^{-1}x)}{dx} = \frac{1}{1-x^2}$$

Parametric Equations – If the parametric equations are $x = f(t)$ and $y = g(t)$

Then $\frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt}$

Find $\frac{dy}{dx}$ when $x = a\cos^3 t$ and $y = a\sin^3 t$

Solution. Here $\frac{dx}{dt} = 3a\cos^2 t(-\sin t)$ and $\frac{dy}{dt} = 3a\sin^2 t\cos t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$$

MEAN VALUE THEOREMS

Rolle's Theorem

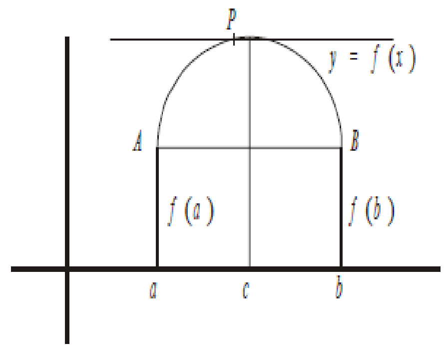
<u>Statement :</u> If $f(x)$ is a function

(i) Continuous in the closed interval $a \le x \le b$

(ii) differentiable in the open interval $a < x < b$ and

(iii) $f(a) = f(b)$ then there exists at least one value c of x such that $f'(c) = 0$ for $a < c < b$.

Geometrical Meaning



A & B are points on the curve such that $f(a) = f(b)$. Tangent at P is parallel to AB such that the slope of the tangent is $f'(c) = 0$

Lagrange's Mean Value Theorem

<u>Statement :</u> If $f(x)$ is a function

(i) Continuous in the closed interval $a \le x \le b$

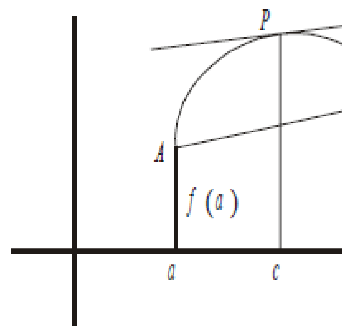
(ii) differentiable in the open interval $a < x < b$

then there exists at least one value c of x in the interval such that $\dfrac{f(b)-f(a)}{b-a} = f'(c)$

Note :- If the interval is $(a, a+h)$ then $f(a+h) - f(a) = f'(c)$

ie $f(a+h) = f(a) + hf'(c)$

Geometrical Meaning



A & B are points on the curve corresponding to $x = a$ & $x = b$. Join AB There will be a tangent at P, parallel to AB so that scope of the tangent is $\dfrac{f(b)-f(a)}{b-a}$ which is $f'(c)$

Cauchy's Mean Value Theorem

__Statement :__ If $f(x)$ & $g(x)$ are two functions which are

(i) continuous in the closed interval $[a, b]$

(ii) differentiable in the open interval (a, b) and

(iii) $g'(x) \neq 0$ for any value of x in (a, b)

then there exists at least one value c of x in (a, b) such that $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$

Taylor's Theorem

If $f(x)$ is a function such that (i) $f(x)$ and its $(n-1)$ derivatives are continuous in the closed interval $[a, a+h]$

ie $a \leq x \leq a+h$ and (ii) n^{th} derivative $f^{(n)}(x)$ exists in the open interval $(a, a+h)$.

Then there exists at least one number θ $(0<\theta<1)$ such that

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{21}f''(a)+\cdots+\cdots+\frac{h^n}{n!}f^{(n)}(a+\theta h)$$

Note: - put $a=0$ & $h=x$, then $f(x)=f(0)+xf'(0)+\frac{x^2}{21}f''(0)+\cdots+\cdots+\frac{x^n}{n!}f^{(n)}(\theta x)$, where $0<\theta<1$

This expression for $f(x)$ is called **Maclaurin's Expansion** and further if $\frac{x^n}{n!}f(\theta x)$ tends to zero as $n\to\infty$.

Then $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots\cdots$ to ∞. This is called **Maclaurin's Series** for $f(x)$.

Examples

(1) Verify Rolle's Theorem for $f(x)=(x+2)^3(x-3)^4$ in $(-2, 3)$ and find c.

Solution: $f(x)$ is continuous in the closed interval $[-2, 3]$ & differentiable in $(-2, 3)$ and further $f(-2)=0$, $f(3)=0$

ie $f(-2)=f(3)$

$\therefore$ There exists a value c in $(-2, 3)$ such that $f'(c)=0$

$$f'(x) = 3(x+2)^2(x-3)^4 + 4(x+2)^3(x-3)^3$$

$$f'(c) = 3(c+2)^2(c-3)^4 + 4(c+2)^3(c-3)^3 = (c+2)^2(c-3)^3[3(c-3)+4(c+2)] = (c+2)^2(c-3)^2(7c-1)$$

$$f'(c) = 0 \Rightarrow 7c-1=0 \Rightarrow c = \frac{1}{7}$$

(2) Find 'c' of the Lagrange's Mean Value Theorem for the function $f(x) = (x-1)(x-2)(x-3)$ in $(0, 4)$

Solution: $f(0) = (-1)(-2)(-3) = -6,\ f(4) = 3\times 2\times 1 = 6$

$$f'(x) = (x-2)(x-3)+(x-1)(x-3)+(x-1)(x-2)$$

$$f'(c) = (c-2)(c-3)+(c-1)(c-3)+(c-1)(c-2) = c^2-5c+b+c^2-4c+3+c^2-3c+2 = 3c^2-12c+11$$

By Lagrange's Theorem $\frac{f(b)-f(a)}{b-a} = f'(c)$

$$\therefore\ \frac{f(4)-f(0)}{4-0} = f'(c)$$

$$\therefore\ \frac{6+6}{4} = 3c^2-12c+11$$

ie $12c^2-48c+44=12$

ie $12c^2-48c+32=0 \Rightarrow 3c^2-12c+8=0$

$$\therefore\ c = \frac{12\pm\sqrt{144-96}}{6} = \frac{12\pm\sqrt{48}}{6} = \frac{12\pm 4\sqrt{3}}{6}$$

$$\therefore\ c = 2\pm\frac{2}{\sqrt{3}}\ \text{ ie }\ c = 2-\frac{2}{\sqrt{3}}\ \ \&\ \ 2+\frac{2}{\sqrt{3}}$$

(3) Using Maclaurin's Series Express $\sin x$ & $\cos x$ as an infinite series.

Solution : Let $f(x) = \sin x$

differentiating n times $f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)$

put $x = 0$, $\therefore$ $f^{(n)}(0) = \sin\frac{n\pi}{2}$

$\therefore$ $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$, $f^{\text{iv}}(0) = 0$

$$\therefore\ f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{\text{iv}}(0) + \cdots\cdots = 0 + x + 0 + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}0$$

$$\therefore\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}\cdots\cdots$$

& let $g(x) = \cos x$, $g^{(n)}(x) = \cos\left(x + \frac{n\pi}{2}\right)$

put $x = 0$, $g(0) = 1$, $g^{(n)}(0) = \cos\frac{n\pi}{2}$

$$\therefore\ g(x) = g(0) + xg_2'(0) + \frac{x^2}{2} g''(0)\cdots\cdots$$

$$\therefore\ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\cdots\cdots$$

(4) Find the Maclaurin's Series for e^x.

Solution: Let $f(x) = e^x,\ f^{(n)}(x) = e^x$

put $x = 0,\ f^{(n)}(0) = e^0 = 1$

$f(0) = 1,\ f'(0) = 1,\ f''(0) = 1, \cdots\cdots$

$$\therefore\ f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots\cdots$$

ie $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$

Exercise

1. Examine the application of Rolle's Theorem for $f(x) = 2 + (x+1)^{2/3}$ in the interval $(0, 2)$

(Answer: Rolle's Theorem donot apply because $f'(1)$ donot exists.)

2. Find 'c' of the Mean Value Theorem for $f(x) = x(x-1)(x-2)$ $a = 0,\ b = \frac{1}{2}$

$\left(\text{Answer: } c = 1 - \frac{\sqrt{21}}{6}\right)$

3. Show that $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\cdots\cdots$ using Maclaurin's Series.

Indeterminate Forms

While evaluating $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ or $\lim\limits_{x\to a}[f(x)-g(x)]$ or $\lim\limits_{x\to a} f(x)^{g(x)}$ when it takes the forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$, 1^{∞}, ∞^{0}, 0^{0} the

are called Indeterminate forms and to evaluate such forms the following rule known as **L' Hospital's Rule** is used.

L' Hospital's Rule

If $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ again if this is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f''(x)}{g''(x)}$ whenever it is of the $\frac{0}{0}$ or $\frac{\infty}{\infty}$ This rule can be applied.

To evaluate $\lim\limits_{x\to a}[f(x)-g(x)]$ when it is of the form $\infty-\infty$, then

Consider $\lim\limits_{x\to a}[f(x)-g(x)]=\lim\limits_{x\to a}\dfrac{\frac{1}{g(x)}-\frac{1}{f(x)}}{\frac{1}{g(x)f(x)}}$ which is of the form $\frac{0}{0}$ **&** hence L' Hospital's Rule can be applied.

Consider $\lim\limits_{x\to a} f(x)^{g(x)}=y$ (say) then $\log y = \lim_{x \to a} \log f(x)^{g(x)} = \lim_{x \to a} g(x) \log f(x) = \lim_{x \to a} \dfrac{\log f(x)}{\dfrac{1}{g(x)}}$

which is of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ and hence L'Hospital's rule can be applied.

Examples

(1) Evaluate $\lim_{x \to 1} \dfrac{\log x}{x^2 - 3x + 2}$

Solution : $\lim_{x \to 1} \dfrac{\log x}{x^2 - 3x + 2}$ this is of the form $\dfrac{0}{0}$ $\therefore$ using L'Hospital's rule

$$= \lim_{x \to 1} \frac{\frac{1}{x}}{2x - 3} = \frac{1}{2 - 3} = -1$$

(2) Evaluate $\lim_{x \to 0} \dfrac{2 \sin x - \sin 2x}{x^3}$

Solution : $\lim_{x \to 0} \dfrac{2 \sin x - \sin 2x}{x^3}$ This is of the form $\dfrac{0}{0}$ $\therefore$ applying L'Hospital's rule

$= \lim_{x \to 0} \dfrac{2 \cos x - 2 \cos 2x}{3x^2}$ again it is of the form $\dfrac{0}{0}$ $\therefore$ applying L'Hospital's rule again

$$= \lim_{x \to 0} \frac{-2 \sin x + 4 \sin 2x}{6x} = \lim_{x \to 0} \left[-\frac{2}{6} \cdot \frac{\sin x}{x} + \frac{4}{3} \frac{\sin 2x}{2x} \right] = -\frac{2}{6} \times 1 + \frac{4}{3} \times 1 \quad \because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

$$= -\frac{1}{3} + \frac{4}{3} = 1$$

(3) Evaluate $\lim_{x\to 0}\frac{\tan x - x}{x^2 \tan x}$

Solution : $\lim_{x\to 0}\frac{\tan x - x}{x^2 \tan x} = \lim_{x\to 0}\frac{\tan x - x}{x^3}\times\frac{x}{\tan x} = \lim_{x\to 0}\frac{\tan x - x}{x^3} \because \lim_{x\to 0}\frac{x}{\tan x} = 1$

This is of the form $\frac{0}{0}$ $\therefore$ using L'Hospital's rule

$$= \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2} = \frac{0}{0} \quad \therefore \text{ using the rule again}$$

$$= \lim_{x\to 0}\frac{2\sec x \cdot \sec x \tan x}{6x} = \lim_{x\to 0}\frac{2\sec^2 x}{6}\times\frac{\tan x}{x} = \frac{2}{6}\times 1 = \frac{1}{3}$$

(4) Evaluate $\lim_{x\to 0}\left[\frac{1}{x} - \frac{1}{e^x - 1}\right]$

Solution : This is of the form $\infty - \infty$

$$\therefore \lim_{x\to 0}\left[\frac{1}{x} - \frac{1}{e^x - 1}\right] = \lim_{x\to 0}\frac{(e^x - 1) - x}{(e^x - 1)x} \text{ which is of the form } \frac{0}{0} \quad \therefore \text{ using L'Hospital's rule}$$

$$= \lim_{x\to 0}\frac{e^x - 1}{e^x - 1 + xe^x}$$

which is again of the form $\frac{0}{0}$ $\therefore$ using the rule again

$$= \lim_{x\to 0}\frac{e^x}{e^x + xe^x + e^x} = \frac{1}{1+0+1} = \frac{1}{2}$$

(5) Evaluate $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\log x}\right)$

Solution : This of the form $\infty-\infty$

$$\therefore \lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\log x}\right)=\lim_{x\to 1}\frac{\log x-\frac{x-1}{x}}{\log x\left(\frac{x-1}{x}\right)}$$

This is of the form $\frac{0}{0}$ $\therefore$ using L' Hospital's rule

$$=\lim_{x\to 1}\frac{\frac{1}{x}-\frac{1}{x^2}}{\log x\times\left(\frac{1}{x^2}\right)+\frac{x-1}{x}\times\frac{1}{x}}=\lim_{x\to 1}\frac{x-1}{\log x+x-1}$$ is $\frac{0}{0}$ form again applying the L' Hospital

$$=\lim_{x\to 1}\frac{1}{\frac{1}{x}+1}=\frac{1}{2}$$

(6) Evaluate $\lim_{x\to\frac{\pi}{2}}(\tan x)^{\cot x}$

Solution : This is of the form ∞^0

$$\text{Let } y = \lim_{x \to \frac{\pi}{2}} (\tan x)^{\cot x}$$

$$\log y = \lim_{x \to \frac{\pi}{2}} \log(\tan x)^{\cot x} = \lim_{x \to \frac{\pi}{2}} \cot x \log \tan x = \lim_{x \to \frac{\pi}{2}} \frac{\log \tan x}{\tan x} \quad \text{using L'Hospital's rule}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\tan x} \cdot \sec^2 x}{\sec^2 x} = \lim_{x \to \frac{\pi}{2}} \frac{1}{\tan x} = \lim_{x \to \frac{\pi}{2}} \cot x = 0$$

$$\therefore\ y = e^0 = 1$$

(7) Evaluate $\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{1/x}$

Solution : This is of the form 1^∞

$$\text{Let } y = \lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{1/x}$$

$$\log y = \lim_{x\to 0}\frac{1}{x}\log\left(\frac{\sin x}{x}\right) = \lim_{x\to 0}\frac{\log\left(\frac{\sin x}{x}\right)}{x}$$

This is of the form $\frac{0}{0}$ $\therefore$ applying the L'Hospital's rule

$$= \lim_{x\to 0}\frac{\frac{x}{\sin x}\times\frac{x\cos x - \sin x}{x^2}}{1} = \lim_{x\to 0}\frac{x}{\sin x}\times\frac{x\cos x - \sin x}{x^2} = 1\times\lim_{x\to 0}\frac{x\cos x - \sin x}{x^2}$$

This is of the form $\frac{0}{0}$ $\therefore$ again applying the L'Hospital's rule

$$= \lim_{x\to 0}\frac{\cos x - x\sin x - \cos x}{2x} = \lim_{x\to 0}\frac{-x\sin x}{2x} = \lim_{x\to 0}\frac{-\sin x}{2} = 0$$

ie $\log y = 0$ $\therefore y = 1$

(8) If $\lim_{x\to 0}\frac{a\sin x - \sin 2x}{\tan^3 x}$ is finite, find the value of a and the limit.

Solution : The given limit is of the form $\frac{0}{0}$ $\therefore$ applying the L'Hospital's rule

$$= \lim_{x\to 0}\frac{a\cos x - 2\cos 2x}{3\tan^2 x\cdot\sec^2 x}$$

limit exists if this is of the form $\frac{0}{0}$

$\therefore$ $a\cos x - 2\cos 2x = 0$ for $x = 0$ $\therefore$ $a = 2$

$$= \lim_{x\to 0}\frac{2\cos x - 2\cos 2x}{3x^2}\times\frac{x^2}{\tan^2 x}\times\frac{1}{\sec^2 x} = \lim_{x\to 0}\frac{3\cos x - 2\cos 2x}{3x^2}\times 1\times 1 \text{ is of form } \frac{0}{0}$$

$\therefore$ using L'Hospital's rule

$$= \lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{6x} = \lim_{x\to 0} \frac{-2\sin x}{6x} + \frac{4}{3} \times \frac{\sin 2x}{2x} = -\frac{2}{6} + \frac{4}{3} = -\frac{1}{3} + \frac{4}{3} = 1$$

$\therefore\ a = 2$ and the limit is 1.

Exercise

Evaluate the following

(1) $\lim_{x\to 0} \frac{\cosh x - \cos x}{x \sin x}$

(2) $\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

(3) $\lim_{x\to 0} \frac{e^x - \sin x - 1}{\log(1+x)}$

(4) $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$

(5) $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$

(6) $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$

(7) $\lim_{x\to 0} (\cos x)^{1/x^2}$

(8) $\lim_{x\to 1} (x)^{\frac{1}{1-x}}$

(9) $\lim_{x\to 0} (\cot x)^{\frac{1}{\log x}}$

(10) $\lim_{x\to 0} \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}}$

Answers: (1) 1 (2) 1 (3) 2 (4) $\frac{3}{2}$ (5) $-\frac{1}{3}$ (6) 0 (7) $\frac{1}{\sqrt{e}}$ (8) $\frac{1}{e}$ (9) $\frac{1}{e}$ (10) $(abc)^{1/3}$

Partial Derivatives

A function of two independent variables and a dependent variable is denoted as $z = f(x, y)$ which is explicit function where x & y are independent variables and z a dependent variable. Implicit function is denoted by $\phi(x, y, z) = C$

If $\lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x, y) - f(x, y)}{\delta x}$ exists then it is called Partial derivative of z or f w.r.t. x and denoted by $\dfrac{\partial z}{\partial x}$ or $\dfrac{\partial f}{\partial x}$

If $\lim\limits_{\delta y \to 0} \dfrac{f(x, y+\delta y) - f(x, y)}{\delta y}$ exists then it is called Partial derivative of z or f w.r.t. y and denoted by $\dfrac{\partial z}{\partial y}$ or $\dfrac{\partial f}{\partial y}$

while obtaining the derivative $\dfrac{\partial z}{\partial x}$ differentiate the given function w.r.t. x treating y as a constant and while finding $\dfrac{\partial z}{\partial y}$ differentiate the given function with respect to y, treating x as a constant.

Eg. (1) If $z = x^2 + xy - y^2$ then $\dfrac{\partial z}{\partial x} = 2x + y + 0 = 2x + y$ **&** $\dfrac{\partial z}{\partial y} = x - 2y$

(2) If $z = x^2 y - x \sin xy$ then $\dfrac{\partial z}{\partial x} = 2xy - x \cos xy \cdot y - \sin xy$ **&** $\dfrac{\partial z}{\partial y} = x^2 - x \cos xy \cdot x = x^2(1 - \cos xy)$

(3) If $z = \tan^{-1}\left(\dfrac{2xy}{x^2 - y^2}\right)$ then $\dfrac{\partial z}{\partial x} = \dfrac{1}{1 + \dfrac{4x^2y^2}{\left(x^2 - y^2\right)^2}} \times \dfrac{(x^2 - y^2)2y - 2xy \cdot 2x}{\left(x^2 - y^2\right)^2}$

$$= \frac{\left(x^2 - y^2\right)^2}{\left(x^2 - y^2\right)^2 + 4x^2y^2} \times \frac{2x^2y - 2y^3 - 4x^2y}{\left(x^2 - y^2\right)^2} = \frac{-2y^3 - 2x^2y}{\left(x^2 + y^2\right)^2} = \frac{-2y(y^2 + x^2)}{\left(x^2 + y^2\right)^2} = \frac{-2y}{x^2 + y^2}$$

& $\dfrac{\partial z}{\partial y} = \dfrac{1}{1 + \dfrac{4x^2y^2}{\left(x^2 - y^2\right)^2}} \times \dfrac{(x^2 - y^2)(2x) - 2xy(-2y)}{\left(x^2 - y^2\right)^2}$

$$= \frac{\left(x^2 - y^2\right)^2}{\left(x^2 - y^2\right)^2 + 4x^2y^2} \times \frac{2x^3y - 2xy^2 + 4xy^2}{\left(x^2 - y^2\right)^2} = \frac{2x^3 + 2xy^2}{\left(x^2 + y^2\right)^2} = \frac{2x(x^2 + y^2)}{\left(x^2 + y^2\right)^2} = \frac{2x}{x^2 + y^2}$$

Successive derivatives

For the function $z = f(x, y)$ $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ are first order partial derivatives, the second order partial derivatives are

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right), \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right), \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right), \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$ which are denoted as $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y \partial x}, \frac{\partial^2 z}{\partial y^2}$ but in general $\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial x}$.

In example (1) $\frac{\partial^2 z}{\partial x^2} = 2,\ \frac{\partial^2 z}{\partial y^2} = -2,\ \frac{\partial^2 z}{\partial x\partial y} = 1$ & $\frac{\partial^2 z}{\partial y\partial x} = 1$

In example (2) $\frac{\partial^2 z}{\partial y\partial x} = 2x + x^2 y \sin xy - x\cos xy - x\cos xy$ & $\frac{\partial^2 z}{\partial x\partial y} = 2x + x^2 y \sin xy - 2x\cos xy$

$$\therefore \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial x}$$

In example (3) $\frac{\partial^2 z}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{-2y}{x^2+y^2}\right) = \frac{(x^2+y^2)(-2)+2y\cdot 2y}{\left(x^2+y^2\right)^2} = \frac{-2x^2-2y^2+4y^2}{\left(x^2+y^2\right)^2} = \frac{-2(x^2-y^2)}{\left(x^2+y^2\right)^2}$

and $\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right) = \frac{(x^2+y^2)2-2x\cdot 2x}{\left(x^2+y^2\right)^2} = \frac{2y^2-2x^2}{\left(x^2+y^2\right)^2} = \frac{-2(x^2-y^2)}{\left(x^2+y^2\right)^2}$

Thus in general, always $\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial x}$

Exercise

(1) Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ for $z = \log(x^2+y^2)$ and show that $\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial x}$

(2) If $x = f(x+ct) + \phi(x-ct)$ show that $\dfrac{\partial^2 z}{\partial t^2} = c^2 \dfrac{\partial^2 z}{\partial x^2}$ where c is a constant.

(3) If $z = e^{ax+by} f(ax-by)$ then show that $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = 2abz$

(4) If $u = \dfrac{x^2+y^2}{x+y}$ then show that $\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right)$

(5) If $u = \sin^{-1}\left(\dfrac{y}{x}\right)$ then show that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

unit – 4 INTEGRAL CALCULUS

Given $\frac{dy}{dx} = f(x)$, the process of finding y is called **'Integration'** and the resulting function is called **'Integral'**. If g

the integral then $\int f(x)dx = g(x)$ is the notation used to represent the process.

In the above notation $f(x)$ is called **'Integrand'** and further $\frac{d}{dx}[g(x)] = f(x)$.

But $\frac{d}{dx}[g(x)+c] = g'(x)$ when c is a constant $\therefore \int f(x)dx = g(x)+c$

Thus integral of a function is not unique and two integrals always differ by a constant.

Properties

(1) $\int [f(x) \pm \phi(x)]dx = \int f(x)dx \pm \int \varphi(x)dx$

(2) $\int Kf(x)dx = K\int f(x)dx$ where K is a constant

(3) $\int 0dx = c$ (a constant)

Standard Integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c \; (n \neq -1) \quad \because \frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + c\right) = x^n$

2. $\int \frac{1}{x} dx = \log_e x + c \quad \because \frac{d}{dx}(\log x + c) = \frac{1}{x}$

3. $\int a^x dx = \frac{a^x}{\log a} + c \quad \because \frac{d}{dx}\left(\frac{a^x}{\log a} + c\right) = a^x$ in particular $\int e^x dx = e^x + c$

4. $\int \sin x \, dx = -\cos x + c$

5. $\int \cos x \, dx = \sin x + c$

6. $\int \sec x \tan x \, dx = \sec x + 1$

7. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

8. $\int \sinh x \, dx = \cosh x + c$

9. $\int \cosh x \, dx = \sinh x + c$

10. $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c$

11. $\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosesh} x + c$

12. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$ or $-\cot^{-1} x + c$

13. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$ or $-\cos^{-1} x + c$

14. $\int \frac{1}{\sqrt{1+x^2}} dx = \sinh^{-1} x + c$

15. $\int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + c$

16. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$ or $-\operatorname{cosec}^{-1} x + c$

17. $\int \frac{1}{x\sqrt{1-x^2}} dx = -\operatorname{sech}^{-1} x + c$

Methods of Integration

There are two methods (1) Integration by substitution & (2) Integration by parts.

1. Integration by substitution

Consider $\int f(x)\,dx$ put $x=\phi(t)$ then $\dfrac{dx}{dt}=\phi'(t)$ ie $dx=\phi'(t)\,dt$

$$\therefore \int f(x)\,dx=\int f[\phi(t)]\phi'(t)\,dt$$

now for the new integrand, we can use the standard forms, ie. we have to make a proper substitution so that the give integrand reduced to a standard one.

Examples

1. $$\int \tan x\,dx=\int \frac{\sin x}{\cos x}\,dx$$

put $\cos x=t$, then $-\sin x=\dfrac{dt}{dx}$ ie $\sin x\,dx=-dt$

$$\therefore \int \tan x\,dx=\int -\frac{dt}{t}=-\log t=-\log\cos x=\log\sec x+c$$

or $\displaystyle\int \tan x\,dx=\int \frac{\tan x\sec x}{\sec x}\,dx$

put $\sec x=t$, differentiating w.r.t. x

$$\sec x\tan x=\frac{dt}{dx} \quad \therefore \sec x\tan x\,dx=dt$$

$$\therefore \int \tan x \, dx = \int \frac{dt}{t} = \log t = \log \sec x + c$$

2. $$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$$

put $\sin x = t,$ differentiating w.r.t. x $\quad \cos x = \frac{dt}{dx}$ ie $\cos x \, dx = dt$

$$\therefore \int \cot x \, dx = \int \frac{dt}{t} = \log t = \log \sin x + c$$

3. $$\int \tanh x \, dx = \log \cosh x + c$$

4. $$\int \coth x \, dx = \log \sinh x + c$$

5. $$\int \sec x \, dx = \log(\sec x + \tan x) + c$$

6. $$\int \operatorname{cosec} x \, dx = \log(\operatorname{cosec} x - \cot x) + c$$

In general $\int \frac{f'(x)}{f(x)} dx = \log f(x) + c$ also $\int [f(x)]^n f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} + c (n \neq -1)$

2. Integration by parts

If u & v are functions of x, we know that, $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$\therefore$ By definition of Integration

$uv = \int\left(u\frac{dv}{dx} + v\frac{du}{dx}\right)dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$ using property (1)

$\therefore \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

The result can be used as the standard result. Out of the two functions of the product, one has to be taken as u & another $\frac{dv}{dx}$ then the RHS after evaluation gives the integral or if both functions have taken as u & v then the result is as follows

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\cdot\int v\,dx\right)dx$$

any one form can be used depending on convenience. The first one can also be written as $\int uv'\,dx = uv - \int u'v\,dx$

Examples

1. $\int xe^x\,dx$ put $u = v,\ v' = e^x,\ u' = 1,\ v = e^x$

$$\therefore \int xe^x\,dx = uv - \int u'v\,dx = xe^x - \int 1\cdot e^x\,dx = xe^x - e^x + c$$

2. $\int x\sin x\,dx = x\int \sin x\,du - \int 1\cdot\int \sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + c$

3. $\int \log x\,dx$ put $u = \log x,\ v' = 1,\ u' = \frac{1}{x},\ v = x$

$$\therefore \int uv'\,dx = uv - \int u'v\,dx$$

ie $\int \log x\,dx = x\log x - \int \frac{1}{x}\cdot x\,dx = x\log x - \int 1\cdot dx = x\log x - x + c$

4. $\int \sin^{-1} x\,dx$ put $u = \sin^{-1} x$, $v' = 1$, $u' = \frac{1}{\sqrt{1-x^2}}$, $v = x$

$$\therefore \int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{1}{\sqrt{1-x^2}}\,x\,dx$$

to evaluate $\int \frac{-x}{\sqrt{1-x^2}}\,dx$ put $1 - x^2 = t^2$ differentiating w.r.t. x

$$-2x\,dx = 2t\,dt \Rightarrow -x\,dx = t\,dt$$

$$\therefore \int \frac{-x\,dx}{\sqrt{1-x^2}} = \int \frac{t\,dt}{\sqrt{t^2}} = \int 1\cdot dt = t = \sqrt{1-x^2}$$

$$\therefore \int \sin^{-1} x\,dx = x\sin^{-1} x - \sqrt{1-x^2} + c$$

Special Types of Integrals

Type I

(1) $\int \frac{dx}{a^2+x^2}$, (2) $\int \frac{dx}{x^2-a^2}$, (3) $\int \frac{dx}{a^2-x^2}$ & (4) $\int \frac{dx}{Ax^2+Bx+C}$

to evaluate (1) put $x = at,\ dx = a\,dt$

$$\therefore \int \frac{dx}{a^2+x^2} = \int \frac{a\,dt}{a^2+a^2t^2} = \frac{1}{a}\int \frac{dt}{1+t^2} = \frac{1}{a}\tan^{-1} t = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

to evaluate (2) & (3) use partial fractions

$$\frac{1}{x^2-a^2} = \frac{1}{(x+a)(x-a)} = \frac{A}{x+a} + \frac{B}{x-a} \quad \text{(Say)}$$

multiply throughout by $x^2 - a^2$

$$1 = A(x-a) + B(x+a)$$

put $x = a,\ 1 = 0 + B\cdot 2a\ \therefore B = \frac{1}{2a}$

put $x = -a,\ 1 = A(-2a) + 0\ \therefore A = \frac{-1}{2a}$

$$\therefore \frac{1}{x^2-a^2} = \frac{\frac{-1}{2a}}{x+a} + \frac{\frac{1}{2a}}{x-a}$$

$$\therefore \int \frac{1}{x^2-a^2}dx = -\frac{1}{2a}\int \frac{dx}{x+a} + \frac{1}{2a}\int \frac{dx}{x-a} = -\frac{1}{2a}\log(x+a) + \frac{1}{2a}\log(x-a) = \frac{1}{2a}\log\frac{x-a}{x+a}$$

$$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a} + c$$

next, $\dfrac{1}{x^2-a^2}=\dfrac{1}{(a+x)(a-x)}=\dfrac{A}{a+x}+\dfrac{B}{a-x}$ (Say)

multiplying throughout by a^2-x^2, then

$$1=A(a-x)+B(a+x)$$

put $x=a$, $1=0+B(2a)\Rightarrow B=\dfrac{1}{2a}$

put $x=-a$, $A(2a)+0\Rightarrow A=\dfrac{1}{2a}$

$$\therefore\ \frac{1}{a^2-x^2}=\frac{\frac{1}{2a}}{a+x}+\frac{\frac{1}{2a}}{a-x}$$

$$\therefore\ \int\frac{1}{a^2-x^2}\,dx=\frac{1}{2a}\int\frac{1}{a+x}dx+\frac{1}{2a}\int\frac{1}{a-x}dx=\frac{1}{2a}\log(a+x)-\frac{1}{2a}\log(a-x)=\frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$$

$$\therefore\ \int\frac{1}{a^2-x^2}dx=\frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)+c$$

to evaluate (4) $\displaystyle\int\frac{dx}{Ax^2+Bx+c}$

to evaluate (4) $\int \frac{dx}{Ax^2+Bx+c}$

$$\text{G.I.}=\frac{1}{A}\int \frac{dx}{x^2+\frac{B}{A}x+\frac{C}{A}}=\frac{1}{A}\int \frac{dx}{\left(x+\frac{B}{2A}\right)^2-\frac{B^2}{4A^2}+\frac{C}{A}}=\frac{1}{A}\int \frac{dx}{\left(x+\frac{B}{2A}\right)^2-\frac{B^2-4AC}{4A^2}}$$

This integral will take any one of (1), (2) or (3) and hence can be evaluated.

Examples

(1) Evaluate $\int \frac{dx}{3x^2-2x+4}$

Solution : $$\int \frac{dx}{3x^2-2x+4}=\frac{1}{3}\int \frac{dx}{x^2-\frac{2}{3}x+\frac{4}{3}}=\frac{1}{3}\int \frac{dx}{\left(x-\frac{1}{3}\right)^2-\frac{4}{9}+\frac{4}{3}}=\frac{1}{3}\int \frac{dx}{\left(x-\frac{1}{3}\right)^2+\frac{-4+12}{9}}$$

$$=\frac{1}{3}\int \frac{dx}{\left(\frac{\sqrt{8}}{3}\right)^2+\left(x-\frac{1}{3}\right)^2}=\frac{1}{3}\int \frac{dx}{\left(\frac{2\sqrt{2}}{3}\right)^2+\left(x-\frac{1}{3}\right)^2}=\frac{1}{3}\times\frac{1}{\frac{2\sqrt{2}}{3}}\tan^{-1}\frac{x-\frac{1}{3}}{\frac{2\sqrt{2}}{3}}=\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{3x-1}{2\sqrt{2}}\right)+c$$

(2) Evaluate $\int \frac{dx}{x^2-10x+21}$

Solution : $$\int \frac{dx}{x^2-10x+21}=\int \frac{dx}{(x-5)^2-25+21}=\int \frac{dx}{(x-5)^2-2^2}=\frac{1}{2\times 2}\log\frac{x-5-2}{x-5+2}=\frac{1}{4}\log\frac{x-7}{x-3}+c$$

(3) Evaluate $\int \frac{dx}{6-4x-2x^2}$

$$\text{Solution}: \int \frac{dx}{6-4x-2x^2} = \frac{1}{2}\int \frac{dx}{3-(x^2+2x)} = \frac{1}{2}\int \frac{dx}{3-(x+1)^2+1} = \frac{1}{2}\int \frac{dx}{2^2-(x+1)^2}$$

$$= \frac{1}{2} \times \frac{1}{2\times 2} \log \frac{2+(x+1)}{2-(x+1)} = \frac{1}{8} \log\left(\frac{3+x}{1-x}\right) + c$$

Type II

$$(1) \int \frac{dx}{\sqrt{a^2-x^2}},\ (2) \int \frac{dx}{\sqrt{a^2+x^2}},\ (3) \int \frac{dx}{\sqrt{x^2-a^2}},\ (4) \int \frac{dx}{\sqrt{Ax^2+Bx+C}}$$

to evaluate $\int \frac{dx}{\sqrt{a^2-x^2}}$ put $x = a\sin\theta,\ dx = a\cos\theta\, d\theta$

$$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}} = \int \frac{a\cos\theta\, d\theta}{a\cos\theta} = \int 1\cdot d\theta = \theta = \sin^{-1}\frac{x}{a}$$

$$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$$

to evaluate $\int \frac{dx}{\sqrt{a^2+x^2}}$ put $x = a\sinh\theta,\ dx = a\cosh\theta\, d\theta$

$$\therefore \int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{a\cosh\theta\, d\theta}{\sqrt{a^2+a^2\sinh^2\theta}} = \int \frac{a\cosh\theta\, d\theta}{a\cosh\theta} = \int 1\cdot d\theta = \theta = \sinh^{-1}\frac{x}{a}$$

$$\therefore \int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} + c$$

to evaluate $\int \frac{dx}{\sqrt{x^2-a^2}}$ put $x = a\cosh\theta,\ dx = a\sinh\theta\, d\theta$

$$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + c$$

$$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sinh\theta\, d\theta}{\sqrt{a^2\cosh^2\theta - a^2}} = \int \frac{a\sinh\theta\, d\theta}{a\sinh\theta} = \int 1\cdot d\theta = \theta = \cosh^{-1}\frac{x}{a}$$

to evaluate $\int \frac{dx}{\sqrt{Ax^2+Bx+C}}$

$$\text{G.I.} = \frac{1}{\sqrt{A}}\int \frac{dx}{\sqrt{x^2+\frac{B}{A}x+\frac{C}{A}}} = \frac{1}{\sqrt{A}}\int \frac{dx}{\sqrt{\left(x+\frac{B}{2A}\right)^2 - \frac{B^2}{4A^2}+\frac{C}{A}}} = \frac{1}{\sqrt{A}}\int \frac{dx}{\sqrt{\left(x+\frac{B}{2A}\right)^2 - \frac{B^2-4AC}{4A^2}}}$$

This will reduce to any one of (1), (2) & (3) and hence can be evaluated.

Examples

(1) Evaluate $\int \frac{dx}{\sqrt{2x-5x^2}}$

$$\text{Solution}: \int \frac{dx}{\frac{1}{\sqrt{5}}\sqrt{\frac{2}{5}x-x^2}} = \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{-\left(x^2-\frac{2}{5}x\right)}} = \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{\left(\frac{1}{5}\right)^2-\left(x-\frac{1}{5}\right)^2}} = \frac{1}{\sqrt{5}}\sin^{-1}\frac{x-\frac{1}{5}}{\frac{1}{5}} = \frac{1}{\sqrt{5}}\times\sin^{-1}(5x-1)+c$$

(2) Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-2x+5}}$

$$\text{Solution}: \int \frac{dx}{\sqrt{x^2-2x+5}} = \int \frac{dx}{\sqrt{(x-1)^2+2^2}} = \int \frac{dx}{\sqrt{2^2+(x-1)^2}} = \sinh^{-1}\frac{x-1}{2}+c$$

(3) Evaluate $\displaystyle\int \frac{dx}{\sqrt{4x^2-12x+8}}$

$$\text{Solution: G.I.} = \frac{1}{\sqrt{4}}\int \frac{dx}{\sqrt{x^2-3x+2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{9}{4}+2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{9-8}{4}}}$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}} = \frac{1}{2}\cosh^{-1}\frac{x-\frac{3}{2}}{\frac{1}{2}} = \frac{1}{2}\cosh^{-1}(2x-3)+c$$

Type III

$$\int \frac{px+q}{Ax^2+Bx+C}dx \text{ and } \int \frac{px+q}{\sqrt{Ax^2+Bx+C}}$$

to evaluate put $px+q = l$ (derivative of $Ax^2+Bx+C)+m = l(2Ax+B)+m$

where l & m are the constants to be found out by equating the co-efficients of corresponding terms on both sides. ie. to solve for m & n from the equations

$$2Al = p \text{ and } lB+m = q$$

then $\int \frac{px+q}{Ax^2+Bx+c}dx = l\int \frac{2Ax+B}{Ax^2+Bx+C}dx + m\int \frac{dx}{Ax^2+Bx+C} = l \cdot \log(Ax^2+Bx+C)+m\int \frac{dx}{Ax^2+Bx+C}$

the second integral in RHS is Type I and hence can be evaluated.

$$\int \frac{px+q}{\sqrt{Ax^2+Bx+C}}dx = l\int \frac{px+q}{\sqrt{Ax^2+Bx+C}}dx + m\int \frac{dx}{\sqrt{Ax^2+Bx+C}} = 2l\sqrt{Ax^2+Bx+C}+m\int \frac{dl}{\sqrt{Ax^2+Bx+C}}$$

the second integral in RHS is Type II and hence can be evaluated.

Examples

(1) Evaluate $\int \frac{2x+3}{3x^2-4x+5}dx$

Solution: Put $2x+3 = l(6x-4)+m = 6lx-4l+m$

$$\therefore\ 6l = 2 \Rightarrow l = \frac{1}{3},\ -4l+m = 3 \text{ ie } m = 3+\frac{4}{3} =$$

$$\therefore \int \frac{2x+3}{3x^2-4x+5}dx = \frac{1}{3}\int \frac{6x-4}{3x^2-4x+5}dx + \frac{13}{3}\int \frac{dx}{3x^2-4x+5} = \frac{1}{3}\mathbf{log}(3x^2-4x+5) + \frac{13}{9}\int \frac{dx}{x^2-\frac{4}{3}x+\frac{5}{3}}$$

$$= \frac{1}{3}\mathbf{log}(3x^2-4x+5) + \frac{13}{9}\int \frac{dx}{\left(x-\frac{2}{3}\right)^2 - \frac{4}{9} + \frac{5}{3}} = \frac{1}{3}\mathbf{log}(3x^2-4x+5) + \frac{13}{9}\int \frac{dx}{\left(x-\frac{2}{3}\right)^2 + \left(\frac{\sqrt{11}}{3}\right)^2}$$

$$= \frac{1}{3}\mathbf{log}(3x^2-4x+5) + \frac{13}{9} \times \frac{3}{\sqrt{11}}\mathbf{tan}^{-1}\left(\frac{x-\frac{2}{3}}{\frac{\sqrt{11}}{3}}\right) = \frac{1}{3}\mathbf{log}(3x^2-4x+5) + \frac{13}{3\sqrt{11}}\mathbf{tan}^{-1}\left(\frac{3x-2}{\sqrt{11}}\right) + c$$

(2) Evaluate $\int \frac{5x-7}{\sqrt{3x-x^2-2}}dx$

Solution: Put $5x-7 = l(3-2x) + m = -2lx + 3l + m$

$$\therefore -2l = 5 \Rightarrow l = -\frac{5}{2} \;\; \& \;\; 3l + m = -7 \Rightarrow m = -7 - 3l = -7 + \frac{15}{2} = \frac{1}{2}$$

$$\therefore \int \frac{5x-7}{\sqrt{3x-x^2-2}}dx = -\frac{5}{2}\int \frac{3-2x}{\sqrt{3x-x^2-2}}dx + \frac{1}{2}\int \frac{dx}{\sqrt{-2-(x^2-3x)}}$$

$$= -\frac{5}{2}\cdot 2\sqrt{3x-x^2-2} + \frac{1}{2}\int \frac{dx}{\sqrt{-2-\left(x-\frac{3}{2}\right)^2+\frac{9}{4}}} = -5\sqrt{3x-x^2-2} + \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}$$

$$= -5\sqrt{3x-x^2-2} + \frac{1}{2}\times \sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{1}{2}}\right) = -5\sqrt{3x-x^2-2} + \frac{1}{2}\sin^{-1}(2x-3)+c$$

Type IV

$$\int \frac{dx}{a\cos x + b\sin x + c}$$

to evalute put $\tan\frac{x}{2}=t$ then differentiating w.r.t. x $\frac{1}{2}\sec^2\frac{x}{2}=\frac{dt}{dx}$

$$\text{ie } dx = \frac{2dt}{\sec^2\frac{x}{2}} = \frac{2dt}{1+\tan^2\frac{x}{2}} = \frac{2dt}{1+t^2} \;\; \& \;\; \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2}$$

$$\& \;\; \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{t}{\sqrt{1+t^2}}\times\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$$

$$\therefore \text{ when } \tan\frac{x}{2}=t,\; dx = \frac{2dt}{1+t^2},\; \cos x = \frac{1-t^2}{1+t^2} \;\; \& \;\; \sin x = \frac{2t}{1+t^2}$$

$$\therefore \int \frac{dx}{a\cos x + b\sin x + c} = \int \frac{\frac{2dt}{(1+t^2)}}{a\frac{(1-t^2)}{(1+t^2)} + b\frac{2t}{(1+t^2)} + c} = \int \frac{2dt}{a(1-t^2) + 2bt + c(1+t^2)} = \int \frac{2dt}{(c-a)t^2 + 2bt + a + c}$$

which is Type I and hence can be evaluated.

Examples

(1) Evaluate $\int \frac{dx}{2\cos x - 3\sin x + 5}$

Solution: Put $\tan\frac{x}{2} = t$, then $dx = \frac{2dt}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ & $\sin x = \frac{2t}{1+t^2}$

$$\text{G.I.} = \int \frac{\frac{2dt}{1+t^2}}{\frac{2(1-t^2)}{1+t^2} - \frac{3\times 2t}{1+t^2} + 5} = \int \frac{2dt}{2(1-t^2) - 6t + 5(1+t^2)} = \int \frac{2dt}{3t^2 - 6t + 7} = \frac{2}{3}\int \frac{dt}{t^2 - 2t + \frac{7}{3}}$$

$$= \frac{2}{3}\int \frac{dt}{(t-1)^2 - 1 + \frac{7}{3}} = \frac{2}{3}\int \frac{dt}{(t-1)^2 + \left(\frac{2}{\sqrt{3}}\right)^2} = \frac{2}{3}\times\frac{1}{\frac{2}{\sqrt{3}}}\tan^{-1}\left(\frac{t-1}{\frac{2}{\sqrt{3}}}\right) = \frac{1}{\sqrt{3}}\tan^{-1}\frac{\sqrt{3}}{2}(t-1)$$

$$\therefore \int \frac{dx}{2\cos x - 3\sin x + 5} = \frac{1}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}}{2}\left(\tan\frac{x}{2} - 1\right) + c$$

(2) Evaluate $\int \frac{dx}{3 - 5\cos x}$

Solution: Put $\tan\frac{x}{2} = t$, then $dx = \frac{2dt}{1+t^2}$ & $\cos x = \frac{1-t^2}{1+t^2}$

$$\therefore \int \frac{dx}{3-5\cos x} = \int \frac{\frac{2dt}{(1+t^2)}}{3 - \frac{5(1-t^2)}{1+t^2}} = \int \frac{2dt}{3(1+t^2) - 5(1-t^2)} = \int \frac{2dt}{8t^2 - 2} = \frac{2}{8}\int \frac{dt}{t^2 - \frac{2}{8}} = \frac{1}{4}\int \frac{dt}{t^2 - \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{4} \times \frac{1}{2 \times \frac{1}{2}} \log \frac{t - \frac{1}{2}}{t + \frac{1}{2}} = \frac{1}{4} \log \frac{\tan\frac{x}{2} - \frac{1}{2}}{\tan\frac{x}{2} + \frac{1}{2}}$$

$$\therefore \int \frac{dx}{3 - 5\cos x} = \frac{1}{4} \log \frac{2\tan\frac{x}{2} - 1}{2\tan\frac{x}{2} + 1} + c$$

(3) Evaluate $\int \frac{dx}{3 + 2\sin x}$

Solution: Put $\tan\frac{x}{2} = t$, then $dx = \frac{2dt}{1+t^2}$ and $\sin x = \frac{2t}{1+t^2}$

$$\therefore \int \frac{dx}{3+2\sin x} = \int \frac{\frac{2dt}{1+t^2}}{3+\frac{4t}{1+t^2}} = \int \frac{2dt}{3(1+t^2)+4t} = \frac{2}{3}\int \frac{dt}{1+t^2+\frac{4}{3}t} = \frac{2}{3}\int \frac{dt}{\left(t+\frac{2}{3}\right)^2 - \frac{4}{9}+1}$$

$$= \frac{2}{3}\int \frac{dt}{\left(t+\frac{2}{3}\right)^2 + \left(\frac{\sqrt{5}}{3}\right)^2} = \frac{2}{3}\times \frac{1}{\frac{\sqrt{5}}{3}} \tan^{-1} \frac{t+\frac{2}{3}}{\frac{\sqrt{5}}{3}} = \frac{2}{\sqrt{5}} \tan^{-1} \frac{3t+2}{\sqrt{5}}$$

$$\therefore \int \frac{dx}{3+2\sin x} = \frac{2}{\sqrt{5}} \tan^{-1}\left(\frac{3\tan\frac{x}{2}+2}{\sqrt{5}}\right) + c$$

Type V

$$\int \frac{a\cos x + b\sin x}{c\sin x + e\cos x}\,dx$$

Solution: to evaluate put $a\cos x + b\sin x = l(\text{Denominator}) + m(\text{derivative of denominator})$

ie $a\cos x + b\sin x = l(c\sin x + e\cos x) + m(c\cos x - e\sin x)$

where l & m are constants to be foundout by equating the co - efficients of $\sin x$ & $\cos x$ separately.

ie from the equations $lc - me = b$ & $le + mc = a$

then $\int \frac{a\cos x + b\sin x}{c\sin x + e\cos x}dx = l\int \frac{c\sin x + e\cos x}{c\sin x + e\cos x}dx + m\int \frac{c\cos x - e\sin x}{c\sin x + e\cos x}dx = lx + m\log(c\sin x + e\cos x) + c$

Examples

Evaluate $\int \frac{3\cos x - 2\sin x}{4\sin x + \cos x}dx$

Solution: Put $3\cos x - 2\sin x = l(4\sin x + \cos x) + m(4\cos x - \sin x)$

$\therefore\ 4l - m = -2$

$l + 4m = 3$

$(1)\times 4 \quad 16l - 4m = -8$

$(2)\times 1 \quad l + 4m = 3$

adding $17l = -5 \Rightarrow l = -\frac{5}{17}$

from (1), $m = 4l + 2 = -\frac{20}{17} + 2 = \frac{-20+34}{17} = \frac{14}{17}$

$$\therefore \int \frac{3\cos x - 2\sin x}{4\sin x + \cos x}dx = -\frac{5}{17}\int \frac{4\sin x + \cos x}{4\sin x + \cos x}dx + \frac{14}{17}\int \frac{4\cos x - \sin x}{4\sin x + \cos x} = -\frac{5}{17}\int 1\cdot dx + \frac{14}{17}\log(4\sin x + \cos x)$$

$$= -\frac{5}{17}x + \frac{14}{17}\log(4\sin x + \cos x) + c$$

Type VI

$$\int f(x)\, e^x\, dx \text{ where } f(x) = \phi(x) + \phi'(x)$$

Solution: $\int f(x)\, e^x\, dx = \int \phi(x)\, e^x\, dx + \int \phi'(x)\, e^x\, dx$

Consider $\int \phi(x)\, e^x\, dx$

put $u = \phi(x),\ v' = e^x,\ u' = \phi(x),\ v = e^x$

$\therefore\ \int \phi(x)\, e^x\, dx = \phi(x)\, e^x - \int \phi'(x)\, e^x\, dx$

substituting this in (1), we have

$$\int f(x)\, e^x\, dx = \phi(x)\, e^x - \int \phi'(x)\, e^x\, dx + \int \phi'(x)\, e^x\, dx = \phi(x)\, e^x + c$$

Examples

(1) Evaluate $\int \frac{xe^x}{(1+x)^2}\, dx$

Solution: $\int \frac{xe^x}{(1+x)^2}\, dx = \int \frac{(1+x-1)\, e^x}{(1+x)^2}\, dx = \int \frac{(1+x)\, e^x}{(1+x)^2}\, dx - \int \frac{e^x}{(1+x)^2}\, dx = \int \frac{e^x}{1+x}\, dx - \int \frac{e^x}{(1+x)^2}\, dx$

Consider $\int \frac{e^x}{(1+x)} dx$

put $u = \frac{1}{1+x}$, $v' = e^x$, $u' = \frac{-1}{(1+x)^2}$, $v = e^x$

$$\therefore \int \frac{e^x}{(1+x)} dx = \frac{e^x}{(1+x)} - \int \frac{-e^x}{(1+x)^2} dx = \frac{e^x}{(1+x)} + \int \frac{e^x}{(1+x)^2} dx$$

substituting in (1)

$$\int \frac{xe^2}{(1+x)^2} dx = \frac{e^x}{(1+x)} + \int \frac{e^x}{(1+x)^2} dx - \int \frac{e^x}{(1+x)^2} dx = \frac{e^x}{1+x} + c$$

(2) Evaluate $\int \frac{x - \sin x}{1 - \cos x} dx$

Solution : $\int \frac{x - \sin x}{1 - \cos x} dx = \int \frac{x}{1 - \cos x} dx - \int \frac{\sin x}{1 - \cos x} dx = \int \frac{x}{2\sin^2 \frac{x}{2}} dx - \int \frac{2\sin \frac{x}{2} \cos \frac{x}{2}}{2\sin^2 \frac{x}{2}} dx$

$$= \int \frac{1}{2} x \operatorname{cosec}^2 \frac{x}{2} dx - \int \cot \frac{x}{2} dx$$

Consider $\int \frac{1}{2} x \operatorname{cosec}^2 x \, dx$

put $u = x$, $v' = \frac{1}{2} \operatorname{cosec}^2 \frac{x}{2}$, $u' = 1$, $v = -\cot \frac{x}{2}$

$$\int \frac{1}{2} x \operatorname{cosec}^2 \frac{x}{2}\, dx = -x \cot \frac{x}{2} + \int \cot \frac{x}{2}\, dx$$

substituting in (1)

$$\int \frac{x - \sin x}{1 - \cos x}\, dx = -x \cot \frac{x}{2} + \int \cot \frac{x}{2}\, dx - \int \cot \frac{x}{2}\, dx = -x \cot \frac{x}{2} + c$$

Other examples

(1) Evaluate $\int \frac{\sin x \cos x}{1 + \sin^4 x}\, dx$

Solution: Put $\sin^2 x = t$, then $2 \sin x \cos x\, dx = dt$

$$\therefore \int \frac{\sin x \cos x}{1 + \sin^4 x}\, dx = \int \frac{\frac{1}{2} dt}{1 + t^2} = \frac{1}{2} \tan^{-1} t = \frac{1}{2} \tan^{-1}(\sin^2 x) + c$$

(2) Evaluate $\int \frac{x^2 + 1}{(x+1)(x^2+2)}\, dx$

Solution: Let $\frac{x^2 + 1}{(x+1)(x^2+2)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 + 2}$

multiplying throughout by $(x+1)(x^2+2)$

then $x^2 + 1 = A(x^2 + 2) + (Bx + C)(x + 1)$

put $x = -1$, $2 = A(1 + 2) + 0 \Rightarrow A = \frac{2}{3}$

put $x = 0$, $1 = 2A + C \Rightarrow C = 1 - \frac{4}{3} = -\frac{1}{3}$

Equating co-efficient of x^2 on both sides

$$A + B = 1 \Rightarrow B = 1 - A = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\therefore \frac{x^2 + 1}{(x+1)(x^2+2)} = \frac{\frac{2}{3}}{x+1} + \frac{\frac{1}{3}x - \frac{1}{3}}{x^2 + 2}$$

$$\therefore \int \frac{x^2+1}{(x+1)(x^2+2)}dx = \frac{2}{3}\int \frac{dx}{x+1} + \frac{1}{3}\int \frac{x-1}{x^2+2}dx = \frac{2}{3}\int \frac{dx}{x+1} + \frac{1}{6}\int \frac{2x}{x^2+2}dx - \frac{1}{3}\int \frac{dx}{x^2+2}$$

$$= \frac{2}{3}\log(x+1) + \frac{1}{6}\log(x^2+2) - \frac{1}{3\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + c$$

Type VII

(1) $\int \sqrt{a^2-x^2}\,dx$ (2) $\int \sqrt{a^2+x^2}\,dx$ (3) $\int \sqrt{x^2-a^2}\,dx$ (4) $\int \sqrt{Ax^2+Bx+C}\,dx$

(5) $\int (px+q)\sqrt{Ax^2+Bx+C}\,dx$

(1) To evaluate $\int \sqrt{a^2-x^2}\,dx$ put $x = a\sin\theta,\ dx = a\cos\theta\,d\theta$

$$\int \sqrt{a^2-x^2}\,dx = \int \sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta = \int a\cos\theta\cdot a\cos\theta\,d\theta = \int a^2\cos^2\theta\,d\theta = \frac{a^2}{2}\int (1+\cos 2\theta)\,d\theta$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\times\frac{\sin 2\theta}{2} = \frac{a^2}{2}\theta + \frac{a^2}{2}\sin\theta\cos\theta = \frac{a^2}{2}\theta + \frac{a^2}{2}\cdot\sin\theta\cdot\sqrt{1-\sin^2\theta}$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{a^2}{2}\cdot\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}} = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{a^2}{2}\cdot\frac{x}{a^2}\sqrt{a^2-x^2}$$

$$\therefore \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

(2) To evaluate $\int \sqrt{a^2+x^2}\,dx$ put $x = a\sinh\theta$, then $dx = a\cosh\theta\,d\theta$

$$\int \sqrt{a^2+x^2}\,dx = \int \sqrt{a^2+a^2\sinh^2\theta}\cdot a\cosh\theta\,d\theta = \int a^2\cosh^2\theta\,d\theta = \frac{a^2}{2}\int (1+\cosh 2\theta)\,d\theta$$

$$= \frac{a^2}{2}\int 1\cdot d\theta + \frac{a^2}{2}\int \cosh 2\theta\,d\theta = \frac{a^2}{2}\theta + \frac{a^2}{2}\cdot\frac{\sinh 2\theta}{2} = \frac{a^2}{2}\theta + \frac{a^2}{2}\times\sinh\theta\cosh\theta$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\sinh\theta\sqrt{1+\sinh^2\theta} = \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{a^2}{2}\cdot\frac{x}{a}\sqrt{1+\frac{x^2}{a^2}} = \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2+x^2}$$

$$\therefore \int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c$$

(3) To evaluate $\int \sqrt{x^2 - a^2}\, dx$ put $x = a\cosh\theta$, then $dx = a\sinh d\theta$

$$\int \sqrt{x^2-a^2}\, dx = \int \sqrt{a^2\cosh^2\theta - a^2} \cdot a\sinh\theta\, d\theta = \int a\sinh\theta \cdot a\sinh\theta\, d\theta = \int a^2\sinh^2\theta\, d\theta = \frac{a^2}{2}\int (\cosh 2\theta - 1)\, d\theta$$

$$= \frac{a^2}{2}\int \cosh 2\theta\, d\theta - \frac{a^2}{2}\int 1 \cdot d\theta = \frac{a^2}{2}\cdot\frac{\sinh 2\theta}{2} - \frac{a^2}{2}\theta = \frac{a^2}{2}\sinh\theta\cosh\theta - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$$

$$= \frac{a^2}{2}\sqrt{\cosh^2\theta - 1}\cosh\theta - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} = \frac{a^2}{2}\sqrt{\frac{x^2}{a^2}-1}\cdot\frac{x}{a} - \frac{a^2}{2}\cosh^{-1}\frac{x}{2} = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$$

$$\therefore \int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c$$

(4) To evaluate $\int \sqrt{Ax^2+Bx+C}\, dx = \sqrt{A}\int \sqrt{x^2 + \frac{B}{A}x + \frac{C}{A}}\, dx$. This will take the form (1), (2) or (3) and hence can be evaluated.

(5) To evaluate $\int (px+q)\sqrt{Ax^2+Bx+C}\, dx$

put $px + q = l(\text{derivative of } Ax^2 + Bx + C) + m = l(2Ax+B) + m$

where l & m are constants to be found out,

then, $\int (px+q)\sqrt{Ax^2+Bx+C}\, dx = l\int (2Ax+B)\sqrt{Ax^2+Bx+C}\, dx + m\int \sqrt{Ax^2+Bx+C}$

$$= \frac{2l}{3}(Ax^2 + Bx + C)^{3/2} + m\sqrt{A}\int\sqrt{x^2 + \frac{B}{A}x + \frac{C}{A}}\,dx$$

the second integral reduces to (1), (2) or (3) and hence can be evaluated.

Type VIII

$$\int\frac{dx}{(px+q)\sqrt{Ax^2 + Bx + C}}$$

put $px + q = \frac{1}{t}$ then $p\,dx = \frac{-1}{t^2}dt$ **&** $x = \frac{1}{p}\left(\frac{1}{t} - q\right)$

$$\therefore \int\frac{dx}{(px+q)\sqrt{Ax^2 + Bx + C}} = \int\frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{\frac{A}{p^2}\left(\frac{1}{t} - q\right)^2 + \frac{B}{p}\left(\frac{1}{t} - q\right) + C}} = \int\frac{-dt}{\sqrt{\frac{A}{p^2}(1 - tq)^2 + \frac{B}{p}(t - qt^2) + Ct^2}}$$

This integral reduces to any one of Type II and hence can be solved.

Type IX

$\int e^{ax}\cos(bx+c)\,dx$ and $\int e^{ax}\sin(bx+c)\,dx$

To evaluatewe have to use integration by parts.

Let $C=\int e^{ax}\cos(bx+c)\,dx$ & $S=\int e^{ax}\sin(bx+c)\,dx$

Consider $C=\int e^{ax}\cos(bx+c)\,dx$

put $u=e^{ax},\ u'=ae^{ax},\ v'=\cos(bx+c),\ v=\dfrac{\sin(bx+c)}{b}$

$$C=\frac{e^{ax}\sin(bx+c)}{b}-\frac{a}{b}\int e^{ax}\sin(bx+c)\,dx$$

$bC=e^{ax}\sin(bx+c)-aS$

$\therefore\ aS+bC=e^{ax}\sin(bx+c)$ (1)

Consider $S=\int e^{ax}\sin(bx+c)\,dx$

put $u=e^{ax},\ u'=ae^{ax},\ v'=\sin(bx+c),\ v=\dfrac{-\cos(bx+c)}{b}$

$$\therefore\ S=\frac{-e^{ax}\cos(bx+c)}{b}+\frac{a}{b}\int e^{ax}\cos(bx+c)\,dx$$

$bS=-ea^{ax}\cos(bx+c)+aC$

ie $bS-aC=-e^{ax}\cos(bx+c)$ (2)

$(1)\times a\quad a^2S+abC=ae^{ax}\sin(bx+c)$

$(2)\times b\quad b^2S-abC=-be^{ax}\cos(bx+c)$

adding $(a^2+b^2)S=e^{ax}\left[a\sin(bx+c)-b\cos(bx+c)\right]$

$$\therefore\ S=\frac{e^{ax}\left[a\sin(bx+c)-b\cos(bx+c)\right]}{(a^2+b^2)}$$

$$\begin{array}{ll}(1)\times b & abS+b^2C=be^{ax}\sin(bx+c)\\ (2)\times a & abS-a^2C=-ae^{ax}\cos(bx+c)\\ \hline \text{subtracting} & (a^2+b^2)C=e^{ax}\left[b\sin(bx+c)+a\cos(bx+c)\right]\end{array}$$

$$\therefore\ C=\frac{e^{ax}\left[a\cos(bx+c)+b\sin(bx+c)\right]}{(a^2+b^2)}$$

Examples

(1) $$\int e^{2x}\sin 3x\cos 2x\,dx=\frac{1}{2}\int e^{2x}[\sin 5x+\sin x]\,dx=\frac{1}{2}\int e^{2x}\sin 5x\,dx+\frac{1}{2}\int e^{2x}\sin dx$$

$$=\frac{1}{2}e^{2x}\frac{(2\sin 5x-5\cos 5x)}{29}+\frac{1}{2}e^{2x}\frac{(2\sin x-\cos x)}{5}+c$$

(2) $$\int e^{3x}\cos^2 x\,dx=\frac{1}{2}\int e^{3x}(1+\cos 2x)\,dx=\frac{1}{2}\int e^{3x}\,dx+\frac{1}{2}\int e^{3x}\cos 2x\,dx=\frac{1}{2}\times\frac{e^{3x}}{3}+\frac{1}{2}e^{3x}\frac{(3\cos 2x+2\sin 2x)}{9+4}$$

$$=\frac{e^{3x}}{6}+\frac{e^{3x}}{26}(3\cos 2x+2\sin 2x)+c$$

Exercise

Integrate the following w.r.t. x

(1) $\sqrt{\dfrac{\sin^{-1}x}{1-x^2}}$ (2) $\dfrac{1}{x\cos^2(\log x)}$ (3) $\dfrac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

(4) $\dfrac{\sec^2 x}{\tan x(2+\tan x)}$ (5) $\dfrac{3x-2}{(x+1)^2(x+3)}$ (6) $\dfrac{x}{(x-1)(x^2+4)}$

(7) $\dfrac{x^3-x-2}{x^2-1}$ (8) $\dfrac{4x+5}{x^2+22x+2}$ (9) $\dfrac{4x+1}{\sqrt{x^2-6x+18}}$

(10) $\dfrac{(1+x)}{(2+x)^2}e^x$ (11) $\dfrac{1}{2+\cos x-\sin x}$ (12) $\dfrac{1}{3+4\cos x}$

(13) $\dfrac{2\sin x + 3\cos x}{4\sin x + 5\cos x}$ (14) $\dfrac{1}{4\cos^2 x + 9\sin^2 x}$ (15) $\dfrac{(1+\sin x)}{(1+\cos x)}e^x$

(16) $\dfrac{1}{x(x^n+1)}$ (17) $\sqrt{6-4x-2x^2}$ (18) $(2x-5)\sqrt{x^2-3x+2}$

(19) $\dfrac{1}{(x+1)\sqrt{2x^2+3x+4}}$ (20) $\dfrac{1}{(x+1)\sqrt{x^2-1}}$ (21) $e^{2x}\sin 4x \sin 2x$

(22) $e^{2x}\cos 3x \cos x$ (23) $e^{3x}\cos^3 x$ (24) $e^{4x}\sin^3 x$

Definite Integrals

Let $f(x)$ be a function defined in the interval (a, b) and $\int f(x)\,dx = g(x) + c$

The value of the integral at $x = b$ minus the value of the integral at $x = a$ ie $[g(b)+c] - [g(a)+c]$

ie $g(b) - g(a)$ is defined as Definite integral and denoted as $\int_a^b f(x)\,dx$

ie If $\int f(x)\,dx = g(x)$ then $\int_a^b f(x)\,dx = g(b) - g(a)$

b is called upper limit and a is called lower limit.

Examples

(1) Evaluate $\int_1^2 (x^3 - 2x^2 + 3)\,dx$

Solution: $\int_1^2 (x^3-2x^2+3)\,dx = \left[\frac{x^4}{4}-2\times\frac{x^3}{3}+3x\right]_1^2 = \left(\frac{2^4}{4}-2\times\frac{2^3}{3}+3\cdot 2\right)-\left(\frac{1}{4}-\frac{2}{3}+3\right)^1 = 4 = \frac{16}{3}+6-\frac{1}{4}+\frac{2}{3}-3$

$$= 7-\frac{16}{3}-\frac{1}{4}+\frac{2}{3} = \frac{84-64-3+8}{12} = \frac{25}{12}$$

(2) Evaluate $\int_0^1 \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}dx$

Solution: Put $\sin^{-1}x = t$ then $\frac{1}{\sqrt{1-x^2}}dx = dt$

when $x=0,\ t=\sin^{-1}0=0$ when $x=1,\ t=\sin^{-1}1=\frac{\pi}{2}$

$$\therefore \int_0^1 \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}dx = \int_0^{\pi/2} t^2\,dt = \left.\frac{t^3}{3}\right]_0^{\pi/2} = \frac{1}{3}\left(\frac{\pi}{2}\right)^3 = \frac{\pi^3}{24}$$

(3) Evaluate $\int_{-1}^1 \frac{dx}{x^2+2x+5}dx$

Solution: $\int_{-1}^1 \frac{dx}{x^2+2x+5}dx = \int_{-1}^1 \frac{dx}{(x+1)^2+2^2} = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)\Bigg]_{-1}^1 = \frac{1}{2}\tan^{-1}\left(\frac{1+1}{2}\right)-\frac{1}{2}\tan^{-1}\left(\frac{-1+1}{2}\right)$

$$= \frac{1}{2}\tan^{-1}1-\frac{1}{2}\tan^{-1}0 = \frac{1}{2}\frac{\pi}{4}-0 = \frac{\pi}{8}$$

(4) Evaluate $\int_0^\pi \frac{dx}{4+3\cos x}$

Solution: Put $\tan\frac{x}{2}=t$ then $dx=\frac{2dt}{1+t^2}$, $\cos x=\frac{1-t^2}{1+t^2}$

when $x=0,\ t=\tan 0=0$ when $x=\pi,\ t=\tan\frac{\pi}{2}=\infty$

$$\int_0^{\pi}\frac{dx}{4+3\cos x}=\int_0^{\infty}\frac{\frac{2dt}{1+t^2}}{4+\frac{3(1-t^2)}{(1+t^2)}}=\int_0^{\infty}\frac{2dt}{4(1+t^2)+3(1-t^2)}=\int_0^{\infty}\frac{2dt}{t^2+\left(\sqrt{7}\right)^2}=\frac{1}{\sqrt{7}}\tan^{-1}\frac{t}{\sqrt{7}}\Bigg]_0^{\infty}$$

$$=\frac{1}{\sqrt{7}}\tan^{-1}\infty-\frac{1}{\sqrt{7}}\tan^{-1}0=\frac{1}{\sqrt{7}}\times\frac{\pi}{2}=\frac{\pi}{2\sqrt{7}}$$

Properties of Definite Integrals

1. $\int_a^b f(x)\,dx=\int_a^b f(y)\,dy$

2. $\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$ where $a<c<b$

4. $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$ also $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$

5. $$\int_{-a}^{a} f(x)\,dx=\begin{cases}2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is an even function}\\ 0 & \text{if } f(x) \text{ is an odd function}\end{cases}$$

6. $$\int_0^{2a} f(x)\,dx=\begin{cases}2\int_0^a f(x)\,dx & \text{if } f(2a-x)=f(x)\\ 0 & \text{if } f(2a-x)=-f(x)\end{cases}$$

Examples

(1) Evaluate $\int_0^{\pi/2}\frac{\sin^n x}{\cos^n x+\sin^n x}dx$

Solution: Let $I=\int_0^{\pi/2}\frac{\sin^n x}{\cos^n x+\sin^n x}dx=\int_0^{\pi/2}\frac{\sin^n\left(\frac{\pi}{2}-x\right)}{\cos^n\left(\frac{\pi}{2}-x\right)+\sin^n\left(\frac{\pi}{2}-x\right)}dx$ using $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$

$$=\int_0^{\pi/2}\frac{\cos^n x}{\sin^n x+\cos^n x}dx$$

$$\therefore\ 2I=\int_0^{\pi/2}\frac{\sin^n x}{\cos^n x+\sin^n x}dx+\int_0^{\pi/2}\frac{\cos^n x}{\sin^n x+\cos^n x}dx=\int_0^{\pi/2}\frac{(\sin^n x+\cos^n x)}{(\cos^n x+\sin^n x)}dx=\int_0^{\pi/2}1\cdot dx=x\Big]_0^{\pi/2}=\frac{\pi}{2}$$

$$\therefore\ I=\frac{\pi}{4}$$

(2) Evaluate $\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx$

Solution: Let $I=\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}dx$ using $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$

ie $I=\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\sin x}dx$

$$\therefore\ 2I=\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx+\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\sin x}dx=\int_0^{\pi}\frac{x\sin x+(\pi-x)\sin x}{1+\sin x}dx=\int_0^{\pi}\frac{\pi\sin x}{1+\sin x}dx=\pi\int_0^{\pi}\frac{\sin x}{1+\sin x}$$

$$=\pi\int_0^{\pi}\frac{\sin x(1-\sin x)}{(1+\sin x)(1-\sin x)}dx=\pi\int_0^{\pi}\frac{\sin x-\sin^2 x}{\cos^2 x}dx=\pi\int_0^{\pi}\sec x\tan x\,dx-\pi\int_0^{\pi}\tan^2 x\,dx$$

$$=\pi\left[\int_0^{\pi}\sec x\tan x-\int_0^{\pi}\sec^2 x\,dx+\int_0^{\pi}1\,dx\right]=\pi\left[\sec x-\tan x+x\right]_0^{\pi}=\pi\left[\sec\pi-\tan\pi+\pi\right]-\pi\left[\sec 0-\tan 0+0\right]$$

$$=\pi[-1+\pi]-\pi[1]=\pi[-2+\pi]$$

$$\therefore\ I=\frac{\pi}{2}(\pi-2)$$

(3) Evaluate $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}$

Solution: Let $I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos x}\,dx}{\sqrt{\cos x}+\sqrt{\sin x}}=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}dx}{\cos\left(\frac{\pi}{2}-x\right)+\sin\left(\frac{\pi}{2}-x\right)}$ using $\int_a^b f(x)\,dx=\int_a^b f(a-x)\,dx$

$$\therefore\ I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}\,dx}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\therefore\ 2I=\int_{\pi/6}^{\pi/3}\left[\frac{\sqrt{\cos x}\,dx}{\sqrt{\cos x}+\sqrt{\sin x}}+\frac{\sqrt{\sin x}\,dx}{\sqrt{\sin x}+\sqrt{\cos x}}\right]dx=\int_{\pi/6}^{\pi/3}\left(\frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right)dx=\int_{\pi/6}^{\pi/3}1\cdot dx=x\Big]_{\pi/6}^{\pi/3}$$

$$=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$$

$$\therefore\ I=\frac{\pi}{12}$$

Exercise

Evaluate the following

(1) $\displaystyle\int_0^{\pi/2}\frac{\sin x}{1+\cos^2 x}dx$ (2) $\displaystyle\int_1^{e}\log x\,dx$ (3) $\displaystyle\int_0^{\pi/4}\frac{e^{\tan x}}{\cos^2 x}dx$

(4) $\displaystyle\int_{-1/2}^{1/2}\frac{dx}{9-x^2}$ (5) $\displaystyle\int_0^{1}x\tan^{-1}x\,dx$ (6) $\displaystyle\int_{-1}^{1}xe^{-x}\,dx$

(7) $\displaystyle\int_0^{\pi/2}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$ (8) $\displaystyle\int_0^{\infty}\frac{x\,dx}{(x+1)(x^2+1)}$ (9) $\displaystyle\int_0^{\pi/4}\log(1+\tan\theta)\,d\theta$

(10) $\displaystyle\int_0^{2a}\frac{f(x)}{f(x)+f(2a-x)}dx$ (11) $\displaystyle\int_0^{\pi/2}\frac{x\,dx}{\sin x+\cos x}$ (12) $\displaystyle\int_0^{1}\frac{\log(1+x)}{(1+x^2)}dx$

Answers: (1) $\frac{\pi}{4}$, (2) 1, (3) $e-1$, (4) $\frac{1}{3}\log\frac{7}{5}$, (5) $\frac{\pi}{4}-\frac{1}{2}$, (6) $-\frac{2}{e}$, (7) $\frac{\pi}{2ab}$, (8) $\frac{\pi}{4}$,

(9) $\frac{\pi}{8}\log 2$ (10) a (11) $\frac{\pi}{2\sqrt{2}}\log\left(\sqrt{2}+1\right)$ (12) $\frac{\pi}{8}\log 2$

Reduction formulae

I. To obtain the reduction formula for $I_n = \int \sin^n x\,dx$ and hence to evaluate $\int_0^{\pi/2} \sin^n x\,dx$

Solution: $I_n = \int \sin^{n-1} x \cdot \sin x\,dx$

put $u = \sin^{n-1} x$ & $v' = \sin x$, $u' = (n-1)\sin^{n-2} x\cos x$ & $v = -\cos x$

$$\therefore\ I_n = -\sin^{n-1} x\cos x + \int (n-1)\sin^{n-2} x\cos^2 x\,dx = -\sin^{n-1}\cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx = -\sin^{n-1}\cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$\therefore\ I_n + (n-1)I_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2}$$

ie $(1+n-1)I_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2}$

ie $I_n = \dfrac{-\sin^{n-1} x\cos x}{n} + \dfrac{n-1}{n} I_{n-2}$

the ultimate integral is I_0 or I_1 according as n is even or odd

If n is even $I_0 = \int 1\,dx = x$ If n is odd $I_1 = \int \sin x\,dx = -\cos x$

If $I_n = \int_0^{\pi/2} \sin^n x\, dx$ then

$$I_n = \left. \frac{-\sin^{n-1} x \cos x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} I_{n-2} = 0 + \frac{n-1}{n} I_{n-2}$$

$$\therefore\ I_n = \frac{n-1}{n} I_{n-2} = \frac{n-1}{n} \times \frac{n-3}{n-2} I_{n-4} = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} I_{n-6}$$

in general

$$I_n = \begin{cases} \dfrac{n-1}{n} \times \dfrac{n-3}{n-2} \times \cdots\cdots \times \dfrac{1}{2} \times \dfrac{\pi}{2} & \text{if } n \text{ is even.} \\ \\ \dfrac{n-1}{n} \times \dfrac{n-3}{n-2} \times \cdots\cdots \times \dfrac{2}{3} \times 1 & \text{if } n \text{ is odd.} \end{cases}$$

Eg. (1) $I_6 = \int_0^{\pi/2} \sin^6 x\, dx = \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{5\pi}{32}$

(2) $I_5 = \int_0^{\pi/2} \sin^5 x\, dx = \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{8}{15}$

II. To obtain the reduction formula for $I_n = \int \cos^n dx$ and to evaluate $\int_0^{\pi/2} \cos^n x\, dx$

Solution : using integration by parts as in I we obtain

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_n$$

and further if $I_n = \int_0^{\pi/2} \cos^n x\, dx = \int_0^{\pi/2} \cos^n \left(\frac{\pi}{2} - x \right) dx$ using $\int_0^a f(x)\, dx = \int_0^a f(a -$

$$= \int_0^{\pi/2} \sin^n x\, dx \quad \text{which is } I$$

$$\therefore \text{ Eg. (1) } \int_0^{\pi/2} \cos^7 x\, dx = \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{16}{35}$$

$$\text{(2) } \int_0^{\pi/2} \cos^8 x\, dx = \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{35\pi}{256}$$

III. To obtain the reduction formula for $I_n = \int \tan^n x\, dx$

Solution: $I_n = \int \tan^{n-2} x \cdot \tan^2 x\, dx = \int \tan^{n-2} x \cdot (\sec^2 x - 1)\, dx = \int \tan^{n-2} x \cdot \sec^2 x\, dx - \int \tan^{n-2} x\, dx$

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2} \quad \text{which is the required formula.}$$

IV. To obtain the reduction formula for $I_n = \int \cot^n x\, dx$

Solution: $I_n = \int \cot^{n-2} x \cdot \cot^2 x\, dx = \int \cot^{n-2} x \cdot (\operatorname{cosec}^2 x - 1)\, dx = \int \cot^{n-2} x \operatorname{cosec}^2 x\, dx - \int \cot^{n-2} x\, dx$

$$I_n = \frac{-\cot^{n-1}x}{n-1} - I_{n-2}$$

V. To obtain the reduction formula for $I_n = \int \sec^n x\,dx$

Solution: $I_n = \int \sec^{n-2} x \cdot \sec^2 x\,dx$

put $u = \sec^{n-2} x$ & $v' = \sec^2 x$, $u' = (n-2)\sec^{n-3} x \cdot \sec x \tan x$ & $v = \tan x$

$$\therefore\ I_n = \sec^{n-2}\tan x - \int (n-2)\sec^{n-2} x \cdot \tan^2 x\,dx = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2}(\sec^2 x - 1)\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$$

$$\therefore\ I_n + (n-2)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

$$(1+n-2)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

$\therefore\ I_n = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$ which is the required reduction formula.

VI. To obtain the reduction formula for $I_n = \int \operatorname{cosec}^n x\,dx$

Solution: $I_n = \int \operatorname{cosec}^{n-2} x \cdot \operatorname{cosec}^2 dx$

put $u = \operatorname{cosec}^{n-2} x$ & $v' = \operatorname{cosec}^2 x$, $u' = -(n-2)\operatorname{cosec}^{n-3} x \cdot \operatorname{cosec} x \cot x$ & $v = -\cot x$

$$\therefore\ I_n = -\operatorname{cosec}^{n-2} x \cot x - \int (n-2)\operatorname{cosec}^{n-2} x \cdot \cot^2 x\,dx = -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^{n-2} x(\operatorname{cosec}^2 x -$$

$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^n x\,dx + (n-2)\int \operatorname{cosec}^{n-2} x\,dx = -\operatorname{cosec}^{n-2} x \cot x - (n-2)I_n + (n-

ie $I_n + (n-2)I_n = -\text{cosec}^{n-2}x\cot x + (n-2)I_{n-2}$

ie $(n-1)I_n = -\text{cosec}^{n-2}x\cot x + (n-2)I_{n-2}$

ie $I_n = \dfrac{-\text{cosec}^{n-2}x\cot x}{(n-1)} + \dfrac{(n-2)}{(n-1)}I_{n-2}$ which is the required reduction formula.

VII. To obtain the reduction formula of $I_{m,n} = \int \sin^m x \cos^n x\,dx$ and hence to evaluate $\int_0^{\pi/2} \sin^m x \cos^n x\,dx$

Solution: $I_{m,n} = \int \sin^m x \cos^{n-1} x \cdot \cos x\,dx$

put $u = \sin^m x \cos^{n-1} x$ & $v' = \cos x$, $u' = m\sin^{m-1} x \cdot \cos^n x - (n-1)\sin^{m+1} x \cdot \cos^{n-2} x$ & $v = \sin x$

$\therefore\ I_{m,n} = \sin^m x \cos^{n-1} x \cdot \sin x - \int \left(m\sin^{m-1} x \cos^n x - (n-1)\sin^{m+1}\cos^{n-2} x\right)\sin x\,dx$

$= \sin^{m+1} x \cos^{n-1} x - m\int \sin^m x \cos^n x\,dx + (n-1)\int \sin^{m+2} x \cdot \cos^{n-2} x\,dx$

$= \sin^{m+1} x \cos^{n-1} x - mI_{m,n} + (n-1)\int \sin^m x(1-\cos^2 x)\cos^{n-2} x\,dx$

$= \sin^{m+1} x \cos^{n-1} x - mI_{m,n} + (n-1)\int (\sin^m x \cos^{n-2} x - \sin^m x \cos^n x)\,dx$

$= \sin^{m+1} x \cos^{n-1} x - mI_{m,n} + (n-1)I_{m,m-2} - (n-1)I_{m,n}$

ie $I_{m,n} + mI_{m,n} + (n-1)I_{m,n} = \sin^{m+1} x \cos x^{n-1} x + (n-1)I_{m,n-2}$

ie $I_{m,n} = \dfrac{\sin^{m+1} x \cos^{n-1} x}{(m+n)} + \dfrac{(n-1)}{m+n}I_{m,n-2}$

ie $I_{m,n} = \dfrac{\sin^{m+1} x \cos^{n-1} x}{(m+n)} + \dfrac{(n-1)}{m+n} I_{m,n-2}$

which is the required reduction formula, if $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$

then $I_{m,n} = \left.\dfrac{\sin^{m+1} x \cos^{n-1} x}{(m+n)}\right]_0^{\pi/2} + \dfrac{n-1}{m+n} I_{m,n-2} = 0 + \dfrac{n-1}{m+n} I_{m,n-2} \quad \therefore\ I_{m,n} = \dfrac{n-1}{m+n} I_{m,n-2}$

applying this reduction formula continuously, we have

$$I_{m,n} = \begin{cases} \dfrac{n-1}{m+n} \times \dfrac{n-3}{m+n-2} \times \cdots\cdots \times \dfrac{2}{m+3} \times \dfrac{1}{m+1} & \text{if } n \text{ is odd \& } m \text{ odd or even} \\ \dfrac{n-1}{m+n} \times \dfrac{n-3}{m+n-2} \times \cdots\cdots \times \dfrac{1}{m+2} \times \dfrac{m-1}{m} \times \dfrac{m-3}{m-2} \times \cdots\cdots \times \dfrac{2}{3} \times 1 & \text{if } n \text{ is even \& } m \text{ is odd} \\ \dfrac{n-1}{m+n} \times \dfrac{n-3}{m+n-2} \times \cdots\cdots \times \dfrac{1}{m+2} \times \dfrac{m-1}{m} \times \dfrac{m-3}{m-2} \times \cdots\cdots \times \dfrac{1}{2} \times \dfrac{\pi}{2} & \text{if } n \text{ is even \& } m \text{ is even} \end{cases}$$

Examples

(1) $\quad I_{5,5} = \int_0^{\pi/2} \sin^5 x \cos^5 x \, dx = \dfrac{4}{10} \times \dfrac{2}{8} \times \dfrac{1}{6} = \dfrac{1}{60}$

(2) $\quad I_{6,5} = \int_0^{\pi/2} \sin^6 x \cos^5 x \, dx = \dfrac{4}{11} \times \dfrac{2}{9} \times \dfrac{1}{7} = \dfrac{8}{693}$

(3) $I_{7,4} = \int_0^{\pi/2} \sin^7 x \cos^4 x\, dx = \frac{3}{11} \times \frac{1}{9} \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{48}{3465}$

(4) $I_{6,6} = \int_0^{\pi/2} \sin^6 x \cos^6 x\, dx = \frac{5}{12} \times \frac{3}{10} \times \frac{1}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{5\pi}{2048}$

(5) Evaluate $\int_0^1 \frac{x^9}{\sqrt{1-x^2}} dx$

Solution: put $x = \sin\theta,\ dx = \cos\theta\, d\theta$ when $x = 0,\ \theta = 0$ when $x = 1,\ \theta = \frac{\pi}{2}$

$$\int_0^1 \frac{x^9}{\sqrt{1-x^2}} dx = \int_0^{\pi/2} \frac{\sin^9\theta \cdot \cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} = \int_0^{\pi/2} \frac{\sin^9\theta \cdot \cos\theta\, d\theta}{\cos\theta} = \int_0^{\pi/2} \sin^9\theta\, d\theta = \frac{8}{9} \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{128}{315}$$

(6) Evaluate $\int_0^{2a} \frac{x^3\, dx}{\sqrt{2ax - x^2}}$

Solution: $\int_0^{2a} \frac{x^3\, dx}{\sqrt{2ax-x^2}} = \int_0^{2a} \frac{x^3\, dx}{\sqrt{a^2-(x-a)^2}}$ put $x - a = a\sin\theta,\ dx = a\cos\theta\, d\theta$

when $x = 0,\ \sin\theta = -1 \Rightarrow \theta = -\frac{\pi}{2}$ when $x = 2a,\ \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \text{ G.I.} = \int_{-\pi/2}^{\pi/2} \frac{(a + a\sin\theta)^3 \cdot a\cos\theta\, d\theta}{\sqrt{a^2 - a^2\sin^2\theta}} = \int_{-\pi/2}^{\pi/2} \frac{a^3(1+\sin\theta)^3 a\cos\theta\, d\theta}{a\cos\theta} = a^3 \int_{-\pi/2}^{\pi/2} (1 + \sin^3\theta + 3\sin\theta + 3\sin^2$$

$$= a^3\left[\int_{-\pi/2}^{\pi/2} 1 d\theta + \int_{-\pi/2}^{\pi/2} \sin^3\theta\, d\theta + 3\int_{-\pi/2}^{\pi/2} \sin\theta\, d\theta + 3\int_{-\pi/2}^{\pi/2} \sin^2\theta\, d\theta\right] = a^3\left[\theta \Big|_{-\pi/2}^{\pi/2} + 0 + 0 + 6\int_0^{\pi/2} \sin^2\theta\, d\theta\right.$$

$$= a^3\left[\frac{\pi}{2} - \left(\frac{\pi}{2}\right) + 6 \times \frac{1}{2} \times \frac{\pi}{2}\right] = a^3\left(\pi + \frac{3\pi}{2}\right) = \frac{5\pi}{2} a^3$$

Exercise

Evaluate the following

(1) $\int_0^{\pi/6} \sin^5 3\theta \, d\theta$ (2) $\int_0^{\pi} x \sin^7 x \, dx$ (3) $\int_0^1 x^4 (1-x^2)^{3/2} dx$

(4) $\int_0^{\infty} \frac{dx}{(1+x^2)^{7/2}}$ (5) $\int_0^1 x^6 \sqrt{1-x^2} \, dx$ (6) $\int_0^2 x^{5/2} \sqrt{2-x} \, dx$

(7) $\int_{\pi/4}^{\pi/2} \cot^4 x \, dx$ (8) $\int_{\pi/6}^{\pi/2} \operatorname{cosec}^5 x \, dx$ (9) $\int_0^{\pi} \sin^2\theta \frac{\sqrt{1-\cos\theta}}{1+\cos\theta} d\theta$

(10) $\int_0^1 \frac{x^7}{\sqrt{1-x^4}} dx$ (11) $\int_0^1 \frac{x^3}{(1+x^2)^4} dx$ (12) $\int_0^a \frac{x^7 dx}{\sqrt{a^2-x^2}}$

Answers: (1) $\frac{8}{45}$, (2) $\frac{16\pi}{35}$, (3) $\frac{3\pi}{256}$, (4) $\frac{8}{15}$, (5) $\frac{5\pi}{256}$, (6) $\frac{5\pi}{8}$, (7) $\frac{3\pi-8}{12}$, (8) $\frac{11\sqrt{3}}{4}+\frac{3}{8}\log(2+\sqrt{3})$, (9) $\frac{8\sqrt{2}}{3}$,

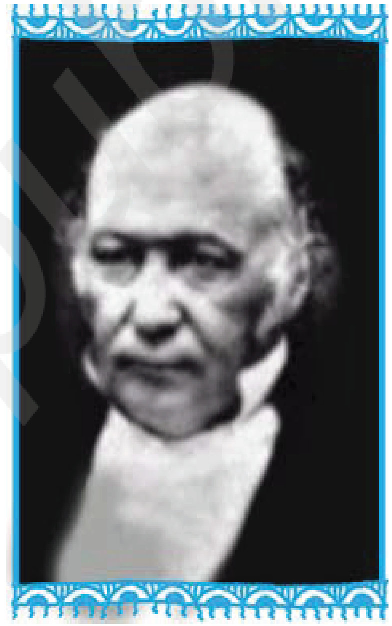
(10) $\frac{1}{3}$, (11) $\frac{1}{24}$, (12) $\frac{16a^7}{35}$

UNIT – 5
VECTOR ALZEBRA

❖ *In most sciences one generation tears down what another has built and what one has established another undoes. In Mathematics alone each generation builds a new story to the old structure. – HERMAN HANKEL* ❖

Introduction

In our day to day life, we come across many queries such as – What is your height? How should a football player hit the ball to give a pass to another player of his team? Observe that a possible answer to the first query may be 1.6 meters, a quantity that involves only one value (magnitude) which is a real number. Such quantities are called *scalars*. However, an answer to the second query is a quantity (called force) which involves muscular strength (magnitude) and direction (in which another player is positioned). Such quantities are called *vectors*. In mathematics, physics and engineering, we frequently come across with both types of quantities, namely, scalar quantities such as length, mass, time, distance, speed, area, volume, temperature, work, money, voltage, density, resistance etc. and vector quantities like displacement, velocity, acceleration, force, weight, momentum, electric field intensity etc.

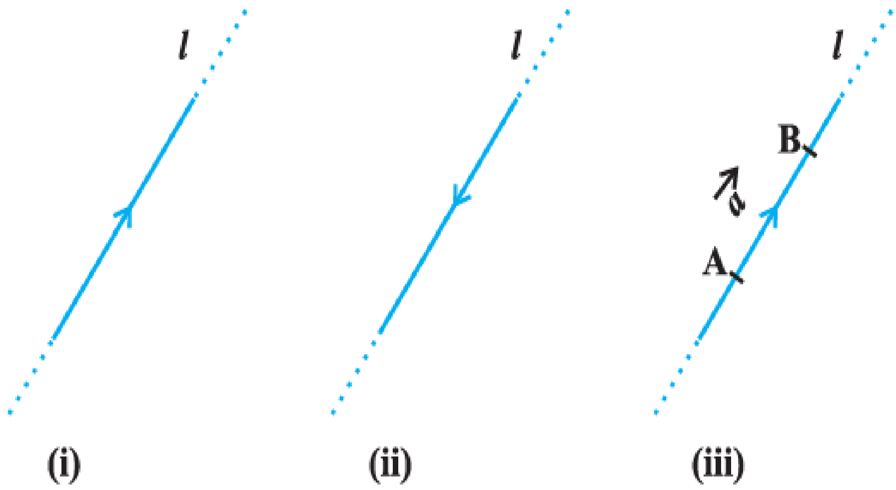


W.R. Hamilton
(1805-1865)

In this chapter, we will study some of the basic concepts about vectors, various operations on vectors, and their algebraic and geometric properties. These two type of properties, when considered together give a full realisation to the concept of vectors, and lead to their vital applicability in various areas as mentioned above.

Some Basic Concepts

Let 'l' be any straight line in plane or three dimensional space. This line can be given two directions by means of arrowheads. A line with one of these directions prescribed is called a *directed line* Fig 6.1 (i) (ii)



Now observe that if we restrict the line l to the line segment AB, then a magnitude is prescribed on the line l with one of the two directions, so that we obtain a *directed line segment* Fig 6.1 (iii)

Thus, a directed line segment has magnitude as well as direction.

Definition 1 A quantity that has magnitude as well as direction is called a vector.

Notice that a directed line segment is a vector (Fig 10.1(iii)), denoted as $\overrightarrow{AB}$ or simply as $\vec{a}$, and read as 'vector $\overrightarrow{AB}$' or 'vector $\vec{a}$'.

The point A from where the vector $\overrightarrow{AB}$ starts is called its *initial point*, and the point B where it ends is called its *terminal point*. The distance between initial and terminal points of a vector is called the *magnitude* (or length) of the vector, denoted as $|\overrightarrow{AB}|$, or $|\vec{a}|$, or a. The arrow indicates the direction of the vector.

Note Since the length is never negative, the notation $|\vec{a}| < 0$ has no meaning.

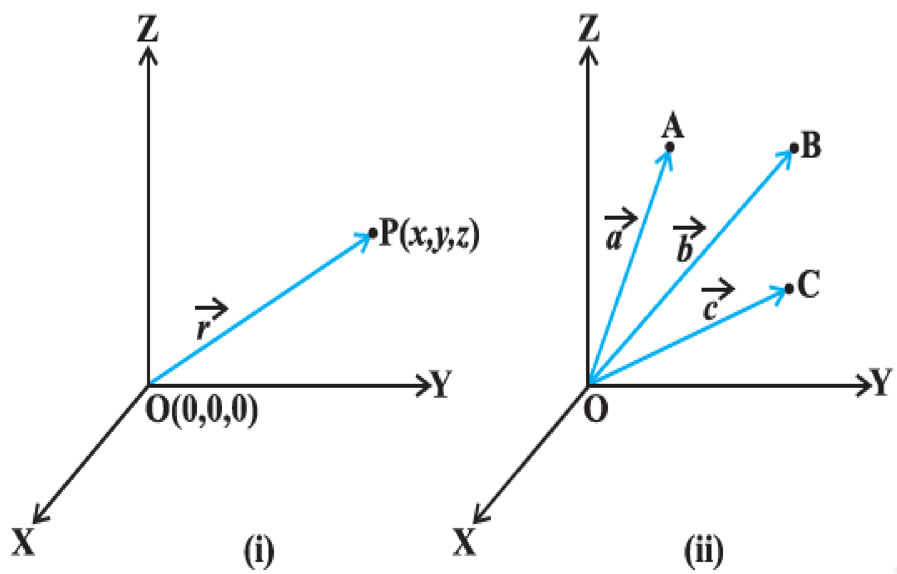
Position Vector

From Class XI, recall the three dimensional right handed rectangular coordinate system Fig 6.2 (i)

Consider a point P in space, having coordinates (x, y, z) with respect to the origin $O(0, 0, 0)$. Then, the vector $\overrightarrow{OP}$ having O and P as its initial and terminal points, respectively, is called the *position vector* of the point P with respect to O. Using distance formula (from Class XI), the magnitude of $\overrightarrow{OP}$ (or $\vec{r}$) is given by

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$

In practice, the position vectors of points A, B, C, etc., with respect to the origin O are denoted by $\vec{a}$, $\vec{b}$, $\vec{c}$, etc., respectively Fig 6.2 (ii)



Example 1

Find unit vector in the direction of vector $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$

Solution The unit vector in the direction of a vector $\vec{a}$ is given by $\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$.

Now $\quad |\vec{a}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$

Therefore $\quad \hat{a} = \frac{1}{\sqrt{14}}(2\hat{i} + 3\hat{j} + \hat{k}) = \frac{2}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{j} + \frac{1}{\sqrt{14}}\hat{k}$

Example 2

Find a vector in the direction of vector $\vec{a} = \hat{i} - 2\hat{j}$ that has magnitude 7 units.

Solution The unit vector in the direction of the given vector $\vec{a}$ is

$$\hat{a} = \frac{1}{|\vec{a}|}\vec{a} = \frac{1}{\sqrt{5}}(\hat{i} - 2\hat{j}) = \frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}$$

Therefore, the vector having magnitude equal to 7 and in the direction of $\vec{a}$ is

$$7\hat{a} = 7\left(\frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}\right) = \frac{7}{\sqrt{5}}\hat{i} - \frac{14}{\sqrt{5}}\hat{j}$$

Example 3

Find the unit vector in the direction of the sum of the vectors, $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + 3\hat{k}$.

Solution The sum of the given vectors is

$$\vec{a} \quad \vec{b} \; (\quad \vec{c}, \text{say}) = 4\hat{i} \quad 3\hat{j} \quad 2\hat{k}$$

and
$$|\vec{c}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$$

Thus, the required unit vector is

$$\hat{c} = \frac{1}{|\vec{c}|}\vec{c} = \frac{1}{\sqrt{29}}(4\hat{i} + 3\hat{j} - 2\hat{k}) = \frac{4}{\sqrt{29}}\hat{i} + \frac{3}{\sqrt{29}}\hat{j} - \frac{2}{\sqrt{29}}\hat{k}$$

Product of Two Vectors

So far we have studied about addition and subtraction of vectors. An other algebraic operation which we intend to discuss regarding vectors is their product. We may recall that product of two numbers is a number, product of two matrices is again a matrix. But in case of functions, we may multiply them in two ways, namely, multiplication of two functions pointwise and composition of two functions. Similarly, multiplication of two vectors is also defined in two ways, namely, scalar (or dot) product where the result is a scalar, and vector (or cross) product where the result is a vector. Based upon these two types of products for vectors, they have found various applications in geometry, mechanics and engineering. In this section, we will discuss these two types of products.

Scalar (or dot) product of two vectors

Definition 2 The scalar product of two nonzero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$, is defined as $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$,

where, θ is the angle between $\vec{a}$ and $\vec{b}$, 0 (Fig 10.19).

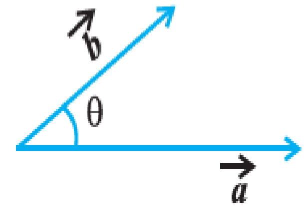


Fig 10.19

If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then θ is not defined, and in this case,

If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then θ is not defined, and in this case,

we define $\vec{a}$ $\vec{b}$ 0

Observations

1. $\vec{a} \cdot \vec{b}$ is a real number.

2. Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors, then $\vec{a} \cdot \vec{b} = 0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular to each other. i.e.

   $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$

3. If $\theta = 0$, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$

   In particular, $\vec{a} \cdot \vec{a} = |\vec{a}|^2$, as θ in this case is 0.

4. If $\theta = \pi$, then $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$

   In particular, $\vec{a}$ ($\vec{a}$) $|\vec{a}|^2$, as θ in this case is π.

5. In view of the Observations 2 and 3, for mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1,$$

$$\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\ \hat{i}\quad 0$$

6. The angle between two nonzero vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\qquad \frac{\vec{a}\ \vec{b}}{|\vec{a}||\vec{b}|},\ \text{or}\ \theta=\cos^{-1}\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

7. The scalar product is commutative. i.e.

$$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}\qquad\text{(Why?)}$$

Two important properties of scalar product

Property 1 (Distributivity of scalar product over addition) Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors, then

$$\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$$

Property 2 Let $\vec{a}$ and $\vec{b}$ be any two vectors, and λ be any scalar. Then

$$(\lambda\vec{a})\cdot\vec{b} = (\ \vec{a})\ \vec{b} \quad (\vec{a}\ \vec{b}) \quad \vec{a}\ (\ \vec{b})$$

If two vectors $\vec{a}$ and $\vec{b}$ are given in component form as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then their scalar product is given as

$$\vec{a}\cdot\vec{b} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$$

$$= a_1\hat{i}\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})+a_2\hat{j}\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})+a_3\hat{k}\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$$

$$= a_1b_1(\hat{i}\cdot\hat{i})+a_1b_2(\hat{i}\cdot\hat{j})+a_1b_3(\hat{i}\cdot\hat{k})+a_2b_1(\hat{j}\cdot\hat{i})+a_2b_2(\hat{j}\cdot\hat{j})+a_2b_3(\hat{j}\cdot\hat{k})$$

$$+ a_3b_1(\hat{k}\cdot\hat{i})+a_3b_2(\hat{k}\cdot\hat{j})+a_3b_3(\hat{k}\cdot\hat{k}) \text{ (Using the above Properties 1 and 2)}$$

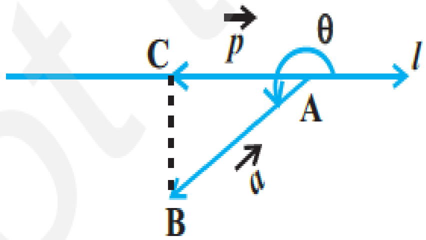
$$= a_1b_1 + a_2b_2 + a_3b_3 \qquad \text{(Using Observation 5)}$$

Thus $\quad \vec{a}\cdot\vec{b} = a_1b_1+a_2b_2+a_3b_3$

Projection of a vector on a line

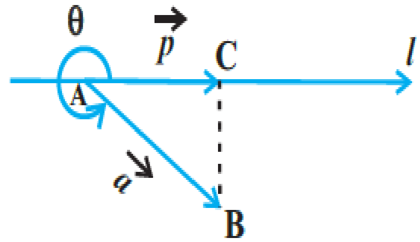
Suppose a vector $\overrightarrow{AB}$ makes an angle θ with a given directed line l (say), in the *anticlockwise direction* (Fig 10.20). Then the projection of $\overrightarrow{AB}$ on l is a vector $\vec{p}$ (say) with magnitude $|\overrightarrow{AB}|\cos\theta$, and the direction of $\vec{p}$ being the same (or opposite) to that of the line l, depending upon whether $\cos\theta$ is positive or negative. The vector $\vec{p}$

$(0^\circ < \theta < 90^\circ)$ (i) $\qquad (90^\circ < \theta < 180^\circ)$ (ii)



$(180^0 < \theta < 270^0)$

(iii)



$(270^0 < \theta < 360^0)$

(iv)

is called the *projection vector*, and its magnitude $|\vec{p}|$ is simply called as the *projection* of the vector $\overrightarrow{AB}$ on the directed line l.

For example, in each of the following figures (Fig 10.20 (i) to (iv)), projection vector of $\overrightarrow{AB}$ along the line l is vector $\overrightarrow{AC}$.

Observations

1. If $\hat{p}$ is the unit vector along a line l, then the projection of a vector $\vec{a}$ on the line l is given by $\vec{a}\ \ \hat{p}$.

2. Projection of a vector $\vec{a}$ on other vector $\vec{b}$, is given by

$$\vec{a}\cdot\hat{b}, \quad \text{or} \quad \vec{a}\cdot\left(\frac{\vec{b}}{|\vec{b}|}\right), \quad \text{or} \quad \frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b})$$

3. If $\theta = 0$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{AB}$ itself and if $\theta = \pi$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{BA}$.

4. If $\theta = \dfrac{\pi}{2}$ or $\theta = \dfrac{3\pi}{2}$, then the projection vector of $\overrightarrow{AB}$ will be zero vector.