1 Forward reasoning example

$$
\begin{aligned}
&\mathit{Input} \leftarrow g = \mathbb{E}(s^n t^c)\\
&\text{while}(n > 0)\{ g_1 = g - g[s/0]\\
&\quad \{n := n-1\} g_2 = g_1 s^{-1}\\
&\quad [1/2]\\
&\quad \{c := c+1\} g_3 = g_1 t\\
&\quad g_4 = \frac{1}{2}(g_2 + g_3)\\
&\}\mathit{Output} \leftarrow \text{fix}\left[g \mapsto g[s/0] + \frac{1}{2}(s^{-1}+t)(g - g[s/0])\right]
\end{aligned}
$$

Finally, we get to solve the LFP.

1. Let $g = \sum_{i=0}^{\infty} s^i h_i$ where $h_i = \frac{1}{i!}\frac{\partial^i g}{\partial s^i}[s/0]$
2. $f = f[s/0] + \frac{1}{2}(s^{-1}+t)(f - f[s/0])$ implies $f = f[s/0]$, every term in the LFP is s-free.
3. The term $s^0 h_0$ becomes h_0 eventually.
4. Consider the term $s^i h_i$ where $i > 0$.
   (a) It gets multiplied by $t/2$ or $s^{-1}/2$, until it becomes a s-free term.
   (b) Suppose that it becomes a s-free term after $i+j$ iterations: The last factor must be $s^{-1}/2$, and j factors among the other $i+j-1$ factors are $t/2$.
   $$s^i h_i \rightsquigarrow h_i 2^{-i}\binom{i+j-1}{j}2^{-j}t^j = \binom{-i}{j}2^{-j}t^j$$
   (c) Sum the results over $j = 0, 1, 2 \dots$
   $$\frac{h_i}{2^i}\sum_{j=0}^{\infty}\binom{-i}{j}2^{-j}t^j = \frac{h_i}{2^i}(1-t/2)^{-i}$$
5. Thus, the least fixed-point is
   $$h_0 + \sum_{i=1}^{\infty}\frac{h_i}{2^i}(1-t/2)^{-i}\sum_{i=0}^{\infty}\frac{h_i}{2^i}(1-t/2)^{-i} = g[s/(1-t/2)^{-1}]$$

$Input \leftarrow g = \mathbb{E}(s^n t^c)$

$\text{if}(n > 0)\{ g_1 = g - g[s/0]$

$c := c + \text{iid}(\text{geometric}(1/2), n); g_2 = g_1[s/s(1 - t/2)^{-1}]$

$n := 0 g_3 = g_2[s/1] = g_1[s/(1 - t/2)^{-1}]$

$\} Output \leftarrow g[s/0] + (g - g[s/0])[s/(1 - t/2)^{-1}] = g[s/(1 - t/2)^{-1}]$

2 Fixed point induction example

Previously, we have proven that the following loop-free program

```
if (n > 0) {
    c := c + iid(geometric(1/2),n);
    n := 0
}
```

is the least fixed point of the following loop

```
while (n > 0) {
    {n := n-1}[1/2]{c := c+1}
}
```

By fixed point induction, the loop is semantically equivalent to

```
if (n > 0) {
    {n := n-1}[1/2]{c := c+1}
    if (n > 0) {
        c := c + iid(geometric(1/2),n);
        n := 0
    }
}
```