Lake Model: Employment and Unemployment

Quantitative Economics with Python

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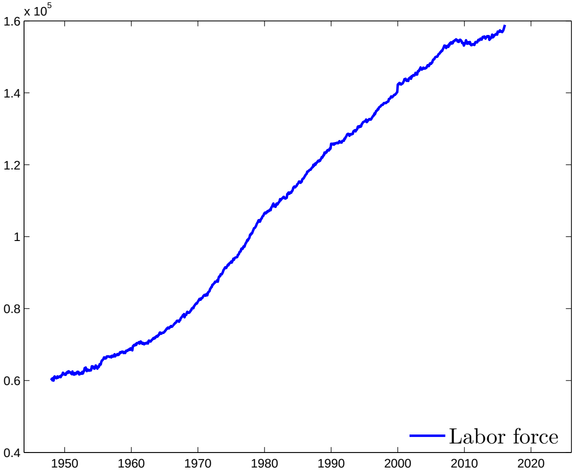
Facts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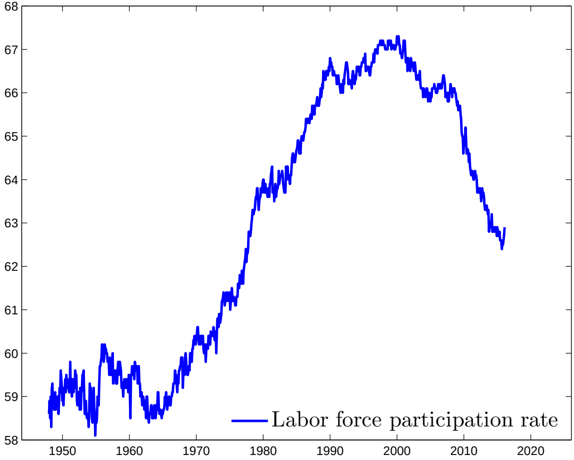
Labor force

Facts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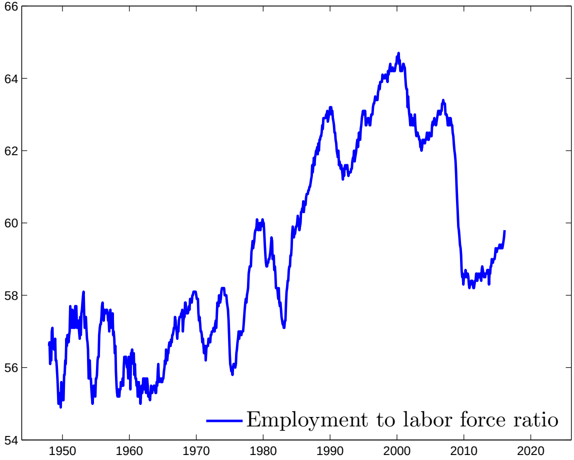
Labor force participation rate

Facts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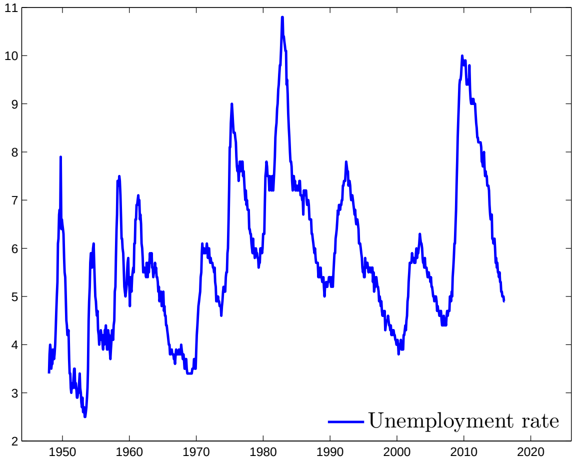
Employment to labor force ratio

Facts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2020
Unemployment rate

Overview

- The lake model is a basic tool for modeling unemployment
- It is a tool for analyzing
  - flows between unemployment and employment
  - how they influence steady state employment and unemployment rates
- It is a good model for interpreting monthly labor department reports on gross and net jobs created and jobs destroyed

Overview

- The "lakes" in the model are the pools of employed and unemployed
- The "flows" between the lakes are caused by
  - firing and hiring
  - entry and exit from the labor force
- First, the parameters governing transitions into and out of unemployment and employment are exogenous
- Later, some of the transition rates are endogenously: McCall search model

Overview

- The **only** knowledge required for this lecture is
  - basic linear algebra
  - elementary concepts of Markov Chains
  - dynamic program
- We'll use some nifty concepts like ergodicity, which provides a fundamental link between cross sectional and long run time series distributions
- These concepts will help us build an equilibrium model of ex ante homogeneous workers whose different luck generates variations in their ex post experiences

Model

- The economy is inhabited by a large number of **ex-ante** identical workers.
  - live forever
  - spend their lives moving between unemployment and employment
- Transition rate between being unemployed and employed are
  - λ: job finding rate for currently unemployed workers
  - α: dismissal rate for currently employed workers
  - b: entry rate into the labor force
  - d: exit rate from the labor force
- The growth rate of the labor force evidently equals $g = b - d$

Aggregates

- We want the dynamics of the following aggregates
  - E_t: total number of employed workers
  - U_t: total number of unemployed workers
  - N_t: number of workers in the labor force
- We also want to know the values of the following objects
  - e_t: employment rate E_t/N_t
  - u_t: unemployment rate U_t/N_t

Laws of Motion

- Of the mass of workers E_t who are employed
  - $(1-d)E_t$ will remain in the labor force
  - $(1-\alpha)(1-d)E_t$ will remain employed
- Of the mass of workers U_t workers who are currently unemployed
  - $(1-d)U_t$ will remain in the labor force
  - $\lambda(1-d)U_t$ will become employed
- The number of workers who will be employed at $t+1$

$$E_{t+1} = (1-d)(1-\alpha)E_t + (1-d)\lambda U_t$$

- The number of workers who will be unemployed at $t+1$

$$U_{t+1} = (1-d)\alpha E_t + (1-d)(1-\lambda)U_t + b(E_t + U_t)$$

Laws of Motion

- The total stock of workers $N_t = E_t + U_t$ evolves as

$$N_{t+1} = (1 + b - d)N_t = (1 + g)N_t$$

- Linear state space

$$X_t = \begin{pmatrix} E_t \\ U_t \end{pmatrix}$$

- Law of motion for X is

$$X_{t+1} = \begin{pmatrix} E_{t+1} \\ U_{t+1} \end{pmatrix} = A \begin{pmatrix} E_t \\ U_t \end{pmatrix} = AX_t$$

where

$$A = \begin{bmatrix} (1-d)(1-\alpha) & (1-d)\lambda \\ (1-d)\alpha + b & (1-d)(1-\lambda) + b \end{bmatrix}$$

Laws of Motion

- Laws of Motion for Rates of Employment and Unemployment

$$\begin{pmatrix} E_{t+1}/N_{t+1} \\ U_{t+1}/N_{t+1} \end{pmatrix} = \frac{1}{1+g} \begin{pmatrix} (1-d)(1-\alpha) & (1-d)\lambda \\ (1-d)\alpha + b & (1-d)(1-\lambda) + b \end{pmatrix} \begin{pmatrix} E_t/N_t \\ U_t/N_t \end{pmatrix}$$

- Define x_t as

$$x_t = \begin{pmatrix} e_t \\ u_t \end{pmatrix} = \begin{pmatrix} E_t/N_t \\ U_t/N_t \end{pmatrix}$$

  or

$$x_{t+1} = \hat{A} x_t \quad \text{where} \quad \hat{A} := \frac{1}{1+g} A$$

- Evidently, $e_t + u_t = 1$ implies that $e_{t+1} + u_{t+1} = 1$

Steady States

- The aggregates E_t and U_t won't converge to steady states because their sum $E_t + U_t$ grows at gross rate $1 + g$

- The vector of employment and unemployment rates x_t can be in a steady state $\bar{x}$ provided that we can find a solution to the matrix equation

$$\bar{x} = \hat{A}\bar{x}$$

  where the components satisfy

$$\bar{e} + \bar{u} = 1$$

- A steady state $\bar{x}$ is an eigenvector of $\hat{A}$ associated with a unit eigenvalue

- We also have $x_t \to \bar{x}$ as $t \to \infty$ provided that the remaining eigenvalues of $\hat{A}$ are in modulus less that 1

Steady States

Let us go to Python notebook...

Lake Model: Workers' Dynamics and Markov Process

- A worker's employment dynamics are governed by a *Markov process*
- The worker can be in one of two states:
  - $s = 0$ means that the worker is unemployed
  - $s = 1$ means that the worker is employed
- The transition matrix between the two states

$$P_{ij} = \text{Prob}(s_{t+1} = j | s_t = i)$$

and

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \lambda & 1-\lambda \end{pmatrix}$$

- P_{ij}: probability for worker moves from state i at t to state j at $t+1$

Questions

- What is the average fraction of time a worker being employed over time?
- What is the fraction of workers who are employed at a particular time?
- What is the link between the answers to the above questions?

To answer these questions, we need some knowledge about Markov chain.

Finite Markov Chains

- Stochastic Matrix and Markov Chain
  - Definition
  - Irreducibility
  - Aperiodicity
- Marginal Distribution
  - Definition
  - Evolution of marginal distribution
  - Stationary marginal distribution
  - Unique stationary distribution
- Ergodicity

Stochastic Matrix (Markov Matrix)

- Let $S = \{s_1, s_2, \ldots, s_n\}$
- A stochastic matrix is an $n \times n$ square matrix $P = P[s, s']$ such that
  - each element $P[s, s']$ is nonnegative, and
  - each row $P[s, \cdot]$ sums to one
- Each row $P[s, \cdot]$ can be regarded as a distribution on S
- Remark: if P is a stochastic matrix, so is k-th power P^k for all $k \in \mathbb{N}$

Stochastic Matrix: Example

- Let's look at the worker's employment dynamics example

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \lambda & 1-\lambda \end{pmatrix}$$

- All elements are positive
- Each row sums to one

Stochastic Matrix: Example

- Stochastic matrix about U.S. economy state (monthly frequency)

$$P = \begin{pmatrix} 0.971 & 0.029 & 0 \\ 0.145 & 0.778 & 0.077 \\ 0 & 0.508 & 0.492 \end{pmatrix}$$

  - the first state represents "normal growth"
  - the second state represents "mild recession"
  - the third state represents "severe recession"

- For example, when the state is normal growth, the state will again be normal growth next month with probability 0.97

- Large values on the main diagonal indicate persistence in the process

Stochastic Matrix: Multiple Step Transition Probability

- The probability of transitioning from s to s' in m step: $P^m[s, s']$.
- A two state example

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

- Probability of moving from state i to state j in one period is p_{ij}
- Probability of moving from state 1 to state 1 in two periods is

$$\begin{pmatrix} p_{11}p_{11} + p_{12}p_{21} & p_{11}p_{12} + p_{12}p_{22} \\ p_{21}p_{11} + p_{22}p_{21} & p_{21}p_{12} + p_{22}p_{22} \end{pmatrix} = P \times P = P^2$$

Markov Chain

- A Markov Chain $\{X_t\}$ is a stochastic process that has the Markov property

$$\mathbb{P}\{X_{t+1} = s' \mid X_t\} = \mathbb{P}\{X_{t+1} = s' \mid X_t, X_{t-1}, \ldots\}$$

- Knowing current state is enough to understand probabilities for future states

- The dynamics of a Markov chain are fully determined by the set of values

$$P[s, s'] = \mathbb{P}\{X_{t+1} = s' \mid X_t = s\} \qquad (s, s' \in S)$$

  - $P[s, s']$ is the probability of going from s to s' in one unit of time (one step)
  - $P[s, \cdot]$ is the conditional distribution of X_{t+1} given $X_t = s$

- It's clear that P is a stochastic matrix

Markov Chain

- With a stochastic matrix P, we can generate a Markov chain $\{X_t\}$
  - draw X_0 from some specified distribution
  - draw X_{t+1} from $P[X_t, \cdot]$
- Let us move to Python to generate some...

Irreducibility

- Let P be a fixed stochastic matrix
- Two states s and s are said to communicate with each other if there exist positive integers j and k such that

$$P^j[s, s'] > 0 \quad \text{and} \quad P^k[s', s] > 0$$

- This means that
  - state s can be reached eventually from state s'
  - state s' can be reached eventually from state s
- The stochastic matrix P is called irreducible if all states communicate

Irreducibility

- Worker's employment dynamics example is irreducible

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \lambda & 1-\lambda \end{pmatrix}$$

- Suppose worker has probability d to die

$$P = \begin{pmatrix} (1-d)(1-\alpha) & (1-d)\alpha & d \\ (1-d)\lambda & (1-d)(1-\lambda) & d \\ 0 & 0 & 1 \end{pmatrix}$$

This is not irreducible. Death is an absorbing state.

Aperidocity

- Loosely speaking, a Markov chain is called periodic if it cycles in a predictible way and aperiodic otherwise

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- The period of a state s is the greatest common divisor of the set of integers

$$D(s) := \{j \geq 1 : P^j[s, s] > 0\}$$

- A stochastic matrix is called aperiodic if the period of every state is 1, and periodic otherwise

Aperidocity

- Consider the following example

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{pmatrix}$$

All the states have period 2, which is a periodic Markov chain.

- Consider the following example

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

All the states have period 1, which is a aperiodic Markov chain.

Marginal Distribution

- Suppose that
  - $\{X_t\}$ is a Markov chain with stochastic matrix P
  - the distribution of X_t is known to be ψ_t
- What is the distribution of X_{t+1}, or, more generally, of X_{t+m}?

Marginal Distribution

- Let ψ_t be the distribution of X_t . We are looking for ψ_{t+1} given ψ_t and P
- To begin, pick any $s' \in S$. The probability that $X_{t+1} = s'$ is:

$$\mathbb{P}\{X_{t+1} = s'\} = \sum_{s \in S} \mathbb{P}\{X_{t+1} = s' \mid X_t = s\} \cdot \mathbb{P}\{X_t = s\}$$

- We account for all ways this can happen and sum their probabilities.
- More compactly

$$\psi_{t+1}[s'] = \sum_{s \in S} P[s, s'] \psi_t[s]$$

- There are n such equations, and the matrix expression is

$$\psi_{t+1} = \psi_t P$$

Marginal Distribution

- To move the distribution forward one unit of time, we postmultiply by P
- By repeating this m times we move forward m steps into the future
- Hence, $\psi_{t+m} = \psi_t P^m$ is also valid
- If ψ_0 is the initial distribution from which X_0 is drawn, then $\psi_0 P^m$ is the distribution of X_m

$$X_0 \sim \psi_0 \quad \Longrightarrow \quad X_m \sim \psi_0 P^m$$

$$X_t \sim \psi_t \quad \Longrightarrow \quad X_{t+m} \sim \psi_t P^m$$

Stationary Distribution

- A distribution ψ^* is called stationary for P if $\psi^* = \psi^* P$

Theorem

Every stochastic matrix P has at least one stationary distribution

Theorem

If P is irreducible and aperiodic, then

1. *P has a unique stationary distribution*
2. *For any initial distribution ψ_0, $\|\psi_0 P^t - \psi^*\| \to 0$ as $t \to \infty$*

Ergodicity

- Under irreducibility, for all $s \in S$,

$$\frac{1}{n} \sum_{t=1}^{n} \mathbf{1}\{X_t = s\} \to \psi^*[s] \quad \text{as } n \to \infty$$

  - $\mathbf{1}\{X_t = s\} = 1$ if $X_t = s$ and zero otherwise
  - convergence is with probability one
  - the result does not depend on the distribution (or value) of X_0

- The fraction of time the chain spends at state s converges to $\psi^*[s]$ as time goes to infinity

McCall Search Model

- The model is about the life an infinitely lived worker and
  - he has the opportunities to work at different wages
  - exogenous events that destroy his current job
  - decide whether to take a job while unemployed
- Key: endogenous decision whether to take a job
  - benefit: earn higher income
  - cost: could wait for a better job

Utility

- Workers try to maximize the lifetime utility

$$\sum_{t=0}^{\infty} \beta^t u(y_t)$$

  - y_t is his wage w_t when employed
  - y_t is unemployment compensation c when he is unemployed
  - $u(\cdot)$ is a utility function, for example, $u(y) = \log y$

Choice

- Wage offers are drawn from a vector of possible wages
  - $\mathbf{w} = \{w_1, w_2, \ldots, w_n\}$ with probabilities in vector $\mathbf{p} = \{p_1, p_2, \ldots, p_n\}$.
- When employed with wage $w \in \{w_1, w_2, \ldots, w_n\}$
  - consumes w, and with prob α of becoming unemployed at the end of the period
- The unemployed worker consumes unemployment benefit c. At the end of a period, with probability λ a worker receives an offer to work next period.
  - conditional on receiving an offer, he receives an offer w_s with prob p_s
  - given offer w_s, the unemployed worker chooses to accept or reject the offer
  - if accepts, he consumes c today and enters next period employed with w_s
  - if rejects, he consumes benefit c and enters next period unemployed

Lifetime utility

- The only decision is when employed, whether to accept a wage offer or not

$$d(s) = 1 \text{ or } d(s) = 0$$

- Let $d^*(s)$ denote the optimal choice of the worker

- Given $d^*(s)$, we can calculate the expected lifetime utility of a worker
  - when the worker is unemployed

$$U = u(c) + \mathbb{E} \sum_{t=1}^{\infty} \beta^t u(y_t)$$

  - when the worker is employed with wage w_s

$$V_s = u(w_s) + \mathbb{E} \sum_{t=1}^{\infty} \beta^t u(y_t)$$

- There are $n+1$ numbers we need to solve, U and $\{V_1, V_2, \ldots, V_n\}$

Value function

- Alternative way to write the lifetime utility

$$V_s = u(w_s) + \beta \left[(1-\alpha) V_s + \alpha U \right]$$

$$U = u(c) + \beta(1-\lambda)U + \beta\lambda \sum_s p_s \max\left\{U, V_s\right\}$$

- How to solve this problem? Iteration.

- Imagine we know the U and $\{V_1, V_2, \ldots, V_n\}$

$$V_s^{(j+1)} = u(w_s) + \beta \left[(1-\alpha) V_s^{(j)} + \alpha U^{(j)} \right]$$

$$U^{(j+1)} = u(c) + \beta(1-\lambda)U^{(j)} + \beta\lambda \sum_s p_s \max\left\{U^{(j)}, V_s^{(j)}\right\}$$

- Stop if the values converge

Reservation wage

- Imagine we have solved the value function

$$V_s = u(w_s) + \beta\left[(1-\alpha)V_s + \alpha U\right]$$

$$U = u(c) + \beta(1-\lambda)U + \beta\lambda\sum_s p_s \max\{U, V_s\}$$

- The optimal choice of the worker is $d^*(s)$

$$d^*(s) = 1 \quad \text{if} \quad V_s > U$$

- Workers will have a reservation wage $\widehat{w}_s$

- The probability that a worker transits from unemployment to employment is

$$\lambda\sum_s d^*(s)p_s$$