Semidefinite Programming

Fin Bauer and Stefan Heidekrüger

21. Mai 2015

2015-05-21

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Outline

1 Introduction

2 Some Theory
- Duality
- Optimality

3 Algorithms
- Interior Points

4 Applications
- Lovasz
- Max Cut
- Machine Learning

What Is Semidefinite Programming?

$$
\begin{aligned}
\min_{X} \quad & \langle C, X \rangle := Tr(CX) = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij} \\
\text{s.t.} \quad & \langle A_i, X \rangle = b_i \\
& X \succeq 0
\end{aligned}
$$

C, X symmetric $X \succeq 0$ positive semidefinite(s.d.p)

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An Easy Example

$$C = \begin{pmatrix} 1 & 2 \\ 2 & 9 \end{pmatrix}, \ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}, \ b_1 = 11, \ b_2 = 19$$

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$$\begin{aligned} \min \quad & x_{11} + 2x_{21} + 2x_{12} + 9x_{22} \\ \text{s.t.} \quad & x_{11} + x_{22} = 11 \\ & 2x_{21} + 2x_{12} + 3x_{22} = 19 \\ & X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \succeq 0 \end{aligned}$$

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Problem Session:
Have I Ever Seen Semidefinite Programming Before?

- Linear Programming

Linear Program

$$
\begin{aligned}
\min_{x} \quad & b_0^T x + c_0 \\
\text{s.t.} \quad & b_i^T x + c_i \leq 0, \quad \text{i in 1,...,n} \\
& x \geq 0
\end{aligned}
$$

Hint

Diagonal Matrix

- Convex Quadratically Constrained Quadratic Programming

CQCQP

$$
\begin{aligned}
\min \quad & x^T A_0 x + b_0^T x + c_0 \\
\text{s.t.} \quad & x^T A_i x + b_i^T x + c_i \leq 0, \quad \text{i in 1,...,n}
\end{aligned}
$$

Hint

$$
\begin{aligned}
\text{Given } & A_i = M_i^T M_i \\
\text{then } & x^T A_i x + b_i^T x + c_i \leq 0 \\
& \begin{pmatrix} I & M_i x \\ x^T M_i^T & -c_i - b_i^T x \end{pmatrix} \succeq 0
\end{aligned}
$$

Problem Session: What is modeled here and is it a SDP?

$$
\begin{aligned}
\max \quad & \rho \\
\text{s.t.} \quad & (a_i - c)^T E (a_i - c) \leq 1 \ \forall i \\
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Problem Session: Model Max-Cut

Problem Session: Lovasz

What's the Dual?

Primal Problem in Standard Form

$$\inf_X \{ Tr(CX);\ Tr(A_iX) = b_i\ (i = 1, \ldots, m),\ X \in \mathcal{S}_n^+ \} \tag{1}$$

Dual Problem in Standard Form

$$\sup_{y,S} \{ b^T y;\ \sum_{i=1}^{m} y_i A_i + S = C,\ S \in \mathcal{S}_n^+, y \in \mathbb{R}^m \} \tag{2}$$

Primal Problem in Standard Form
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Weak and Strong Duality

Example with Duality Gap

When is the Solution Optimal?

Optimality Conditions

$$
\begin{aligned}
Tr(A_i X) &= b_i, \quad X \succeq 0, \quad i = 1, \ldots, m \\
\sum_{i=1}^{m} y_i A_i + S &= C, \quad S \succeq 0 \\
XS &= 0
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bla

bla

What is MAX CUT?

Group presentation time.

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$$\max \frac{1}{2} \sum_{i<j} w_{ij}(1 - y_i y_j), \qquad s.t. y_i \in \{1, -1\}$$

What is MAX CUT?

IQP Model

Problem (MC) should be on the board now.

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An SDP-approximation algorithm (Goermans-Williamson)

Outline

- Relax (MC) into a QP (P)
- Find approximation bound of QP
- Show: equivalent SQP (SQ) to (P)
- Strong duality holds for (SQ) (ommitted)
- Solve SQP's dual in polynomial time

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- 1995
- Approx: 0.87856
- eq. algorithm already existed, but bound wasn't known.

QP Relaxation of (MC)

(P)

$$W_P^* := \max \frac{1}{2} \sum_{i<j} w_{ij}(1 - v_i^T v_j)$$

$$s.t. \ v_i \in \mathbb{S}^n \qquad \forall i \in V$$

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relaxation gives upper bound

A randomized algorithm using (P)

1. Solve (P) to get vectors v_i
2. Sample $r \sim UNIFORM(\mathbb{S}^n)$
3. Set $S := \{i | v_i^T r \geq 0\}$

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For cut W obtained this way:

$$E[W] = \sum_{i<j} w_{ij} \frac{\arccos(v_i^T v_j)}{\pi}$$

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A bound given by $E[W]$

Theorem

$$E[W] \geq \alpha \frac{1}{2} \sum_{i<j} w_{ij}(1 - v_i^T v_j)$$

with

$$\alpha := \min_{0 \leq \Theta \leq \pi} \frac{2}{\pi} \frac{\Theta}{1 - \cos\Theta} > .87856\ldots$$

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Corollary

$$E[W] \geq \alpha W_P^* \geq \alpha W_{MC}^*$$

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Corollary
$E[W] \geq \alpha W_P^* \geq \alpha W_{MC}^*$

SDP formulation of (P)

(SD)

$$W_P^* := \max \frac{1}{2} \sum_{i<j} w_{ij}(1 - y_{ij})$$

$$s.t.\ y_{ii} = 1 \qquad \forall i \in V$$

$$Y \text{ sym. pos. sem. def.}$$

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How is this an SDP? Rewrite the objective!

$$= \frac{1}{4} \sum_{i \in V} \sum_{j \in V} w_{ij}(1 - y_{ij})$$

$$= \frac{1}{2} W_{tot} - \frac{1}{4} \langle W, Y \rangle$$

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Classification - Using SDP to tell two things apart (1)

Ellipsoid

$$\mathcal{E} = \{x \in \mathbb{R}^n;\ (x-c)^T E(x-c) <= 1,\ E \text{ is p.s.d.}\}$$

First idea for SDP

$$\begin{aligned} \max \quad & \rho \\ \text{s.t.} \quad & (a_i - c)^T E(a_i - c) \leq 1\ \forall i \\ & (b_j - c)^T E(b_j - c) \geq \rho^2\ \forall j \\ & E \in \mathbb{S}^n_+ \end{aligned}$$

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First idea for SDP
max ρ
s.t. $(a_i - c)^T E(a_i - c) \leq 1\ \forall i$
$(b_j - c)^T E(b_j - c) \geq \rho^2\ \forall j$
$E \in \mathbb{S}^n_+$

Classification - Using SDP to tell two things apart (2)

$$
\begin{aligned}
\max \quad & \rho \\
\text{s.t.} \quad & (1, a_i)^T \bar{E} (1, a_i) \leq 1 \ \forall i \\
& (1, b_j)^T \bar{E} (1, b_j) \geq \rho^2 \ \forall j \\
& E \in \mathbb{S}_+^{n+1}
\end{aligned}
$$

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\max \quad & \rho \\
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