Semidefinite Programming

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Fin Bauer and Stefan Heidekroger

21. Mai 2015

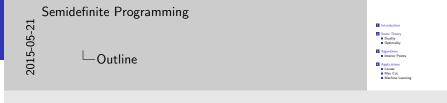
Semidefinite Programming

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Outline

- 1 Introduction
- 2 Some Theory
 - Duality
 - Optimality
- 3 Algorithms
 - Interior Points
- 4 Applications
- - Lovasz
 - Max Cut
 - Machine Learning



What Is Semidefinite Programming?

$$\min_{X} \langle C, X \rangle := Tr(CX) = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$$
s.t. $\langle A_i, X \rangle = b_i$

$$X \succeq 0$$

C, X symmetric $X \succeq 0$ positive semidefinite(s.d.p)

Semidefinite Programming -Introduction What Is Semidefinite Programming? C, X symmetric $X \succeq 0$ positive semidefinite(s.d.p)

 $\min_{Y} \ \langle C, X \rangle := Tr(CX) = \sum_{n}^{n} \sum_{i=1}^{n} C_{ij}X_{ij}$



$$C = \begin{pmatrix} 1 & 2 \\ 2 & 9 \end{pmatrix}, \ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}, \ b_1 = 11, \ b_2 = 19$$

Semidefinite Programming

—An Easy Example

-Introduction

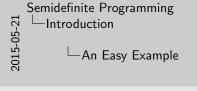
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An Easy Example

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min
$$x_{11} + 2x_{21} + 2x_{12} + 9x_{22}$$

s.t. $x_{11} + x_{22} = 11$
 $2x_{21} + 2x_{12} + 3x_{22} = 19$
 $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \succeq 0$





 $C = \begin{pmatrix} 1 & 2 \\ 2 & 9 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}$, $b_1 = 11$, $b_2 = 19$

Problem Session: Have I Ever Seen Semidefinite Programming Before?

Linear Programming

Linear Program

 $\min_{x} b_0^T x + c_0$ s.t. $b_i^T x + c_i \le 0$, i in 1,...,n x > 0

Hint

 ${\sf Diagonal\ Matrix}$

Convex Quadratically Constrained Quadratic Programming

CQCQP

min $x^{T}A_{0}x + b_{0}^{T}x + c_{0}$ s.t. $x^{T}A_{i}x + b_{i}^{T}x + c_{i} \le 0$, i in 1,...,n

Hint

Given $A_i = M_i^T M_i$ then $x^T A_i x + b_i^T x + c_i \le 0$ $\begin{pmatrix} I & M_i x \\ x^T M_i^T & -c_i - b_i^T x \end{pmatrix} \succeq 0$ Introduction

Introduction

Problem Session:

Have I Ever Seen Semidefinite Programming

Pofore?

then $x^T A x + b_1^T x + c_1 \le 0$ $\begin{pmatrix} I & M x \\ x^T M^T & -c_1 - b_1^T x \end{pmatrix} \succeq 0$

Semidefinite Programming

Problem Session: What is modeled here and is it a SDP?

max
$$ho$$

s.t. $(a_i-c)^T E(a_i-c) \leq 1 \ \forall i$
 $(b_j-c)^T E(b_j-c) \geq
ho^2 \ \forall j$
 $E \in \mathbb{S}^n_+$

s.t. $(a_i - c)^T E(a_i - c) \le 1 \forall i$ Problem Session: What is modeled here and is it a SDP?

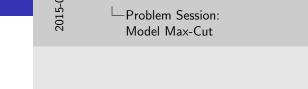
 $(b_j - c)^T E(b_j - c) \ge \rho^2 \forall j$ $E \in \mathbb{S}^n_+$

Semidefinite Programming

-Introduction

Model Max-Cut

Problem Session:



Semidefinite Programming

-Introduction

Lovasz

Problem Session:



Semidefinite Programming

-Introduction

What's the Dual?

Primal Problem in Standard Form

$$\inf_{X} \{ Tr(CX); Tr(A_{i}X) = b_{i} \ (i = 1, ..., m), \ X \in \mathcal{S}_{n}^{+} \}$$
 (1)

Dual Problem in Standard Form

$$\sup_{y,S} \{b^{T}y; \sum_{i=1}^{m} y_{i}A_{i} + S = C, \ S \in \mathcal{S}_{n}^{+}, y \in \mathbb{R}^{m}\}$$
 (2)

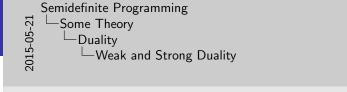
Semidefinite Programming

Semidefinite Programming

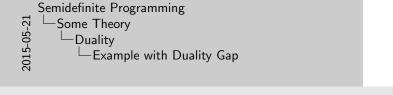
Duality
What's the Dual?



Weak and Strong Duality



Example with Duality Gap



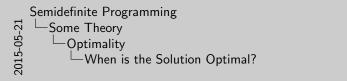
When is the Solution Optimal?

Optimality Conditions

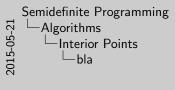
$$Tr(A_iX) = b_i, \quad X \succeq 0, \quad i = 1, ..., m$$

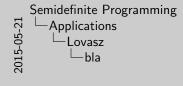
$$\sum_{i=1}^m y_i A_i + S = C, \quad S \succeq 0$$

$$XS = 0$$









What is MAX CUT?

Group presentation time.

Semidefinite Programming
Applications
Max Cut
What is MAX CUT?

Group presentation time.

$$\max \frac{1}{2} \sum_{i < j} w_{ij} (1 - y_i y_j), \qquad s.t. y_i \in \{1, -1\}$$

What is MAX CUT?

IQP Model

Problem (MC) should be on the board now.

Semidefinite Programming

Applications

Max Cut

What is MAX CUT?

IQP Model
Problem (MC) should be on the board now.

$$\max \frac{1}{2} \sum_{i < j} w_{ij} (1 - y_i y_j), \qquad s.t. y_i \in \{1, -1\}$$

An SDP-approximation algorithm (Goermans-Williamson)

Outline

- Relax (MC) into a QP (P)
- Find approximation bound of QP
- Show: equivalent SQP (SQ) to (P)
- Strong duality holds for (SQ) (ommitted)
- Solve SQP's dual in polynomial time



Outline

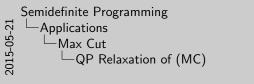
Relax (MC) into a QP (P)

- Find approximation bound of
- Strong duality holds for (SQ) (ommit

- 1995
- Approx: 0.87856
- eq. algorithm already existed, but bound wasn't known.

QP Relaxation of (MC)

$$W_P^* := \max rac{1}{2} \sum_{i < j} w_{ij} (1 - v_i^{\mathsf{T}} v_j)$$
 $s.t. \ v_i \in \mathbb{S}^n \qquad orall i \in V$

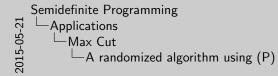




relaxation gives upper bound

A randomized algorithm using (P)

- 1. Solve (P) to get vectors v_i
- 2. Sample $r \sim \mathit{UNIFORM}(\mathbb{S}^n)$
- 3. Set $S := \{i | v_i^T r \ge 0\}$



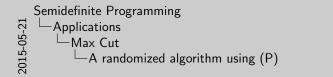
```
    Solve (P) to get vectors v;
    Sample r ~ UNIFORM(S")
    Set S := {i|v<sub>i</sub><sup>T</sup> r ≥ 0}
```

A randomized algorithm using (P)

- 1. Solve (P) to get vectors v_i
- 2. Sample $r \sim UNIFORM(\mathbb{S}^n)$
- 3. Set $S := \{i | v_i^T r \ge 0\}$

For cut W obtained this way:

$$E[W] = \sum_{i < j} w_{ij} \frac{\operatorname{arccos}(v_i^T v_j)}{\pi}$$





 $E[W] = \sum_{i < j} w_{ij} \frac{\arccos(v_i^T v_j)}{\pi}$

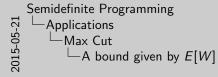
A bound given by E[W]

Theorem

$$E[W] \ge \alpha \frac{1}{2} \sum_{i < j} w_{ij} (1 - v_i^T v_j)$$

with

$$\alpha := \min_{0 \le \Theta \le \pi} \frac{2}{\pi} \frac{\Theta}{1 - \cos \Theta} > .87856\dots$$





A bound given by E[W]

Theorem

$$E[W] \ge \alpha \frac{1}{2} \sum_{i < j} w_{ij} (1 - v_i^T v_j)$$

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Corollary

$$E[W] \ge \alpha W_P^* \ge \alpha W_{MC}^*$$

Semidefinite Programming

Applications

Max Cut

A bound given by E[W]



SDP formulation of (P)

(SD)

$$W_P^* := \max \frac{1}{2} \sum_{i < j} w_{ij} (1 - y_{ij})$$
 $s.t. \ y_{ii} = 1 \quad \forall i \in V$

Y sym. pos. sem. def.



SDP formulation of (P)

(SD)

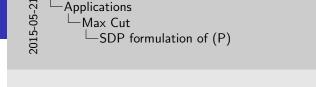
$$egin{aligned} W_P^* := \max rac{1}{2} \sum_{i < j} w_{ij} (1 - y_{ij}) \ & ext{s.t. } y_{ii} = 1 \qquad orall i \in V \end{aligned}$$

 $Y_{ii} = 1$ $\forall i \in V$ Y sym. pos. sem. def.

How is this an SDP? Rewrite the objecti

How is this an SDP? Rewrite the objective!
$$= \frac{1}{4} \sum_{i \in V} \sum_{j \in V} w_{ij} (1 - y_{ij})$$

$$= \frac{1}{2} W_{tot} - \frac{1}{4} \langle W, Y \rangle$$



Semidefinite Programming



 $W_P^* := \max \, \frac{1}{2} \sum_i w_{ij} (1-y_{ij})$

Classification - Using SDP to tell two things apart (1)

Ellipsoid

$$\mathcal{E} = \{ x \in \mathbb{R}^n; (x - c)^T E(x - c) <= 1, E \text{ is p.s.d.} \}$$

First idea for SDP

$$egin{array}{ll} \mathsf{max} &
ho \ & \mathsf{s.t.} & (\mathsf{a}_i-c)^\mathsf{T} \mathsf{E} (\mathsf{a}_i-c) \leq 1 \ orall i \ & (b_j-c)^\mathsf{T} \mathsf{E} (b_j-c) \geq
ho^2 \ orall j \ & \mathsf{E} \in \mathbb{S}_+^n \end{array}$$

Classification - Using SDP to tell two things apart (2)

max
$$\rho$$

s.t. $(1, a_i)^T \bar{E}(1, a_i) \leq 1 \ \forall i$
 $(1, b_j)^T \bar{E}(1, b_j) \geq \rho^2 \ \forall j$
 $E \in \mathbb{S}^{n+1}_+$

