CQF Exam One

January 2024 Cohort

Instructions

All questions must be attempted. Requested mathematical and full computational workings must be provided to obtain maximum credit. Books, lecture notes, CQF material may be referred to. Help from others is not permitted. Exam One computation in Excel is acceptable but we strongly recommend to implement in Python.

1. Your upload must be two files: E1_YOURNAME_REPORT.pdf and E1_YOURNAME_CODE.zip (to include code, any other files). YOURNAME as registered on CQF Portal. The signed declaration can be included inside ZIP, or page inserted at the start of PDF.

2. You must prepare PDF REPORT that integrates workings, numerical answers and plots in the order of questions. Python notebook can be edited into a report, but 'Python code+output only' will receive marks deduction.

3. Where tasks explicitly ask to organise results into a table or for the specific plot – that must be provided and explained.

Python notebook – remove unnecessary output, draft code and draft comments; save Python notebook to HTML first, and then print as PDF. Excel with handwritten inserts has to be printed as PDF report.

Implementing exam questions is part of the task, that includes making operational sense of the question. Please make a good use of lecture material, tutorials and exercise solutions. Where formula not given, it is up to your quant judgement to identify which formula to use. Tutor is unable to discuss your numerical answers or provide hints beyond ones given on the exam paper.

Exam submissions and file names which do not follow these instructions might take extra processing time.

Marking Scheme: **Q1.1** 10% **Q1.2** 16% **Q2** 10% **Q3** 20% **Q4.1** 14% **Q4.2** 10% **Q5** 20%. Total is 100%.

Optimal Portfolio Allocation

An investment universe of the following risky assets with a dependence structure (correlation) applies to all questions below as relevant:

Asset	$\boldsymbol{\mu}$	$\boldsymbol{\sigma}$	$\boldsymbol{w}$
A	0.02	0.05	w_1
B	0.07	0.12	w_2
C	0.15	0.17	w_3
D	0.20	0.25	w_4

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0.3 & 0.3 & 0.3 \\ 0.3 & 1 & 0.6 & 0.6 \\ 0.3 & 0.6 & 1 & 0.6 \\ 0.3 & 0.6 & 0.6 & 1 \end{pmatrix}$$

Question 1.1. Consider **the min variance portfolio with a target return** m.

$$\underset{\boldsymbol{w}}{\operatorname{argmin}} \frac{1}{2}\boldsymbol{w'\Sigma w} \qquad \text{s.t. } \boldsymbol{w'1} = 1, \qquad \mu_\Pi = \boldsymbol{w'\mu} = m$$

- Formulate the Lagrangian and give its partial derivatives.
- Write down the analytical solution for $\boldsymbol{w}^*$ optimal allocations – no derivation required.
- Compute allocations $\boldsymbol{w}^*$ and portfolio risk $\sigma_\Pi = \sqrt{\boldsymbol{w'\Sigma w}}$, for $m = 4.5\%$.

Question 1.2. Instead of computing other optimal allocations by formula, let's conduct an experiment.

- Generate above 700 random allocation sets: 4×1 vectors. Each set has to satisfy the constraint $w'\mathbf{1} = 1$. In fact, once you generate three random numbers w_1, w_2, w_3, the 4th can be computed.
- Weights will not be optimal and can be negative.
- Compute $\mu_\Pi = \boldsymbol{w'\mu}$ and $\sigma_\Pi = \sqrt{\boldsymbol{w'\Sigma w}}$ for each set.
- Plot points with coordinates μ_Π on the vertical axis and σ_Π on the horistonal axis.
  Identify the shape and explain this plot.

Question 2. VaR and ES sensitivities are computed *with regard to* each asset i, individually. For the given allocated portfolio below, provide a summary table with computed $\frac{\partial \text{VaR}(w)}{\partial w_i}$ and $\frac{\partial \text{ES}(w)}{\partial w_i}$.

Asset	$\boldsymbol{\mu}$	$\boldsymbol{\sigma}$	$\boldsymbol{w}$
1	0	0.30	50%
2	0	0.20	20%
3	0	0.15	30%

$$\boldsymbol{Corr} = \begin{pmatrix} 1 & 0.8 & 0.5 \\ 0.8 & 1 & 0.3 \\ 0.5 & 0.3 & 1 \end{pmatrix}$$

$$\frac{\partial \text{VaR}(w)}{\partial w_i} = \mu_i + \text{Factor} \times \frac{(\boldsymbol{\Sigma w})_i}{\sqrt{\boldsymbol{w^T \Sigma w}}} \quad \text{and} \quad \frac{\partial \text{ES}(w)}{\partial w_i} = \mu_i - \frac{\phi(\text{Factor})}{1-c} \times \frac{(\boldsymbol{\Sigma w})_i}{\sqrt{\boldsymbol{w^T \Sigma w}}}$$

Confidence $c = 99\%$, and $\Phi^{-1}(1-c)$ Factor is computed with *Normal icdf.* Signs in formulae are as intended and will not be discussed.

Hint: this is a task on reading the formulae in order to compute. Bold notation means vector computation and $()_i$ refers to i-th element, no further hints given.

Products and Market Risk

Question 3. Implement the multi-step binomial method as described in Binomial Method lecture with the following variables and parameters: stock $S = 100$, interest rate $r = 0.05$ (continuously compounded) for a call option with strike $E = 100$, and maturity $T = 1$. European payoff.

- Use any suitable parametrisation for up and down moves uS, vS.
- Plot 1: compute option value for a range of volatilities $[0.05, \ldots, 0.80]$ and plot the result (volatility at axe X, and option value axe Y). Trees to have a minimum four time steps.
- Plot 2: now, set $\sigma_{imp} = 0.2$ and compute and plot the value of one option, as you increase the number of time steps $NTS = 4, 5, \ldots, 50$.

Hint: This is a computational problem best coded in Python. You can visualise 1-2 trees (optional) but especially in case of Excel computation, do not provide multiple pages of individual pricing trees.

Question 4.1. Answer this question with step-by-step mathematical derivation and make no omitted transformations, variable changes: each next line must mathematically follow from the previous line.

Begin with the definition of Expected Shortfall as a conditional expectation (which implies integration) and obtain ES computation formula for the Normal Distribution

$$\mathrm{ES}_c(X) \quad = \quad \mathbb{E}[\, X \mid X \le \mathrm{VaR}_c(X)]$$

where $\mathrm{VaR}(X) = \mu + \Phi^{-1}(1-c) \times \sigma$.

Handwritten working must not be a rough work: use ample and clear spacing between lines, no crossings/corrections. Submitting copied in formulae will score less than 25% of marks. If not using Markdown LaTeX, insdert a scan/photo of workings (pdf) into Python notebook or final PDF report.

Question 4.2. Compute the standardised value of Expected Shortfall for the range of percentiles $[99.95; 99.75; 99.5; 99.25; 99; 98.5; 98; 97.5]$. Organise results into a table.

$$\mathrm{ES}_c(X) \quad = \quad \mu - \sigma\, \frac{\phi\left(\Phi^{-1}(1-c)\right)}{1-c}$$

Hints: Normal *pdf* is indifferent to the input sign, so the results are the same for upper or lower percentiles.

Example: $\phi(\Phi^{-1}(1-c))$ and $\phi(\Phi^{-1}(c))$ give $\phi(-2.33)$ and $\phi(2.33)$. $1-c$ refers to $1-0.9995$ and so on.

Question 5. Use S&P500 index data to implement the backtesting of 99%/10day Value at Risk and report the following:

(a) The count and percentage of VaR breaches.

(b) The count and percentage of consecutive VaR breaches. Example: $1, 1, 1$ indicates two consecutive occurrences.

(c) Provide a plot which identifies the breaches visually, with crosses or other marks.

$$\text{VaR}_{10D,t} = \text{Factor} \times \sigma_t \times \sqrt{10}$$

- Compute the rolling standard deviation σ_t from 21 daily returns.
- Timescale of that σ_t remains 'daily' regardless of how many returns are in the sample. To make projection, use the additivity of variance $\sigma_{10D} = \sqrt{\sigma_t^2 \times 10}$.
- A breach occurs when the forward realised 10-day return is below the VaR_t quantity.

$$r_{10D,t+10} < \text{VaR}_{10D,t} \qquad \text{means breach, given both numbers are negative.}$$

VaR is fixed at time t and compared to the return realised from t to $t + 10$, computed $\ln(S_{t+10}/S_t)$. Alternatively, you can compare to $\ln(S_{t+11}/S_{t+1})$ but state this assumption in your report upfront.