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Final Project

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The Boussinesq Equations

Derivation of the Boussinesq Equations

Assume two-dimensional motion with the x-axis horizontal, coinciding with the undisturbed water level, and the z-axis pointing vertically upwards. Moreover, assume a bottom at $z^* = -h^*(x^*)$, where $*$ denotes a magnitude with dimension. To make the variables involved dimensionless, apply the maximum depth, H, and a characteristic wavelength, l, as vertical and horizontal scales, respectively. Furthermore, the field dimension scales with an additional factor $\alpha = \frac{a}{H}$, where a is a designation for the amplitude. Linearization demands that α is small. Long wave generation is constructed in that $\epsilon \equiv \frac{H^2}{l^2}$ is small. The non-dimensioning for all the relevant variables is thus done in the following manner:

$$x = \frac{1}{l}x^* \qquad z = \frac{1}{H}z^* \qquad t = \frac{\sqrt{gH}}{l}t^*$$

$$h = \frac{1}{H}h^* \qquad \eta = \frac{1}{\alpha H}\eta^* \qquad p = \frac{1}{\rho g H}p^*$$

$$u = \frac{1}{\alpha\sqrt{gH}}u^* \quad w = \frac{1}{\sqrt{\epsilon}\alpha\sqrt{gH}}w^*$$

Assuming that the fluid is homogeneous, frictionless, and incompressible and that the motion is irrotational, the flow velocity can be deduced from a potential ϕ and $\mathbf{v} = \nabla\phi$. The dynamic and kinematic boundary conditions at the surface $z = \alpha\eta$ can therefore be written

$$p = 0, \qquad \eta_t + \alpha u \eta_x = w.$$

At the bottom zero throughput is demanded, which gives

$$w = -h_x u.$$

In the fluid, it follows from the continuity equation that

$$u_x + w_z = 0,$$

and from Euler's equations of motion that

$$\begin{aligned} u_t + \alpha u u_x + \alpha w u_z &= -\alpha^{-1} p_x, \\ \epsilon(w_t + \alpha u w_x + \alpha w w_z) &= -\alpha^{-1}(p_z + 1). \end{aligned}$$

Long wave approximations may be developed as expansions in the parameters α and ϵ assuming they both are small. This means keeping all the terms of order α and retaining the leading orders in ϵ. Now, define a vertical average

$$\bar{u} = (h + \alpha\eta)^{-1} \int_{-h}^{\alpha\eta} u \, \mathrm{d}z.$$

An average depth continuity equation can thus be written

$$\eta_t = -\left[(h + \alpha\eta)\bar{u}\right]_x . \tag{1}$$

Together with

$$\bar{u}_t + \alpha\bar{u}\bar{u}_x = -\eta_x + \epsilon\frac{h}{2}\left[(h\bar{u}_t)_{xx} - \frac{h}{3}\bar{u}_{xxt}\right] \tag{2}$$

this constitutes the *Boussinesq equations*, which is a set of two equations in x and t for the determination of the unknowns η and $\bar{u}$. Sometimes the equations based on the averaged velocity potential, ϕ, are preferable, that is

$$\eta_t = -\left[(h + \alpha\eta)\phi_x + \epsilon\frac{h}{3}\left(\frac{1}{2}\eta_t - h_x\phi_x\right)h_x\right]_x , \tag{3}$$

$$\phi_t + \alpha\phi\phi_x = -\eta + \epsilon\frac{h}{2}\left[(h\phi_{tx})_x - \frac{h}{3}\phi_{txx}\right]. \tag{4}$$

When $\alpha, \epsilon \to 0$ both sets of equations simplify to the linear shallow water equations.

A finite difference method

To solve the Boussinesq equations numerically, one may discretize by centered differences in time and space. Divide the time domain, $0 < t \leq T$, into discrete time levels with time step Δt, such that $t_n = n\Delta t$. Similarly, the spatial domain Ω may be divided into a grid with spatial step Δx, such that $x_i = i\Delta x$. By employing a staggered grid the discrete unknowns may be defined according to

$$\eta^n_{i-\frac{1}{2}} \approx \eta((i-\frac{1}{2})\Delta x, n\Delta t), \qquad u^{n+\frac{1}{2}}_i \approx u(i\Delta x, (n+\frac{1}{2})\Delta t).$$

In the following a little bit of formalism is employed to avoid too lengthy expressions. Let the numerical approximation to a quantity q at a grid-point with coordinates $(i\Delta x, n\Delta t)$ be denoted by q^n_i. The discrete Boussinesq equations primarily builds up by replacing the unknowns by their discrete counterparts, or averages thereof, and the derivatives by finite differences. Introduce the symmetric difference operator D_x, the average operator $\bar{\ }^x$, and a temporal geometric mean square $^{(*2)}$, and group terms of identical indices inside square brackets, such that

$$\begin{aligned}
[D_x q]^n_i &= \frac{q^n_{i+\frac{1}{2}} - q^n_{i-\frac{1}{2}}}{\Delta x}, \\
[\bar{q}^x]^n_i &= \frac{q^n_{i+\frac{1}{2}} + q^n_{i-\frac{1}{2}}}{2}, \\
[q^{(*2)}]^n_i &= q^{n-\frac{1}{2}}_i q^{n+\frac{1}{2}}_i .
\end{aligned}$$

Difference and average operators with respect to t are defined similarly. The continuity equation (1) and the momentum equation (2) are evaluated at grid-point $(i+\frac{1}{2}, n-\frac{1}{2})$ and (i, n), respectively. To avoid non-linearity in the implicit equations at each time step, the non-linear term is discretized by the geometrical square. The discretized Boussinesq equations expressed in terms of the surface elevation and fluid velocity are thus

$$\left[D_t\eta = -D_x(\bar{h}^x + \alpha\bar{\eta}^{xt})u\right]^{n-\frac{1}{2}}_{i+\frac{1}{2}}, \tag{5}$$

$$\left[D_t u + \frac{\alpha}{2} D_x \overline{u^{*2}}^x = -D_x\eta + \epsilon\frac{h}{2}\left(D_x D_x \bar{h}^x D_t u - \frac{h}{3} D_t D_x D_x u\right)\right]^n_i, \tag{6}$$

where the bar designating the vertical average has been omitted. The Boussinesq equations give tri-diagonal linear sets of equations to be solved for the new values of η and u, respectively.

Finite element methods

A mixed method for the BEV model

A Boussinesq model based on the wave surface Elevation and horizontal Velocities (BEV) formulation can be obtained from (1) and (2). In the following the Boussinesq equations will be discretized by centered differences in time and Galerkin finite elements in space. Employing a staggered grid in time, the surface elevation, η, is sought at integer time levels n, while the horizontal velocity, u, is sought at half-integer time levels $n + \frac{1}{2}$. If the depth is assumed constant for simplicity, the time discretization provide the continuity equation in the form

$$\left[D_t \eta = -\frac{\partial}{\partial x} \left((h + \alpha \bar{\eta}^{xt}) u \right) \right]^{n-\frac{1}{2}},$$

and the momentum equation in the form

$$\left[D_t u + \frac{\alpha}{2} \frac{\partial}{\partial x} \overline{u^{*2}}^{x} = -\frac{\partial \eta}{\partial x} + \epsilon \frac{h^2}{3} \frac{\partial^2}{\partial x^2} D_t u \right]^{n}.$$

A standard method for the BEP model

A Boussinesq model based on the wave surface Elevation and velocity Potential (BEP) formulation can be obtained from (3) and (4). In the following the Boussinesq equations will be discretized by centered differences in time and Galerkin finite elements in space. Employing a staggered grid in time, the surface elevation, η, is sought at integer time levels n, while the velocity potential, ϕ, is sought at half-integer time levels $n + \frac{1}{2}$. Time discretization provide the continuity equation in the form

$$\left[D_t \eta = -\frac{\partial}{\partial x} \left((h + \alpha \bar{\eta}^{t}) \frac{\partial \phi}{\partial x} + \epsilon \frac{h}{3} \left(\frac{1}{2} D_t \eta - \frac{\partial h}{\partial x} \frac{\partial \phi}{\partial x} \right) \frac{\partial h}{\partial x} \right) \right]^{n-\frac{1}{2}},$$

and the momentum equation in the form

$$\left[D_t \phi + \frac{\alpha}{2} \left(\frac{\partial \phi}{\partial x} \right)^{(*2)} = -\eta + \epsilon \frac{h}{2} \left(\frac{\partial}{\partial x} \left(h D_t \frac{\partial \phi}{\partial x} \right) - \frac{h}{3} \frac{\partial^2}{\partial x^2} D_t \phi \right) \right]^{n}.$$

In addition to centered differences in time, the arithmetic- and geometric mean has been applied to the η and $\frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x}$ term, respectively. The latter provides a linear set of equations. Now, the originally coupled system (3) and (4) can be solved in sequence. That is, first η^n is found by means of η^{n-1} and $\phi^{n-\frac{1}{2}}$, values obtained at the previous time level, and then $\phi^{n+\frac{1}{2}}$ is found by means of the recently computed η^n and the previous $\phi^{n-\frac{1}{2}}$.

Furthermore, the Galerkin method with a test function $V \in \Omega$ provide the time discrete weak form of the equation of continuity as

$$\int_\Omega \left[D_t \eta V = \left((h + \alpha \bar{\eta}^t) \frac{\partial \phi}{\partial x} + \epsilon \frac{h}{3} \left(\frac{1}{2} D_t \eta - \frac{\partial h}{\partial x} \frac{\partial \phi}{\partial x} \right) \frac{\partial h}{\partial x} \right) \frac{\partial V}{\partial x} \right]^{n-\frac{1}{2}} \mathrm{d}\Omega,$$

and the time discrete weak form of momentum equation as

$$\int_\Omega \left[D_t \phi V + \frac{\alpha}{2} \left(\frac{\partial \phi}{\partial x} \right)^{(*2)} V = -\eta V - \epsilon \frac{1}{2} \left(\frac{\partial}{\partial x} (hV) h \frac{\partial}{\partial x} (D_t \phi) - \frac{1}{3} \frac{\partial}{\partial x} (h^2 V) \frac{\partial}{\partial x} (D_t \phi) \right) \right]^n \mathrm{d}\Omega.$$