# Main Formulation

$$\min \sum_{t} \sum_{i \in \mathcal{N}} \sum_{j \in \Delta_i} \sum_{\omega \in \Omega} \left\{ \frac{1}{2|\Omega|} \left( r_{ij}(\omega,t) I_{ij}(\omega,t) \right) \right\} \tag{1}$$

- We have current in both directions, so we divide by 2 for loss.

Where:

$$\begin{aligned}
\mathcal{N} &= \{1,2,\ldots,i,\ldots,N\} \quad \text{set of nodes,} \\
\Delta_i &= \{j = l,\ldots,k\} \quad \text{set of neighboring nodes of node } i, \\
\Omega &= \{1,2,\ldots,\omega,\ldots,\bar{\Omega}\} \quad \text{set of scenarios,} \\
r_{ij}(\omega,t) &= \text{resistance of the line between node } i \text{ and } j \text{ in scenario } \omega \text{ and time t ,} \\
I_{ij}(\omega,t) &= \text{squared current between nodes } i \text{ and } j \text{ in scenario } \omega \text{ and time t .}
\end{aligned}$$

## Subject to: DistFlow Constraints for specific time t

$$\sum_{k \in \Delta_i} P_{ik}(\omega,t) = P_i(\omega,t) - u_i P_{Gi}(\omega,t) \qquad \forall i \in \mathcal{N}, \omega \in \Omega \tag{2a}$$

$$\sum_{k \in \Delta_i} Q_{ik}(\omega,t) = Q_i(\omega,t) - u_i Q_{Gi}(\omega,t) \qquad \forall i \in \mathcal{N}, \omega \in \Omega \tag{2b}$$

$$I_{ij}(\omega,t) V_i(\omega,t) \geq (P_{ij}(\omega,t))^2 + (Q_{ij}(\omega,t))^2 \quad \forall i \in \mathcal{N}, \forall j \in \Delta_i, \forall \omega \in \Omega \tag{2c}$$

$$V_i(\omega,t) - V_j(\omega,t) - 2(r_{ij} P_{ij}(\omega,t) + x_{ij} Q_{ij}(\omega,t)) + (r_{ij}^2 + x_{ij}^2) I_{ij}(\omega,t) = 0 \qquad \forall i \in \mathcal{N}, \forall j \in \Delta_i, \forall \omega \in \Omega \tag{2d}$$

Where:

- $u_i$: defines if node $i$ is HV/MV substation (parameter)
- $P_i(\omega,t), Q_i(\omega,t)$: power generated by external grid
- $P_{ik}(\omega,t)$: active power flow from node $i$ to node $k$ at scenario $\omega$

$$\begin{aligned}
P_{ik}(\omega,t) \geq 0 \quad &\Rightarrow \quad \text{flow from } i \text{ to } k \\
P_{ki}(\omega,t) \leq 0 \quad &\Rightarrow \quad \text{flow from } k \text{ to } i
\end{aligned}$$

- $Q_{ik}(\omega,t)$: reactive power from node $i$ to node $k$
- $V_i(\omega,t)$: square of voltage at node $i$
- $x_{ij}$: reactance of the line from node $i$ to node $j$
- $P_i(\omega,t)$: net consumption at node $i$

## Radiality of Distribution Network

**Index sets.**

$N$ : set of buses (nodes),
$L$ : set of undirected lines (edges),
$S \subseteq L$ : subset of lines equipped with switches,
$nS = L \setminus S$ : lines without switches,
$C \subseteq L \times N \times N$ : $\{(\ell, i, j)\}$ directed candidate arcs for each line.

**Parameters.**

$\texttt{slack\_node} \in N$ : index of the root (slack) bus,
$\varepsilon > 0$ : small constant (default 1) for numerical stability.

**Decision variables.**

$\delta_\ell \in \{0,1\}$ (or $[0,1]$ if relaxed) $\quad \forall\, \ell \in S$ : switch-status (1=closed),
$f_{\ell,i,j} \in \mathbb{R}$ $\quad \forall\, (\ell, i, j) \in C$ : artificial "flow" on arc $(\ell, i, j)$,
$\theta \in \mathbb{R}$ : Benders-cut auxiliary variable.

**1. Flow–Balance (one parent per node).**

$$\sum_{\substack{(\ell,i,j)\in C \\ j=n}} f_{\ell,i,j} = \begin{cases} -\varepsilon, & n \in N \setminus \{\texttt{slack\_node}\}, \\ \geq \varepsilon \left(|N| - 1\right), & n = \texttt{slack\_node} \end{cases} \qquad \forall\, n \in N \tag{1}$$

**2. Orientation (zero net on each physical line).**

$$\sum_{(\ell,i,j)\in C} f_{\ell,i,j} = 0 \qquad \forall\, \ell \in L \tag{2}$$

**3. Switch–Propagation (flow only if switch closed).**

$$f_{\ell,i,j} \leq \varepsilon\, |N|\, \delta_\ell, \tag{3}$$

$$f_{\ell,i,j} \geq -\, \varepsilon\, |N|\, \delta_\ell \qquad \forall\, (\ell, i, j) \in C,\ \ell \in S \tag{4}$$

**4. Number of Closed Switches (spanning-tree size).**

$$\sum_{\ell \in S} \delta_\ell = |N| - |nS| - 1 \tag{5}$$

## Power & Current Relation with State of Switches

$$-M\delta_{ij} \leq P_{ij}(\omega,t) \leq M\delta_{ij} \qquad \forall(i,j) \in \text{Lines With Switches} \tag{4a}$$
$$-M\delta_{ij} \leq Q_{ij}(\omega,t) \leq M\delta_{ij} \qquad \forall(i,j) \in \text{Lines With Switches} \tag{4b}$$
$$0 \leq I_{ij}(\omega,t) \leq M\delta_{ij} \qquad \forall(i,j) \in \text{Lines With Switches} \tag{4c}$$

$$V_i(\omega,t) - V_j(\omega,t) - 2(r_{ij}P_{ij}(\omega,t) + x_{ij}Q_{ij}(\omega,t)) + (r_{ij}^2 + x_{ij}^2)I_{ij}(\omega,t) + M(1-\delta_{ij}) \geq 0 \tag{4d}$$
$$V_i(\omega,t) - V_j(\omega,t) - 2(r_{ij}P_{ij}(\omega,t) + x_{ij}Q_{ij}(\omega,t)) + (r_{ij}^2 + x_{ij}^2)I_{ij}(\omega,t) - M(1-\delta_{ij}) \leq 0 \tag{4f}$$
$$\forall(i,j) \in \text{Lines With Switches}$$

## Constraints Limiting:

$$V_i^{min} \leq V_i(\omega,t) \leq V_i^{max} \qquad \forall i \in \mathcal{N} \tag{6}$$
$$0 \leq I_{ij}(\omega,t) \leq I_{ij}^{max} \qquad \forall i \in \mathcal{N}, \forall j \in \Delta_i \tag{7}$$
$$V_i(\omega,t) = \tilde{V}_i(\omega,t) \qquad \forall i \in \text{Substation Nodes},\ \forall\omega \in \Omega \tag{8}$$

Where:

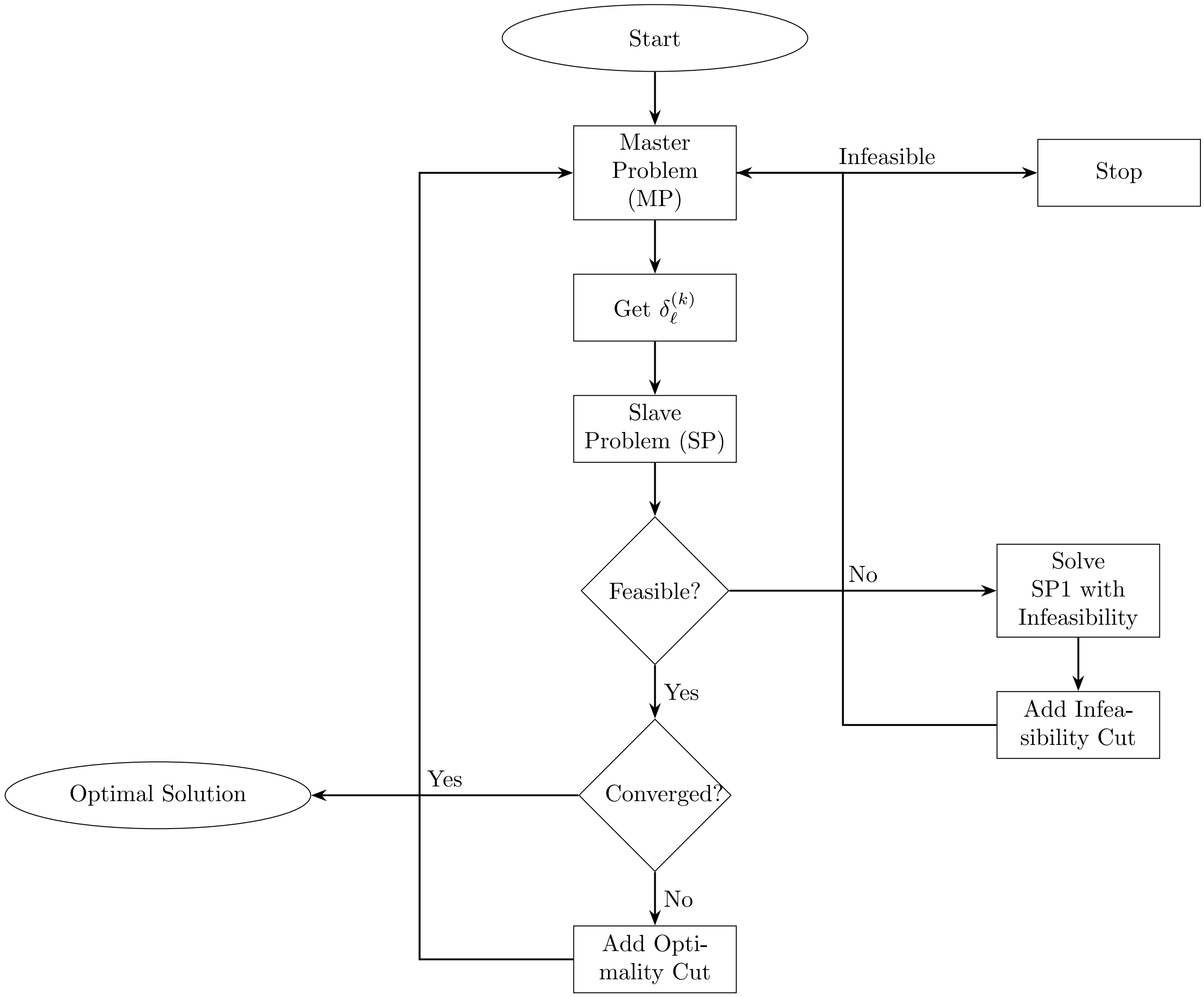
- $\underline{V}_i$, $\overline{V}_i$ are lower/upper levels of voltage.
- $\tilde{V}_i(\omega,t)$ is the voltage of substation at scenario $\omega$ (imposed by external grid).

**Sets:**

- $\mathcal{N}$: set of nodes
- $\Delta_i$: set of neighboring nodes of node $i$
- $\Omega$: set of scenarios
- $\mathcal{LS}$: set of lines with switches
- $\mathcal{SN}$: set of substation nodes

**Decision Variables and Parameters:**

| **Symbol** | Description | Index Domain |
|---|---|---|
| $I_{ij}(\omega,t)$ | Squared current from node $i$ to $j$ in scenario $\omega$ | $\forall i \in \mathcal{N}, j \in \Delta_i, \omega \in \Omega$ |
| $P_{ij}(\omega,t)$ | Active power flow from $i$ to $j$ in scenario $\omega$ | Same as above |
| $Q_{ij}(\omega,t)$ | Reactive power flow from $i$ to $j$ in scenario $\omega$ | Same as above |
| $V_i(\omega,t)$ | Squared voltage at node $i$ in scenario $\omega$ | $\forall i \in \mathcal{N}, \omega \in \Omega$ |
| $P_i(\omega,t)$ | Net consumption at node $i$ (parameter) | $\forall i \in \mathcal{N}, \omega \in \Omega$ |
| $Q_i(\omega,t)$ | Net reactive power at node $i$ (parameter) | Same as above |
| $P_{Gi}(\omega,t)$ | Power generated by external grid at node $i$ | $\forall i \in \text{ExternalPoints}, \omega \in \Omega$ |
| $Q_{Gi}(\omega,t)$ | Reactive power from external grid | Same as above |
| $d_{ij}$ | Binary variable for direction between $i$ and $j$ | $\forall(i,j) \in \text{LinesWithSwitches}$ |
| $\delta_{ij}$ | Binary variable for line/switch status | Same as above |


Start
Master Problem (MP)
Infeasible
Stop
Get $\delta_\ell^{(k)}$
Slave Problem (SP)
Feasible?
No
Solve SP1 with Infeasibility
Yes
Add Infeasibility Cut
Optimal Solution
Yes
Converged?
No
Add Optimality Cut

## Slave Subproblem $\mathrm{SP}^{\omega}$ for each scenario $\omega \in W$

**Index sets.**

$N$ : set of buses, $\quad L$ : set of undirected lines,
$S \subseteq L$ : lines equipped with switches, $\quad C \subseteq L \times N \times N$ : directed candidate arcs,
$W$ : set of scenarios.

**Given from the master:** $\delta^*_\ell \in \{0,1\}$ for each $\ell \in S$.

**Variables (per $\omega$):**

$v_n^2(\omega,t),\ p_{ij}(\omega,t),\ q_{ij}(\omega,t),\ I_{ij}(\omega,t),$ (and if infeasible: slack variables $s_{vn}(\omega,t),\ s_{Iij})(\omega,t)$ .

**Objective.**

$$\min_{v^2,p,q,I} \sum_{(i,j)\in C} \tfrac{1}{2}\, r_{ij}\, I_{ij}(\omega,t) \quad \longrightarrow \text{ optimality cost } \mathrm{obj}^{\mathrm{feas},(k)}(\omega,t),$$

or when infeasible

$$\min \sum_{n\in N} (s^+_{vn}(\omega,t) + s^-_{vn}(\omega,t)) \ + \sum_{(i,j)\in C} s_{Iij}(\omega,t) \ \longrightarrow \ \mathrm{obj}^{\mathrm{infeas},(k)}(\omega,t).$$

**Constraints (for each $\omega$):**

1. **Switch-status propagation:**
$$\delta_{\ell\omega} = \delta^*_\ell\,, \quad \forall\, \ell \in S.$$
2. **Slack-bus voltage:**
$$V(\omega,t) = V_n(\omega,t), \quad \text{if } n = \texttt{slack\_node}, \quad \text{else skipped.}$$
3. **Nodal active-power balance:**
$$\sum_{(i,j)\in C:\, i=n} p_{ij}(\omega,t) = \begin{cases} -P^{\mathrm{slack}}(\omega,t), & n = \texttt{slack\_node}, \\ -P_n(\omega,t), & \text{otherwise.} \end{cases}$$
4. **Nodal reactive-power balance:**
$$\sum_{(i,j)\in C:\, i=n} \Big(q_{ij}(\omega,t) - \tfrac{1}{2}\, b_{ij}\, V_i(\omega,t)\Big) = \begin{cases} -Q^{\mathrm{slack}}(\omega,t), & n = \texttt{slack\_node}, \\ -Q_n(\omega,t), & \text{otherwise.} \end{cases}$$
5. **Branch voltage-drop (DistFlow):**
$$V_i(\omega,t) - V_j(\omega,t) - 2\big(r_{ij}p_{ij}(\omega,t) + x_{ij}q_{ij}(\omega,t)\big) + (r_{ij}^2 + x_{ij}^2)\, I_{ij}(\omega,t) \begin{cases} = 0, & \ell \notin S, \\ \geq -M_{ij}(1-\delta_\ell(\omega,t)), & \\ \leq +M_{ij}(1-\delta_\ell(\omega,t)), & \ell \in S. \end{cases}$$

6. **Rotated-cone current constraint (no switch):**

$$p_{ij}^2(\omega,t) + q_{ij}^2(\omega,t) \;\le\; I_{ij}(\omega,t)\,\frac{V_i(\omega,t)}{n_{ij}^2}, \quad \forall\,\ell \notin S.$$

   If $\ell \in S$, simply $I_{ij}(\omega,t) = 0$.

7. **Branch power-balance (Kirchhoff):**

$$\sum_{(i,j)\in C:\,\ell=ij} \big(p_{ij}(\omega,t) - \tfrac{1}{2}\,r_{ij} I_{ij}\big) = 0, \qquad \sum_{(i,j)\in C:\,\ell=ij} \big(q_{ij}(\omega,t) - \tfrac{1}{2}\,x_{ij} I_{ij}(\omega,t)\big) = 0.$$

8. **Switch $P/Q$ bounds ($\ell \in S$):**

$$-M_{ij}\,\delta_\ell(\omega,t) \;\le\; p_{ij}(\omega,t) \;\le\; M_{ij}\,\delta_\ell(\omega,t), \quad -M_{ij}\,\delta_\ell(\omega,t) \;\le\; q_{ij}(\omega,t) \;\le\; M_{ij}\,\delta_\ell(\omega,t).$$

9. **Physical limits (only in optimal subproblem):**

$$0 \;\le\; I_{ij}(\omega,t) \;\le\; I_{ij}^{\max}, \quad V_n^{\min}(\omega,t) \le v_n^2(\omega,t) \le V_n^{\max}(\omega,t).$$

**Infeasible Subproblem** $\mathrm{SP}_{\text{infeas}}(\omega,t)$

**Objective (infeasibility penalties):**

$$\min_{V,\,p,\,q,\,I,\,s_v^{\pm},\,s_I} \quad \sum_{n\in N} \big(s_{vn}^{+}(\omega,t) + s_{vn}^{-}(\omega,t)\big) \;+ \sum_{(i,j)\in C} s_{Iij}(\omega,t) \;\;\longrightarrow\;\; \mathrm{obj}^{\text{infeas},(k)}(\omega,t) \tag{9}$$

**Relaxed physical-limit constraints (per $\omega$):**

$$V_n(\omega,t) \le V_n^{max} \;+\; s_{vn}^{+}(\omega,t), \qquad \forall n \in N, \tag{10}$$

$$V_n(\omega,t) \ge V_n^{min} \;-\; s_{vn}^{-}(\omega,t), \qquad \forall n \in N, \tag{11}$$

$$I_{ij}(\omega,t) \le \overline{I}_{ij}^{max} \;+\; s_{Iij}(\omega,t), \qquad \forall (i,j) \in C, \tag{12}$$

$$s_{vn}^{+}(\omega,t),\; s_{vn}^{-}(\omega,t),\; s_{Iij}(\omega,t) \ge 0, \qquad \forall n \in N,\; \forall (i,j) \in C. \tag{13}$$

All of the network constraints

$$\{\text{Slack-voltage},\ \text{Power-balances},\ \text{Voltage-drops},\ \text{Conic-current},\ \text{Switch-bounds}, \ldots\}$$

are assembled in `slave_model_constraints(model)` and thus appear in both the *optimal* and *infeasible* subproblem models. The only differences between them are considering the slack variable in the physical limitation of current and voltage

## Master Problem at Iteration $k$

$$\min_{\{\theta_\omega\}_{\omega\in\Omega},\,\{\delta_\ell\}_{\ell\in S},\,\{f_{\ell,i,j}\}_{(\ell,i,j)\in C}} \quad \sum_{\omega\in\Omega}\theta_\omega$$

$$\text{s.t.} \quad \underbrace{\sum_{\substack{(\ell,i,j)\in C\\ j=n}} f_{\ell,i,j} = \begin{cases} -\,\varepsilon, & n\in N\setminus\{\texttt{slack\_node}\},\\ \geq \varepsilon\,(|N|-1), & n=\texttt{slack\_node}, \end{cases}}_{\text{(1) Flow–Balance}} \quad \forall\, n\in N,$$

$$\underbrace{\sum_{\substack{(\ell,i,j)\in C\\ \ell=\ell'}} f_{\ell,i,j} = 0}_{\text{(2) Orientation}} \quad \forall\,\ell'\in L,$$

$$\underbrace{-\,\varepsilon\,|N|\,\delta_\ell \;\leq\; f_{\ell,i,j} \;\leq\; \varepsilon\,|N|\,\delta_\ell}_{\text{(3) Switch–Propagation}} \quad \forall\,(\ell,i,j)\in C,\ \ell\in S,$$

$$\underbrace{\sum_{\ell\in S}\delta_\ell = |N|-|nS|-1}_{\text{(4) Spanning-Tree Size}}$$

$$\underbrace{\theta_\omega \;\geq\; \varphi_\omega^{(m)} \;+\; \sum_{\ell\in S}\pi_{\ell\omega}^{(m)}\left(\delta_\ell-\delta_\ell^{(m)}\right)}_{\substack{\text{Optimality cuts}\\ m=1,\ldots,k-1}} \quad \forall\,\omega\in\Omega,$$

$$\underbrace{0 \;\geq\; \psi_\omega^{(m)} \;+\; \sum_{\ell\in S}\rho_{\ell\omega}^{(m)}\left(\delta_\ell-\delta_\ell^{(m)}\right)}_{\substack{\text{Infeasibility cuts}\\ m=1,\ldots,k-1}} \quad \forall\,\omega\in\Omega,$$

$$\delta_\ell\in\{0,1\},\ \forall\,\ell\in S, \quad f_{\ell,i,j}\in\mathbb{R},\ \forall\,(\ell,i,j)\in C, \quad \theta_\omega\in\mathbb{R},\ \forall\,\omega\in\Omega.$$

**Notes on the Benders-cut notation:** - $\varphi_\omega^{(m)}$, $\pi_{\ell\omega}^{(m)}$ are the dual-derived intercept and slope coefficients from the *optimality* subproblem for scenario $\omega$ at previous iteration $m$. - $\psi_\omega^{(m)}$, $\rho_{\ell\omega}^{(m)}$ likewise come from an *infeasibility* cut for scenario $\omega$, iteration $m$. - $\delta_\ell^{(m)}$ is the value of $\delta_\ell$ at iteration $m$.