Curry and Howard Meet Borel

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Abstract

We show that an intuitionistic version of counting propositional logic corresponds, in the sense of Curry and Howard, to an expressive type system for the event probabilistic λ-calculus, the latter a vehicle calculus in which both call-by-name and call-by-value evaluation of discrete randomized functional programs can be simulated. Remarkably, proofs (respectively, types) do not only guarantee that validity (respectively, termination) holds, but also *reveal* the underlying probability. We finally show that by endowing the type system with an intersection operator, one obtains a system precisely capturing the probabilistic behavior of lambda-terms.

1 Introduction

Among the many ways in which mathematical logic influenced programming language theory, the so-called Curry-Howard correspondence is certainly among the most intriguing and meaningful ones. Traditionally, the correspondence identified by Curry [14] and formalized by Howard [33] (CHC in the following) relates propositional intuitionistic logic and the simply-typed λ-calculus. It is well-known, though, that the correspondence holds in other contexts, too. Indeed, in the last fifty years more sophisticated type systems have been put in relation with logical formalisms: from polymorphism [26, 27] to various forms of session typing [12, 53], from control operators [46] to dependent types [39, 51].

Nevertheless, there is a class of programming languages and type systems for which a correspondence in the style of the CHC has not yet been found. We are talking about languages with probabilistic effects, for which type-theoretic accounts have recently been put forward in various ways, e.g. type systems based on sized types [16], intersection types [10] or type systems in the style of so-called amortized analysis [57]. In all the aforementioned cases, a type system was built by modifying well-known type systems for deterministic languages *without* being guided by logic, and instead incepting inherently quantitative concepts directly from probability theory. Is there any logical system behind all this? What kind of logic could possibly play the role of propositional logic in suggesting meaningful and expressive type systems for a λ-calculus endowed with probabilistic choice effects?

A tempting answer is to start from modal logic, which is known to correspond in the Curry-Howard sense to staged computation and to algebraic effects [13, 19, 61]. However, there is one aspect of randomized computation that modal logic fails to capture[1], namely the *probability* of certain notable events, typically termination. In many of the probabilistic type systems mentioned above, for example, a term t receives a type that captures the fact that t has a certain probability q, perhaps strictly smaller than 1, of reducing to a value. This probability is an essential part of what we want to observe with respect to t and, as such, has to be captured by its type, at least if one wants the type system to be expressive.

Recently, *counting quantifiers* [2] have been proposed as a family of operators capable of expressing probabilities within a logical language. Counting quantifiers, unlike standard ones, determine not only *the existence* of an assignment of values to variables with certain characteristics, but rather check *how many* of those assignments exist. Recent works show that classical propositional logic, enriched with counting quantifiers, corresponds to Wagner's hierarchy of counting complexity classes [56] (itself intimately linked to probabilistic complexity), and that Peano Arithmetics, enriched with similar quantitative quantifiers, yields a system capable of speaking of randomized computation in the same sense in which standard PA models deterministic computation [4]. Can all this scale to something like a proper CHC?

The aim of this work is to give a positive answer to the aforementioned question, at the same time highlighting a few remarkable consequences of our study of probabilistic computation through the lens of logic. More specifically, the contributions of this paper are threefold:

- First of all, we introduce an intuitionistic version of Antonelli et al.'s counting propositional logic [2], together with a Kripke-style semantics based on the Borel σ-algebra of the Cantor space and a sound and complete proof-theory. Then, we identify a "computational fragment" of this logic and we design a proof system in natural deduction for it, showing that proof normalization simulates well-known evaluation strategies for probabilistic programs. This is in Section 3.
- We then show that derivations can be decorated with terms of the *event probabilistic λ-calculus*, a λ-calculus for randomized computation introduced by Dal Lago et al. in [17]. This gives rise to a type-system called $\mathrm{C}\lambda_{\rightarrow}^{\{\}}$. Remarkably, the correspondence scales to the underlying dynamics, i.e. proof normalization relates to reduction in the event probabilistic λ-calculus. This is in Section 4 and in Section 5.

[1] With a few notable exceptions, e.g. [24].

- We complete the picture by giving an intersection type assignment system, derived from $\mathbf{C}\lambda^{\{\}}_{\rightarrow}$, and prove that it precisely captures the normalization probability of terms of the event probabilistic λ-calculus. This is in Section 6.

Space limits prevent us from being comprehensive, meaning that many technical details will be unfortunately confined to the Appendix.

2 From Logic to Counting and Probability: a Roadmap

In this section, we explain how a CHC can be defined for probabilistic λ-calculi, at the same time sketching the route we will follow in the rest of the paper.

2.1 Randomized Programs and Counting Quantifiers

The first question we should ask ourselves concerns the kind of programs we are dealing with. What is a probabilistic functional program? Actually, it is a functional program with the additional ability of sampling from some distributions, or to proceed by performing some form of discrete probabilistic choice.[2] This has a couple of crucial consequences: program evaluation becomes an essentially stochastic process, while programs satisfy a given specification *up to* a certain probability. As an example, consider the λ-term,

$$\Xi_{\mathsf{half}} := \lambda x.\lambda y.x \oplus y$$

where $\oplus$ is a binary infix operator for fair probabilistic choice. When applied to two arguments t and u, the evaluation on Ξ_{half} results in either t, with probability one half, or in u, again with probability one half.

But now, if we try to take Ξ_{half} as a proof of, say, a propositional logic formula, we see that the latter is simply not rich enough to capture the behaviour above. Indeed, given that Ξ_{half} is a function of two arguments, it is natural to see it as a proof of an implication $A \rightarrow B \rightarrow C$, namely (following the BHK interpretation) as a function turning a proof of A and a proof of B into a proof of C. What is C, then? Should it be A or should it be B? Actually, it could be both, with some degree of uncertainty, but propositional logic is not able to express all this.

At this point, the recent work on *counting quantifiers* in propositional logic [2] comes to the rescue. Another way to look at discrete probabilistic programs is as programs which are allowed to sample an element ω from the *Cantor space* $2^{\mathbb{N}}$. For example, a probabilistic Turing machine M can be described as a 2-tape machine where the second tape is read-only and sampled from the Cantor space at each run. Similarly, the execution of Ξ_{half} applied to programs t and u can be described as the result of sampling ω and returning either t or u depending on some read value, e.g. $\omega(0)$. Crucially, for each input x and each possible output y of a probabilistic program f, the set $S_{x,y}$ of elements of $2^{\mathbb{N}}$, which make $f(x)$ produce y, is a *Borel* set, so it makes sense to say that the probability that $f(x)$ yields y coincides with the *measure* of $S_{x,y}$ (more on Borel sets and measures below).

Yet, what has all this to do with logic and counting? The fundamental observation is that formulas of classical propositional logic (from now on *Boolean formulas* $\mathscr{b}, \mathscr{c}, \ldots$) provide ways of denoting Borel sets, namely the sets $\{\omega \in 2^{\mathbb{N}} \mid \omega \vDash \mathscr{b}\}$ of valuations satisfying the formula. Hence, for any Boolean formula $\mathscr{b}$, it makes sense to define a *new* formula like e.g. $\mathbf{C}^{\frac{1}{2}}\mathscr{b}$, which is true when *at least* $\frac{1}{2}$ of its models satisfies $\mathscr{b}$, i.e when the Borel set associated with $\mathscr{b}$ has measure greater than $\frac{1}{2}$. For instance, the formula $\mathbf{C}^{\frac{1}{2}}(\mathscr{x}_0 \vee \mathscr{x}_1)$ is true, since at least $\frac{1}{2}$ of its models satisfy $\mathscr{x}_0 \vee \mathscr{x}_1$. Instead, $\mathbf{C}^{\frac{1}{2}}(\mathscr{x}_0 \wedge \mathscr{x}_1)$ is false, since only $\frac{1}{4}$ of its models satisfy $\mathscr{x}_0 \wedge \mathscr{x}_1$.

Counting propositional logic (from now on CPL) is defined by enriching classical propositional logic with counting quantifiers $\mathbf{C}^q$, with $q \in (0, 1] \cap \mathbb{Q}$. Following our sketch, CPL admits a natural semantics in the Borel σ-algebra of the Cantor space, together with a sound and complete sequent calculus [2]. Notice that measuring a Boolean formula actually amounts at *counting* its models, that is, to a purely recursive operation, albeit one which needs not be doable in polynomial time (indeed, CPL is deeply related to the hierarchy of counting complexity classes [2]).

Going back to the term Ξ_{half}, what seems to be lacking in intuitionistic logic is precisely a way to express that C could be $\mathbf{C}^{\frac{1}{2}}A$, namely that it should be A *with probability at least* $\frac{1}{2}$. Similarly, C could be $\mathbf{C}^{\frac{1}{2}}B$.

In Section 3 we introduce an intuitionistic version of CPL, called iCPL, which enriches intuitionistic logic with Boolean variables as well as the counting quantifier $\mathbf{C}^q$. Intuitively, if a proof of a formula A can be seen as a deterministic program satisfying the specification A, a proof of $\mathbf{C}^q A$ will correspond to a probabilistic program that satisfies the specification A with probability q. Our main result will be to show that proofs in iCPL correspond to functional probabilistic programs and, most importantly, that normalization in this logic describes probabilistic evaluation.

2.2 Can CbN and CbV Evaluation Coexist?

In extensions of the λ-calculus with probabilistic choice, two different evaluation strategies are generally considered: the *call-by-name* (CbN) strategy might duplicates choices before evaluating them, while the *call-by-value* (CbV) strategy evaluates choices before possibly duplicating their outcomes. Importantly, the probability of termination of a program might differ depending on the chosen strategy. For example, consider the application of the term $\mathbf{2} := \lambda yx.y(yx)$ (i.e. the second Church numeral) to $I \oplus \Omega$, where $I = \lambda x.x$ and Ω is the diverging term $(\lambda x.xx)\lambda x.xx$. Under CbN, the redex $\mathbf{2}(I \oplus \Omega)$ first produces $\lambda x.(I \oplus \Omega)((I \oplus \Omega)x)$, then reduces

[2] Here we are not at all concerned with sampling from continuous distributions, nor with capturing any form of conditioning.

to any of the terms $\lambda x.u(vx)$, with u, v chosen from $\{I, \Omega\}$, each with probability $\frac{1}{4}$. Since $\lambda x.u(vx)$ converges only when $u = v = I$, the probability of convergence is thus $\frac{1}{4}$. Under CbV, in $\mathbf{2}(I \oplus \Omega)$ one first has to evaluate $I \oplus \Omega$, and then pass the result to $\mathbf{2}$, hence returning either the converging term $I(Ix)$ or the diverging term $\Omega(\Omega x)$, each with probability $\frac{1}{2}$.

If we now try to think of the Church numeral $\mathbf{2}$ as a proof of some counting quantified formula, we see that, depending on the reduction strategy we have in mind, it must prove a *different* formula. Indeed, given that $I \oplus \Omega$ proves $\mathbf{C}^{\frac{1}{2}}(A \to A)$, in the CbN case, $\mathbf{2}$ proves $\mathbf{C}^{\frac{1}{2}}(A \to A) \to A \to \mathbf{C}^{\frac{1}{4}}A$, since only in one over four cases it yields a proof of A, while in the CbV case, $\mathbf{2}$ proves the formula $\mathbf{C}^{\frac{1}{2}}(A \to A) \to A \to \mathbf{C}^{\frac{1}{2}}A$, as it yields a proof of A in one case over two.

In the literature on probabilistic λ-calculi, the apparent incompatibility of CbN and CbV evaluation is usually resolved by restricting to calculi with one or the other strategy. However, the observation above suggests that, if functional programs are typed using counting quantifiers, it should become possible to make the two evaluation strategies coexist, by assigning them different types.

Actually, a few recent approaches [15, 17, 20, 22] already suggest ways to make CbN and CbV evaluation live together. In particular, in the *probabilistic event λ-calculus* [17] the choice operator $\oplus$ is decomposed into two different operators, yielding a confluent calculus: a *choice* operator $t \oplus_a u$, depending on some probabilistic event $a \in \{0, 1\}$, and a *probabilistic event generator* $\nu a.t$ which actually "flips the coin". In this language, the CbN and CbV applications of $\mathbf{2}$ to $I \oplus \Omega$ are encoded by two *distinct* terms $\mathbf{2}(\nu a.I \oplus_a \Omega)$ and $\nu a.\mathbf{2}(I \oplus_a \Omega)$, crucially distinguishing between generating a probabilistic choice *before* or *after* a duplication takes place.

This calculus constitutes then an ideal candidate for our CHC. Indeed, the rules for the counting quantifier $\mathbf{C}^q$ naturally give rise to typing rules for the event generator νa. In Section 4 we introduce a variant $\Lambda_{\mathsf{PE}}^{\{\}}$ of the calculus from [17], with the underlying probability space $\{0, 1\}$ replaced by the Cantor space, and in Section 5 we introduce a type system $\mathbf{C}\lambda_{\to}^{\{\}}$ for simple types with counting quantifiers, showing that natural deduction derivations translate into typing derivations in $\mathbf{C}\lambda_{\to}^{\{\}}$, with normalization precisely corresponding to reduction in $\Lambda_{\mathsf{PE}}^{\{\}}$.

2.3 Capturing Probability of Normalization via Types

As observed in the Introduction, a fundamental quantitative property we would like to observe using types is the probability of termination. However, given that the reduction of $\Lambda_{\mathsf{PE}}^{\{\}}$ is purely deterministic, what notions of probabilistic termination should we actually observe?

Rather than evaluating programs by implementing probabilistic choices, reduction in the probabilistic event λ-calculus has the effect of progressively generating the *full tree* of outcomes of (sequences of) probabilistic choices, giving rise to a *distribution* of values. Therefore, given a term t, rather than asking whether *some* or *all* reductions of t terminate, it makes sense to ask *with what probability* a normal form is found by generating all probabilistic outcomes of t.

In Section 6 we will first show that when the type $\mathbf{C}^q\sigma$ is assigned to a program t, the value q provides a lower bound for the actual probability of finding a (head) normal form in the development of t. Then we show that, by extending the type system with an intersection operator, one can attain an upper bound and thus fully characterize the distribution of values associated with a term.

2.4 Preliminaries on the Cantor Space

Throughout the paper, we exploit some basic facts about the Cantor space, its Borel σ-algebra, and their connections with Boolean logic, that we briefly recall here.

We consider a countably infinite set $\mathscr{A}$ of *names*, noted $a, b, c, \dots$. For any finite subset $X \subseteq \mathscr{A}$, we let $\mathbf{B}_X$ (resp. $\mathbf{B}_{\mathscr{A}}$) indicate the *Borel σ-algebra* on the X-th product of the Cantor space $(2^{\mathbb{N}})^X$ (resp. on the $\mathscr{A}$-th product $(2^{\mathbb{N}})^{\mathscr{A}}$), that is, the smallest σ-algebra containing all open sets under the product topology. There exists a unique measure μ of $\mathbf{B}_{\mathscr{A}}$ such that $\mu(C_{a,i}) = \frac{1}{2}$ for all *cylinders* $C_{a,i} = \{\omega \mid \omega(a)(i) = 1\}$. μ restricts to a measure μ_X on $\mathbf{B}_X$ by letting $\mu_X(S) = \mu(S \times (2^{\mathbb{N}})^{\mathscr{A}-X})$.

Boolean formulas with names in $\mathscr{A}$ are defined by:

$$\mathscr{b} ::= \top \mid \bot \mid x_a^i \mid \neg\mathscr{b} \mid \mathscr{b} \wedge \mathscr{b} \mid \mathscr{b} \vee \mathscr{b}$$

where $a \in \mathscr{A}$ and $i \in \mathbb{N}$. We let $\mathsf{FN}(\mathscr{b}) \subseteq \mathscr{A}$ be the set of names occurring in $\mathscr{b}$. For all Boolean formula $\mathscr{b}$ and $X \supseteq \mathsf{FN}(\mathscr{b})$, we let $[\![\mathscr{b}]\!]_X$ indicate the Borel set $\{\omega \in (2^{\mathbb{N}})^X \mid \omega \vDash \mathscr{b}\}$. The value $\mu_X([\![\mathscr{b}]\!]_X) \in [0,1] \cap \mathbb{Q}$ is independent from the choice of $X \supseteq \mathsf{FN}(\mathscr{b})$, and we will note it simply as $\mu(\mathscr{b})$.

3 Intuitionistic Counting Propositional Logic

In this section we introduce a constructive version of CPL, that we call iCPL, which extends standard intuitionistic logic with Boolean variables and counting quantifiers. The logic iCPL combines constructive reasoning (corresponding, under CHC, to functional programming) with semantic reasoning on Boolean formulas and their models (corresponding, as we have seen, to discrete probabilistic reasoning). The formulas of iCPL are *hybrid*, that is, comprise both a countable set $\mathscr{P} = \{\mathscr{p}, \mathscr{q}, \dots\}$ of intuitionistic propositional variables, and the Boolean propositional variables x_a^i. For example, consider the formula A below:

$$A := \mathscr{p} \to \mathscr{q} \to (x_a^0 \wedge \mathscr{p}) \vee (\neg x_a^0 \wedge \mathscr{q})$$

Intuitively, proving A amounts to showing that, whenever $\mathscr{p}$ and $\mathscr{q}$ hold, given that either x_a^0 or $\neg x_a^0$ holds, in the first case $\mathscr{p}$ holds, and in the second case $\mathscr{q}$ holds.

Suppose we test A against some element ω from the Cantor space. The following algorithm could then describe a way of showing that A holds: given a proof Π of p and a proof Σ of q, output Π if $\omega(0) = 1$ (i.e. if $\boldsymbol{x}_a^0$ holds), and Σ if $\omega(0) = 0$ (i.e. if $\neg\boldsymbol{x}_a^i$ holds). In other words, a proof of A, in the "environment" ω, could be something like $\lambda xy.x \oplus_a y$.

Suppose now that ω is randomly generated. What are the chances that our strategy will actually yield a proof of A? Well, there are two possible cases, and in both cases we get a proof of one of the disjuncts $\boldsymbol{x}_a^0 \wedge p$ and $\neg\boldsymbol{x}_a^0 \wedge q$, and thus a proof of A. Since, by freely generating our ω, we get a proof of A with probability 1, we can conclude that we have a proof of $\mathbf{C}_a^1 A$, not depending on any "environment", and that written as a term looks precisely as $\nu a.\lambda xy.x \oplus_a y$.

Consider now yet another formula B:

$$B := p \to q \to \boldsymbol{x}_a^0 \wedge p$$

Given some ω, an algorithm for proving B should output Π when $\omega(0) = 1$, but when $\omega(0) = 0$, it could do anything, since we have nothing to prove in this case. A proof of B should then be something like $\lambda xy.x \oplus_a ?$, where ? can be any program. By freely generating our ω, we get then a proof of B only in one half of the cases, i.e. we get a proof of $\mathbf{C}_a^{\frac{1}{2}} B$.

3.1 The Semantics and Proof-Theory of iCPL.

The formulas of the logic just sketched are defined by:

$$A ::= \top \mid \bot \mid \boldsymbol{x}_a^i \mid p \mid A \wedge A \mid A \vee A \mid A \to A \mid \mathbf{C}_a^q A$$

where $q \in (0, 1] \cap \mathbb{Q}$. A rather intuitive semantics for iCPL-formulas can be given in terms of Kripke-like structures:

Definition 3.1. A iCPL-*structure* is a triple $\mathscr{M} = (W, \preceq, \mathbf{i})$ where W is a countable set, $\preceq$ is a preorder on W, and $\mathbf{i} : \mathscr{P} \to W^{\uparrow}$, where $W^{\uparrow}$ is the set of upper subsets of W.

The interpretation of iCPL formulas in Kripke structures combines a set W of worlds (for the interpretation of intuitionistic propositional variables) with the choice of an element of the Cantor space (for the interpretation of Boolean variables): for any iCPL-structure $\mathscr{M} = (W, \preceq, \mathbf{i})$ and finite set X, we define the relation $w, \omega \Vdash_{\mathscr{M}}^X A$ (where $w \in W$, $\omega \in (2^{\mathbb{N}})^X$ and $\mathsf{FN}(A) \subseteq X$) by induction as follows:

- $w, \omega \nVdash_{\mathscr{M}}^X \bot$ and $w, \omega \Vdash_{\mathscr{M}}^X \top$;
- $w, \omega \Vdash_{\mathscr{M}}^X \boldsymbol{x}_a^i$ iff $\omega(a)(i) = 1$;
- $w, \omega \Vdash_{\mathscr{M}}^X p$ iff $w \in \mathbf{i}(p)$;
- $w, \omega \Vdash_{\mathscr{M}}^X A \wedge B$ iff $w, \omega \Vdash_{\mathscr{M}}^X A$ and $w, \omega \Vdash_{\mathscr{M}}^X B$;
- $w, \omega \Vdash_{\mathscr{M}}^X A \vee B$ iff $w, \omega \Vdash_{\mathscr{M}}^X A$ or $w, \omega \Vdash_{\mathscr{M}}^X B$;
- $w, \omega \Vdash_{\mathscr{M}}^X A \to B$ iff for all $w' \succeq w$, $w', \omega \Vdash_{\mathscr{M}}^X A$ implies $w', \omega \Vdash_{\mathscr{M}}^X B$;
- $w, \omega \Vdash_{\mathscr{M}}^X \mathbf{C}_a^q A$ iff

$$\mu\left(\left\{\omega' \in 2^{\mathbb{N}} \mid w, \omega + \omega' \Vdash_{\mathscr{M}}^{X \cup \{a\}} A\right\}\right) \geq q$$

Using properties of the Borel σ-algebra $\mathbf{B}_X$, it can be shown that for any $w \in W$ and formula A, the set $\{\omega \in (2^{\mathbb{N}})^X \mid w, \omega \Vdash_{\mathscr{M}}^X A\}$ is a Borel set, and thus measurable.

We write $\Gamma \Vdash_{\mathscr{M}}^X A$ when for all $w \in W$ and $\omega \in (2^{\mathbb{N}})^X$, whenever $w, \omega \Vdash_{\mathscr{M}}^X \Gamma$ holds, also $w, \omega \Vdash_{\mathscr{M}}^X A$ holds. We write $\Gamma \vDash A$ when for any iCPL-structure $\mathscr{M}$ and $X \supseteq \mathsf{FN}(A)$, $\Gamma \Vdash_{\mathscr{M}}^X A$ holds.

A sound and complete proof-theory for iCPL can be defined. Indeed, starting from usual natural deduction for intuitionistic logic, one obtains a proof system $\mathbf{ND}_{\mathsf{iCPL}}$ for iCPL by adding the excluded middle $\boldsymbol{x}_a^i \vee \neg\boldsymbol{x}_a^i$ as an axiom for Boolean variables, as well as suitable rules and axioms for counting quantifiers (details can be found in the Appendix).

Theorem 3.2. $\Gamma \vDash A$ *iff* $\Gamma \vdash_{\mathbf{ND}_{\mathsf{iCPL}}} A$.

For example, $\mathbf{C}^q$ admits an introduction rule as below

$$\frac{\Gamma, \boldsymbol{d} \vdash A \qquad \mu(\boldsymbol{d}) \geq q}{\Gamma \vdash \mathbf{C}_a^q A} \quad \text{(C-intro)}$$

with the proviso that a does not occur in Γ, and is the only name occurring in $\boldsymbol{d}$. Intuitively, this rule says that if we can prove A under assumptions Γ and $\boldsymbol{d}$, and if a randomly chosen valuation of a has chance q of being a model of $\boldsymbol{d}$, then we can build a proof of $\mathbf{C}_a^q A$ from Γ. Observe that the rule (CI) has a semantic premise $\mu(\boldsymbol{d}) \geq q$: we ask to an oracle to count the models of $\boldsymbol{d}$ for us (this is similar to what happens in sequent calculi for CPL, see [2]).

3.2 The Computational Fragment of iCPL.

From the perspective of the CHC, however, the natural deduction system just sketched is not what we are looking for. As iCPL contains Boolean logic, in order to relate proofs and programs, one should first choose among the several existing constructive interpretations of classical logic. Yet, in our previous examples Boolean formulas were never those to be proved; rather, they were used as *semantic constraints* that programs may or may not satisfy (for example, when saying that a program, depending on some event ω, yields a proof of A *when* ω *satisfies* $\boldsymbol{b}$, or that a program has q chances of yielding a proof of A *when* $\boldsymbol{b}$ *has measure at least* q).

Would it be possible, then, to somehow *separate* purely constructive reasoning from Boolean semantic reasoning within formulas and proofs of iCPL? The following lemma suggests that this is indeed possible.

Lemma 3.3 (Decomposition Lemma)**.** *For any formula A of* iCPL *there exist Boolean formulas $\boldsymbol{b}_v$ and purely intuitionistic formulas A_v (i.e. formulas containing no Boolean variables), where v varies over all possible valuations of the Boolean variables in A, such that*

$$\vDash A \leftrightarrow \bigvee_v \boldsymbol{b}_v \wedge A_v$$

Proof sketch. The idea is to let $\boldsymbol{b}_v$ be the formula characterizing v, i.e. the conjunction of all variables true in v and of all negations of variables false in v, and let A_v be obtained from A by replacing each Boolean variable by either $\top$ or $\bot$, depending on its value under v. □

By Lemma 3.3, any sequent $\Gamma \vdash A$ of iCPL can be associated with a family of intuitionistic sequents of the form $\Gamma_v, \mathscr{b}_v \vdash A_v$ where v ranges over all valuations of the Boolean variables of Γ and A, and $\Gamma_v, \mathscr{b}_v, A_v$ are as in Lemma 3.3. We will note such sequents as $\Gamma_v \vdash \mathscr{b}_v \rightarrowtail A_v$ to highlight the special role played by the Boolean formula $\mathscr{b}_v$.

The sequents of the form $\Gamma \vdash \mathscr{b} \rightarrowtail A$ have a natural computational interpretation: they express program specifications in the form "Π yields a proof of A from Γ whenever its sampled function satisfies $\mathscr{b}$". This suggests that a computational interpretation of counting quantifiers can be obtained by focusing on sequents of this form, i.e. to purely intuitionistic formulas. By the way, Lemma 3.3 (and Prop. 3.6 below), ensure that, *modulo* Boolean reasoning, logical arguments in iCPL can be reduced to (families of) arguments of this kind.

Let then iCPL_0 be the fragment of iCPL defined by:

$$A ::= \mathscr{p} \mid A \to A \mid \mathbf{C}^q A$$

where $q \in (0,1] \cap \mathbb{Q}$. For simplicity, and since this is enough for the CHC, we take here implication as the only propositional connective, yet all other propositional connectives could be added. As formulas do not contain Boolean variables, counting quantifiers in iCPL_0 are not named. We use $\mathbf{C}^{q_1 * \cdots * q_n} A$ as an abbreviation for $\mathbf{C}^{q_1} \dots \mathbf{C}^{q_n} A$.

The rules of the natural deduction system $\mathbf{ND}_{\mathsf{iCPL}_0}$ for iCPL are illustrated in Fig. 1 (where in (CI) it is assumed that $\mathsf{FN}(\mathscr{b}) \cap \mathsf{FN}(\mathscr{d}) = \emptyset$). A few rules involve *semantic premises* of the forms $\mathscr{b} \vDash \mathscr{c}$ and $\mu(\mathscr{b}) \geq q$. This reflects our demand that Boolean semantic constraints should not be objects of proof but should rather be checked by an external oracle.

Beyond standard intuitionistic rules, $\mathbf{ND}_{\mathsf{iCPL}_0}$ comprises structural rules to manipulate Boolean formulas, and introduction and elimination rules for the counting quantifier. The rule $(\bot)$ yields *dummy* proofs of any formula, i.e. proofs which are correct for *no* possible event; the rule (m) combines two proofs Π_1, Π_2 of the same formula into one single proof Π', with the choice depending on the value of some Boolean variable $\mathscr{x}_a^i$ (Π' is thus something like $\Pi_1 \oplus_a \Pi_2$). The introduction rule for $\mathbf{C}^q$ is similar to rule (C-intro). It is explained as follows: if Π, in the "environment" $\omega + \omega' \in (2^{\mathbb{N}})^{X \cup \{a\}} \simeq (2^{\mathbb{N}})^X \times 2^{\mathbb{N}}$, yields a proof of A whenever $\omega + \omega'$ satisfies the two *independent* constraints $\mathscr{b}$ and $\mathscr{d}$ (i.e. $\omega \vDash \mathscr{b}$ and $\omega' \vDash \mathscr{d}$), then by randomly choosing $\omega' \in 2^{\mathbb{N}}$, we have at least $q \geq \mu(\mathscr{d})$ chances of getting a proof of A (Π' is thus something like $\nu a.\Pi$). Finally, the elimination rule (CE) essentially turns a proof of $A \to B$ into a proof of $\mathbf{C}^q A \to \mathbf{C}^{qs} B$. As we'll see, this rule captures CbV function application.

Example 3.4. In Fig. 2 is illustrated a proof $\Pi_{\frac{1}{2}\mathrm{id}}$ of $\mathbf{C}^{\frac{1}{2}}(A \to A)$ obtained by first "mixing" an exact proof of $A \to A$ with a dummy one, and then introducing the counting quantifier.

Identity Rule

$$\frac{}{\Gamma, A \vdash \mathscr{b} \rightarrowtail A}\ (\mathrm{id})$$

Structural Rules

$$\frac{\mathscr{b} \vDash \bot}{\Gamma \vdash \mathscr{b} \rightarrowtail A}\ (\bot)$$

$$\frac{\Gamma \vdash \mathscr{c} \rightarrowtail A \qquad \Gamma \vdash \mathscr{d} \rightarrowtail A \qquad \mathscr{b} \vDash (\mathscr{c} \wedge \mathscr{x}_a^i) \vee (\mathscr{d} \wedge \neg \mathscr{x}_a^i)}{\Gamma \vdash \mathscr{b} \rightarrowtail A}\ (\mathrm{m})$$

Logical Rules

$$\frac{\Gamma, A \vdash \mathscr{b} \rightarrowtail B}{\Gamma \vdash \mathscr{b} \rightarrowtail (A \to B)}\ (\to\mathrm{I})$$

$$\frac{\Gamma \vdash \mathscr{b} \rightarrowtail (A \to B) \qquad \Gamma \vdash \mathscr{b} \rightarrowtail A}{\Gamma \vdash \mathscr{b} \rightarrowtail B}\ (\to\mathrm{E})$$

Counting Rules

$$\frac{\Gamma \vdash \mathscr{b} \wedge \mathscr{d} \rightarrowtail A \qquad \mu(\mathscr{d}) \geq q}{\Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^q A}\ (\mathrm{CI})$$

$$\frac{\Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^q A \qquad \Gamma, A \vdash \mathscr{b} \rightarrowtail B}{\Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^{qs} B}\ (\mathrm{CE})$$

Figure 1. Rules of $\mathbf{ND}_{\mathsf{iCPL}_0}$.

Example 3.5. In Fig. 3 we illustrate derivations of $\mathbf{C}^q(A \to A) \to A \to \mathbf{C}^{q*q} A$ and $\mathbf{C}^q(A \to A) \to A \to \mathbf{C}^q B$, which intuitively correspond to the "CbV" and "CbN" versions of the Church numeral 2: the first one uses the probabilistic assumption $\mathbf{C}^q(A \to A)$ twice (hence duplicating choice), while the second one uses it once.

The proof systems of iCPL and iCPL_0 are related by:

Proposition 3.6. *For any set of formulas Γ and formula A of* iCPL*, $\Gamma \vdash A$ is provable in $\mathbf{ND}^-_{\mathsf{iCPL}}$[3] iff for any valuation v, $\Gamma_v \vdash \mathscr{b}_v \rightarrowtail A_v$ is provable in $\mathbf{ND}_{\mathsf{iCPL}_0}$.*

3.3 Normalization in iCPL_0

From the CHC perspective, natural deduction proofs correspond to programs, and normalization corresponds to execution. Let us look at normalization in iCPL_0, then.

The two main normalization steps are ($\to$I/$\to$E) and (CI/CE), and correspond to the CbN and CbV variants of function application. The cuts ($\to$I/$\to$E) are eliminated, as usual, by means of an admissible substitution rule

$$\frac{\Gamma \vdash \mathscr{b} \rightarrowtail A \qquad \Gamma, A \vdash \mathscr{b} \rightarrowtail B}{\Gamma \vdash \mathscr{b} \rightarrowtail B}\ (\mathrm{subst})$$

When the minor premiss of ($\to$E) is a conclusion $\mathbf{C}^q A$ of a (CI) rule (as illustrated in Fig. 4a), normalization duplicates a proof of $\mathbf{C}^q A$ (i.e. it duplicates choice). For instance, if we cut the proof Π_{CbN} with $\Pi_{\frac{1}{2}\mathrm{id}}$ (by letting $q = \frac{1}{2}$), normalization duplicates the choice between the correct and the dummy proof of $A \to A$, yielding a proof of $A \to \mathbf{C}^{\frac{1}{2} * \frac{1}{2}} A$.

[3] $\mathbf{ND}^-_{\mathsf{iCPL}}$ is $\mathbf{ND}_{\mathsf{iCPL}}$ deprived of two "non constructive" axioms.

$$\Pi_{\frac{1}{2}\mathsf{id}} \quad = \quad \dfrac{\dfrac{\dfrac{A \vdash x^i_a \rightarrowtail A}{\vdash x^i_a \rightarrowtail (A \to A)}\,(\to\mathrm{I}) \qquad \dfrac{}{\vdash \bot \rightarrowtail (A \to A)}\,(\bot) \qquad x^i_a \vDash (x^i_a \wedge x^i_a) \vee (\neg x^i_a \wedge \bot)}{\vdash x^i_a \rightarrowtail (A \to A)}\,(\mathsf{m}) \qquad \mu(x^i_a) \geq \frac{1}{2}}{\vdash \mathbf{C}^{\frac{1}{2}}(A \to A)}\,(\mathrm{CI})$$

Figure 2. A derivation of $\mathbf{C}^{\frac{1}{2}}(A \to A)$ in $\mathbf{ND}_{\mathsf{iCPL}_0}$.

$$\Pi_{\mathrm{CbN}} \quad = \quad \dfrac{\dfrac{\dfrac{\mathbf{C}^q(A \to A), A \vdash \mathbf{C}^q(A \to A) \qquad \dfrac{A \to A, A \vdash A \to A \qquad A \to A, A \vdash A}{A \to A, A \vdash A}\,(\to\mathrm{E})}{\mathbf{C}^q(A \to A), A \vdash \mathbf{C}^q A}\,(\mathrm{CE}) \qquad \dfrac{\mathbf{C}^q(A \to A), A, A \vdash \mathbf{C}^q(A \to A) \qquad \dfrac{A \to A, A, A \vdash A \to A \qquad A \to A, A, A \vdash A}{A \to A, A, A \vdash A}\,(\to\mathrm{E})}{\mathbf{C}^q(A \to A), A, A \vdash \mathbf{C}^q A}\,(\mathrm{CE})}{\mathbf{C}^q(A \to A), A \vdash \mathbf{C}^{q*q} A}\,(\mathrm{CE})}{\vdash \mathbf{C}^q(A \to A) \to A \to \mathbf{C}^{q*q} A}\,(\to\mathrm{I})$$

$$\Pi_{\mathrm{CbV}} \quad = \quad \dfrac{\dfrac{\mathbf{C}^q(A \to A), A \vdash \mathbf{C}^q(A \to A) \qquad \dfrac{A \to A, A \vdash A \to A \qquad \dfrac{A \to A, A \vdash A \to A \qquad A \to A, A \vdash A}{A \to A, A \vdash A}\,(\to\mathrm{E})}{A \to A, A \vdash A}\,(\to\mathrm{E})}{\mathbf{C}^q(A \to A), A \vdash \mathbf{C}^q A}\,(\mathrm{CE})}{\vdash \mathbf{C}^q(A \to A) \to A \to \mathbf{C}^q A}\,(\to\mathrm{I})$$

Figure 3. The "CbN" and "CbV" Church numerals as derivations in $\mathbf{ND}_{\mathsf{iCPL}_0}$.

The normalization step (CI/CE), illustrated in Fig. 4b, permutes the rule (CI) downwards (i.e. choices are postponed but not duplicated): the probabilistic rule (CI) is permuted downwards and it is the proof Π of A which is substituted in Σ. For instance, if we cut the proof Π_{CbV} with $\Pi_{\frac{1}{2}\mathrm{id}}$, normalization now produces a proof of $A \to \mathbf{C}^{\frac{1}{2}} A$: the choice between a correct proof of $A \to A$ and a dummy one is not duplicated, but just postponed.

The other normalization steps permute (m) with other rules, and correspond to *permuting reductions* for the probabilistic λ-calculus introduced in the next section.To make the study of normalization as simple as possible, we did not consider a "multiplication rule" to pass from $\mathbf{C}^q\mathbf{C}^s A$ to $\mathbf{C}^{qs} A$, as this would introduce other normalization steps. However, in the Appendix we study an alternative "CbN" proof-system $\mathbf{ND}_{\mathsf{iCPL}_0}$ also comprising this rule.

As a by-product of the CHC developed in the following sections, we will obtain a strong normalization theorem for iCPL_0 (Corollary 5.5).

4 The Event Probabilistic Lambda Calculus

In this section we introduce the computational side of the CHC, that is, a variant of the probabilistic event λ-calculus Λ_{PE} from [17], with choices depending on events from the Cantor space. We will then discuss how terms of Λ_{PE} yield distributions of values, and we define two notions of probabilistic normalization for such distributions. We finally introduce a further variant $\Lambda^{\{\}}_{\mathsf{PE}}$ of Λ_{PE} which provides a smoother representation of CbV functions.

4.1 A λ-Calculus Sampling from the Cantor Space

The terms of Λ_{PE} are defined by the grammar below:

$$t ::= x \mid \lambda x.t \mid tt \mid t \oplus^i_a u \mid \nu a.t$$

with $a \in \mathscr{A}$, and $i \in \mathbb{N}$. The intuition is that $\nu a.$ samples some function ω from the Cantor space, and $t \oplus^i_a u$ yields either t or u depending on the value $\omega(a)(i) \in \{0, 1\}$. In the following we let $t \oplus^i u$ be an abbreviation for $\nu a.t \oplus^i_a u$ (supposing a does not occur free in either t or u).

For any term t, finite set X and $\omega \in (2^{\mathbb{N}})^X$, let $\pi^\omega_X(t)$, the "application of ω to t through X", be defined as:

$$\begin{aligned}
\pi^\omega_X(x) &= x \\
\pi^\omega_X(\lambda x.t) &= \lambda x.\pi^\omega_X(t) \\
\pi^\omega_X(tu) &= \pi^\omega_X(t)\pi^\omega_X(u) \\
\pi^\omega_X(t \oplus^i_a u) &= \begin{cases} \pi^\omega_X(t) & \text{if } a \in X \text{ and } \omega(a)(i) = 1 \\ \pi^\omega_X(u) & \text{if } a \in X \text{ and } \omega(a)(i) = 0 \\ \pi^\omega_X(t) \oplus^i_a \pi^\omega_X(u) & \text{otherwise} \end{cases} \\
\pi^\omega_X(\nu b.t) &= \nu b.\pi^\omega_X(t)
\end{aligned}$$

In usual randomized λ-calculi, program execution is defined so as to be inherently probabilistic: for example a term $t \oplus u$ can reduce to either t or u, with probability $\frac{1}{2}$. In this way, chains of reduction can be described as *stochastic Markovian sequences* [48], leading to formalize the idea of *normalization with probability* $r \in [0, 1]$ (see [9]).

By contrast, reduction in Λ_{PE} is fully deterministic: beyond the usual (and un-resticted) β-rule $(\lambda x.t)u \rightarrowtriangle_\beta t[u/x]$, it comprises a *permutative reduction* $t \rightarrowtriangle_{\mathsf{p}} u$ defined by the rules in Fig. 5 (where $(a, i) < (b, j)$ if either νb occurs in the scope of νa, or $a = b$ and $i < j$). Intuitively, permutative reductions implement probabilistic choices by computing the full tree of possible choices. For example, given terms t_1, t_2, u_1, u_2, one can see that the term $\nu a.(t_1 \oplus^0_a t_2)(u_1 \oplus^1_a u_2)$ reduces to $\nu a.(t_1 u_1 \oplus^1_a t_1 u_2) \oplus^0_a (t_2 u_1 \oplus^1_a t_2 u_2)$, hence displaying all possible alternatives.

The fundamental properties of Λ_{PE} are the following:

Theorem 4.1 ([17]). *$\rightarrowtriangle_{\mathsf{p}}$ is confluent and strongly normalizing.* Full reduction $\rightarrowtriangle := \rightarrowtriangle_\beta \cup \rightarrowtriangle_{\mathsf{p}}$ *is confluent.*

$$\dfrac{\dfrac{\overset{\Pi}{\Gamma, \mathbf{C}^q A \vdash \mathscr{b} \rightarrowtail B}}{\Gamma \vdash \mathscr{b} \rightarrowtail (\mathbf{C}^q A \to B)}\,(\to\mathrm{I}) \qquad \dfrac{\overset{\Sigma}{\Gamma \vdash \mathscr{b} \wedge \mathscr{d} \rightarrowtail A} \quad \mu(\mathscr{d}) \geq q}{\Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^q A}\,(\mathrm{CI})}{\Gamma \vdash \mathscr{b} \rightarrowtail B}\,(\to\mathrm{E}) \quad \rightsquigarrow \quad \dfrac{\overset{\Pi}{\Gamma, \mathbf{C}^q A \vdash \mathscr{b} \rightarrowtail B} \qquad \dfrac{\overset{\Sigma}{\Gamma \vdash \mathscr{b} \wedge \mathscr{d} \rightarrowtail A} \quad \mu(\mathscr{d}) \geq q}{\Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^q A}\,(\mathrm{CI})}{\Gamma \vdash \mathscr{b} \rightarrowtail B}\,(\mathrm{subst})$$

(a) (→I)/(→E).

$$\dfrac{\dfrac{\overset{\Sigma}{\Gamma \vdash \mathscr{b} \wedge \mathscr{d} \rightarrowtail A} \quad \mu(\mathscr{d}) \geq q}{\Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^q A}\,(\mathrm{CI}) \qquad \overset{\Pi}{\Gamma, A \vdash \mathscr{b} \rightarrowtail B}}{\Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^{qs} B}\,(\mathrm{CE}) \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\Sigma}{\Gamma \vdash \mathscr{b} \wedge \mathscr{d} \rightarrowtail A} \qquad \overset{\Pi[\mathscr{b} \mapsto \mathscr{b} \wedge \mathscr{d}]}{\Gamma, A \vdash \mathscr{b} \wedge \mathscr{d} \rightarrowtail B}}{\Gamma \vdash \mathscr{b} \wedge \mathscr{d} \rightarrowtail B}\,(\mathrm{subst}) \quad \mu(\mathscr{d}) \geq qs}{\Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^{qs} B}\,(\mathrm{CI})$$

(b) (CI)/(CE).

Figure 4. Main normalization steps of $\mathbf{ND}_{\mathsf{iCPL}_0}$.

$$\begin{array}{rcllr}
t \oplus_a^i t & \rightharpoonup_{\mathsf{p}} & t & & (\mathrm{i}) \\
(t \oplus_a^i u) \oplus_a^i v & \rightharpoonup_{\mathsf{p}} & t \oplus_a^i v & & (\mathrm{c}_1) \\
t \oplus_a^i (u \oplus_a^i v) & \rightharpoonup_{\mathsf{p}} & t \oplus_a^i v & & (\mathrm{c}_2) \\
\lambda x.(t \oplus_a^i u) & \rightharpoonup_{\mathsf{p}} & (\lambda x.t) \oplus_a^i (\lambda x.u) & & (\oplus\lambda) \\
(t \oplus_a^i u)v & \rightharpoonup_{\mathsf{p}} & (tu) \oplus_a^i (uv) & & (\oplus\mathrm{f}) \\
t(u \oplus_a^i v) & \rightharpoonup_{\mathsf{p}} & (tu) \oplus_a^i (tv) & & (\oplus\mathrm{a}) \\
(t \oplus_a^i u) \oplus_b^j v & \rightharpoonup_{\mathsf{p}} & (t \oplus_b^j v) \oplus_a^i (u \oplus_b^j v) & ((a,i) < (b,j)) & (\oplus\oplus_1) \\
t \oplus_b^j (u \oplus_a^i v) & \rightharpoonup_{\mathsf{p}} & (t \oplus_b^j u) \oplus_a^i (t \oplus_b^j v) & ((a,i) < (b,j)) & (\oplus\oplus_2) \\
\nu b.(t \oplus_a^i u) & \rightharpoonup_{\mathsf{p}} & (\nu b.t) \oplus_a^i (\nu b.u) & (a \neq b) & (\oplus\nu) \\
\nu a.t & \rightharpoonup_{\mathsf{p}} & t & (a \notin \mathsf{FN}(t)) & (\neg\nu) \\
\lambda x.\nu a.t & \rightharpoonup_{\mathsf{p}} & \nu a.\lambda x.t & & (\nu\lambda) \\
(\nu a.t)u & \rightharpoonup_{\mathsf{p}} & \nu a.(tu) & & (\nu\mathrm{f})
\end{array}$$

Figure 5. Permutative reductions.

The existence and unicity of normal forms for $\rightharpoonup_{\mathsf{p}}$ (that we call *permutative normal forms*, PNFs for short) naturally raises the question of what these normal forms represent.

Let $\mathcal{T}$ indicate the set of PNFs containing no free name occurrence. For any $t \in \mathcal{T}$, the PNF of t can be of two forms: either t starts with a generator, i.e. $t = \nu a.t'$, and t' is a tree of a-labeled choices $\oplus_a^i$ whose leaves form a finite set of $\mathcal{T}$ (the *support* of t', $\mathrm{supp}(t)$). Otherwise, t is of the form $\lambda x_1.\ldots.\lambda x_n.t' t_1 \ldots t_p$, where t' is either a variable or a λ. We call these last terms *pseudo-values*, and we let $\mathcal{V} \subseteq \mathcal{T}$ indicate the set formed by them.

Using this decomposition, any $t \in \mathcal{T}$ can be associated in a unique way with a *(sub-)distribution* of pseudo-values $\mathcal{D}_t : \mathcal{V} \longrightarrow [0,1]$ by letting $\mathcal{D}_t(v) = \delta_t$ when $t \in \mathcal{V}$, and

$$\mathcal{D}_t(v) = \sum_{u \in \mathrm{supp}(t')} \mathcal{D}_u(v) \cdot \mu(\{\omega \in 2^{\mathbb{N}} \mid \pi^{\omega}_{\{a\}}(t') = u\})$$

if $t = \nu a.t'$. Intuitively, $\mathcal{D}_t(v)$ measures the probability of finding v by iteratively applying to t random choices of events from the Cantor space any time a ν is found.

4.2 Probabilistic (Head) Normalization.

Given a term $t \in \mathcal{T}$, the questions "is t in normal form?" and "does t reduce to a normal form?" have univocal yes/no answers, because $\rightharpoonup$ is deterministic. However, if we think of t rather as $\mathcal{D}_t$, the relevant questions become "with what probability is t in normal form?" and "with what probability does t reduce to normal form?".

To answer this kind of questions we will introduce functions $\mathrm{HNV}_{\rightharpoonup}(t)$, $\mathrm{NF}_{\rightharpoonup}(t)$ measuring the probability that t reduces to a normal form.

Let us consider head-normal forms, first. A *head-reduction* $t \rightharpoonup_{\mathsf{h}} u$ is either a $\rightharpoonup_{\mathsf{p}}$-reduction or a $\rightharpoonup_{\beta}$-reduction of the form $\mathsf{R}[\lambda\vec{x}.(\lambda y.t)uu_1 \ldots u_n] \rightharpoonup_{\beta} \mathsf{R}[\lambda\vec{x}.t[u/x]u_1 \ldots u_n]$, where R is a *randomized context*, defined by the grammar

$$\mathsf{R}[\,] ::= [\,] \mid \mathsf{R}[\,] \oplus_a^i u \mid t \oplus_a^i \mathsf{R}[\,] \mid \nu a.\mathsf{R}[\,]$$

A *head normal value* (in short, HNV) is a $\rightharpoonup_{\mathsf{h}}$-normal term which is also a pseudo-value, i.e. it is of the form $\lambda\vec{x}.yu_1 \ldots u_n$. We let HNV indicate the set of such terms.

Definition 4.2. For any $t \in \mathcal{T}$, $\mathrm{HNV}(t) := \sum_{v \in \mathrm{HNV}} \mathcal{D}_t(v)$ and $\mathrm{HNV}_{\rightharpoonup}(t) := \sup\{\mathrm{HNV}(u) \mid t \rightharpoonup_{\mathsf{h}}^* u\}$. When $\mathrm{HNV}_{\rightharpoonup}(t) \geq q$, we say that t *yields a HNV with probability at least* q.

For example, if $t = \nu a.(\lambda x \lambda y.(y \oplus_a^i I)x)u$, where $u = I \oplus^j \Omega$, then $\mathrm{HNV}_{\rightharpoonup}(t) = \frac{3}{4}$. Indeed, we have

$$t \rightharpoonup_{\mathsf{h}}^* \nu a.(\lambda y.y(\nu b.I \oplus_b^j \Omega)) \oplus_a^i (\nu b'.I \oplus_{b'}^j \Omega)$$

and three over the four possible choices (corresponding to choosing between either left or right for both νa and $\nu b'$) yield a HNV. Observe that the choice about νb does not matter, since $\lambda y.yu$ is already a HNV.

Let us now consider normal forms. The first idea might be to define a similar function $\mathrm{NF}(t) = \sum_{v \text{ normal form}} \mathcal{D}_t(v)$. However, with this definition a term like $t = \lambda x.x(\nu a.I \oplus_a^0 \Omega)$ would have probability 0 of yielding a normal form. Instead, our guiding intuition here is that t should yield a normal form with probability $\frac{1}{2}$, i.e. depending on a choice for a. This leads to the following definition:

Definition 4.3. For any $t \in \mathcal{T}$, $\mathrm{NF}(t)$ is defined by:

- if $t = \lambda\vec{x}.yu_1 \dots u_n \in \mathrm{HNV}$, then $\mathrm{NF}(t) := \prod_{i=1}^{n} \mathrm{NF}(u_i)$;
- otherwise $\mathrm{NF}(t) := \sum_{u \in \mathrm{HNV}} \mathrm{NF}(u) \cdot \mathscr{D}_t(u)$.

We let $\mathrm{NF}_{\rightarrowtriangle}(t) = \sup\{\mathrm{NF}(u) \mid t \rightarrowtriangle^* u\}$ and, if $\mathrm{NF}_{\rightarrowtriangle}(t) \geq q$, we say that t *yields a normal form with probability at least* q.

For example, for the term t considered above, $\mathrm{NF}_{\rightarrowtriangle}(t) = \frac{4}{8} = \frac{1}{2}$: four over the eight possible choices for νa, νb and $\nu b'$ yield a normal form (i.e. either choose left for νa and νb and choose anything for $\nu b'$, or choose right for νa, left for $\nu b'$, and choose anything for νb).

4.3 Extending Λ_{PE} with CbV Functions

Λ_{PE} makes it possible to encode a CbV redex like $\nu a.2(I \oplus_a^0 \Omega)$, as we have seen. However, in view of the functional interpretation of iCPL_0, it would be convenient also to be able to represent the CbV *functions* mapping $\nu a.(u \oplus_a v)$ onto $\nu a.2(u \oplus_a^0 v)$. A simple way to do this is by enriching the language of Λ_{PE} with a "CbV application" operator $\{t\}u$, with suitable permutative rules. Let $\Lambda_{\mathsf{PE}}^{\{\}}$ indicate this extension of terms of Λ_{PE} with $\{\}$. While β-reduction for $\Lambda_{\mathsf{PE}}^{\{\}}$ is the same as for Λ_{PE}, permutative reduction $\rightarrowtriangle_{\mathsf{p}\{\}}$ is defined by all rules in Fig. 5 except for $(\neg\nu)$, together with the three permutations below

$$
\begin{aligned}
\{t\}\nu a.u &\rightarrowtriangle_{\mathsf{p}\{\}} \nu a.tu && (\{\}\nu)\\
\{t \oplus_a^i u\}v &\rightarrowtriangle_{\mathsf{p}\{\}} \{t\}v \oplus_a^i \{t\}v && (\{\}\oplus_1)\\
\{t\}(u \oplus_a^i v) &\rightarrowtriangle_{\mathsf{p}\{\}} \{t\}u \oplus_a^i \{t\}v && (\{\}\oplus_2)
\end{aligned}
$$

For instance, a CbV Church numeral can be encoded in $\Lambda_{\mathsf{PE}}^{\{\}}$ as $2^{\mathrm{CbV}} := \lambda f.\{2\}f$, since one has $2^{\mathrm{CbV}}(\nu a.u \oplus_a^i v) \rightarrowtriangle_\beta \{2\}\nu a.u \oplus_a^i v \rightarrowtriangle_{\mathsf{p}\{\}} \nu a.2(u \oplus_a^i v)$.

The fundamental properties of Λ_{PE} extend to $\Lambda_{\mathsf{PE}}^{\{\}}$:

Proposition 4.4. *$\rightarrowtriangle_{\mathsf{p}\{\}}$ is confluent and strongly normalizing. Full reduction $\rightarrowtriangle_{\{\}} := \rightarrowtriangle_\beta \cup \rightarrowtriangle_{\mathsf{p}\{\}}$ is confluent.*

In the Appendix it is shown how the definitions of $\mathscr{D}_t$ and $\mathrm{HNV}_{\rightarrowtriangle}(t)$ extend to terms of $\Lambda_{\mathsf{PE}}^{\{\}}$.

5 The Correspondence, Statically and Dynamically

In this section we present the core of the CHC. First, we introduce two type systems $\mathbf{C}\lambda_{\rightarrow}$ and $\mathbf{C}\lambda_{\rightarrow}^{\{\}}$ for Λ_{PE} and $\Lambda_{\mathsf{PE}}^{\{\}}$, respectively, which extend the simply typed λ-calculus with counting quantifiers. Then, we show that each proof Π in iCPL_0 can be associated with a typing derivation in $\mathbf{C}\lambda_{\rightarrow}^{\{\}}$ of some probabilistic term t^Π, in such a way that normalization of Π corresponds to reduction of t^Π. Observe that translating the rule (CE) requires the CbV application operator $\{\}$. In the Appendix we show that an alternative "CbN" proof-system for iCPL_0 can be translated without $\{\}$, and thus into $\mathbf{C}\lambda_{\rightarrow}$.

5.1 Two Type Systems with Counting Quantifiers

Both type systems $\mathbf{C}\lambda_{\rightarrow}$ and $\mathbf{C}\lambda_{\rightarrow}^{\{\}}$ extend the simply typed λ-calculus with counting quantifiers, but in a slightly different way: in $\mathbf{C}\lambda_{\rightarrow}$ types are of the form $\mathbf{C}^q\mathfrak{s}$, i.e. prefixed by *exactly one* counting quantifier, while in $\mathbf{C}\lambda_{\rightarrow}^{\{\}}$ types are of the form $\mathbf{C}^{\vec{q}}\mathfrak{s}$, i.e. prefixed by a (possibly empty) list of counting quantifiers.

More precisely, the types $\mathfrak{s}$ of $\mathbf{C}\lambda_{\rightarrow}$ and $\mathbf{C}\lambda_{\rightarrow}^{\{\}}$ are generated by the grammars below:

$$
\begin{aligned}
&\mathfrak{s} ::= \mathbf{C}^q\sigma && \sigma ::= o \mid \mathfrak{s} \Rightarrow \sigma && (\mathbf{C}\lambda_{\rightarrow})\\
&\mathfrak{s} ::= \mathbf{C}^{\vec{q}}\sigma && \sigma ::= o \mid \mathfrak{s} \Rightarrow \sigma && (\mathbf{C}\lambda_{\rightarrow}^{\{\}})
\end{aligned}
$$

where in both cases the qs are chosen in $(0, 1] \cap \mathbb{Q}$.

Judgements in both systems are of the form $\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathfrak{s}$, where Γ is a set of type declarations $x_i : \mathfrak{s}_i$ of pairwise distinct variables, t is a term of Λ_{PE} (resp. of $\Lambda_{\mathsf{PE}}^{\{\}}$), $\mathscr{b}$ a Boolean formula, and X a finite set of names with $\mathsf{FN}(t), \mathsf{FN}(\mathscr{b}) \subseteq X$.

The intuitive reading of $\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathfrak{s}$ is that, whenever $\omega \in (2^{\mathbb{N}})^X$ satisfies $\mathscr{b}$, then $\pi_X^\omega(t)$ is a program from Γ to $\mathfrak{s}$.

The *typing rules* of $\mathbf{C}\lambda_{\rightarrow}^{\{\}}$, which are essentially derived from those of iCPL_0, are illustrated in Fig. 6, where in the rule (μ) it is assumed that $\mathsf{FN}(\mathscr{b}) \subseteq X$, $\mathsf{FN}(\mathscr{d}) \subseteq \{a\}$ and $a \notin X$. The rule $(\vee)$ allows one to merge n typing derivations for the same term; in particular, with $n = 0$, one has that $\Gamma \vdash^X t : \bot \rightarrowtail \mathfrak{s}$ holds for *any* term t. The rule $(\oplus)$ is reminiscent of the rule (m) of iCPL_0. The rule (μ) is reminiscent of the rule (CI) of iCPL_0, while the rule $(\{\})$ for CbV application is reminiscent of the rule (CE).

The typing rules of $\mathbf{C}\lambda_{\rightarrow}$ coincide with those of $\mathbf{C}\lambda_{\rightarrow}^{\{\}}$ (with $\mathbf{C}^{\vec{q}}$ replaced everywhere by $\mathbf{C}^q$), except for the rule $(\{\})$, which is obviously absent, and for the rule (μ), which adapted as follows to the presence of exactly one counting quantifier:

$$
\frac{\Gamma \vdash^{X \cup \{a\}} t : \mathscr{b} \wedge \mathscr{d} \rightarrowtail \mathbf{C}^q\sigma \qquad \mu(\mathscr{d}) \geq s}{\Gamma \vdash^{X \cup \{a\}} t : \mathscr{b} \wedge \mathscr{d} \rightarrowtail \mathbf{C}^{qs}\sigma}\,(\mu')
$$

Both systems enjoy the subject reduction property:

Proposition 5.1 (Subject Reduction). *If $\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathfrak{s}$ in $\mathbf{C}\lambda_{\rightarrow}$ (resp. in $\mathbf{C}\lambda_{\rightarrow}^{\{\}}$) and $t \rightarrowtriangle u$ (resp. $t \rightarrowtriangle_{\{\}} u$), then $\Gamma \vdash^X u : \mathscr{b} \rightarrowtail \mathfrak{s}$.*

The choice of considering arrow types of the form $\mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \sigma)$ (hence of never having a counting quantifier *to the right* of $\Rightarrow$, as in e.g. $\mathfrak{s} \Rightarrow \mathbf{C}^q\sigma$) was made to let the rule (λ) be permutable over (μ), as required by the permuting rule $(\nu\lambda)$.

Example 5.2. In Fig. 7 we illustrate typing derivations in $\mathbf{C}\lambda_{\rightarrow}^{\{\}}$ for a version of the CbN Church numeral $2^{\mathrm{CbN}} = \lambda y.\lambda x.\{y\}(yx)$, with type $\mathbf{C}^{q*q}(\mathbf{C}^q(o \Rightarrow o) \Rightarrow (o \Rightarrow o))$, and for the CbV Church numeral 2^{CbV}, with type $\mathbf{C}^q(\mathbf{C}^q(o \Rightarrow o) \Rightarrow (o \Rightarrow o))$.

Both $\mathbf{C}\lambda_{\rightarrow}$ and $\mathbf{C}\lambda_{\rightarrow}^{\{\}}$ can type *non-normalizable* terms. For example, one can type all terms of the form $I \oplus^i \Omega$ with $\mathbf{C}^{\frac{1}{2}}(o \Rightarrow o)$ in $\mathbf{C}\lambda_{\rightarrow}^{\{\}}$ and with $\mathbf{C}^{\frac{1}{2}}(\mathbf{C}^1 o \Rightarrow o)$ in $\mathbf{C}\lambda_{\rightarrow}$.

Identity rule:

$$\frac{\mathrm{FN}(\mathscr{b}) \subseteq X}{\Gamma, x : \mathfrak{s} \vdash^{X} x : \mathscr{b} \rightarrowtail \mathfrak{s}}\ (\mathrm{id})$$

Structural rule:

$$\frac{\left\{\Gamma \vdash^{X} t : \mathscr{b}_i \rightarrowtail \mathfrak{s}\right\}_{i=1,\dots,n} \qquad \mathscr{b} \vDash^{X} \bigvee_i \mathscr{b}_i}{\Gamma \vdash^{X} t : \mathscr{b} \rightarrowtail \mathfrak{s}}\ (\vee)$$

Plus rule:

$$\frac{\Gamma \vdash^{X\cup\{a\}} t : \mathscr{c} \rightarrowtail \mathfrak{s} \qquad \Gamma \vdash^{X\cup\{a\}} u : \mathscr{d} \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash (\mathscr{x}^i_a \wedge \mathscr{c}) \vee (\neg \mathscr{x}^i_a \wedge \mathscr{d})}{\Gamma \vdash^{X\cup\{a\}} t \oplus^i_a u : \mathscr{b} \rightarrowtail \mathfrak{s}}\ (\oplus)$$

Arrow rules:

$$\frac{\Gamma, x : \mathfrak{s} \vdash^{X} t : \mathscr{b} \rightarrowtail \mathbf{C}^{\vec{q}}\tau}{\Gamma \vdash^{X} \lambda x.t : \mathscr{b} \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \tau)}\ (\lambda)$$

$$\frac{\Gamma \vdash^{X} t : \mathscr{c} \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \tau) \qquad \Gamma \vdash^{X} u : \mathscr{d} \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash^{X} \mathscr{c} \wedge \mathscr{d}}{\Gamma \vdash^{X} tu : \mathscr{b} \rightarrowtail \mathbf{C}^{\vec{q}}\tau}\ (@)$$

$$\frac{\Gamma \vdash^{X} t : \mathscr{c} \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \tau) \qquad \Gamma \vdash^{X} u : \mathscr{d} \rightarrowtail \mathbf{C}^{r}\mathfrak{s} \qquad \mathscr{b} \vDash^{X} \mathscr{c} \wedge \mathscr{d}}{\Gamma \vdash^{X} \{t\}u : \mathscr{b} \rightarrowtail \mathbf{C}^{rs}\mathbf{C}^{\vec{q}}\tau}\ (\{\})$$

Counting rule:

$$\frac{\Gamma \vdash^{X\cup\{a\}} t : \mathscr{b} \wedge \mathscr{d} \rightarrowtail \mathfrak{s} \qquad \mu(\mathscr{d}) \geq q}{\Gamma \vdash^{X} \nu a.t : \mathscr{b} \rightarrowtail \mathbf{C}^{q}\mathfrak{s}}\ (\mu)$$

Figure 6. Typing rules of $\mathbf{C}\lambda^{\{\}}_{\rightarrow}$.

Actually, the failure of normalization for typable programs can be ascribed to the rule $(\vee)$, as shown by the result below (where we let $\Gamma \vdash^{X}_{\neg\vee} t : \mathscr{b} \rightarrowtail \mathfrak{s}$ indicate that $\Gamma \vdash^{X} t : \mathscr{b} \rightarrowtail \mathfrak{s}$ is deduced without using the rule $(\vee)$).

Theorem 5.3 (Deterministic Normalization)**.** *In both* $\mathbf{C}\lambda_{\rightarrow}$ *and* $\mathbf{C}\lambda^{\{\}}_{\rightarrow}$, *if* $\Gamma \vdash_{\neg\vee} t : \mathscr{b} \rightarrowtail \mathfrak{s}$, *then* t *is strongly normalizable.*

As observed in the previous section, restricting to terms of Λ_{PE} having a normal form excludes the most interesting part of the calculus, which is made of terms for which normalization is inherently probabilistic. Similarly, restricting to type derivations without $(\vee)$ trivializes the most interesting features of $\mathbf{C}\lambda_{\rightarrow}$ and $\mathbf{C}\lambda^{\{\}}_{\rightarrow}$, that is, their ability to estimate probabilities of termination. We will explore the expressiveness of these systems in this sense in the next section.

5.2 Translating iCPL_0 into $\mathbf{C}\lambda^{\{\}}_{\rightarrow}$.

We now show how derivations in iCPL_0 translate into typing derivations in $\mathbf{C}\lambda^{\{\}}_{\rightarrow}$. In the Appendix a similar translation of a "CbN" proof-system for iCPL_0 into $\mathbf{C}\lambda_{\rightarrow}$ is shown.

Let us define, for any formula A of iCPL_0, a corresponding type $\mathfrak{s}_A$ by letting $\mathfrak{s}_p := o$, $\mathfrak{s}_{A\rightarrow \mathbf{C}^{\vec{q}}B} := \mathbf{C}^{\vec{q}}(\mathfrak{s}_A \Rightarrow \mathfrak{s}_B)$ and $\mathfrak{s}_{\mathbf{C}^{q}A} := \mathbf{C}^{q}\mathfrak{s}_A$. In Fig. 8 it is shown how a derivation Π of $\Gamma \vdash \mathscr{b} \rightarrowtail A$ in iCPL translates into a typing derivation D^{Π} of $\mathfrak{s}_\Gamma \vdash t^{\Pi} : \mathscr{b} \rightarrowtail \mathfrak{s}_A$ in $\mathbf{C}\lambda^{\{\}}_{\rightarrow}$, with $\mathsf{FN}(t^{\Pi}) \subseteq \mathsf{FN}(\mathscr{b})$, by induction on Π. Notice that we exploit a special constant $\mathbf{c}$ to translate the rule $(\bot)$.

$$\frac{\Gamma \vdash^{X} t : \mathscr{c} \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash \mathscr{c}}{\Gamma \vdash^{X} t : \mathscr{b} \rightarrowtail \mathfrak{s}}\ (\vDash)$$

Moreover, the rule $(\rightarrow\mathrm{E})$ translates as CbN application tu, while the rule (CE) translates as CbV application $\{t\}u$.

As required by the CHC, normalization steps of iCPL_0 are simulated by $\rightarrow_{\{\}}$-reductions:

Proposition 5.4 (Stability Under Normalization)**.** *If* $\Pi \rightsquigarrow \Pi'$, *then* $t^{\Pi} \rightarrow^{*}_{\{\}} t^{\Pi'}$.

Proof sketch. The normalization steps in Fig. 4 translate into the following CbN and CbV reductions:

$$(\lambda x^{\mathbf{C}^{q}\mathfrak{s}_A}.t^{\Pi})\nu a.t^{\Sigma} \rightarrow_{\beta} t^{\Pi}[\nu a.t^{\Sigma}/x]$$

$$\{\lambda x^{\mathfrak{s}_A}.t^{\Pi}\}\nu a.t^{\Sigma} \rightarrow_{\mathsf{p}\{\}} \nu a.(\lambda x^{\mathfrak{s}_A}.t^{\Pi})t^{\Sigma} \rightarrow_{\beta} \nu a.t^{\Pi}[t^{\Sigma}/x]$$

All other normalization steps translate into $\rightarrow_{\mathsf{p}\{\}}$-reductions. ◻

By observing that the only use of $(\vee)$ coming from the translation introduces a constant $\mathbf{c}$, from Theorem 5.3 we deduce, as promised, the following:

Corollary 5.5. *Normalization in* iCPL_0 *is strongly normalizing.*

6 From Type Soundness to Type Completeness: Intersection Types

In this section we first show that derivations in $\mathbf{C}\lambda_{\rightarrow}$ and $\mathbf{C}\lambda^{\{\}}_{\rightarrow}$ provide sound approximations of $\mathrm{HNV}(t)$ and $\mathrm{NF}(t)$. In order to achieve completeness, we then introduce an extension $\mathbf{C}\lambda_{\rightarrow,\cap}$ of $\mathbf{C}\lambda_{\rightarrow}$ with intersection types and we show that this system fully captures both deterministic and probabilistic notions of termination for Λ_{PE}.

6.1 From Types to Probability

We already observed, through examples, that if a term t has a type like, e.g. $\mathbf{C}^{\frac{1}{2}}(o \Rightarrow o)$, then t has one chance over two of yielding a "correct" program for $o \Rightarrow o$. The result below makes this intuition precise, by showing that the probabilities derived in $\mathbf{C}\lambda_{\rightarrow}$ and $\mathbf{C}\lambda^{\{\}}_{\rightarrow}$ are lower bounds for the function $\mathrm{HNV}_{\rightarrow}(t)$, that is, for the actual probability of finding a head normalizable term in the distribution $\mathscr{D}_t$.

Theorem 6.1. *If* $\vdash_{\mathbf{C}\lambda_{\rightarrow}} t : \top \rightarrowtail \mathbf{C}^{q}\sigma$, *then* $\mathrm{HNV}_{\rightarrow}(t) \geq q$. *If* $\vdash_{\mathbf{C}\lambda^{\{\}}_{\rightarrow}} t : \top \rightarrowtail \mathbf{C}^{q_1 * \dots * q_n}\sigma$, *then* $\mathrm{HNV}_{\rightarrow}(t) \geq \prod_{i=1}^{n} q_i$.

What about reduction to normal form, i.e. the function $\mathrm{NF}_{\rightarrow}(t)$? A result like Theorem 6.1 cannot hold in this case. Indeed, consider the term $t = \lambda y.y(I \oplus^{i} \Omega)$. While $\mathrm{NF}(t) = \frac{1}{2}$, $\mathbf{C}\lambda^{\{\}}_{\rightarrow}$ types t with $\mathfrak{s} = \mathbf{C}^{1}(\mathbf{C}^{1}(\mathbf{C}^{\frac{1}{2}}\sigma \Rightarrow \sigma) \Rightarrow \sigma)$, with $\sigma = o \Rightarrow o$. The problem in this example is that the type $\mathfrak{s}$ contains the "unbalanced" assumption $\mathbf{C}^{1}(\mathbf{C}^{\frac{1}{2}}\sigma \Rightarrow \sigma)$ (in logical terms, $\mathbf{C}^{\frac{1}{2}}A \rightarrow \mathbf{C}^{1}A$), i.e. it exploits the assumption of the existence of a function turning a $\frac{1}{2}$-correct input into a 1-correct output. Notice that such a function f can only be one that erases its input, and these are the only functions such that tf can reduce to a normal form.

$$\dfrac{y : \mathsf{C}^q(o \Rightarrow o), x : \mathsf{C}^q o \vdash y : \top \rightarrowtail \mathsf{C}^q(o \Rightarrow o) \qquad \dfrac{y : \mathsf{C}^q(o \Rightarrow o), x : \mathsf{C}^q o \vdash y : \top \rightarrowtail \mathsf{C}^q(o \Rightarrow o) \qquad y : \mathsf{C}^q(o \Rightarrow o), x : \mathsf{C}^q o \vdash x : \top \rightarrowtail o}{y : \mathsf{C}^q(o \Rightarrow o), x : \mathsf{C}^q o \vdash yx : \top \rightarrowtail \mathsf{C}^q o}\,(@)}{\dfrac{y : \mathsf{C}^q(o \Rightarrow o), x : \mathsf{C}^q o \vdash \{y\}(yx) : \top \rightarrowtail \mathsf{C}^{q*q} o}{\vdash 2^{\mathrm{CbN}} : \top \rightarrowtail \mathsf{C}^{q*q}(\mathsf{C}^q(o \Rightarrow o) \Rightarrow (o \Rightarrow o))}\,(\lambda)}\,(\{\})$$

$$\dfrac{\dfrac{\vdots}{y : \mathsf{C}^q(o \Rightarrow o) \vdash 2 : \top \rightarrowtail (o \Rightarrow o) \Rightarrow (o \Rightarrow o)} \qquad y : \mathsf{C}^q(o \Rightarrow o) \vdash y : \top \rightarrowtail \mathsf{C}^q(o \to o)}{\dfrac{y : \mathsf{C}^q(o \Rightarrow o) \vdash \{2\}y : \top \rightarrowtail \mathsf{C}^q(o \Rightarrow o)}{\vdash 2^{\mathrm{CbV}} : \top \rightarrowtail \mathsf{C}^q(\mathsf{C}^q(o \Rightarrow o) \Rightarrow (o \Rightarrow o))}\,(\lambda)}\,(\{\})$$

Figure 7. Typings of the CbN and CbV Church numerals in $\mathsf{C}\lambda_{\to}^{\{\}}$.

$$\dfrac{}{\Gamma, A \vdash \mathscr{b} \rightarrowtail A}\,(\mathrm{id}) \quad\mapsto\quad \dfrac{}{\mathfrak{s}_\Gamma, x : \mathfrak{s}_A \vdash x : \mathscr{b} \rightarrowtail \mathfrak{s}_A}\,(\mathrm{id})$$

$$\dfrac{\mathscr{b} \vDash \bot}{\Gamma \vdash \mathscr{b} \rightarrowtail A}\,(\bot) \quad\mapsto\quad \dfrac{\mathscr{b} \vDash \bot}{\mathfrak{s}_\Gamma \vdash \mathbf{c} : \mathscr{b} \rightarrowtail \mathfrak{s}_A}\,(\vee)$$

$$\dfrac{\overset{\Pi}{\Gamma \vdash \mathscr{c} \rightarrowtail A} \qquad \overset{\Pi'}{\Gamma \vdash \mathscr{d} \rightarrowtail A} \qquad \mathscr{b} \vDash (\mathscr{c} \wedge \mathscr{x}_a^i) \vee (\mathscr{d} \wedge \neg \mathscr{x}_a^i)}{\Gamma \vdash \mathscr{b} \rightarrowtail A}\,(\mathrm{m}) \quad\mapsto\quad \dfrac{\overset{D^{\Pi}}{\mathfrak{s}_\Gamma \vdash t^{\Pi} : \mathscr{c} \rightarrowtail A} \qquad \overset{D^{\Pi'}}{\mathfrak{s}_\Gamma \vdash t^{\Pi'} : \mathscr{d} \rightarrowtail A} \qquad \mathscr{b} \vDash (\mathscr{c} \wedge \mathscr{x}_a^i) \vee (\mathscr{d} \wedge \neg \mathscr{x}_a^i)}{\mathfrak{s}_\Gamma \vdash t^{\Pi} \oplus_a^i t^{\Pi'} : \mathscr{b} \rightarrowtail \mathfrak{s}_A}\,(\oplus)$$

$$\dfrac{\overset{\Pi}{\Gamma, A \vdash \mathscr{c} \rightarrowtail B}}{\Gamma \vdash \mathscr{b} \rightarrowtail (A \to B)}\,(\to\mathrm{I}) \quad\mapsto\quad \dfrac{\overset{D^{\Pi}}{\mathfrak{s}_\Gamma, x : \mathfrak{s}_A \vdash t^{\Pi} : \mathscr{c} \rightarrowtail \mathfrak{s}_B}}{\mathfrak{s}_\Gamma \vdash \lambda x.t^{\Pi} : \mathscr{b} \rightarrowtail (\mathfrak{s}_{A \to B})}\,(\lambda)$$

$$\dfrac{\overset{\Pi}{\Gamma \vdash \mathscr{c} \rightarrowtail (A \to \mathsf{C}^{\vec{q}}B)} \qquad \overset{\Sigma}{\Gamma \vdash \mathscr{b} \rightarrowtail A}}{\Gamma \vdash \mathscr{b} \rightarrowtail \mathsf{C}^{\vec{q}}B}\,(\to\mathrm{E}) \quad\mapsto\quad \dfrac{\overset{D^{\Pi}}{\mathfrak{s}_\Gamma \vdash t^{\Pi} : \mathscr{b} \rightarrowtail \mathsf{C}^{\vec{q}}(\mathfrak{s}_A \to \mathfrak{s}_B)} \qquad \overset{D^{\Sigma}}{\mathfrak{s}_\Gamma \vdash t^{\Sigma} : \mathscr{b} \rightarrowtail \mathfrak{s}_A}}{\mathfrak{s}_\Gamma \vdash t^{\Pi} t^{\Sigma} : \mathscr{b} \rightarrowtail \mathsf{C}^{\vec{q}}\mathfrak{s}_B}\,(@)$$

$$\dfrac{\overset{\Pi}{\Gamma \vdash \mathscr{b} \wedge \mathscr{d} \rightarrowtail A} \qquad \mu(\mathscr{d}) \geq q}{\Gamma \vdash \mathscr{b} \rightarrowtail \mathsf{C}^q A}\,(\mathrm{CI}) \quad\mapsto\quad \dfrac{\overset{D^{\Pi}}{\mathfrak{s}_\Gamma \vdash t^{\Pi} : \mathscr{b} \wedge \mathscr{d} \rightarrowtail \mathfrak{s}_A} \qquad \mu(\mathscr{d}) \geq q}{\mathfrak{s}_\Gamma \vdash \nu a.t^{\Pi} : \mathscr{b} \rightarrowtail \mathfrak{s}_{\mathsf{C}^q A}}\,(\mu)$$

$$\dfrac{\overset{\Pi}{\Gamma \vdash \mathscr{b} \rightarrowtail \mathsf{C}^q A} \qquad \overset{\Sigma}{\Gamma, A \vdash \mathscr{b} \rightarrowtail \mathsf{C}^{\vec{s}} B}}{\Gamma \vdash \mathscr{b} \rightarrowtail \mathsf{C}^{qs}\mathsf{C}^{\vec{s}} B}\,(\mathrm{CE}) \quad\mapsto\quad \dfrac{\overset{D^{\Pi}}{\mathfrak{s}_\Gamma \vdash t^{\Pi} : \mathscr{b} \rightarrowtail \mathsf{C}^q \mathfrak{s}_A} \qquad \dfrac{\overset{D^{\Sigma}}{\mathfrak{s}_\Gamma, x : \mathfrak{s}_A \vdash t^{\Sigma} : \mathscr{b} \rightarrowtail \mathsf{C}^{\vec{s}} \mathfrak{s}_B}}{\mathfrak{s}_\Gamma \vdash \lambda x.t^{\Sigma} : \mathscr{b} \rightarrowtail \mathsf{C}^{\vec{s}}(\mathfrak{s}_A \Rightarrow \mathfrak{s}_B)}\,(\lambda)}{\mathfrak{s}_\Gamma \vdash \{\lambda x.t^{\Sigma}\} t^{\Pi} : \mathscr{b} \rightarrowtail \mathfrak{s}_{\mathsf{C}^{qs}\mathsf{C}^{\vec{s}} B}}\,(\{\})$$

Figure 8. Translation $\Pi \mapsto D^{\Pi}$ from iCPL to $\mathsf{C}\lambda_{\to}^{\{\}}$.

Nevertheless, soundness with respect to $\mathrm{NF}(t)$ can be proved, for $\mathsf{C}\lambda_{\to}$, by restricting to those types not containing such "unbalanced" assumptions, i.e. to types whose programs cannot increase probabilities.

Definition 6.2. A type $\mathsf{C}^q(\mathfrak{s}_1 \Rightarrow \ldots \Rightarrow \mathfrak{s}_n \Rightarrow o)$ of $\mathsf{C}\lambda_{\to}$ is *balanced* if all $\mathfrak{s}_i$ are balanced and $q \leq \prod_{i=1}^n \lceil \mathfrak{s}_i \rceil$ (where for any type $\mathfrak{s} = \mathsf{C}^q\sigma$, $\lceil \mathfrak{s} \rceil = q$).

Theorem 6.3. *If* $\vdash t : \top \rightarrowtail \mathsf{C}^q\sigma$ *holds in* $\mathsf{C}\lambda_{\to}$*, where* σ *is balanced, then* $\mathrm{NF}_{\to}(t) \geq q$.

Theorems 6.1 and 6.3 are proved by adapting the standard technique of *reducibility predicates* to the quantitative notion of probabilistic normal form.

6.2 From Probability to (Intersection) Types

To achieve a type-theoretic characterization of $\mathrm{HNV}_{\to}(t)$ and $\mathrm{NF}_{\to}(t)$, we introduce an extension of $\mathsf{C}\lambda_{\to}$ with intersection types. Like those of $\mathsf{C}\lambda_{\to}$, the types of $\mathsf{C}\lambda_{\to,\cap}$ are of the form $\mathfrak{s} = \mathsf{C}^q\sigma$, but σ is now defined as:

$$\sigma ::= o \mid \mathsf{n} \mid \mathsf{hn} \mid \mathfrak{M} \Rightarrow \sigma \qquad \mathfrak{M} ::= [\mathfrak{s}, \ldots, \mathfrak{s}]$$

where $[a_1, \ldots, a_n]$ indicates a finite set. While $\mathfrak{M}$ intuitively stands for a finite intersection of types, the new ground types n and hn intuitively correspond to the types of normalizable and head-normalizable programs.

We introduce a preorder $\sigma \preceq \tau$ over types by $\alpha \preceq \alpha$, for $\alpha = o, \mathsf{n}, \mathsf{hn}$, $\mathsf{C}^q\sigma \preceq \mathsf{C}^s\tau$ if $q \leq s$ and $\sigma \preceq \tau$, and $(\mathfrak{M} \Rightarrow \sigma) \preceq (\mathfrak{N} \Rightarrow \tau)$ if $\sigma \preceq \tau$ and $\mathfrak{N} \preceq^* \mathfrak{M}$, where $[\mathfrak{s}_1, \ldots, \mathfrak{s}_n] \preceq^* [\mathfrak{t}_1, \ldots, \mathfrak{t}_m]$ holds is there exists an injective function $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that $\mathfrak{s}_{f(i)} \preceq \mathfrak{t}_i$.

A type judgment of $\mathsf{C}\lambda_{\to,\cap}$ is of the form $\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathfrak{s}$, where Γ is made of declarations of the form $x_i : \mathfrak{M}_i$. The typing rules of $\mathsf{C}\lambda_{\to,\cap}$ are illustrated in Fig. 9. We omit the $(\vee)$- and $(\oplus)$-rules, which are as in $\mathsf{C}\lambda_{\to}$. In the rule (μ_Σ) it is

Identity rule:

$$\frac{\mathfrak{s}_i \preceq \mathfrak{t} \qquad \mathrm{FN}(\mathscr{b}) \subseteq X}{\Gamma, x : [\mathfrak{s}_1, \dots, \mathfrak{s}_n] \vdash^X x : \mathscr{b} \rightarrowtail \mathfrak{t}}\ (\mathrm{id}_{\preceq})$$

Ground types rules:

$$\frac{\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathsf{C}^q\sigma}{\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathsf{C}^q\mathsf{hn}}\ (\mathsf{hn}) \qquad \frac{\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathsf{C}^q\sigma \qquad \sigma \text{ safe}}{\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathsf{C}^q\mathsf{n}}\ (\mathsf{n})$$

Arrow rules:

$$\frac{\Gamma, x : \mathfrak{M} \vdash^X t : \mathscr{b} \rightarrowtail \mathsf{C}^q\tau}{\Gamma \vdash^X \lambda x.t : \mathscr{b} \rightarrowtail \mathsf{C}^q(\mathfrak{M} \Rightarrow \tau)}\ (\lambda)$$

$$\frac{\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathsf{C}^q([\mathfrak{s}_1, \dots, \mathfrak{s}_n] \Rightarrow \tau) \qquad \Big\{\Gamma \vdash^X u : \mathscr{b} \rightarrowtail \mathfrak{s}_i\Big\}_{i=1,\dots,n}}{\Gamma \vdash^X tu : \mathscr{b} \rightarrowtail \mathsf{C}^q\tau}\ (@_{\cap})$$

Counting rule:

$$\frac{\Big\{\Gamma \vdash^{X \cup \{a\}} t : \mathscr{b} \wedge \mathscr{d}_i \rightarrowtail \mathsf{C}^{q_i}\sigma\Big\}_{i=1,\dots,n+1} \qquad \mu(\mathscr{d}_i) \geq s_i}{\Gamma \vdash^X \nu a.t : \mathscr{b} \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}\sigma}\ (\mu_\Sigma)$$

Figure 9. Typing rules of $\mathsf{C}\lambda_{\rightarrow,\cap}$.

assumed that a does not occur in $\mathscr{b}$ and is the only name in the $\mathscr{d}_i$, and that for $i \neq j$, $\mathscr{d}_i \wedge \mathscr{d}_j \vDash \bot$.

The two rules (hn) and (n) are justified by Proposition 6.5 and Theorem 6.6 below. As rule (n) must warrant a bound on normal forms, following Theorem 6.3, σ has to be *safe*, i.e. balanced[4] and $\{[], \mathsf{hn}\}$-free. The rule $(@_\cap)$ is a standard extension of rule (@) of $\mathsf{C}\lambda_{\rightarrow}$ to finite intersections.

The counting rule (μ_Σ) requires some discussion. The rule admits $n+1$ major premisses expressing typings for t which depend on pairwise disjoint events (the Boolean formulas $\mathscr{d}_i$). This is needed to cope with situations as follows: let

$$t[a,b] = \Big((I \oplus_b^1 \Omega) \oplus_b^0 \Omega\Big) \oplus_a^0 \Big(\Omega \oplus_b^0 I\Big)$$

$t[a,b]$ can be given type $\sigma = (\mathsf{C}^1 o) \Rightarrow o$ under either of the two *disjoint* Boolean constraints $\mathscr{d}_1 := \mathscr{x}_a^0 \wedge (\mathscr{x}_b^0 \wedge \mathscr{x}_b^1)$ and $\mathscr{d}_2 := \neg\mathscr{x}_a^0 \wedge \neg\mathscr{x}_b^0$. In $\mathsf{C}\lambda_{\rightarrow}$, the best we can achieve to measure the probability that $\nu a.\nu b.t[a,b]$ has type σ is $\frac{1}{4}$. Indeed, the rule (μ') forces us to approximate $\mu(\mathscr{x}_b^0 \wedge \mathscr{x}_b^1)$ and $\mu(\neg\mathscr{x}_b^1)$ to a *common* lower bound, i.e. $\frac{1}{4}$, in order to apply a $(\vee)$-rule as illustrated in Fig. 10. Instead, using (μ_Σ) we can compute (again, see Fig. 10), the *actual* probability $\mu(\mathscr{x}_a^0)\mu(\mathscr{x}_b^0 \wedge \mathscr{x}_b^1) + \mu(\neg\mathscr{x}_a^0)\mu(\neg\mathscr{x}_b^0) = \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{8}$.

Thanks to the rule (μ_Σ), the generalized counting rule (μ^*) below becomes admissible in $\mathsf{C}\lambda_{\rightarrow,\cap}$:

$$\frac{\Gamma \vdash^{\{a_1,\dots,a_n\}} t : \mathscr{b} \rightarrowtail \mathsf{C}^q\sigma}{\Gamma \vdash^{\emptyset} \nu a_1.\dots.\nu a_n.t : \top \rightarrowtail \mathsf{C}^{q\cdot\mu(\mathscr{b})}\sigma}\ (\mu^*)$$

This rule plays an essential role in the completeness results below, together with the standard result that both *subject reduction* and *subject expansion* hold for intersection types.

[4] Def. 6.2 extends to the types of $\mathsf{C}\lambda_{\rightarrow,\cap}$ by letting $\lceil\mathsf{hn}\rceil = \lceil[]\rceil = 0$, $\lceil\mathsf{n}\rceil = 1$ and $\lceil[\mathfrak{s}_1, \dots, \mathfrak{s}_{n+1}]\rceil = \max\{\lceil\mathfrak{s}_i\rceil\}$.

Proposition 6.4 (Subject Reduction/Expansion)**.** *If $\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathfrak{s}$ and either $t \twoheadrightarrow u$ or $u \twoheadrightarrow t$, then $\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathfrak{s}$.*

We will now discuss how typings in $\mathsf{C}\lambda_{\rightarrow,\cap}$ capture both deterministic and probabilistic properties of terms. First we have the following facts, which show that the types hn and n capture deterministic termination.

Proposition 6.5 (Deterministic Completeness)**.** *For any closed term t,*
(i.) t is head-normalizable iff $\vdash_{\neg\vee} t : \top \rightarrowtail \mathsf{C}^1\mathsf{hn}$;
(ii.) t is normalizable iff $\vdash_{\neg\vee} t : \top \rightarrowtail \mathsf{C}^1\mathsf{n}$.
(iii.) t is strongly normalizable iff $\vdash_{\neg\vee} t : \top \rightarrowtail \mathsf{C}^1\mathsf{n}$ and all types in the derivation are safe.

Proof sketch. Using standard intersection types arguments, it is shown that $\vdash_{\neg\vee} t : \top \rightarrowtail \mathsf{C}^1\mathsf{hn}$ holds for any head-normal t. The first half of (i.) is deduced then using Prop. 6.4. The second half follows from a normalization argument similar to that of Theorem 5.3. Cases (ii.) and (iii.) are similar. ◻

The probabilistic normalization theorems 6.1 and 6.3 (which extend smoothly to $\mathsf{C}\lambda_{\rightarrow,\cap}$) ensure that if t has type $\mathsf{C}^q\mathsf{hn}$ (resp. $\mathsf{C}^q\mathsf{n}$), then $\mathrm{HNV}_{\twoheadrightarrow}(t) \geq q$ (resp. $\mathrm{NF}_{\twoheadrightarrow}(t) \geq q$). Conversely, $\mathrm{HNV}_{\twoheadrightarrow}(t)$ and $\mathrm{NF}_{\twoheadrightarrow}(t)$ can be bounded by means of derivations in $\mathsf{C}\lambda_{\rightarrow,\cap}$, in the following sense:

Theorem 6.6 (Probabilistic Completeness)**.** *For any closed term t,*

$$\mathrm{HNV}_{\twoheadrightarrow}(t) = \sup\{q \mid \vdash t : \top \rightarrowtail \mathsf{C}^q\mathsf{hn}\}$$
$$\mathrm{NF}_{\twoheadrightarrow}(t) = \sup\{q \mid \vdash t : \top \rightarrowtail \mathsf{C}^q\mathsf{n}\}$$

Proof sketch. Suppose w.l.o.g. that $t = \nu a_1.\dots.\nu a_k.t'$. For any $u \in \mathrm{HNV}$ such that $\mathscr{D}_t(u) > 0$, we can deduce $\vdash u : \top \rightarrowtail \mathsf{hn}$. The sequence of probabilistic choices leading to u is finite, and thus captured by a Boolean formula $\mathscr{b}_{t\mapsto u}$. Using subject reduction/expansion we thus deduce $\vdash t' : \mathscr{b}_{t\mapsto u} \rightarrowtail \mathsf{hn}$. Hence, for any finite number of head normal forms $u_1, \dots, u_n$, we deduce $\vdash t' : \mathscr{b}_{t\mapsto u_i} \rightarrowtail \mathsf{hn}$. Using $(\vee)$ and the generalized counting rule (μ^*) we deduce then $\vdash t : \top \rightarrowtail \mathsf{C}^s\mathsf{hn}$, where $s = \sum_{i=1}^n \mu(\mathscr{b}_{t\mapsto u_i}) = \mu(\bigvee_{i=1}^n \mathscr{b}_{t\mapsto u_i})$. The argument for $\mathrm{NF}(t)$ is similar. ◻

7 Related Work

As discussed in the introduction, our results provide the first clear correspondence between a proof system and a probabilistic extension of the λ-calculus. This is not to say that our logic and calculi come from nowhere.

Different kinds of measure-theoretic quantifiers have been investigated in the literature, with the intuitive meaning of "A is true *for almost all x*" (see [42, 52] and more recently [40, 41]), or "A is true for the *majority of x*" [45, 59, 60]. Our use of the term "counting quantifier" comes from [2], where an extension of classical propositional logic with such quantifiers is studied and related to Wagner's counting operators on classes of languages [54–56]. To our knowledge, our

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash^{\{a,b\}} t[a,b] : \boldsymbol{x}^0_a \wedge (\boldsymbol{x}^0_b \wedge \boldsymbol{x}^1_b) \rightarrowtail \mathbf{C}^1\sigma \qquad \mu(\boldsymbol{x}^0_b \wedge \boldsymbol{x}^1_b) \geq 1/4}{\Gamma \vdash^{\{a\}} \nu b.t[a,b] : \boldsymbol{x}^0_a \rightarrowtail \mathbf{C}^{\frac{1}{4}}\sigma}\,(\mu') \qquad \dfrac{\Gamma \vdash^{\{a,b\}} t[a,b] : \neg\boldsymbol{x}^0_a \wedge \neg\boldsymbol{x}^0_b \rightarrowtail \mathbf{C}^1\sigma \qquad \mu(\neg\boldsymbol{x}^0_b) \geq 1/4}{\Gamma \vdash^{\{a\}} \nu b.t[a,b] : \neg\boldsymbol{x}^0_a \rightarrowtail \mathbf{C}^{\frac{1}{4}}\sigma}\,(\mu')}{\Gamma \vdash^{\{a\}} \nu b.t[a,b] : \boldsymbol{x}^0_a \vee \neg\boldsymbol{x}^0_a \rightarrowtail \mathbf{C}^{\frac{1}{4}}\sigma}\,(\vee) \qquad \mu(\boldsymbol{x}^0_a \vee \neg\boldsymbol{x}^0_a) \geq 1}{\Gamma \vdash^{\emptyset} \nu a.\nu b.t[a,b] : \top \rightarrowtail \mathbf{C}^{\frac{1}{4}}\sigma}\,(\mu')$$

$$\dfrac{\dfrac{\Gamma \vdash^{\{a,b\}} t[a,b] : \boldsymbol{x}^0_a \wedge (\boldsymbol{x}^0_b \wedge \boldsymbol{x}^1_b) \rightarrowtail \mathbf{C}^1\sigma \qquad \mu(\boldsymbol{x}^0_b \wedge \boldsymbol{x}^1_b) \geq 1/4}{\Gamma \vdash^{\{a\}} \nu b.t[a,b] : \boldsymbol{x}^0_a \rightarrowtail \mathbf{C}^{\frac{1}{4}}\sigma}\,(\mu_\Sigma) \qquad \dfrac{\Gamma \vdash^{\{a,b\}} t[a,b] : \neg\boldsymbol{x}^0_a \wedge \neg\boldsymbol{x}^0_b \rightarrowtail \mathbf{C}^1\sigma \qquad \mu(\neg\boldsymbol{x}^0_b) \geq 1/2}{\Gamma \vdash^{\{a\}} \nu b.t[a,b] : \neg\boldsymbol{x}^0_a \rightarrowtail \mathbf{C}^{\frac{1}{2}}\sigma}\,(\mu_\Sigma) \qquad \mu(\boldsymbol{x}^0_a), \mu(\neg\boldsymbol{x}^0_a) \geq 1/2}{\Gamma \vdash^{\emptyset} \nu a.\nu b.t[a,b] : \top \rightarrowtail \mathbf{C}^{\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}}\sigma}\,(\mu_\Sigma)$$

Figure 10. Comparing probabilities derived with the rules (μ') and (μ_Σ).

work is the first to apply some form of measure quantifier to typed probabilistic functional programs.

Despite the extensive literature on logical systems enabling (in various ways and for different purposes) some forms of probabilistic reasoning, there is not much about logics tied to computational aspects, as iCPL is. Most of the recent logical formalisms have been developed in the realm of modal logic, like e.g. [5–7, 23, 30, 31, 43, 44]. Another class of probabilistic modal logics have been designed to model Markov chains and similar structures [32, 37, 38]. With the sole exception of *Riesz modal logic* [24], we are not aware of proof-systems for probability logic.

Intuitionistic modal logic has been related in the Curry-Howard sense to monadic extensions of the λ-calculus [1, 8, 13, 19, 61]. However, in these correspondences modal operators are related to *qualitative* properties of programs (typically, tracing algebraic effects), as opposed to the *quantitative* properties expressed by counting quantifiers. Our Kripke structures for iCPL can be related to standard Kripke structures for intuitionistic modal logic [47, 50]. These are based on a set W with *two* pre-order relations $\leq$ and R enjoying a suitable "diamond" property $R; \leq \subseteq \leq; R$. We obtain a similar structure by taking worlds to be pairs w, ω made of a world and an outcome from the Cantor space, with $(w, \omega) \leq (w', \omega)$ whenever $w \leq w'$, and $(w, \omega)R(w, \omega + \omega')$. The clause for $\mathbf{C}^q A$ can then be seen as a quantitative variant of the corresponding clause for $\Diamond A$. This is not very surprising, given the similarity between the introduction and elimination rules for $\mathbf{C}^q$ and those for $\Diamond$ (see e.g. [1, 8]).

On the other hand, quantitative semantics arising from linear logic have been largely used in the study of probabilistic λ-calculi (e.g. [15, 20, 22]). Notably, *probabilistic coherence spaces* [20, 21, 28] have been shown to provide a fully abstract model of probabilistic PCF. While we are not aware of correspondences relating probabilistic programs with proofs in linear logic, it seems that the proof-theory of counting quantifiers could be somehow related to that of *bounded exponentials* [18, 29] and, more generally, to the theory of *graded* monads and comonads [11, 25, 34, 35].

The calculus Λ_{PE} derives from [17], which also introduces a simple type system ensuring strong normalization, although the typings do not provide any quantitative information. Beyond this, several type systems for probabilistic λ-calculi have been introduced in the recent literature. Among these, systems based on *type distributions* [16], i.e. where a single derivation assigns several types to a term, each with some probability, and systems based on *oracle intersection types* [10], where type derivations capture single evaluations as determined by an oracle. Our type systems sit in between these two approaches: like the first (and unlike the second), typing derivations can capture a finite number of different evaluations, although without using distributions of types; like the second, typings reflect the dependency of evaluation on oracles, although the latter are manipulated in a collective way by means of Boolean constraints.

Finally, in [58] dependent type theory is enriched with a probabilistic choice operator, yielding a calculus with both term and type distributions. Interestingly, a fragment of this system enjoys a sort of CHC with so-called *Markov Logic Networks* [49], a class of probabilistic graphical models specified by means of first-order logic formulas.

8 Conclusions

The main contribution of this work consists in defining a Curry-Howard correspondence between a logic with counting quantifiers and a type system that expresses probability of termination. Moreover, in analogy with what happens in the deterministic case, extending the type system with an intersection operator leads to a full characterization of probability of termination. Even though intersection types do not have a clear logical counterpart, the existence of this extension convinces us that the correspondence introduced is meaningful. The possibility of defining a Curry-Howard correspondence relating algebraic effects, on the program side, with a modal operator, on the logic side, is certainly not surprising. Instead, it seems to us that the new and surprising contribution of this work is showing that the peculiar features of probabilistic effects can be managed in an elegant way using ideas coming from logic.

Among the many avenues of research that this work opens, the study of the problem of type inference must certainly be mentioned, as well as the extension of the correspondence to polymorphic types or to control operators. Particularly intriguing, then, is the possibility of studying the system of intersection types introduced to support the synthesis, always in analogy with what is already known in deterministic calculations.

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Appendices

A Details about Intuitionistic Counting Propositional Logic

In this section we introduce a proof-system $\mathbf{ND}_{\mathsf{iCPL}}$ for iCPL and describe the correspondence between proofs in iCPL and families of proofs iCPL_0; then, we establish the soundness and completeness of $\mathbf{ND}_{\mathsf{iCPL}}$ with respect to iCPL-structures. Finally, we provide some more details about normalization in the "computational fragment" iCPL_0, and we describe an alternative "CbN" proof-system for iCPL_0, whose derivations can be decorated with terms of Λ_{PE}, yielding a CHC with the type system $\mathbf{C}\lambda_{\to}$.

A.1 The Proof-Theory of iCPL.

The natural deduction system $\mathbf{ND}_{\mathsf{iCPL}}$ for iCPL is formed by the rules illustrated in Fig. 11, with the following proviso:

- it is everywhere assumed that in a sequent $\Gamma \vdash A$, $\mathsf{FN}(\Gamma), \mathsf{FN}(A) \subseteq X$;
- in the rule (CI) and (CE_3) it is assumed that $\mathsf{FN}(\mathscr{d}) \subseteq \{a\}$.

together with all instances of the two axiom schema below:

$$\mathbf{C}^q_a(A \vee B) \to A \vee (\mathbf{C}^q_a A) \qquad (a \notin \mathsf{FN}(A)) \qquad (\mathbf{C}\vee)$$

$$\neg \mathbf{C}^q_a \mathscr{b} \qquad (\mathsf{FN}(\mathscr{b}) \subseteq \{a\}, \mu(\mathscr{b}) < q) \qquad (\mathbf{C}\bot)$$

As usual, we take $\neg A$ as an abbreviation for $A \supset \bot$.

We let $\mathbf{ND}^-_{\mathsf{iCPL}}$ indicate $\mathbf{ND}_{\mathsf{iCPL}}$ without ($\mathbf{C}\vee$) and ($\mathbf{C}\bot$).

Remark 1. *The reason for distinguishing between the systems $\mathbf{ND}_{\mathsf{iCPL}}$ and $\mathbf{ND}^-_{\mathsf{iCPL}}$ is somehow analogous to what happens in intuitionistic modal logic (IML). Indeed, standard axiomatizations of IML include two axioms*

$$\Diamond(A \vee B) \to \Diamond A \vee \Diamond B \qquad (\Diamond\vee)$$

$$\neg\Diamond\bot \qquad (\Diamond\bot)$$

Classical Identity

$$\frac{}{\Gamma \vdash \mathbf{x}_a^i \vee \neg \mathbf{x}_a^i}\ (\mathrm{Cid})$$

Intuitionistic Identity

$$\frac{}{\Gamma, A \vdash A}\ (\mathrm{Iid})$$

Logical Rules

$$\frac{}{\Gamma \vdash \top}\ (\top\mathrm{I}) \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ (\bot\mathrm{E})$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash A}{\Gamma \vdash A \wedge B}\ (\wedge)\mathrm{I} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ (\wedge\mathrm{E}_1) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ (\wedge\mathrm{E}_2)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee\mathrm{I}_1) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee\mathrm{I}_2) \qquad \frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}\ (\vee\mathrm{E})$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\to\mathrm{I}) \qquad \frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ (\to\mathrm{E})$$

Counting Rules

$$\frac{\Gamma, \mathscr{d} \vdash A \qquad \mu(\mathscr{d}) \geq q}{\Gamma \vdash \mathbf{C}_a^q A}\ (\mathrm{CI})\ a \notin \Gamma$$

$$\frac{\Gamma \vdash \mathbf{C}_a^q A}{\Gamma \vdash A}\ (\mathrm{CE}_1)\ a \notin A \qquad \frac{\Gamma \vdash \mathbf{C}_a^q A \qquad \Gamma, A \vdash C}{\Gamma \vdash \mathbf{C}_a^{qs} C}\ (\mathrm{CE}_2)\ a \notin \Gamma$$

Figure 11. Rules of $\mathbf{ND}_{\mathsf{iCPL}}$.

which do not have a clear computational interpretation. Instead, a Curry-Howard correspondence can be defined for an axiomatization of IML (usually referred to as constructive modal logic*) which does not include these two axioms.*

In a similar way, we will show that provability in $\mathbf{ND}^{-}_{\mathsf{iCPL}}$ corresponds, under the decomposition provided by Lemma 3.3 to provability in the Curry-Howard proof-system $\mathbf{ND}_{\mathsf{iCPL}_0}$, while axioms ($\mathsf{C}\vee$) and ($\mathsf{C}\bot$) cannot be similarly interpreted.

A formula A of iCPL is *purely classical* if A contains no intuitionistic propositional variable, and *purely intuitionistic* if A contains no classical propositional variable.

Let us list a few properties of iCPL.

Lemma A.1. *For any purely Boolean formula $\mathscr{b}$, $\vdash \mathscr{b}$ is derivable in $\mathbf{ND}_{\mathsf{iCPL}}$ iff $\mathscr{b}$ is a tautology.*

Lemma A.2. *The following are theorems of* iCPL*:*
(i.) $\mathbf{C}_a^q \bot \leftrightarrow \bot$ and $\mathbf{C}_a^q \top \leftrightarrow \top$;
(ii.) if $a \notin \mathsf{FN}(A)$, $\mathbf{C}_a^q(A \to B) \to (A \to \mathbf{C}_a^q B)$.
(iii.) if $a \notin \mathsf{FN}(A)$, $\mathbf{C}_a^q(A \wedge B) \leftrightarrow A \wedge \mathbf{C}_a^q B$.
iv. if $a \notin \mathsf{FN}(A)$, then $\mathbf{C}_a^q A \leftrightarrow A$.

Lemma A.3. *For any Boolean formula $\mathscr{b}$ with $\mathsf{FN}(\mathscr{b}) \subseteq \{a\}$, $\mathbf{C}_a^q \mathscr{b} \vee (\neg \mathbf{C}_a^q \mathscr{b})$ is provable.*

Proof. Two possibilities arise: if $\mu(\mathscr{b}) \geq q$, then from $\mathscr{b} \vdash \mathscr{b}$ we deduce, by (CI), $\vdash \mathbf{C}_a^q \mathscr{b}$ as well as $\vdash \mathbf{C}_a^q \mathscr{b} \vee (\neg \mathbf{C}_a^q \mathscr{b})$. if $\mu(\mathscr{b}) > q$, then we deduce, using axiom $\mathsf{C}\bot$, $\vdash \neg \mathbf{C}_a^q \mathscr{b}$, as well as $\vdash \mathbf{C}_a^q \mathscr{b} \vee (\neg \mathbf{C}_a^q \mathscr{b})$. ◻

A.2 Relating iCPL and iCPL_0.

Let us first provide a complete proof of the decomposition lemma (Lemma 3.3).

For any formula C and $b \in \{0,1\}$, let $\neg^b C$ indicate the formula C if $b = 0$ and $\neg C$ if $b = 1$. For any $\omega \in (2^{\mathbb{N}})^{\mathscr{A}}$, let the *theory of ω* be the following set of Boolean formulas:

$$\mathsf{Th}(\omega) = \{\neg^{(1-\omega(a)(i))} \mathbf{x}_a^i \mid a, i \in \mathbb{N}\}$$

Moreover, for all $K \in \mathbb{N}$ and finite set X, let $\mathsf{Th}_X^K(\omega)$ be the Boolean formula below

$$\mathsf{Th}_X^K(\omega) = \bigwedge_{a \in A, i \leq K} \neg^{(1-\omega(a)(i))} \mathbf{x}_a^i$$

Observe that the formula $\mathsf{Th}_X^K(\omega)$ only depends on a finite amount of information of ω, namely, on the unique $v \in (2^{\{1,\dots,K\}})^X$ such that $v(a)(i) = \omega(a)(i)$. When v is clear from the context, we will indicate $\mathsf{Th}_X^K(\omega)$ simply as $\mathsf{Th}(v)$.

The decomposition lemma can be reformulated then as follows:

Lemma A.4 (decomposition lemma, syntactic formulation)**.** *For any formula A of* iCPL *there exist intuitionistic formulas A_v, where v varies over all possible valuations of the Boolean variables in A, such that* $\vdash A \leftrightarrow \bigvee_v \mathsf{Th}(v) \wedge A_v$.

Proof. The formula A_v is defined by induction on A as follows:

- if $A = \top$, $A = \bot$, or $A = \mathscr{p}$, then $A_v = A$;
- if $A = \mathscr{x}^i_a$, then $A_v = \top$ if $v(a)(i) = 1$, and $A_v = \bot$ if $v(a)(i) = 0$;
- if $A = BcC$, where $c = \wedge, \vee, \rightarrow$, then $A_v = B_v c C_v$;
- if $A = \mathbf{C}^q_a B$, then we consider two cases:
  - if a does not occur in B, then we let $A_v := B_v$;
  - if a does occur in B, then for all valuation w of the variables of name a in B, we can suppose that the formulas B_{v+w} are well-defined. We let then $A_v = \mathbf{C}^q_a \bigvee_w B_{v+w}$.

Let us show that for any formula A and valuation v of its Boolean variables, $\mathsf{Th}(v) \vdash_{\mathsf{iCPL}} A \leftrightarrow A_v$. We argue by induction on A. Indeed, the only non-trivial case is that of $A = \mathbf{C}^q_a B$. If a does not occur in B, then we can conclude by the IH. Otherwise, by the IH we have that for any valuation w of the Boolean variables of B of name a, $\mathsf{Th}(v+w) \vdash B \leftrightarrow B_{v+w}$. Using the fact that $\bigvee_w \mathsf{Th}(w)$ is provable and that $\mathsf{Th}(v+w) \leftrightarrow (\mathsf{Th}(v) \wedge \mathsf{Th}(w))$, we deduce then $\mathsf{Th}(v) \vdash B \leftrightarrow \bigvee_w B_{v+w}$, from which we can deduce $\mathsf{Th}(v) \vdash A \leftrightarrow A_v$.

Using $\mathsf{Th}(v) \vdash A \leftrightarrow A_v$ and $\vdash \bigvee_v \mathsf{Th}(v)$, we can conclude $\vdash A \leftrightarrow \bigvee_v \mathsf{Th}(v) \wedge A_v$. ◻

If A is a formula of iCPL containing no Boolean variable, we let $|A|$ indicate the corresponding formula of iCPL_0 obtained by deleting names from counting quantifiers, i.e. replacing $\mathbf{C}^q_a$ by $\mathbf{C}^q$.

In the following, we consider as $\mathbf{ND}_{\mathsf{iCPL}_0}$ the proof-system obtained by enriching the rules from Section 3 with all standard rules for intuitionistic connectives (straightforwardly adapted to the sequents of $\mathbf{ND}_{\mathsf{iCPL}_0}$ by adding everywhere a fixed Boolean formula $\dots \vdash \mathscr{b} \rightarrowtail \dots$).

Lemma A.5. *In* $\mathbf{ND}_{\mathsf{iCPL}_0}$, *if* $\Gamma \vdash \mathscr{b} \wedge \mathscr{c} \rightarrowtail A$, *where* $\mathscr{b} \wedge \mathscr{c}$ *is satisfiable, then* $\Gamma \vdash \mathscr{b} \rightarrowtail A$.

Proof. By induction on the rules of $\mathbf{ND}_{\mathsf{iCPL}_0}$. ◻

The following is a more precise formulation of Proposition 3.6.

Proposition A.6. $\Gamma \vdash_{\mathbf{ND}^-_{\mathsf{iCPL}}} A$ *iff for all valuations* v, $|\Gamma_v| \vdash_{\mathbf{ND}_{\mathsf{iCPL}_0}} \mathsf{Th}(v) \rightarrowtail |A_v|$.

Proof. The if part follows from the decomposition lemma (Lemma 3.3) and from the fact that $|\Gamma| \vdash_{\mathbf{ND}_{\mathsf{iCPL}_0}} \mathscr{b} \rightarrowtail |A|$ implies $\Gamma, \mathscr{b} \vdash_{\mathbf{ND}^-_{\mathsf{iCPL}}} A$ (easily checked by induction on the rules of $\mathbf{ND}_{\mathsf{iCPL}_0}$).

For the only-if part we argue by induction on the rules of $\mathbf{ND}_{\mathsf{iCPL}}$. All propositional cases are straightforward, so we only focus here on the rules for counting quantifiers.

(CI)

$$\dfrac{\Gamma, \mathscr{d} \vdash A \qquad \mu(\mathscr{d}) \geq q}{\Gamma \vdash \mathbf{C}^q_a A} \quad \mapsto \quad \dfrac{\dfrac{\left\{\dfrac{|\Gamma_v| \vdash \mathsf{Th}(v) \wedge \mathsf{Th}(w) \rightarrowtail |A_{v+w}|}{|\Gamma_v| \vdash \mathsf{Th}(v) \wedge \mathsf{Th}(w) \rightarrowtail \bigvee_w |A_{v+w}|}\,(\vee\mathrm{I})\right\}_{w \vDash \mathscr{d}}}{|\Gamma_v| \vdash \mathsf{Th}(v) \wedge \mathscr{d} \rightarrowtail \bigvee_w |A_{v+w}|}\,(\mathrm{m}) \qquad \mu(\mathscr{d}) \geq q}{|\Gamma_v| \vdash \mathsf{Th}(v) \rightarrowtail |(\mathbf{C}^q_a A)_v|}\,(\mathrm{CI})$$

(CE)$_1$ Since $a \notin \mathsf{FN}(A)$, it follows that $(\mathbf{C}^q_a A)_v = A_v$, so we can conclude by the IH.

(CE)$_2$

$$\dfrac{\Gamma \vdash \mathbf{C}^q_a \qquad \Gamma, A \vdash B}{\Gamma \vdash \mathbf{C}^{qs}_a B} \quad \mapsto \quad \dfrac{|\Gamma_v| \vdash \mathsf{Th}(v) \rightarrowtail |(\mathbf{C}^q_a A)_v| \qquad \dfrac{\left\{\dfrac{\dfrac{|\Gamma_v|, |A_{v+w}| \vdash \mathsf{Th}(v) \wedge \mathsf{Th}(w) \rightarrowtail |B_{v+w}|}{|\Gamma_v|, |A_{v+w}| \vdash \mathsf{Th}(v) \rightarrowtail |B_{v+w}|}\,[\text{Lemma A.5}]}{|\Gamma_v|, |A_{v+w}| \vdash \mathsf{Th}(v) \rightarrowtail (\bigvee_w |B_{v+w}|)}\,(\vee\mathrm{I})\right\}_w}{|\Gamma_v|, \bigvee_w |A_{v+w}| \vdash \mathsf{Th}(v) \rightarrowtail (\bigvee_w |B_{v+w}|)}\,(\vee\mathrm{E})}{|\Gamma_v| \vdash \mathsf{Th}(v) \rightarrowtail |(\mathbf{C}^{qs}_a B)_v|}\,(\mathrm{CE})$$

◻

A.3 Normalization in $\mathbf{ND}^-_{\mathsf{iCPL}}$.

It is possible to define a normalization procedure for $\mathbf{ND}^-_{\mathsf{iCPL}}$. Cuts formed by propositional intro-elim rules are reduced as in standard intuitionistic logic. The cuts formed by counting rules are reduced as follows:

(CI/CE$_1$)

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ \Gamma, d \vdash A\end{array} \qquad \mu(d) \geq q}{\Gamma \vdash \mathbf{C}^q_a A}\,(\mathrm{CI})}{\Gamma \vdash A}\,(\mathrm{CE})_1 \quad \rightsquigarrow \quad \begin{array}{c}\Pi^*\\ \Gamma \vdash A\end{array}$$

where the derivation Π^* is obtained from Lemma A.7 below, easily established by induction.

Lemma A.7. *If $\Gamma, b \vdash A$ is derivable in $\mathbf{ND}^-_{\mathsf{iCPL}}$, where $\mathsf{FN}(b) \subseteq \{a\}$, and a does not occur in either Γ nor A, then $\Gamma \vdash A$ is derivable with a derivation of same length and using the same rules.*

(CI/CE$_2$)

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ \Gamma, d \vdash A\end{array} \qquad \mu(d) \geq q}{\Gamma \vdash \mathbf{C}^q_a A}\,(\mathrm{CI}) \qquad \begin{array}{c}\Sigma\\ \Gamma, A \vdash B\end{array}}{\Gamma \vdash \mathbf{C}^{qs}_a B}\,(\mathrm{CE})_2 \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\Pi\\ \Gamma, d \vdash A\end{array} \qquad \begin{array}{c}\Sigma\\ \Gamma, A \vdash B\end{array}}{\Gamma, d \vdash B}\,(\mathrm{subst}) \qquad \mu(d) \geq qs}{\Gamma \vdash \mathbf{C}^{qs}_a B}\,(\mathrm{CI})$$

where the admissibility of the rule (subst) is easily checked by induction.

The proof of Theorem A.6 yields a way to associate each proof Π of $\Gamma \vdash A$ in $\mathbf{ND}^-_{\mathsf{iCPL}}$ with a family of proofs Π_v of $|\Gamma_v| \vdash \mathsf{Th}(v) \rightarrowtail |A_v|$ in $\mathbf{ND}^-_{\mathsf{iCPL}_0}$. This association preserves normalization in the following sense:

Lemma A.8. *For any two proofs Π, Σ of $\Gamma \vdash A$ in $\mathbf{ND}^-_{\mathsf{iCPL}}$ and any valuation v of the Boolean variables in Γ and A, if $\Pi \rightsquigarrow^* \Sigma$, then $\Pi_v \rightsquigarrow^* \Sigma_v$.*

The lemma above, easily checked, can be used to deduce the strong normalization of $\mathbf{ND}^-_{\mathsf{iCPL}}$ from that of $\mathbf{ND}_{\mathsf{iCPL}_0}$:

Theorem A.9. *$\mathrm{ND}^-_{\mathsf{iCPL}}$ is strongly normalizing.*

A.4 Soundness and Completeness of $\mathrm{ND}_{\mathsf{iCPL}}$.

Let us first establish a few properties of Kripke Semantics. For this we first need to recall a fundamental property of Borel sets.

For any Borel set $S \in \mathbf{B}_{X \cup Y}$ and $\omega \in (2^{\mathbb{N}})^X$, let

$$\Pi^{\omega}(S) = \{\omega' \in (2^{\mathbb{N}})^Y \mid \omega + \omega' \in S\} \subseteq (2^{\mathbb{N}})^X$$

Notice that $\Pi^{\omega}(S)$ is an *analytic* set and needs not be Borel. However, since the Lebesgue measure is defined on all analytic sets, the values $\mu(\Pi^{\omega}(S))$, for S Borel (or more generally, analytic), are always defined. Moreover the following holds:

Lemma A.10. *[[36], Theorem 14.11 + Theorem 29.26] For any $S \in \mathbf{B}_{X \cup Y}$, with $X \cap Y = \emptyset$, and $r \in [0,1]$, $\{\omega \in (2^{\mathbb{N}})^X \mid \mu(\Pi^f(S)) \geq r\} \in \mathbf{B}_X$.*

For any $S \subseteq \mathbf{B}_X$ and $Y \supseteq X$, let $S^{\uparrow Y} := S \times (2^{\mathbb{N}})^{Y-X}$.

Using Lemma A.10 we can show that for any iCPL-structure $\mathcal{M} = (W, \preceq, \mathbf{i})$ and world $w \in W$, the set of functions $\omega \in (2^{\mathbb{N}})^X$ such that $w, \omega \Vdash^X_{\mathcal{M}} A$ is a Borel set.

Lemma A.11. *Given a iCPL-structure $\mathcal{M} = (W, \preceq, \mathbf{i})$, for any finite set X, $w \in W$ and formula A with $\mathsf{FN}(A) \subseteq X$, the set*

$$\mathrm{mod}_{\mathcal{M}}(A, X, w) = \{\omega \in (2^{\mathbb{N}})^X \mid w, \omega \Vdash^X_{\mathcal{M}} A\}$$

is Borel.

Proof. We argue by induction on A:

- if $A = \bot$, then $\mathrm{mod}_{\mathcal{M}}(A, X, w) = \emptyset$ is Borel; similarly, if $A = \top$, then $\mathrm{mod}_{\mathcal{M}}(A, X, w) = (2^{\mathbb{N}})^X$ is Borel;
- if $A = x^i_a$, then $\mathrm{mod}_{\mathcal{M}}(A, X, w)$ is the cylinder $\{f \mid f(a)(i) = 1\}$, so it is Borel;
- if $A = p$, then $\mathrm{mod}_{\mathcal{M}}(A, X, w)$ is either $(2^{\mathbb{N}})^X$ or the empty set, which are both Borel;
- if $A = B \wedge C$, then $\mathrm{mod}_{\mathcal{M}}(A, X, w) = \mathrm{mod}_{\mathcal{M}}(B, X, w) \cap \mathrm{mod}_{\mathcal{M}}(C, X, w)$, so we conclude by the I.H.;
- if $A = B \vee C$, then $\mathrm{mod}_{\mathcal{M}}(A, X, w) = \mathrm{mod}_{\mathcal{M}}(B, X, w) \cup \mathrm{mod}_{\mathcal{M}}(C, X, w)$, so we conclude by the I.H.;
- if $A = B \to C$, then $\mathrm{mod}_{\mathcal{M}}(A, X, w) = \bigcap_{w' \geq w} \overline{\mathrm{mod}_{\mathcal{M}}(B, X, w')} \cup \mathrm{mod}_{\mathcal{M}}(C, X, w')$ is, by the I.H., a countable intersection of Borel sets, so it is Borel;
- if $A = \mathbf{C}^q_a B$, then by the I.H. , for all $w' \in W$, $\mathrm{mod}_{\mathcal{M}}(B, X \cup \{a\}, w')$ is Borel; using Lemma A.10 we then have that the set

$$\mathrm{mod}_{\mathcal{M}}(A, X, w) = \bigcap_{w' \geq w} \left\{\omega \in (2^{\mathbb{N}})^X \mid \mu\left(\Pi^{\omega}\left(\mathrm{mod}_{\mathcal{M}}(B, X \cup \{a\}, w')\right)\right) \geq q\right\}$$

is Borel.

$\square$

Lemma A.12 (monotonicity). *If $w, \omega \Vdash^X_{\mathscr{M}} A$ and $w \leq w'$, then $w', \omega \Vdash^X_{\mathscr{M}} A$.*

Proof. We argue by induction on A.

If $A = \mathscr{p}$ the claim follows from the fact that $\mathbf{i}(\mathscr{p})$ is an upper set;

If $A = \mathbf{x}^i_a$ the claim is immediate.

If $A = B \wedge C$, the claim follows from the I.H.

If $A = B \to C$ and $w, \omega \Vdash^X_{\mathscr{M}} A$, then for all $w'' \geq w$, $w'', \omega \Vdash^X_{\mathscr{M}} B$ implies $w'', \omega \Vdash^X_{\mathscr{M}} C$. Hence, in particular, for all $w'' \geq w' \geq w$, $w'', \omega \Vdash^X_{\mathscr{M}} B$ implies $w'', \omega \Vdash^X_{\mathscr{M}} C$, and thus $w', \omega \Vdash^X_{\mathscr{M}} A$.

If $A = \mathbf{C}^q_a B$ then by the I.H. for all $\omega' \in 2^{\mathbb{N}}$, $w, \omega + \omega' \Vdash^{X \cup \{a\}}_{\mathscr{M}} B$ implies that for all $w' \geq w$, $w', \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} B$; we deduce then that for for all $w' \geq w$, for all $w'' \geq w' \geq w$, the set $\left\{\omega' \in 2^{\mathbb{N}} \mid w'', \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} B\right\}$ has measure $\geq q$, which implies $w', \omega \Vdash^X_{\mathscr{M}} A$. $\square$

Lemma A.13. *Let $\mathscr{M} = \langle W, \preceq, \alpha\rangle$ be a* iCPL*-structure. For any finite set X and $a \notin X$, for all $u \in W$ and $\mathscr{b} \in \mathbf{B}_X$, and for any formula $A \in$* Formulae$_X$*, if $u, \omega \Vdash^X_{\mathscr{M}} A$ holds, then $u, \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} A$ holds for any $\omega' \in \{\omega\}^{\uparrow X \cup \{a\}}$.*

Proof. By induction on the rules. $\square$

Lemma A.14. *Let $\mathscr{M} = \langle W, \preceq, \mathbf{i}\rangle$ be a* iCPL*-structure. For any finite set X, $a \notin X$, formula $A \in$* Formulae$_X$*, $q \in (0, 1]$ and $\omega \in \mathbf{B}_X$,*

$$w, \omega \Vdash^X_{\mathscr{M}} \mathbf{C}^q_a A \quad \Rightarrow \quad w, \omega \Vdash^X_{\mathscr{M}} A$$

Proof. $w', \omega \Vdash^X_{\mathscr{M}} \mathbf{C}^q_a A$ implies that $\mu(\{\omega' \in 2^{\mathbb{N}} \mid u, \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} A\}) \geq q > 0$. From $u, \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} A$ one can deduce by induction on A that $u, \omega \Vdash^X_{\mathscr{M}} A$. Since there exists at least one such f, we conclude that $u, \omega \Vdash^X_{\mathscr{M}} A$ holds.

$\square$

We are now in a position to establish, by induction on the rules, the soundness of $\mathbf{ND}_{\mathsf{iCPL}}$.

Proposition A.15 (soundness). *If $\Gamma \vdash_{\mathbf{ND}_{\mathsf{iCPL}}} A$, then $\Gamma \vDash A$.*

Proof. Let $\mathscr{M} = \langle W, \preceq, \mathbf{i}\rangle$ be a iCPL-structure. We argue by induction on the rules of $\mathbf{ND}_{\mathsf{iCPL}}$ (selected rules):

- if the last rule is $\overline{\Gamma, A \vdash A}$, then for all $w \in W$ and $\omega \in \mathbf{B}_X$, if $w, \omega \Vdash^X_{\mathscr{M}} \Gamma, A$, then $w, \omega \Vdash^X_{\mathscr{M}} A$.
- if the last rule is
$$\frac{\Gamma, \mathscr{d} \vdash^{X\cup\{a\}} A \qquad \mu(\mathscr{d}) \geq q}{\Gamma \vdash \mathbf{C}^q_a A}$$
then by the I.H. together with Lemma A.12, for all $w \in W$, for all $\omega \in \mathbf{B}_X$ and $\omega' \in 2^{\mathbb{N}}$, if $w, \omega \Vdash^X_{\mathscr{M}} \Gamma$ and $w, \omega' \Vdash^{\{a\}} \mathscr{d}$ (where the latter condition only depends on ω'), then for all $w' \geq w$, we have $w', \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} A$.
Since $\mu(\mathscr{d}) \geq q$, we deduce that for all $w' \geq w$,
$$\left\{\omega' \mid w', \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} A\right\}$$
has measure greater than q, and we conclude then that $w, \omega \Vdash^X_{\mathscr{M}} \mathbf{C}^q_a A$.
- if the last rule is
$$\frac{\Gamma \vdash \mathbf{C}^q_a A \qquad \Gamma, A \vdash^{X\cup\{a\}} C}{\Gamma \vdash C}\ (\mathrm{CE}_1)\ \ a \notin \mathsf{FN}(\Gamma, C)$$
let $w \in W$, $\omega \in (2^{\mathbb{N}})^X$ and $w, \omega \Vdash^X_{\mathscr{M}} \Gamma$. By I.H. we deduce that the set
$$S := \left\{\omega' \mid w, \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} A\right\}$$
has measure greater than $q > 0$, and it is thus non-empty. Let ω' be an element of S; again, by the I.H. we deduce that $w, \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} C$; since $a \notin \mathsf{FN}(C)$, by Lemma A.14, we conclude then that $w, \omega \Vdash^X_{\mathscr{M}} C$.
- if the last rule is
$$\frac{\Gamma \vdash \mathbf{C}^q_a A \qquad \Gamma, A \vdash^{X\cup\{a\}} C}{\Gamma \vdash \mathbf{C}^{qs}_a C}\ (\mathrm{CE}_2)\ \ a \notin \mathsf{FN}(\Gamma)$$
let $w \in W$, $\omega \in (2^{\mathbb{N}})^X$ and $w, \omega \Vdash^X_{\mathscr{M}} \Gamma$. By I.H. we deduce that the set
$$S := \left\{\omega' \mid w, \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} A\right\}$$

has measure greater than $q \geq qs$ and is contained in the set $S' := \left\{\omega' \mid w, \omega + \omega' \Vdash_{\mathcal{M}}^{X \cup \{a\}} C\right\}$. We can conclude then that $w, \omega \Vdash_{\mathcal{M}}^{X} \mathbf{C}_a^{qs} C$.

- Axiom ($\mathbf{C}\vee$) is valid: suppose $w, \omega \Vdash_{\mathcal{M}}^{X} \mathbf{C}_a^{q}(A \vee B)$, where $a \notin \mathsf{FN}(A)$. Then the set $\{\omega' \mid w, \omega + \omega' \Vdash_{\mathcal{M}}^{X \cup \{a\}} A \vee B\}$ has measure greater than q. Observe that $w, \omega + \omega' \Vdash_{\mathcal{M}}^{X \cup \{a\}} A$ holds iff $w, \omega \Vdash_{\mathcal{M}}^{X} A$; so suppose $w, \omega \Vdash_{\mathcal{M}}^{X} A$ does not hold; then for any ω', $w, \omega + \omega' \Vdash_{\mathcal{M}}^{X \cup \{a\}} A$ does not hold, and thus the set $\{\omega' \mid w, \omega + \omega' \Vdash_{\mathcal{M}}^{X \cup \{a\}} B\}$ must have measure greater than q. We have thus proved that either $w, \omega \Vdash_{\mathcal{M}}^{X} A$ holds or $w, \omega \Vdash_{\mathcal{M}}^{X} \mathbf{C}_a^{q} B$ holds, and thus that $w, \omega \Vdash^{X} A \vee \mathbf{C}_a^{q} B$.
- Axiom ($\mathbf{C}\bot$) is valid: suppose $\mathsf{FN}(\mathscr{b}) \subseteq \{a\}$ and $\mu(\mathscr{b}) < q$; for any $w \in W$, the set $\{\omega' \mid w, \omega' \vdash^{\{a\}} \mathscr{b}\}$ coincides with $[\![\mathscr{b}]\!]$, and has thus measure $< q$ by hypothesis; we deduce that for any $w' \geq w$, $w', * \Vdash_{\mathcal{M}}^{\emptyset} \mathbf{C}_a^{q} \mathscr{b}$ does not hold, and we conclude that $w, * \Vdash_{\mathcal{M}}^{\emptyset} \neg \mathbf{C}_a^{q} \mathscr{b}$.

∎

To prove the completeness theorem we first need to study provability in iCPL a bit further.

Definition A.16. For any formula A and finite set X, $|A|_X := \max\{i \mid \exists a \in X \cap \mathsf{FN}(A) \text{ s.t. } \mathbf{x}_a^i \text{ occurs in } A\}$. We use $|A|$ as a shorthand for $|A|_{\mathsf{FN}(A)}$. For any $\omega \in (2^{\mathbb{N}})^{\mathscr{A}}$, $A^{\omega} := \mathsf{Th}_{\mathsf{FN}(A)}^{|A|}(\omega) \to A$.

We will exploit the following relativized version of the decomposition lemma (Lemma 3.3), which is proved in a similar way:

Lemma A.17. *Let A be a formula with $\mathsf{FN}(A) \subseteq X \cup Y$, where $X \cap Y = \emptyset$. Then there exist formulas A_w^X, with $\mathsf{FN}(A_w^X) \subseteq X$, where w ranges over the valuations of the variables of name in Y in A, such that $\vdash A \leftrightarrow (\bigvee_w \mathsf{Th}(w) \wedge A_w^X)$.*

Definition A.18. For each formula A, with $\mathsf{FN}(A) \subseteq \{a\}$, let $\mathscr{b}_A := \bigvee_v \{\mathsf{Th}(v) \mid A_v \nvdash_{\mathsf{ND}_{\mathsf{iCPL}}} \bot\}$.

From $A \vdash \bigvee_v \mathsf{Th}(v) \wedge A_v$ and $A \vdash \neg \mathsf{Th}(w)$ for all $\mathsf{Th}(w)$ not occurring in $\mathscr{b}_A$, we can deduce $A \vdash \mathscr{b}_A$. The following result shows that measuring the formula $\mathscr{b}_A$ provides a test to know if $\mathbf{C}_a^q A$ is consistent:

Corollary A.19. *Let A be a formula with $\mathsf{FN}(A) \subseteq \{a\}$. Then for all $q \in (0, 1]$,*

$$\mu(\mathscr{b}_A) < q \quad \Rightarrow \quad \mathbf{C}_a^q A \vdash \bot$$

Proof. The claim follows from the observation that $A \vdash \mathscr{b}_A$ by applying Axiom ($\mathbf{C}\bot$) and the rule (CE_2). ∎

Remark 2 (relativizing $\mathscr{b}_A$ to ω)**.** *Suppose A is a formula with $\mathsf{FN}(A) \subseteq X \cup \{a\}$ (with $a \notin X$). Then for each $\omega \in (2^{\mathbb{N}})^{\mathscr{A}}$, let A_{ω}^X be A_w^X, where w is the restriction of ω to $(2^{|A|})^X$. Then, by applying the construction from Lemma A.17 to A_{ω}^X we can define a Boolean formula $\mathscr{b}_A^{\omega}$ with $\mathsf{FN}(\mathscr{b}_A^{\omega}) \subseteq \{a\}$ such that $\mathsf{Th}_X^{|A|}(\omega) \vdash A \to \mathscr{b}_A^{\omega}$ and such that $\mu(\mathscr{b}_A^{\omega}) < q \Rightarrow \mathsf{Th}_X^{|A|}(\omega) \vdash \neg \mathbf{C}_a^q A$.*

The proof of the completeness theorem relies on the construction of a suitable "canonical model" based on sets of formulas. Let us first introduce some terminology. Let Γ be a (possibly infinite) set of formulas. We say that Γ is

- *A-consistent* if $\Gamma \nvdash A$;
- *consistent* if it is $\bot$-consistent;
- *super-consistent* if for all $\omega \in (2^{\mathbb{N}})^{\mathscr{A}}$, $\mathsf{Th}(\omega) \cup \Gamma$ is consistent;
- *closed* if $A_1, \dots, A_n \in \Gamma$ and $A_1, \dots, A_n \vdash_{\mathsf{ND}_{\mathsf{iCPL}}} A$ implies $A \in \Gamma$, and if $(\mathscr{d} \to A)^{\omega} \in \Gamma$ and $\mu(\mathscr{d}) \geq q$ then $(\mathbf{C}_a^q A)^{\omega} \in \Gamma$.

Let Θ be the set of all super-consistent and closed sets of formulas Γ satisfying the following conditions, for all $\omega \in (2^{\mathbb{N}})^{\mathscr{A}}$:

$$(A \vee B)^{\omega} \in \Gamma \quad \Rightarrow \quad A^{\omega} \in \Gamma \text{ or } B^{\omega} \in \Gamma \qquad (\vee\text{-closure})$$

$$(\mathbf{C}_a^q A)^{\omega} \in \Gamma \quad \Rightarrow \quad \exists \mathscr{d} \text{ s.t. } \mu(\mathscr{d}) \geq q \text{ and } (\mathscr{d} \to A)^{\omega} \in \Gamma \qquad (\text{C-closure})$$

The fundamental ingredient for the completeness theorem is the lemma below, which will be used to lift any consistent set of formulas to an element of Θ.

Lemma A.20 (saturation lemma)**.** *Let Γ be a super-consistent set and let $\omega_0 \in (2^{\mathbb{N}})^{\mathscr{A}}$ be such that $\mathsf{Th}(\omega_0) \cup \Gamma$ is A-consistent. Then there exists a set $\Gamma^* \supseteq \Gamma$ such that $\Gamma^* \in \Theta$ and $\mathsf{Th}_X(\omega_0) \cup \Gamma^*$ is A-consistent.*

Proof. Let us fix an enumeration $(a_i)_{i \in \mathbb{N}}$ of all names. For any $p, q \in \mathbb{N}$, Let $\mathscr{F}_{p,q}$ be the set of all formulas B such that $\mathsf{FN}(B) \subseteq \{a_0, \dots, a_{q-1}\}$ and $|B| \leq p$. Let us fix, for all $p, q \in \mathbb{N}$ an enumeration $(C_n^{p,q})_{n \in \mathbb{N}}$ of $\mathscr{F}_{p,q}$.

For any natural number N, let $[N] := \{0, \dots, N-1\}$. Given $p \leq p'$ and $q \leq q'$, and finite matrices $s \in (2^{[p+1]})^{[q+1]}$ $s' \in (2^{[p'+1]})^{[q'+1]}$, let $s \sqsubseteq s'$ if for all $i \leq p$ and $j \leq q$ $s(j)(i) = s'(j)(i)$. We will often indicate as p_s and q_s the (unique) natural numbers such that $s \in (2^{[p+1]})^{[q+1]}$.

Moreover, for all $\omega \in (2^{\mathbb{N}})^X$, let $s \sqsubseteq \omega$ hold if for all $i \leq p_s$ and $j \leq q_s$, $s(j)(i) = \omega(a_j)(i)$.

Observe that if $s \sqsubseteq \omega$, $\mathsf{Th}(\omega) \vdash \mathsf{Th}(s)$.

For all p, q and $s \in (2^{[p+1]})^{[q+1]}$, we define a set of formulas $\Gamma^{\langle p,q,s\rangle} \subseteq \mathscr{F}_{p,q}$ such that for all $p \leq p'$ and $q \leq q'$, $s \in (2^{[p+1]})^{[q+1]}$ and $s' \in (2^{[p'+1]})^{[q'+1]}$, $s \sqsubseteq s'$ implies $\Gamma^{\langle p,q,s\rangle} \subseteq \Gamma^{\langle p',q',s'\rangle}$.

We let $\Gamma^{\langle p,q,s\rangle} := \bigcup_n \Gamma_n^{\langle p,q,s\rangle}$, where the sets $\Gamma_n^{\langle p,q,s\rangle}$ are defined by a triple induction on p, q and n as follows:

- if $p = q = 0$ and $b = \omega_0(0)$,
  - $\Gamma_0^{\langle 0,0,b\rangle} := \Gamma \cup \{\neg^{(1-b)} x_0^0\}$;
  - $\Gamma_{n+1}^{\langle 0,0,b\rangle} := \Gamma_n \cup \{C_n^{0,0}\}$, if this is A-consistent, and $\Gamma_{n+1}^{\langle 0,0,b\rangle} := \Gamma_n^{\langle 0,0,b\rangle}$ otherwise.

  If $b \neq \omega_0(0)$, the definition is the same, with $\bot$ in place of A;
- if $p > 0$ and $q = 0$, $\Gamma^{\langle p,0,s\rangle} = \bigcup_n \Gamma_n^{\langle p,0,s\rangle}$, where if $s \sqsubseteq \omega_0$,
  - $\Gamma_0^{\langle p,0,s\rangle} := \Gamma^{\langle p-1,0,s|_{p-1,0}\rangle} \cup \{\neg^{(1-s(p)(0))} x_p^0\}$;
  - $\Gamma_{n+1}^{\langle p,0,s\rangle} := \Gamma_n^{\langle p,0,s\rangle} \cup \{C_n^{p,0}\}$ if this is A-consistent, and $\Gamma_{n+1}^{\langle p,0,s\rangle} := \Gamma_n^{\langle p,0,s\rangle}$ otherwise;

  and if $s \not\sqsubseteq \omega_0$, the definition is similar, with $\bot$ in place of A;
- if $p = 0$ and $q > 0$, $\Gamma^{\langle 0,q,s\rangle} = \bigcup_n \Gamma_n^{\langle 0,q,s\rangle}$, where if $s \sqsubseteq \omega_0$,
  - $\Gamma_0^{\langle 0,q,s\rangle} := \Gamma^{\langle 0,q-1,s|_{0,q-1}\rangle} \cup \{\neg^{(1-s(0)(q))} x_0^q\}$;
  - $\Gamma_{n+1}^{\langle 0,q,s\rangle} := \Gamma_n^{\langle 0,q,s\rangle} \cup \{C_n^{0,q}\}$ if this is A-consistent, and $\Gamma_{n+1}^{\langle 0,q,s\rangle} := \Gamma_n^{\langle 0,q,s\rangle}$ otherwise;

  and if $s \not\sqsubseteq \omega_0$, the definition is similar, with $\bot$ in place of A;
- if both $p > 0$ and $q > 0$, $\Gamma^{\langle p,q,s\rangle} = \bigcup_n \Gamma_n^{\langle p,q,s\rangle}$, where if $s \sqsubseteq \omega_0$,
  - $\Gamma_0^{\langle p,q,s\rangle} := \Gamma^{\langle p-1,q,s|_{p-1,q}\rangle} \cup \Gamma^{\langle p,q-1,s|_{p,q-1}\rangle} \cup \{\neg^{(1-s(p)(q))} x_p^q\}$;
  - $\Gamma_{n+1}^{\langle p,q,s\rangle} := \Gamma_n^{\langle p,q,s\rangle} \cup \{C_n^{p,q}\}$ if this is A-consistent, and $\Gamma_{n+1}^{\langle p,q,s\rangle} := \Gamma_n^{\langle p,q,s\rangle}$ otherwise;

  and if $s \not\sqsubseteq \omega_0$, the definition is similar, with $\bot$ in place of A.

In the following, whenever this creates no confusion, we will indicate $\Gamma^{\langle p,q,s\rangle}$ simply as Γ^s.

For all finite matrices s, s', the following hold:

a. $\mathsf{Th}(s) \subseteq \Gamma^s$;
b. if $s \sqsubseteq \omega_0$, Γ^s is A-consistent;
c. if $s \not\sqsubseteq \omega_0$, Γ^s is consistent;
d. if $s \sqsubseteq s'$, $\Gamma^s \subseteq \Gamma^{s'}$;
e. if $\Gamma^s \vdash B$, then for all $t \sqsupseteq s$ such that $B \in \mathscr{F}_{p_t,q_t}$, $B \in \Gamma^t$.

Facts a.-d. are verified by construction, so we only prove e.: suppose there exist $B_1, \dots, B_n \in \Gamma^s$ such that $B_1, \dots, B_n \vdash_{\mathbf{ND}_{\mathsf{iCPL}}} B$ and let $t \sqsupseteq s$ be such that $B \in \mathscr{F}_{p_t,q_t}$. Suppose $B \notin \Gamma^t$: then for some $k \in \mathbb{N}$, $\Gamma_k^t \cup \{B\} \vdash \bot$; yet, from d. it follows that $B_1, \dots, B_n \in \Gamma^t$, and thus $\Gamma^t \vdash \bot$, contradicting c.

Given matrices $s_1, \dots, s_n \sqsubseteq \omega$, let $\bigvee^\omega\{s_1, \dots, s_n\}$ indicate the smallest sub-matrix of ω extending all $s_1, \dots, s_n$ (i.e. the restriction of ω to $p = \max\{p_{s_1}, \dots, p_{s_n}\}$ and $q = \max\{q_{s_1}, \dots, q_{s_n}\}$.

For any matrix s, let $\Gamma^{\dagger s} = \{A^s \mid A \in \Gamma^s\} = \{\mathsf{Th}(s) \to A \mid A \in \Gamma^s\}$. Let $\Gamma^\dagger = \bigcup_s \Gamma^{\dagger s}$ and let Γ^* be the deductive closure of $\Gamma^\dagger$. We will establish the following properties of Γ^*:

α. Γ^* is super-consistent;
β. $\mathsf{Th}(\omega_0) \cup \Gamma^*$ is A-consistent;
γ. $\Gamma \subseteq \Gamma^*$;
δ. if $B^\omega \notin \Gamma^*$ and $B^\omega = B^s$, then $\mathsf{Th}(s) \cup \Gamma^* \cup \{B\} \vdash \bot$; moreover, if $B^{\omega_0} \notin \Gamma^*$ and $B^{\omega_0} = B^s$, then $\mathsf{Th}(s) \cup \Gamma^* \cup \{B\} \vdash A$;
ϵ. Γ^* is $\vee$-closed;
η. Γ^* is C-closed.

This will conclude the proof of the theorem.

Let us preliminarily observe that any formula B^ω can be written as B^s for a unique $s \sqsubseteq \omega$ such that $p_s = |B|$ and q_s is minimum with the property that $\mathsf{FN}(B) \subseteq \{a_0, \dots, a_{q_s-1}\}$.

α. Let us show that $\Gamma^\dagger$ is super-consistent. This immediately implies that Γ^* is super-consistent too. Suppose $\mathsf{Th}(\omega) \cup \Gamma^\dagger \vdash \bot$; then there exist $s_1, \dots, s_n$ and $B_1 \in \Gamma^{s_1}, \dots, B_n \in \Gamma^{s_n}$, such that $\mathsf{Th}(\omega), B_1^{s_1}, \dots, B_n^{s_n} \vdash \bot$; we can suppose w.l.o.g. that $s_1, \dots, s_n \sqsubseteq \omega$, since if $s_i \not\sqsubseteq \omega$, $\mathsf{Th}(\omega) \vdash \neg\mathsf{Th}(s_i)$, and thus $\mathsf{Th}(\omega) \vdash B_i^{s_i}$.
We have then $B_i^{s_i} = B_i^\omega$; by letting $s = \bigvee^\omega\{s_1, \dots, s_n\}$ we thus have $B_i^{s_i} = B_i^s$ and $\mathsf{Th}(s), B_1^s, \dots, B_n^s \vdash \bot$, which implies $\mathsf{Th}(s), B_1, \dots, B_n \vdash \bot$, and since $\mathsf{Th}(s) \cup \{B_1, \dots, B_n\} \subseteq \Gamma^s$, we deduce $\bot \in \Gamma^s$, contradicting a.
β. The argument is similar to the one for α.

γ. Let $B \in \Gamma$ and s be such that $B^{\omega} = B^{s}$, $p_s = |B|$ and $\mathsf{FN}(B) \subseteq \{a_0, \dots, a_{q_s - 1}\}$. For any $s' \in (2^{[p_s+1]})^{[q_s+1]}$, $B^{s'} \in \Gamma^{\dagger s'} \subseteq \Gamma^{\dagger}$. Hence, using the fact that $\vdash \bigvee_{s \in (2^{[p_s+1]})^{[q_s+1]}} \mathsf{Th}(s)$, we deduce that $\{B^{s} \mid s \in (2^{[p_s+1]})^{[q_s+1]}\} \vdash B$, and thus that $\Gamma^{\dagger} \vdash B$, which implies $B \in \Gamma^{*}$.

δ. Let s be such that $B^{\omega} = B^{s}$. We consider the case of $\omega = \omega_0$; the case $\omega \neq \omega_0$ is proved similarly with $\bot$ in place of A. We will prove the contrapositive, i.e. that if $\mathsf{Th}(s) \cup \Gamma^{\dagger} \cup \{B\}$ is A-consistent, then $B^{\omega_0} = B^{s} \in \Gamma^{*}$, from which δ. follows. Suppose that $\mathsf{Th}(\omega) \cup \Gamma^{\dagger} \cup \{B\}$ is A-consistent. Suppose $B \notin \Gamma^{s}$: this implies $B \notin \Gamma$ and that for some $k \in \mathbb{N}$, $\Gamma^{s}_{k} \cup \{B\} \vdash A$. But this forces then $\Gamma^{s} \cup \{B\} \vdash A$; since for all $F \in \Gamma^{s}$, $F^{s} \in \Gamma^{\dagger}$, and $\mathsf{Th}(s), F^{s} \vdash F$, we deduce then $\mathsf{Th}(s) \cup \Gamma^{\dagger} \cup \{B\} \vdash A$, which is absurd. We conclude then that $B \in \Gamma^{s}$, and thus $B^{s} = B^{\omega} \in \Gamma^{\dagger} \subseteq \Gamma^{*}$.

ϵ. Again, we consider the case of $\omega = \omega_0$; the case $\omega \neq \omega_0$ is proved similarly with $\bot$ in place of A. Let $s \sqsubseteq \omega_0$ be such that $(B \vee C)^{\omega} = (B \vee C)^{s}$ and suppose $(B \vee C)^{s} \in \Gamma^{*}$ but neither $B^{s} \in \Gamma^{*}$ nor $C^{s} \in \Gamma^{*}$; then by δ., $\mathsf{Th}(s) \cup \Gamma^{*} \cup \{B\} \vdash A$ and $\mathsf{Th}(s) \cup \Gamma^{*} \cup \{C\} \vdash A$, hence $\mathsf{Th}(s) \cup \Gamma^{*} \cup \{B \vee C\} \vdash A$; since $(B \vee C)^{s} \in \Gamma^{*}$ we have $\mathsf{Th}(s) \cup \Gamma^{*} \vdash B \vee C$, so we deduce $\mathsf{Th}(s) \cup \Gamma^{*} \vdash A$, and since $\mathsf{Th}(s) \subseteq \mathsf{Th}(\omega_0)$, we have $\mathsf{Th}(\omega_0) \cup \Gamma^{*} \vdash A$, contradicting β.

η. Once more, we consider the case of $\omega = \omega_0$; the case $\omega \neq \omega_0$ is proved similarly with $\bot$ in place of A.
Let $s \sqsubseteq \omega_0$ be such that $(\mathbf{C}^{q}_{a}B)^{\omega_0} = (\mathbf{C}^{q}_{a}B)^{s}$ and suppose $(\mathbf{C}^{q}_{a}B)^{s} \in \Gamma^{*}$.
We can suppose w.l.o.g. that $a = a_q$ with $q > q_s$. Hence we can suppose that the formula B^{ω_0} is of the form $B^{s+s'+s''}$, for some finite matrices s', s'', with $s'' \in 2^{[|B|+1]}$.
By Lemma A.17 and Remark 2 we have that $\Gamma^{s} \vdash B \leftrightarrow (\bigvee_{v \in 2^{[|B|+1]}} \mathsf{Th}(v) \wedge B_{s+v})$, where B_{s+v} has no free name. Let $S = \{v \in 2^{[|B|+1]} \mid B_{s+v} \in \Gamma^{s}\}$ and $\mathscr{d} = \bigvee_{v \in S} \mathsf{Th}(v)$. Observe that for any $v \in S$, $\Gamma^{s+s'} \vdash \mathsf{Th}(v) \to B$. Let $v \in S$ and ω_v be such that $v \subseteq \omega_v$. Then $\Gamma^{s+s'} \vdash \mathsf{Th}(\omega_v) \to B$. Using e. we have then that for all $t \in 2^{[|B|+1]}$, $\mathsf{Th}(\omega_v) \to B \in \Gamma^{s+s'+t}$, so in particular, $\mathsf{Th}(\omega_v) \to B \in \Gamma^{s+s'+s''}$, so $(\mathsf{Th}(\omega_v) \to B)^{\omega_0} \in \Gamma^{\dagger} \subseteq \Gamma^{*}$. We thus have that for all $v \in S$, $\Gamma^{*} \vdash (\mathsf{Th}(v) \to B)^{\omega_0}$, which implies $\Gamma^{*} \vdash (\mathscr{d} \to B)^{\omega_0}$ and thus $(\mathscr{d} \to B)^{\omega_0} \in \Gamma^{*}$.
At this point, if $\mu(\mathscr{d}) \geq q$ we are done. Otherwise, suppose $\mu(\mathscr{d}) < q$; if $v \notin S$, then $B_{s+v} \notin \Gamma^{s}$, which implies that for some k, $\Gamma^{s}_{k} \cup \{B_{s+v}\} \vdash A$; this implies that $\Gamma^{s} \vdash (\mathsf{Th}(v) \wedge B_{s+v}) \to A$; we deduce then that $\Gamma^{s} \vdash B \to ((\bigvee_{v \in S} \mathsf{Th}(v) \wedge B_{s+v}) \vee A)$, and in particular that $\Gamma^{s} \vdash B \to (\mathscr{d} \vee A)$; this means that there exist formulas $B_1, \dots, B_n \in \Gamma^{s} \subseteq \mathscr{F}_{p_s, q_s}$, hence containing no occurrences of the name a, such that $B_1, \dots, B_n \vdash B \to (\mathscr{d} \vee A)$; from this we deduce first $\mathsf{Th}(s), B^{s}_{1}, \dots, B^{s}_{n} \vdash B \to (\mathscr{d} \vee A)$, and then, using (CE_2), $\mathsf{Th}(s), \mathbf{C}^{q}_{a}B, B^{s}_{1}, \dots, B^{s}_{n} \vdash \mathbf{C}^{q}_{a}(\mathscr{d} \vee A)$. Now, since $s \subseteq \omega_0$, $\mathbf{C}^{q}_{a}A, B^{s}_{1}, \dots, B^{s}_{n} \in \Gamma^{*}$ and $\mathbf{C}^{q}_{a}(\mathscr{d} \vee A) \vdash (\mathbf{C}^{q}_{a}\mathscr{d}) \vee A$ (Axiom $(\mathrm{C}\vee)$, since we can suppose w.l.o.g. that $a \notin \mathsf{FN}(A)$), we have $\mathsf{Th}(\omega_0) \cup \Gamma^{*} \vdash (\mathbf{C}^{q}_{a}\mathscr{d}) \vee A$; moreover, from $\mu(\mathscr{d}) < q$ we get (by Axiom $(\mathrm{C}\bot)$) $\vdash \neg \mathbf{C}^{q}_{a}\mathscr{d}$ and thus $(\mathbf{C}^{q}_{a}\mathscr{d}) \vee A \vdash A$, so we can conclude $\mathsf{Th}(\omega_0) \cup \Gamma^{*} \vdash A$, which contradicts β.

□

Lemma A.21. *Let Γ be an A-consistent set of formulas. Then there exists a super-consistent set Δ and $\omega \in (2^{\mathbb{N}})^{\mathscr{A}}$ such that $\mathsf{Th}(\omega) \cup \Delta$ is A-consistent and its closure contains Γ.*

Proof. Let us define sets of formulae $\Gamma_{a,i}$ together with $\omega(\langle a, i\rangle)$ as follows:

- let $\omega(\langle 0, 0\rangle) = 1$ if $\Gamma \cup \{\mathscr{x}^{0}_{0}\}$ is A-consistent, and $\omega(\langle 0, 0\rangle) = 0$ otherwise; moreover, let $\Gamma_{0,0} = \Gamma$;
- let $\omega(\langle a, i+1\rangle) = 1$ if $\Gamma_{a,i} \cup \{\mathscr{x}^{i+1}_{a}\}$ is A-consistent, and $\omega(\langle a, i+1\rangle) = 0$ otherwise; moreover, let $\Gamma_{a,i+1} = \Gamma_{a,i} \cup \{\neg^{(1-\omega(\langle a, i+1\rangle))}\mathscr{x}^{i+1}_{a}\}$;
- let $\omega(\langle a+1, 0\rangle) = 1$ if $\bigcup_i \Gamma_{a,i} \cup \{\mathscr{x}^{0}_{a+1}\}$ is A-consistent, and $\omega(\langle a, i+1\rangle) = 0$ otherwise; moreover, let $\Gamma_{a+1,0} = \bigcup_i \Gamma_{a,i} \cup \{(\neg^{1-\omega(\langle a+1, 0\rangle))}\mathscr{x}^{0}_{a+1}\}$.

By construction we then have that $\mathsf{Th}(\omega) \cup \Gamma = \bigcup_{a,i} \Gamma_{a,i}$; let us show that $\bigcup_{a,i} \Gamma_{a,i}$ is A-consistent:

- $\Gamma_{0,0} = \Gamma$ is A-consistent by hypothesis;
- suppose $\Gamma_{a,i}$ is A-consistent; if $\Gamma_{a,i+1} = \Gamma_{a,i} \cup \{\mathscr{x}^{i+1}_{a}\}$ then by construction $\Gamma_{a,i+1}$ is A-consistent; if $\Gamma_{a,i+1} = \Gamma_{a,i} \cup \{\neg \mathscr{x}^{i+1}_{a}\}$ then by construction $\Gamma_{a,i} \cup \{\mathscr{x}^{i}_{a}\} \vdash A$; if moreover $\Gamma_{a,i} \cup \{\neg \mathscr{x}^{i}_{a}\} \vdash A$, the $\Gamma_{a,i} \cup \{\mathscr{x}^{i}_{a} \vee \neg \mathscr{x}^{i}_{a}\} \vdash A$ and thus $\Gamma_{a,i} \vdash A$, which is absurd. We conclude then that $\Gamma_{a,i+1}$ is A-consistent.
- suppose $\Gamma_{a,i}$ is A-consistent for all $i \in \mathbb{N}$; first observe that $\Gamma_{a} := \bigcup_i \Gamma_{a,i}$ is A-consistent: if $\Gamma_a \vdash A$, then there exist $B_1 \in \Gamma_{a,i_1}, \dots, B_n \in \Gamma_{a,i_n}$ such that $B_1, \dots, B_n \vdash A$; since for $j \leq k$, $\Gamma_{a,j} \subseteq \Gamma_{a,k}$, we deduce that $\Gamma_{a,\max\{i_1,\dots,i_n\}} \vdash A$, which is absurd.
Now, if $\Gamma_{a+1,0} = \Gamma_a \cup \{\mathscr{x}^{0}_{a+1}\}$ then by construction $\Gamma_{a+1,0}$ is A-consistent; if $\Gamma_{a+1,0} = \Gamma_a \cup \{\neg \mathscr{x}^{0}_{a+1}\}$ then $\Gamma_a \cup \{\mathscr{x}^{0}_{a+1}\} \vdash A$; hence, if $\Gamma_{a+1,0} \vdash A$, we deduce $\Gamma_a \cup \{\mathscr{x}^{0}_{a+1} \vee \neg \mathscr{x}^{0}_{a+1}\} \vdash A$, so we conclude $\Gamma_a \vdash A$, which is absurd.

Now, for each formula $B \in \Gamma$ let us define a formula B^{*} as follows:

$$B^{*} := B \vee \left(\bigvee_{a \in \mathsf{FN}(B),\, i \leq |B|} \neg^{\omega(\langle a, i\rangle)} \mathscr{x}^{i}_{a} \right)$$

Observe that $\mathsf{Th}(\omega) \cup \{B^*\} \vdash B$ and that for all $\omega' \in (2^{\mathbb{N}})^{\mathscr{A}}$, $\mathsf{Th}(\omega') \cup \{B^*\}$ is consistent.

Let then $\Delta = \{B^* \mid B \in \Gamma\}$. It is clear that $\mathsf{Th}(\omega) \cup \Delta \vdash B$ for all $B \in \Gamma$. Suppose $\mathsf{Th}(\omega) \cup \Delta \vdash A$, then there exists Boolean formulas $\mathscr{e}_1, \dots, \mathscr{e}_k \in \mathsf{Th}(\omega)$ and formulas $B_1, \dots, B_n \in \Gamma$ such that $\mathscr{e}_1, \dots, \mathscr{e}_k, B_1^*, \dots, B_n^* \vdash A$. Since $B_i \vdash B^*$, this implies then $\mathscr{e}_1, \dots, \mathscr{e}_k, B_1, \dots, B_n \vdash A$, and thus $\mathsf{Th}(\omega) \cup \Gamma$ is not A-consistent, which is absurd. □

We now have all elements to proceed to the proof of the completeness theorem.

Theorem A.22 (completeness). *If $\Gamma \vDash A$, then $\Gamma \vdash_{\mathsf{ND}_{\mathsf{iCPL}}} A$.*

Proof. Let $\mathscr{T} = \langle \Theta, \subseteq, \mathbf{i} \rangle$, where $\mathbf{i}(\mathscr{p}) = \{\Gamma \mid \mathscr{p} \in \Gamma\}$.

We will prove that for all $\Gamma \in \Theta$ and $\omega \in \mathbf{B}_X$, $\Gamma, \omega \vdash^X_{\mathscr{T}} A$ iff $A^\omega \in \Gamma$. From this claim the theorem is proved as follows: suppose $\Gamma \nvdash A$; by Lemma A.21 we obtain a super-consistent set Δ and $\omega_0 \in (2^{\mathbb{N}})^{\mathscr{A}}$ such that $\mathsf{Th}(\omega_0) \cup \Delta$ is A-consistent and its closure contains Γ. By Lemma A.20 Δ extends to $\Gamma^* \in \Theta$ such that $\Gamma^* \supseteq \Gamma$ and $\mathsf{Th}(\omega_0) \cup \Gamma^*$ is A-consistent. Then using the claim, from $\Gamma \subseteq \Gamma^*$, we deduce that $\Gamma^*, \omega_0 \vdash^X_{\mathscr{T}} \Gamma$, and from $A \notin \Gamma^*$ we deduce that $\Gamma^*, \omega_0 \nvdash^X_{\mathscr{T}} A$; we can conclude then that $\Gamma \nvdash^X_{\mathscr{T}} A$, and thus that $\Gamma \nvDash A$.

Let us now prove the claim, by arguing by induction on A:

- if $A = \bot$, from the vacuous assumption $\Gamma, \omega \vdash^X_{\mathscr{T}} \bot$ we can freely deduce $\bot^\omega \in \Gamma$; conversely, from the vacuous assumption $\bot^\omega \in \Gamma$ we can freely deduce $\Gamma, \omega \vdash^X_{\mathscr{T}} \bot$;
- if $A = \top$, since $\Gamma, \omega \vdash^X_{\mathscr{T}} \top$ always holds and $\top^\omega \in \Gamma$ always holds (since Γ is closed), we can conclude;
- if $A = \mathscr{p}$, $\Gamma, \omega \vdash^X_{\mathscr{T}} \mathscr{p}$ iff $\mathscr{p} \in \Gamma$ holds by definition (since $\mathscr{p} \notin \mathsf{Th}(\omega)$);
- if $A = \mathscr{x}^i_a$, if $\Gamma, \omega \vdash^X_{\mathscr{T}} A$ holds then $\omega(a)(i) = 1$, hence $\mathsf{Th}(\omega) \vdash \mathscr{x}^i_a$, and thus A^ω is logically valid, which implies $A^\omega \in \Gamma$ by closure; conversely, if $A^\omega \in \Gamma$, since $\mathsf{Th}(\omega) \cup \Gamma$ is consistent, the only possibility is that $\omega(a)(i) = 1$, which forces $\Gamma, \omega \vdash^X_{\mathscr{T}} A$.
- if $A = B \wedge C$, then $\Gamma, \omega \vdash^X_{\mathscr{T}} A$ iff $\Gamma, \omega \vdash^X_{\mathscr{T}} B$ and $\Gamma, \omega \vdash^X_{\mathscr{T}} C$, which by the I.H. is equivalent to $B^\omega \in \Gamma$ and $C^\omega \in \Gamma$, which is in turn equivalent to $(B \wedge C)^\omega \in \Gamma$.
- if $A = B \vee C$, and $\Gamma, \omega \vdash^X_{\mathscr{T}} A$, then either $\Gamma, \omega \vdash^X_{\mathscr{T}} B$ or $\Gamma, \omega \vdash^X_{\mathscr{T}} C$ holds; if the first holds, then by the I.H. $B^\omega \in \Gamma$, which implies $(B \vee C)^\omega \in \Gamma$, and one can argue similarly if the second holds; conversely, if $(B \vee C)^\omega \in \Gamma$, by $\vee$-closure either $B^\omega \in \Gamma$ or $C^\omega \in \Gamma$, so by the I.H. in each case we deduce $\Gamma, \omega \vdash^X_{\mathscr{T}} A$.
- if $A = B \to C$, then suppose that $\Gamma, \omega \vdash^X_{\mathscr{T}} A$; then for all $\Gamma' \in \Theta$ such that $\Gamma' \supseteq \Gamma$ and $\Gamma', \omega \vdash^X_{\mathscr{T}} B$, also $\Gamma', \omega \vdash^X_{\mathscr{T}} C$ holds. Suppose $\Gamma \cup \{B^\omega\}$ is super-consistent.
  Furthermore, suppose $\mathsf{Th}(\omega) \cup \Gamma \cup \{B^\omega\} \nvdash C^\omega$; then, by Lemma A.20 (with $\omega_0 = \omega$ and $A = C$) there exists $\Gamma' \in \Theta$ with $\Gamma' \supseteq \Gamma \cup \{B^\omega\}$ such that $C^\omega \notin \Gamma'$. Using the I.H. we deduce then that $\Gamma', \omega \vdash^X_{\mathscr{T}} B$ but $\Gamma', \omega \nvdash^X_{\mathscr{T}} C$, against the assumption. We conclude then that $\mathsf{Th}(\omega) \cup \Gamma \cup \{B^\omega\} \vdash C^\omega$, and thus that $\Gamma \vdash (B \to C)^\omega$, which implies by closure $(B \to C)^\omega \in \Gamma$.
  Suppose now that $\Gamma \cup \{B^\omega\}$ is not super-consistent. There exists ω' such that $\mathsf{Th}(\omega') \cup \Gamma \cup \{B^\omega\} \vdash \bot$. Since $\mathsf{Th}(\omega') \cup \Gamma$ is consistent the only possibility is that $\mathsf{Th}(\omega') \vdash \mathsf{Th}^{|B|}_{\mathsf{FN}(B)}(\omega)$ (which forces $\omega|_{|B|,\mathsf{FN}(B)}$ to coincide with $\omega'|_{|B|,\mathsf{FN}(B)}$) and $\mathsf{Th}(\omega') \cup \Gamma \cup \{B\} \vdash \bot$; but then we also have that $\mathsf{Th}(\omega) \cup \Gamma \cup \{B\} \vdash \bot$, which implies $\mathsf{Th}(\omega) \cup \Gamma \vdash B \to C$ and thus $\Gamma \vdash (B \to C)^\omega$; by closure this implies then $(B \to C)^\omega \in \Gamma$.
  Conversely, suppose $(B \to C)^\omega \in \Gamma$ and let $\Gamma' \in \Theta$ be such that $\Gamma' \supseteq \Gamma$ and $\Gamma', \omega \vdash^X_{\mathscr{T}} B$; by the I.H. this implies $B^\omega \in \Gamma'$, and since $(B \to C)^\omega \in \Gamma \subseteq \Gamma'$ we also deduce by closure $C^\omega \in \Gamma'$, which by the I.H. implies $\Gamma', \omega \vdash^X_{\mathscr{T}} C$.
- if $A = \mathbf{C}^q_a B$, then suppose that $\Gamma, \omega \vdash^X_{\mathscr{T}} A$; then by the I.H. the Borel set

$$S = \left\{\omega' \mid B^{\omega+\omega'} \in \Gamma\right\} \supseteq \left\{\omega' \mid \Gamma, \omega + \omega' \Vdash^{X \cup \{a\}}_{\mathscr{T}} B\right\}$$

  has measure greater than q. Observe that for all $\omega' \in 2^{\mathbb{N}}$, $B^{\omega+\omega'}$ is equivalent to

$$\mathsf{Th}^{|B|_X}_X(\omega|_X) \to \left(\mathsf{Th}^{|B|_{\{a\}}}_{\{a\}}(\omega') \to B\right)$$

  Let $\mathscr{d} := \bigvee\{\mathsf{Th}^{|B|_{\{a\}}}_{\{a\}}(\omega') \mid \omega' \in S\}$; since Γ is closed, we deduce then that $(\mathscr{d} \to B)^\omega \in \Gamma$ and that $\mu(\mathscr{d}) \geq q$; again, by closure, this implies $(\mathbf{C}^q_a B)^\omega \in \Gamma$.
  For the converse direction, suppose $(\mathbf{C}^q_a B)^\omega \in \Gamma$ and let $\Gamma' \supseteq \Gamma$; since Γ is $\mathbf{C}$-closed, there exists a Boolean formula $\mathscr{d}$ with $\mu(\mathscr{d}) \geq q$ such that $(\mathscr{d} \to B)^\omega \in \Gamma \subseteq \Gamma'$. This implies that for all $\omega' \in [\![\mathscr{d}]\!]$, $B^{\omega+\omega'} \in \Gamma'$; hence, by the I.H. for all $\omega' \in [\![\mathscr{d}]\!]$, $\Gamma', \omega + \omega' \Vdash^{X \cup \{a\}}_{\mathscr{T}} B$. Since $\mu(\mathscr{d}) \geq q$, the set of ω' such that $\Gamma', \omega + \omega' \Vdash^{X \cup \{a\}}_{\mathscr{T}} B$ has measure greater than q, and we can conclude $\Gamma, \omega \Vdash^X_{\mathscr{T}} A$.

□

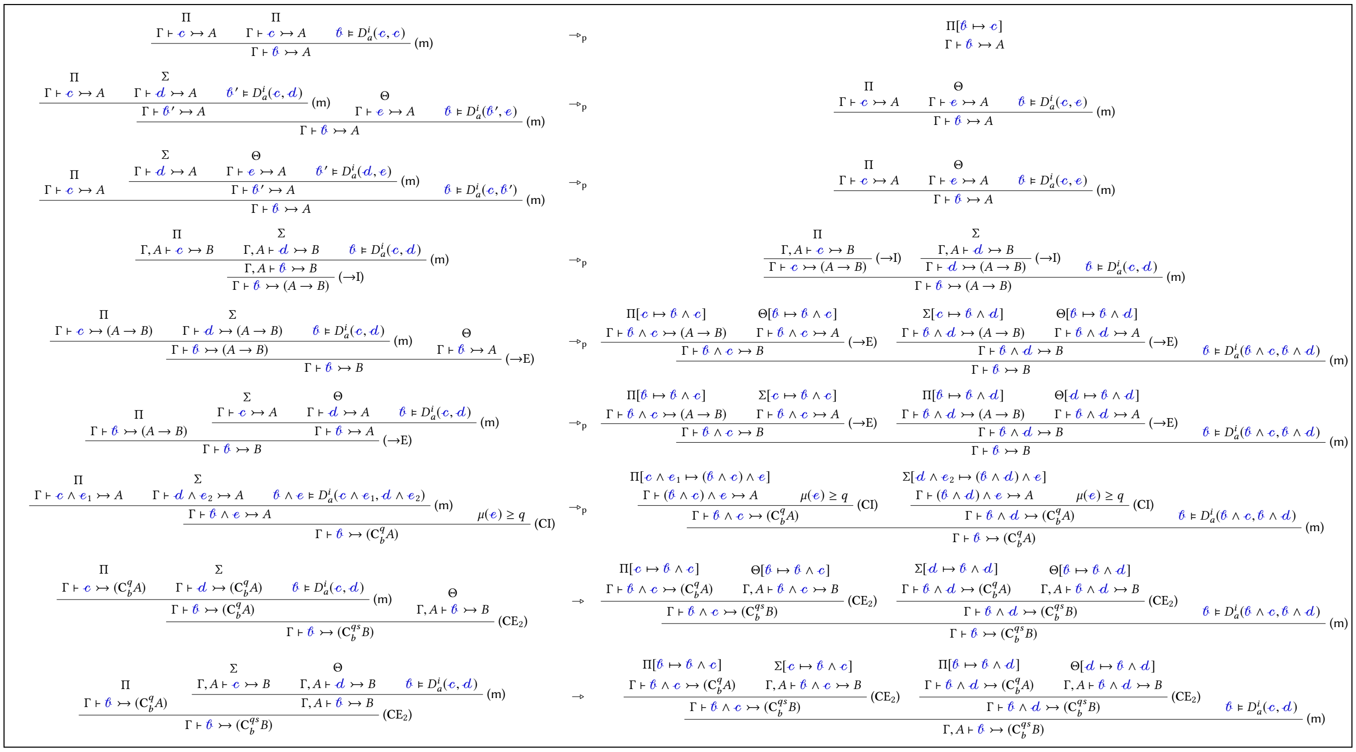

Figure 12. Permutative rules of iCPL_0.

A.5 Permutative Rules in iCPL_0.

We complete the picture of the normalization rules of iCPL_0 by considering permutative rules for (m), illustrated in Fig. 12, where we let $D^i_a(\mathscr{b}, \mathscr{c})$ be an abbreviation for $(\mathscr{x}^i_a \wedge \mathscr{b}) \vee (\neg \mathscr{x}^i_a \wedge \mathscr{c})$.

It is easily checked that if $\Pi \rightarrowtriangle \Sigma$, $t^{\Pi} \rightarrowtriangle_{\{\}} t^{\Sigma}$, since the rules closely correspond to the permutation rules for $\oplus$ in $\Lambda^{\{\}}_{\mathsf{PE}}$.

A.6 A "CbN" Proof-System.

The CHC described in Section 5 relates the proof-system $\mathbf{ND}_{\mathsf{iCPL}_0}$ with the type system $\mathsf{C}\lambda^{\{\}}_{\rightarrow}$. In this subsection we describe a "CbN" variant $\mathbf{ND}^{\mathrm{CbN}}_{\mathrm{iCPL0}}$ of the proof-system $\mathbf{ND}_{\mathsf{iCPL}_0}$, for which it is possible to describe a CHC with the type system $\mathsf{C}\lambda_{\rightarrow}$.

In the correspondence from Section 5, CbV application $\{t\}u$ plays a fundamental role, as it translates the elimination rule of the counting quantifier. To obtain a translation into Λ_{PE} we thus need to restrict the rule (CE).

A sequent of $\mathbf{ND}^{\mathrm{CbN}}_{\mathrm{iCPL0}}$ is of the form $\Phi; \Gamma \vdash \mathscr{b} \rightarrowtail A$, where Φ and Γ are two sets of formulas, and Φ contains *at most* one formula. The fundamental intuition is that the formula, if any, in Φ, has to be used *linearly* in the proof.

The rules of $\mathbf{ND}^{\mathrm{CbN}}_{\mathrm{iCPL0}}$ are illustrated in Fig. 14. Observe that, contrarily to $\mathbf{ND}_{\mathsf{iCPL}_0}$ (see Section 3), the rules include a "multiplication rule" ($\mathbf{C}\times$) to passes from $\mathbf{C}^q\mathbf{C}^s A$ to $\mathbf{C}^{qs}A$.

$\mathbf{ND}^{\mathrm{CbN}}_{\mathrm{iCPL0}}$ proves *less* formulas than $\mathbf{ND}_{\mathsf{iCPL}_0}$. Indeed, the restricted (CE)-rule allows to deduce $\mathbf{C}^q B$ from $\mathbf{C}^q A$ only when B can be deduced *linearly* from A. For example, one cannot reproduce in $\mathbf{ND}^{\mathrm{CbN}}_{\mathrm{iCPL0}}$ the proof of $\mathbf{C}^q(A \rightarrow A) \rightarrow (A \rightarrow \mathbf{C}^q A)$ illustrated in Section 3, since the hypothesis $\mathbf{C}^q(A \rightarrow A)$, which is used as major premiss in a (CE)-rule, should be used *twice*. From the programming perspective, this means that one cannot encode in $\mathbf{ND}^{\mathrm{CbN}}_{\mathrm{iCPL0}}$ the non-linear "CbV" function $\lambda y.\lambda x.\{\lambda f.f(fx)\}y$. For similar reasons, it seems that one cannot prove $\mathbf{C}^q A \rightarrow (A \rightarrow A \rightarrow B) \rightarrow \mathbf{C}^q B$ in $\mathbf{ND}^{\mathrm{CbN}}_{\mathrm{iCPL0}}$, while this can be proved in $\mathbf{ND}_{\mathsf{iCPL}_0}$ as shown in Fig. 13. From the programming perspective, this means that one cannot encode the non-linear "CbV" function $\lambda x.\lambda y.\{\lambda y.yxx\}x$.

The normalization steps are as in $\mathbf{ND}_{\mathsf{iCPL}_0}$, except for (CI/CE), which is illustrated in Fig. 15 and exploits the admissibility of the following rule:

$$\frac{;\Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^{s_1 * \dots * s_n} A \qquad A; \Gamma \vdash \mathscr{b} \rightarrowtail C}{;\Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^{s_1 * \dots * s_n} C}\ (\mathrm{subst}^*)$$

Notice that the (CI/CE)-step now includes a finite number of internal "multiplications" ($\mathbf{C}\times$).

$$\dfrac{\dfrac{\mathbf{C}^q A, A \to A \to A \vdash \mathbf{C}^q A \qquad \dfrac{\dfrac{A, A \to A \to A \vdash A \to A \to B \qquad A, A \to A \to A \vdash A}{A, A \to A \to A \vdash A \to B}\,(\to\mathrm{E}) \qquad A, A \to A \to A \vdash A}{A, A \to A \to A \vdash B}\,(\to\mathrm{E})}{\mathbf{C}^q A, A \to A \to A \vdash \mathbf{C}^q B}\,(\mathrm{CE})}{\vdash \mathbf{C}^q A \to (A \to A \to A) \to \mathbf{C}^q B}\,(\to\mathrm{I})$$

Figure 13. Proof in $\mathbf{ND}_{\mathsf{iCPL}}$ of $\mathbf{C}^q A \to (A \to A \to A) \to \mathbf{C}^q A$.

Identity Rules

$$\dfrac{}{;\Gamma, A \vdash \mathcal{b} \rightarrowtail A}\,(\mathrm{id}) \qquad \dfrac{}{A; \Gamma \vdash \mathcal{b} \rightarrowtail A}\,(\mathrm{id}_{\mathrm{lin}})$$

Structural Rules

$$\dfrac{\mathcal{b} \vDash \bot}{;\Gamma \vdash \mathcal{b} \rightarrowtail A}\,(\bot)$$

$$\dfrac{;\Gamma \vdash \mathcal{c} \rightarrowtail A \qquad ;\Gamma \vdash \mathcal{d} \rightarrowtail A \qquad \mathcal{b} \vDash (\mathcal{c} \wedge x_a^i) \vee (\mathcal{d} \wedge \neg x_a^i)}{;\Gamma \vdash \mathcal{b} \rightarrowtail A}\,(\mathrm{m})$$

Logical Rules

$$\dfrac{\Phi; \Gamma, A \vdash \mathcal{b} \rightarrowtail B}{\Phi; \Gamma \vdash \mathcal{b} \rightarrowtail (A \to B)}\,(\to\mathrm{I}) \qquad \dfrac{\Phi; \Gamma \vdash \mathcal{b} \rightarrowtail (A \to B) \qquad ;\Gamma \vdash \mathcal{b} \rightarrowtail A}{\Phi; \Gamma \vdash \mathcal{b} \rightarrowtail B}\,(\to\mathrm{E})$$

Counting Rules

$$\dfrac{;\Gamma \vdash \mathcal{b} \wedge \mathcal{d} \rightarrowtail A \qquad \mu(\mathcal{d}) \geq q}{;\Gamma \vdash \mathcal{b} \rightarrowtail \mathbf{C}^q A}\,(\mathrm{CI}) \qquad \dfrac{\Phi; \Gamma \vdash \mathcal{b} \rightarrowtail \mathbf{C}^q A \qquad A; \Gamma \vdash \mathcal{b} \rightarrowtail B}{\Phi; \Gamma \vdash \mathcal{b} \rightarrowtail \mathbf{C}^q B}\,(\mathrm{CE})$$

$$\dfrac{\Phi; \Gamma \vdash \mathcal{b} \rightarrowtail \mathbf{C}^q \mathbf{C}^s A}{\Phi; \Gamma \vdash \mathcal{b} \rightarrowtail \mathbf{C}^{qs} A}\,(\mathrm{C}_\times)$$

Figure 14. Rules of $\mathbf{ND}^{\mathsf{CbN}}_{\mathsf{iCPL0}}$.

$$\dfrac{\dfrac{\dfrac{\overset{\Sigma}{;\Gamma \vdash \mathcal{b} \wedge \mathcal{d} \rightarrowtail \mathbf{C}^{s_1 * \cdots * s_n} A} \qquad \mu(\mathcal{d}) \geq q}{;\Gamma \vdash \mathcal{b} \rightarrowtail \mathbf{C}^{q * s_1 * \cdots * s_n} A}\,(\mathrm{CI})}{;\Gamma \vdash \mathcal{b} \rightarrowtail \mathbf{C}^{q \prod_i s_i} A}\,(\mathrm{C}\times) \qquad \overset{\Pi}{A; \Gamma \vdash \mathcal{b} \rightarrowtail B}}{;\Gamma \vdash \mathcal{b} \rightarrowtail \mathbf{C}^{q \prod_i s_i} B}\,(\mathrm{CE}) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\Sigma}{;\Gamma \vdash \mathcal{b} \wedge \mathcal{d} \rightarrowtail \mathbf{C}^{s_1 * \cdots * s_n} A} \qquad \overset{\Pi[\mathcal{b} \mapsto \mathcal{b} \wedge \mathcal{d}]}{A; \Gamma \vdash \mathcal{b} \wedge \mathcal{d} \rightarrowtail B}}{;\Gamma \vdash \mathcal{b} \wedge \mathcal{d} \rightarrowtail \mathbf{C}^{s_1 * \cdots * s_n} B}\,(\mathrm{subst}^*) \qquad \mu(\mathcal{d}) \geq q}{;\Gamma \vdash \mathcal{b} \rightarrowtail \mathbf{C}^{q * s_1 * \cdots * s_n} B}\,(\mathrm{CI})}{;\Gamma \vdash \mathcal{b} \rightarrowtail \mathbf{C}^{q \prod_i s_i} B}\,(\mathrm{C}\times)$$

(a) (CI)/(CE).

Figure 15. Normalization steps of $\mathbf{ND}^{\mathsf{CbN}}_{\mathsf{iCPL0}}$.

The translation from $\mathbf{ND}^{\mathsf{CbN}}_{\mathsf{iCPL0}}$ to $\mathsf{C}\lambda_{\to}$ relies on properties of *head contexts* in Λ_{PE}. These are defined by the grammar below:

$$\mathsf{H}[\,] ::= [\,] \mid \lambda x.\mathsf{H}[\,] \mid \mathsf{H}[\,]u$$

The fundamental property of head contexts is that they naturally behave as CbV functions, due to the following lemma:

Lemma A.23. *For any head context* $\mathsf{H}[\,]$ *and term* t, $\mathsf{H}[\nu a.t] \twoheadrightarrow^*_{\mathsf{p}} \nu a.\mathsf{H}[t]$.

Proof. By induction on $\mathsf{H}[\,]$:

- if $\mathsf{H}[\,] = [\,]$, the claim is immediate;
- if $\mathsf{H}[\,] = \lambda x.\mathsf{H}'[\,]$, then we have $\mathsf{H}[\nu a.t] = \lambda x.\mathsf{H}'[\nu a.t] \overset{\mathrm{IH}}{\twoheadrightarrow^*_{\mathsf{p}}} \lambda x.\nu a.\mathsf{H}'[t] \twoheadrightarrow_{\mathsf{p}} \nu a.\lambda x.\mathsf{H}'[t] = \nu a.\mathsf{H}[t]$;
- if $\mathsf{H}[\,] = \mathsf{H}'[\,]u$, then we have $\mathsf{H}[\nu a.t] = \mathsf{H}'[\nu a.t]u \overset{\mathrm{IH}}{\twoheadrightarrow^*_{\mathsf{p}}} (\nu a.\mathsf{H}'[t])u \twoheadrightarrow_{\mathsf{p}} \nu a.\mathsf{H}'[t]u = \nu a.\mathsf{H}[t]$.

□

In other words, whenever t is a function of the form $\lambda x.\mathsf{H}[x]$ for some head context $\mathsf{H}[\,]$, CbN and CbV application of t coincide, since $t(\nu a.v)$ and $\nu a.tv$ have the same normal form.

This property is reflected in the following:

$$\frac{}{;\Gamma, A \vdash \mathscr{b} \rightarrowtail A}\,(\mathrm{id}) \quad\mapsto\quad \frac{}{\mathfrak{s}_\Gamma, y : \mathfrak{s}_A \vdash y : \mathscr{b} \rightarrowtail \mathfrak{s}_A}\,(\mathrm{id})$$

$$\frac{}{A; \Gamma \vdash \mathscr{b} \rightarrowtail A}\,(\mathrm{id}_{\mathrm{lin}}) \quad\mapsto\quad \frac{}{x : \mathfrak{s}_A, \mathfrak{s}_\Gamma \vdash x : \mathscr{b} \rightarrowtail \mathfrak{s}_A}\,(\mathrm{id})$$

$$\frac{\mathscr{b} \vDash \bot}{;\Gamma \vdash \bot \rightarrowtail A}\,(\bot) \quad\mapsto\quad \frac{\mathscr{b} \vDash \bot}{\mathfrak{s}_\Gamma \vdash \mathbf{c} : \mathscr{b} \rightarrowtail \mathfrak{s}_A}\,(\vee)$$

$$\frac{\overset{\Pi}{;\Gamma \vdash \mathscr{c} \rightarrowtail A} \qquad \mathscr{b} \vDash \mathscr{c}}{;\Gamma \vdash \mathscr{b} \rightarrowtail A}\,(\vDash) \quad\mapsto\quad \frac{\overset{D^{\Pi}}{\mathfrak{s}_\Gamma \vdash t^{\Pi} : \mathscr{c} \rightarrowtail \mathfrak{s}_A} \qquad \mathscr{b} \vDash \mathscr{c}}{\mathfrak{s}_\Gamma \vdash t^{\Pi} : \mathscr{b} \rightarrowtail \mathfrak{s}_A}\,(\vDash)$$

$$\frac{\overset{\Pi}{;\Gamma \vdash \mathscr{c} \rightarrowtail A} \qquad \overset{\Pi'}{;\Gamma \vdash \mathscr{d} \rightarrowtail A} \qquad \mathscr{b} \vDash (\mathscr{c} \wedge \boldsymbol{x}_a^i) \vee (\mathscr{d} \wedge \neg \boldsymbol{x}_a^i)}{;\Gamma \vdash \mathscr{b} \rightarrowtail A}\,(\mathrm{m}) \quad\mapsto\quad \frac{\overset{D^{\Pi}}{\mathfrak{s}_\Gamma \vdash t^{\Pi} : \mathscr{c} \rightarrowtail A} \qquad \overset{D^{\Pi'}}{\mathfrak{s}_\Gamma \vdash t^{\Pi'} : \mathscr{d} \rightarrowtail A} \qquad \mathscr{b} \vDash (\mathscr{c} \wedge \boldsymbol{x}_a^i) \vee (\mathscr{d} \wedge \neg \boldsymbol{x}_a^i)}{\mathfrak{s}_\Gamma \vdash t^{\Pi} \oplus_a^i t^{\Pi'} : \mathscr{b} \rightarrowtail \mathfrak{s}_A}\,(\oplus)$$

$$\frac{\overset{\Pi}{\Phi; \Gamma, A \vdash \mathscr{c} \rightarrowtail B}}{\Phi; \Gamma \vdash \mathscr{b} \rightarrowtail (A \to B)}\,(\to\mathrm{I}) \quad\mapsto\quad \frac{\overset{D^{\Pi}}{\mathfrak{s}_\Phi, \mathfrak{s}_\Gamma, y : \mathfrak{s}_A \vdash t^{\Pi} : \mathscr{c} \rightarrowtail \mathbf{C}^{|B|}\sigma_B}}{\mathfrak{s}_\Phi, \mathfrak{s}_\Gamma \vdash \lambda y.t^{\Pi} : \mathscr{b} \rightarrowtail \mathbf{C}^{|B|}(\mathfrak{s}_A \Rightarrow \sigma_B)}\,(\lambda)$$

$$\frac{\overset{\Pi}{\Phi; \Gamma \vdash \mathscr{c} \rightarrowtail (A \to B)} \qquad \overset{\Sigma}{\Phi; \Gamma \vdash \mathscr{b} \rightarrowtail A}}{\Phi; \Gamma \vdash \mathscr{b} \rightarrowtail B}\,(\to\mathrm{E}) \quad\mapsto\quad \frac{\overset{D^{\Pi}}{\mathfrak{s}_\Phi; \mathfrak{s}_\Gamma \vdash t^{\Pi} : \mathscr{b} \rightarrowtail \mathbf{C}^{|B|}(\mathfrak{s}_A \to \sigma_B)} \qquad \overset{D^{\Sigma}}{\mathfrak{s}_\Phi; \mathfrak{s}_\Gamma \vdash t^{\Sigma} : \mathscr{b} \rightarrowtail \mathfrak{s}_A}}{\mathfrak{s}_\Phi; \mathfrak{s}_\Gamma \vdash t^{\Pi} t^{\Sigma} : \mathscr{b} \rightarrowtail \mathbf{C}^{|B|}\sigma_B}\,(@)$$

$$\frac{\overset{\Pi}{;\Gamma \vdash \mathscr{b} \wedge \mathscr{d} \rightarrowtail A} \qquad \mu(\mathscr{d}) \geq q}{;\Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^q A}\,(\mathrm{CI}) \quad\mapsto\quad \frac{\overset{D^{\Pi}}{\mathfrak{s}_\Gamma \vdash t^{\Pi} : \mathscr{b} \wedge \mathscr{d} \rightarrowtail \mathfrak{s}_A} \qquad \mu(\mathscr{d}) \geq q}{\mathfrak{s}_\Gamma \vdash \nu a.t^{\Pi} : \mathscr{b} \rightarrowtail \mathfrak{s}_{\mathbf{C}^q A}}\,(\mu)$$

$$\frac{\overset{\Pi}{\Phi; \Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^q A} \qquad \overset{\Sigma}{A; \Gamma \vdash \mathscr{b} \rightarrowtail B}}{\Phi; \Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^q B}\,(\mathrm{CE}) \quad\mapsto\quad \frac{\overset{D^{\Pi}}{\mathfrak{s}_\Phi, \mathfrak{s}_\Gamma \vdash t^{\Pi} : \mathscr{b} \rightarrowtail \mathbf{C}^{q \cdot |A|}\sigma_A} \qquad \overset{D^{\Sigma}}{x : \mathbf{C}^{|A|}\sigma_A, \mathfrak{s}_\Gamma \vdash t^{\Sigma}[x] : \mathscr{b} \rightarrowtail \mathbf{C}^{|B|}\sigma_B}}{\mathfrak{s}_\Phi, \mathfrak{s}_\Gamma \vdash t^{\Sigma}[t^{\Pi}] : \mathscr{b} \rightarrowtail \mathbf{C}^{q \cdot |B|}\sigma_B}\,(\text{head-subst})$$

Figure 16. Translation $\Pi \mapsto D^{\Pi}$ from $\mathbf{ND}^{\mathrm{CbN}}_{\mathrm{iCPL0}}$ to $\mathbf{C}\lambda_{\to}$.

Lemma A.24. *For any head context* $\mathsf{H}[\ ]$*, the following rule is derivable in* $\mathbf{C}\lambda_{\to}$*:*

$$\frac{\Gamma \vdash t : \mathscr{b} \rightarrowtail \mathbf{C}^{qs}\sigma \qquad x : \mathbf{C}^{q}\sigma, \Gamma \vdash \mathsf{H}[x] : \mathscr{b} \rightarrowtail \mathbf{C}^{r}\tau}{\Gamma \vdash \mathsf{H}[t] : \mathscr{b} \rightarrowtail \mathbf{C}^{rs}\tau}\,(\textit{head-subst})$$

Proof. By induction on $\mathsf{H}[\ ]$:

- if $\mathsf{H}[\ ] = [\ ]$, then it must be $q = r$, and the claim is then immediate;
- if $\mathsf{H}[\ ] = \lambda y.\mathsf{H}'[\ ]$, then $\tau = \mathrm{t} \Rightarrow \tau'$ and we must have $x : \mathbf{C}^q\sigma, \Gamma, y : \mathrm{t} \vdash \mathsf{H}'[x] : \mathscr{b} \rightarrowtail \mathbf{C}^q\tau'$. Then by IH we deduce $\Gamma, y : \mathrm{t} \vdash \mathsf{H}'[t] : \mathscr{b} \rightarrowtail \mathbf{C}^{rs}\tau'$, and finally $\Gamma \vdash \mathsf{H}[t] : \mathscr{b} \rightarrowtail \mathbf{C}^{rs}\tau$.
- if $\mathsf{H}[\ ] = \mathsf{H}'[\ ]u$, then we must have $x : \mathbf{C}^q\sigma, \Gamma \vdash \mathsf{H}'[x] : \mathscr{b} \rightarrowtail \mathbf{C}^q(\mathrm{t} \Rightarrow \tau)$ and $x : \mathbf{C}^q\sigma, \Gamma \vdash u : \mathscr{b} \rightarrowtail \mathrm{t}$. Then by IH we deduce $\Gamma \vdash \mathsf{H}'[t] : \mathscr{b} \rightarrowtail \mathbf{C}^{rs}(\mathrm{t} \Rightarrow \tau)$, and we can thus conclude $\Gamma \vdash \mathsf{H}[t] : \mathscr{b} \rightarrowtail \mathbf{C}^{rs}\tau$.

□

To any formula A of iCPL_0, we associate a non-quantified type σ_A and a positive real $|A| \in (0,1] \cap \mathbb{Q}$ of $\mathbf{C}\lambda_{\to}$ as follows:

$$\sigma_{\mathscr{p}} := o \qquad\qquad |o| := 1$$
$$\sigma_{A \to B} := (\mathbf{C}^{|A|}\sigma_A) \Rightarrow \sigma_B \qquad\qquad |A \to B| = |B|$$
$$\sigma_{\mathbf{C}^q A} := \sigma_A \qquad\qquad |\mathbf{C}^q A| = q \cdot |A|$$

We let then $\mathfrak{s}_A := \mathbf{C}^{|A|}\sigma_A$ (observe that $\mathfrak{s}_{\mathbf{C}^q A} = \mathbf{C}^{q \cdot |A|}\sigma_A$).

The translation of a derivation Π of $\Phi; \Gamma \vdash \mathscr{b} \rightarrowtail A$ into a typing derivation D^{Π} of $\mathfrak{s}_\Phi, \mathfrak{s}_\Gamma \vdash t^{\Pi} : \mathscr{b} \rightarrowtail \mathfrak{s}_A$ in $\mathbf{C}\lambda_{\to}$ is illustrated in Fig. 16, where we exploit the fact that if $\Phi = \{A\}$ is non-empty, then $t^{\Pi} = t^{\Pi}[x : \mathfrak{s}_A]$ is a head context (as it can be checked by induction on the construction). We omit the case of the rule $(\mathbf{C}_\times)$, as it follows immediately from the induction hypothesis, since $\mathfrak{s}_{\mathbf{C}^q\mathbf{C}^s A} = \mathfrak{s}_{\mathbf{C}^{qs} A}$.

The stability of the translation under normalization is easily checked using Lemma A.23.

B Details about the Probabilistic Event λ-Calculus.

B.1 The Language Λ_{PE}.

Let us first comment on how Theorem 4.1 (i.e. confluence and strong normalization of $\rightarrowtriangle_{\mathsf{p}}$, and confluence of $\rightarrowtriangle$) follows from confluence of $\rightarrowtriangle$ in the calculus from [17]. The λ-calculus in [17] slightly differs from our presentation of Λ_{PE}, since choice operators do not depend on indexes $i \in \mathbb{N}$. However, if φ is any bijection from $\mathbb{N}^2$ to $\mathbb{N}$, one can define an invertible embedding $t \mapsto t^{\varphi}$ from Λ_{PE} to the calculus in [17] by replacing $t \oplus_a^i u$ with $t^{\varphi} \overset{\varphi(a,i)}{\oplus} u^{\varphi}$ and $(\nu a.t)^{\varphi} = \nu\varphi(a,0).\dots.\nu\varphi(a,\mathrm{ord}_a(t)).t^{\varphi}$, where $\mathrm{ord}_a(t)$ is the maximum i s.t. $\oplus_a^i$ occurs in t. Since the permutative rules in [17] translate into those in Fig. 5 under this translation, the results from [17] can be transported to our language.

We now discuss PNFs in Λ_{PE}.

Definition B.1. Let S be a set of name-closed λ-terms and a be a name. For all $i \in \mathbb{N} \cup \{-1\}$, the set of (S,a)-*trees of level* i is defined as follows:

- any $t \in S$ is a (S,a)-tree of level -1;
- if t, u are (S,a)-trees of level j and k, respectively, and $j,k < i$, then $t \oplus_a^i u$ is a (S,a)-tree of level i.

The *support* of a (S,a)-tree t, indicated as $\mathrm{Supp}(t)$, is the finite set of terms in S which are leaves of t.

Definition B.2. The sets $\mathcal{T}$ and $\mathcal{V}$ of name-closed terms are defined inductively as follows:

- all variable $x \in \mathcal{V}$;
- if $t \in \mathcal{V}$, $\lambda x.t \in \mathcal{V}$;
- if $t \in \mathcal{V}$ and $u \in \mathcal{T}$, then $tu \in \mathcal{V}$;
- if $t \in \mathcal{V}$, then $t \in \mathcal{T}$;
- if t is a $(\mathcal{T},a)$-tree, then $\nu a.t \in \mathcal{T}$.

Lemma B.3. *For all name-closed term $t \in \mathcal{T}$, $t \in \mathcal{V}$ iff it does not start with ν.*

Proof. First observe that if $t = \nu a.b$, then $t \notin \mathcal{V}$. For the converse direction, we argue by induction on $t \neq \nu a.b$:

- if $t = x$ then $t \in \mathcal{V}$;
- if $t = \lambda x.u$ then by IH $u \in \mathcal{V}$, so $t \in \mathcal{V}$;
- if $t = uv$, then the only possibility is that $t \in \mathcal{V}$;
- if $t = u \oplus_a^i v$, then t is not name-closed, again the hypothesis.

□

For all term t and $a \in \mathsf{FN}(t)$, we let $I_a(t)$ be the maximum index i such that $\oplus_a^i$ occurs in t, and $I_a(t) = -1$ if $\oplus_a^i$ does not occur in t for all index i.

We now show that Definition B.2 precisely captures PNF.

Lemma B.4. *A name-closed term t is in PNF iff $t \in \mathcal{T}$.*

Proof. **($\Rightarrow$)** We argue by induction on t. If t has no bound name, then t is obviously in $\mathcal{T}$. Otherwise:

- if $t = \lambda x.u$, then u is also name-closed. Hence, by IH $u \in \mathcal{T}$; observe that u cannot start with ν (as t would not be normal) so by Lemma B.3, $u \in \mathcal{V}$, which implies $t \in \mathcal{T}$;
- if $t = uv$, then u and v are both name-closed, and so by induction $u, v \in \mathcal{T}$; if u started with ν, then t would not be normal, hence, by Lemma B.3, $u \in \mathcal{V}$, and we conclude then that $t \in \mathcal{T}$.
- if $t = u \oplus_a^i v$, then it cannot be name-closed;
- if $t = \nu a.u$, then we show, by a sub-induction on u, that u is a $(\mathcal{T},a)$-tree of level $I_a(u)$: first note that $I_a(u)$ cannot be -1, since otherwise $\nu a.u \rightarrowtriangle_{\mathsf{p}} u$, so u would not be normal. We now consider all possible cases for u:
  - u cannot be a variable, or $I_a(u)$ would be -1;
  - If $u = \lambda x.v$, then $I_a(v) = I_a(u)$ and so by sub-IH v is a $(\mathcal{T},a)$-tree of level $I_a(u)$, which implies that $u = \lambda x.v_1 \oplus_a^{I_a(u)} v_2$, which is not normal. Absurd.
  - if $u = vw$, then let $J = I_a(u) = \max\{I_a(v), I_a(w)\}$ cannot be -1, since otherwise $I_a(u)$ would be -1. Hence $J \geq 0$ is either $I_a(v)$ or $I_a(w)$. We consider the two cases separately:
    - if $J = I_a(v)$, then by sub-IH, $v = v_1 \oplus_a^J v_2$, and thus $u = (v_1 \oplus_a^J v_2)w$ is not normal;
    - if $J = I_a(w)$, then by sub-IH, $w = w_1 \oplus_a^J w_2$, and thus $u = v(w_1 \oplus_a^J w_2)$ is not normal.

    In any case we obtain an absurd conclusion.
  - if $u = u_1 \oplus_a^i u_2$, then if u_1, u_2 are both in $\mathcal{V}$, we are done, since u is a $(\mathcal{T},a)$-tree of level $i = I_a(u)$. Otherwise, if $i < I_a(u)$, then $J = I_a(u) = \max\{I_a(v), I_a(w)\}$, so we consider two cases:

- if $J = I_a(v) > i$, then by sub-IH, $v = v_1 \oplus_a^J v_2$, and thus $u = (v_1 \oplus_a^J v_2) \oplus_a^i u_2$ is not normal;
- if $J = I_a(w) > i$, then by sub-IH, $w = w_1 \oplus_a^J w_2$, and thus $u = v \oplus_a^i (w_1 \oplus_a^J w_2)$ is not normal.

We conclude then that $i = I_a(u)$; we must then show that $I_a(v), I_a(w) < i$. Suppose first $I_a(v) \geq i$, then $(v_1 \oplus_a^J v_2) \oplus_a^i u_2$ is not normal. In a similar way we can show that $I_a(w) < i$.

- if $u = \nu b.v$, then $I_a(u) = I_a(v)$, so by sub-IH $v = v_1 \oplus_a^{I_a(u)} v_2$, and we conclude that $u = \nu b.v_1 \oplus_a^i v_2$ is not normal.

(⇐) It suffices to check by induction on $t \in \mathcal{T}$ that it is in PNF.

□

Corollary B.5. *A name-closed term of the form $\nu a.t$ is in PNF iff t is a $(\mathcal{T}, a)$-tree of level $I_a(t) \geq 0$.*

Let us conclude by discussing *randomized trees* and *head-reduction*. With any term t we associate a labeled finitely branching tree $RBT(t)$ as follows:

- if $t \rightarrowtriangle_{\mathsf{h}}^* u$, where $u = \lambda x_1. \dots .\lambda x_n.yu_1 \dots u_n$ is a HNV, then $RBT(t)$ only consists of one node labeled u;
- if $t \rightarrowtriangle_{\mathsf{h}}^* u$, where u is a PNF of the form $\nu a.T$, T is a $(\mathcal{T}, a)$-tree and $\mathrm{supp}(T) = \langle u_1, \dots, u_n \rangle$, then $RBT(t)$ has root labeled $\nu a.T$ and coincides with the syntactic tree of T, with leaves replaced by $RBT(u_1), \dots, RBT(u_n)$;
- otherwise, $RBT(t)$ has a root labeled Ω and no sub-trees.

Proposition B.6. *If t reduces to u by either $\rightarrowtriangle_{\mathsf{p}}$ or head β-reduction, then $RBT(t) \sqsubseteq RBT(u)$.*

A *randomized path in $RBT(t)$* is any path π in $RBT(t)$ starting from the root and ending in a leaf. A randomized path in $RBT(t)$ is thus of one of the following two forms:

1. a finite path $\pi = \langle \nu a_1.T_1, \nu a_2.T_2, \dots, \nu a_N.T_N, \lambda x_1 \dots x_n.y \rangle$, where $v \rightsquigarrow \nu a_{i+1}.T_{i+1}$ holds for some $v \in \mathrm{supp}(T_i)$;
2. an infinite path $\pi = \langle \nu a_1.T_1, \nu a_2.T_2, \dots, \nu a_n.T_n, \dots \rangle$, where $v \rightsquigarrow \nu a_{i+1}.T_{i+1}$ holds for some $v \in \mathrm{supp}(T_i)$.

With any such path π we can associate an (either finite or infinite) list of words $b_i^\pi \in \{0,1\}^*$, where b_i^π is the list of choices leading from $\nu a.T_i$ to the unique term $v \in \mathrm{supp}(T_i)$ such that $v \rightsquigarrow \nu a_{i+1}.T_{i+1}$.

If π is a finite randomized path in $RBT(t)$ ending in some $u \in \mathcal{V}$, we say that π is a randomized path *from t to u* (noted $\pi : t \mapsto u$).

Let us consider the (unique) Borel σ-algebra μ on $2^{\mathbb{N}\times\mathbb{N}}$ satisfying $\mu(C_{i,j}) = 1/2$, where $C_{i,j}$ is the cylinder $\{\omega \mid \omega(i,j) = 1\}$.

Any randomized path π yields a Borel set $B_\pi := \bigcap_{i,j} C_{i,(b_i^\pi)_j}$, so that two distinct paths $\pi \neq \pi'$ are such that $B_\pi \cap B_{\pi'} = \emptyset$. Moreover, if π is finite, the Borel set B_π is captured by the Boolean formula $\mathscr{b}_\pi := \bigwedge_i \mathscr{b}_\pi^i$, where $\mathscr{b}_\pi^i = \bigwedge_j (\neg)^{1+(b_i^\pi)_j} \mathscr{x}_{a_i}^j$.

Lemma B.7. *For all terms $t \in \mathcal{T}$ and $u \in \mathcal{V}$, $\mathcal{D}_t(u) = \mu(\bigcup\{B_\pi \mid \pi : t \mapsto u\}) = \sum_{\pi : t \mapsto u} \mu(\mathscr{b}_\pi)$.*

Lemma B.8. *For all $t \in \mathcal{T}$, the following are equivalent:*
(i.) t has a head-normal form;
(iii.) $RBT(t)$ is finite.

Proof. (i.)⇒ (iii.) is proved by induction on the length of a reduction of t to a head-normal form. For (iii.)⇒ (i.), observe that $RBT(t)$ contains an infinite path only if t admits an infinite head-reduction. From (iii.) we deduce then that all head reductions of t are finite, so t has a head-normal form. □

B.2 The Language $\Lambda_{\mathsf{PE}}^{\{\}}$.

In this section we study some properties of reduction $\rightarrowtriangle_{\{\}}$ in the calculus $\Lambda_{\mathsf{PE}}^{\{\}}$. Most arguments closely resemble arguments from [17], so we omit several details.

Lemma B.9. *$\rightarrowtriangle_{\mathsf{p}\{\}}$ is strongly normalizing.*

Proof. We follow the proof of [[17], Lemma 7], we define the partial order $\prec_M$ by

$$\begin{aligned} \oplus_a^i &\prec_M \oplus_b^j && \text{if } (a,i) \prec_M (b,j) \\ \oplus_a^i &\prec_M \nu b && \text{for all labels } a, b \\ \nu a &\prec_M @, \lambda x && \text{fo any label } b \\ @, \lambda x &\prec_M \{\} \end{aligned}$$

One can check then that the well-founded recursive path ordering $<$ defined by

$$t < u \;\Leftrightarrow\; \begin{cases} [t_1, \dots, t_n] < [u_1, \dots, u_m] & \text{if } f = g \\ [t_1, \dots, t_n] < [u] & \text{if } f \prec_M g \\ [t] \leq [u_1, \dots, u_m] & \text{if } f \not\prec_B g \end{cases}$$

where $t = f(t_1, \dots, t_n)$ and $u = g(u_1, \dots, u_n)$, with $f, g \in \{\oplus^i_a, \nu a., @, \lambda x, \{\}\}$, is such that $t \rightarrowtriangle_{\mathsf{p}\{\}} u$ implies $u < t$. □

Lemma B.10. *$\rightarrowtriangle_{\mathsf{p}\{\}}$ is confluent.*

Proof. By Lemma B.9 and Newman's Lemma, it suffices to check local confluence. All rules from (c_1) to ($\oplus\nu$) from Fig. 5, as well as rules ($\{\}\oplus_1$) and ($\{\}\oplus_2$), can be written under the general form

$$\mathsf{C}[t \oplus^i_a u] \rightarrowtriangle_{\mathsf{p}\{\}} \mathsf{C}[t] \oplus^i_a \mathsf{C}[u] \qquad (\oplus\star)$$

where $\mathsf{C}[\ ]$ is defined by the grammar

$$\mathsf{C}[\ ] ::= [\ ] \mid \lambda x.\mathsf{C}[\ ] \mid \mathsf{C}[\ ]u \mid t\mathsf{C}[\ ] \mid \mathsf{C}[\ ] \oplus^i_a u \mid t \oplus^i_a \mathsf{C}[\ ] \mid \nu a.\mathsf{C}[\ ] \mid \{\mathsf{C}[\ ]\}u \mid \{t\}\mathsf{C}[\ ]$$

We consider all rules against each other. All cases involving ($\oplus\star$) can be treated as in the proof of [[17], Lemma 9]. Beyond these, the only new case with respect to those treated in [[17], Lemma 9] is the one below:

$$\begin{tikzcd}
\{t\}\nu a.u \oplus^i_b v \arrow[r, "(\oplus\nu)"] \arrow[d, "(\{\}\nu)"] & \{t\}(\nu a.u) \oplus^i_b (\nu a.v) \arrow[d, "(\{\}\oplus_1)"] \\
\nu a.t(u \oplus^i_b v) \arrow[d, "(\oplus\mathsf{a})"] & (\{t\}\nu a.u) \oplus^i_b (\{t\}\nu a.v) \arrow[d, "(\{\}\nu)"] \\
\nu a.(tu) \oplus^i_b (tv) \arrow[r, "(\oplus\nu)"'] & (\nu a.tu) \oplus^i_b (\nu a.tv)
\end{tikzcd}$$

□

Theorem B.11. *$\rightarrowtriangle_{\{\}}$ is confluent.*

Proof. The argument closely follows the one from [[17], pp. 9-12], using the observation that none of the permutations ($\{\}\nu$)-($\{\}\oplus_2$) can either block a β-redex or figure in a critical pair with a β-redex (in other words, β-reduction commutes with ($\{\}\nu$)-($\{\}\oplus_2$)). □

To conclude, let us study head-reduction $\rightarrowtriangle_{\{\}h}$ in $\Lambda^{\{\}}_{\mathsf{PE}}$.

Randomized contexts $\mathsf{R}[\ ]$ for $\Lambda^{\{\}}_{\mathsf{PE}}$ are defined as for Λ_{PE}. *Head-contexts* $\mathsf{H}[\ ]$ for Λ_{PE} are defined by the grammar below:

$$\mathsf{H}[\ ] := [\ ] \mid \lambda x.\mathsf{H}[\ ] \mid \mathsf{H}[\ ]u \mid \{\mathsf{H}[\ ]\}u$$

Head-reductions $t \rightarrowtriangle_{\{\}h} u$ are defined inductively as being either a $\rightarrowtriangle_{\{\}}$-reduction or a β-reduction of one of the following two forms

$$\mathsf{R}[\mathsf{H}[(\lambda y.t)u]] \rightarrowtriangle_{\beta} \mathsf{R}[\mathsf{H}[t[u/y]]]$$
$$\mathsf{R}[\mathsf{H}[\{t\}u]] \rightarrowtriangle \mathsf{R}[\mathsf{H}[\{t\}u']]$$

where $u \rightarrowtriangle_{\{\}h} u'$.

Let $\mathcal{T}^{\{\}}$ be the set of PNF with respect to $\rightarrowtriangle_{\mathsf{p}\{\}}$. A pseudo-value t is a PNF which is either a λ, a variable, an application tu or a CbV application $\{t\}u$. We let $\mathcal{V}^{\{\}}$ indicate the set of pseudo-values.

The definition of $\pi^{\omega}_X(t)$ is extended from Λ_{PE} to $\Lambda^{\{\}}_{\mathsf{PE}}$ simply by adding the condition $\pi^{\omega}_X(\{t\}u) = \{\pi^{\omega}_X(t)\}\pi^{\omega}_X(u)$. In this way, a sub-distribution $\mathcal{D}_t : \mathcal{V}^{\{\}} \to [0,1]$ can be defined as in Section 3.

Lemma B.12. *A pseudo-value $t \in \mathcal{V}^{\{\}}$ is in head normal form iff either $t = \mathsf{H}[x]$ or $t = \mathsf{H}[\{t\}u]$, where $u \in \mathcal{V}^{\{\}}$ and is $\rightarrowtriangle_{\{\}}$-normal.*

A head normal value is defined as in Λ_{PE} as a pseudo-value in head normal form. We let $\mathrm{HNV}^{\{\}}$ indicate the set of head normal values. Using Lemma B.12 we can define, in analogy with the case of Λ_{PE}, the functions $\mathrm{HNV}(t) := \sum_{v \in \mathrm{HNV}^{\{\}}} \mathcal{D}_t(v)$, and $\mathrm{HNV}_{\rightarrowtriangle}(t) := \sup\{\mathrm{HNV}(u) \mid t \rightarrowtriangle^*_{\{\}h} u\}$.

C Details about $C\lambda^{\{\}}_{\rightarrow}$.

C.1 Subject Reduction.

The goal of this section is to establish the following result:

Proposition C.1 (Subject Reduction). *If $\Gamma \vdash^{X} t : \mathscr{b} \rightarrowtail \mathsf{s}$ and $t \rightarrow_{\{\}} u$, then $\Gamma \vdash^{X} u : \mathscr{b} \rightarrowtail \mathsf{s}$.*

With the goal of making proof slightly simpler, in the formulation of $C\lambda^{\{\}}_{\rightarrow}$ we replace the rule $(\oplus)$ with the two rules $(\oplus l)$ and $(\oplus r)$ below:

$$\frac{\Gamma \vdash^{X\cup\{a\}} t : \mathscr{c} \rightarrowtail \mathsf{s} \qquad \mathscr{b} \vDash \mathscr{c} \wedge \mathscr{x}^{i}_{a}}{\Gamma \vdash^{X\cup\{a\}} t \oplus^{i}_{a} u : \mathscr{b} \rightarrowtail \mathsf{s}}\ (\oplus l) \qquad \frac{\Gamma \vdash^{X\cup\{a\}} u : \mathscr{c} \rightarrowtail \mathsf{s} \qquad \mathscr{b} \vDash \mathscr{c} \wedge \neg\mathscr{x}^{i}_{a}}{\Gamma \vdash^{X\cup\{a\}} t \oplus^{i}_{a} u : \mathscr{b} \rightarrowtail \mathsf{s}}\ (\oplus r)$$

It is easily checked that, in presence of the rule $(\vee)$, having the rule $(\oplus)$ is equivalent to having the rules $(\oplus l)$ and $(\oplus r)$.

To establish the subject reduction property, we first need to establish a few auxiliary lemmas.

Lemma C.2. *If $\Gamma \vdash^{X} t : \mathscr{b} \rightarrowtail \mathsf{s}$ and $X \subseteq Y$, then $\Gamma \vdash^{Y} t : \mathscr{b} \rightarrowtail \mathsf{s}$.*

Proof. By induction on a type derivation of t. □

Lemma C.3. *If $\Gamma \vdash^{X} t : \mathscr{b} \rightarrowtail \mathsf{s}$ holds and $\mathscr{c} \vDash^{X} \mathscr{b}$, then $\Gamma \vdash^{X} t : \mathscr{c} \rightarrowtail \mathsf{s}$ is derivable by a derivation of the same length.*

Proof. By induction on a type derivation of t. □

Lemma C.4 (substitution lemma). *The following rule is derivable:*

$$\frac{\Gamma, x : \mathsf{s} \vdash^{X} t : \mathscr{c} \rightarrowtail \mathsf{t} \qquad \Gamma \vdash^{X} u : \mathscr{d} \rightarrowtail \mathsf{s} \qquad \mathscr{b} \vDash \mathscr{c} \wedge \mathscr{d}}{\Gamma \vdash^{X} t[u/x] : \mathscr{b} \rightarrowtail \mathsf{t}}\ (subst)$$

Proof. We argue by induction on the typing derivation of t:

- if the last rule is
$$\frac{\mathsf{FN}(\mathscr{b}) \subseteq X}{\Gamma, x : \mathsf{s} \vdash^{X} x : \mathscr{c} \rightarrowtail \mathsf{s}}\ (\mathrm{id})$$
then $t[u/x] = u$, so the claim can be deduced using Lemma C.3.
- if the last rule is
$$\frac{\{\Gamma, x : \mathsf{s} \vdash^{X} t : \mathscr{c}_i \rightarrowtail \mathsf{t}\}_i \qquad \mathscr{c} \vDash^{X} \bigvee_i \mathscr{c}_i}{\Gamma, x : \mathsf{s} \vdash^{X} x : \mathscr{c} \rightarrowtail \mathsf{t}}\ (\vee)$$
Then, by IH, we deduce $\Gamma, x : \mathsf{s} \vdash^{X} t[u/x] : \mathscr{c}_i \wedge \mathscr{d} \rightarrowtail \mathsf{t}$, and since $\mathscr{b} \vDash^{X} \bigvee_i (\mathscr{b}_i \wedge \mathscr{d})$ we conclude by applying an instance of $(\vee)$.
- if the last rule is
$$\frac{\Gamma, x : \mathsf{s} \vdash^{X\cup\{a\}} t_1 : \mathscr{c}' \rightarrowtail \mathsf{t} \qquad \mathscr{c} \vDash^{X\cup\{a\}} \mathscr{x}^{i}_{a} \wedge \mathscr{c}'}{\Gamma, x : \mathsf{s} \vdash^{X\cup\{a\}} t_1 \oplus^{i}_{a} t_2 : \mathscr{c} \rightarrowtail \mathsf{t}}\ (\oplus l)$$
Then, by IH, we deduce $\Gamma \vdash^{X\cup\{a\},q} t_1[u/x] : \mathscr{c}' \wedge \mathscr{d}$. From $t[u/x] = (t_1[u/x]) \oplus^{i}_{a} (t_2[u/x])$ and the fact that $\mathscr{b} \vdash^{X\cup\{a\}} \mathscr{x}^{i}_{a} \wedge (\mathscr{c}' \wedge \mathscr{d})$, we deduce the claim by an instance of the same rule.
- if the last rule is
$$\frac{\Gamma, x : \mathsf{s} \vdash^{X\cup\{a\}} t_2 : \mathscr{c}' \rightarrowtail \mathsf{t} \qquad \mathscr{c} \vDash^{X\cup\{a\}} \neg\mathscr{x}^{i}_{a} \wedge \mathscr{c}'}{\Gamma, x : \mathsf{s} \vdash^{X\cup\{a\}} t_1 \oplus^{i}_{a} t_2 : \mathscr{c} \rightarrowtail \mathsf{t}}\ (\oplus r)$$
we can argue similarly to the previous case.
- if the last rule is
$$\frac{\Gamma, y : \mathsf{t}', x : \mathsf{s} \vdash^{X} t' : \mathscr{c} \rightarrowtail \mathsf{C}^{\vec{q}}\tau}{\Gamma, x : \mathsf{s} \vdash^{X} \lambda y.t' : \mathscr{c} \rightarrowtail \mathsf{C}^{\vec{q}}(\mathsf{t}' \Rightarrow \tau)}\ (\lambda)$$
Then, by IH, we deduce $\Gamma, y : \mathsf{t}' \vdash^{X} t'[u/x] : \mathscr{b} \rightarrowtail \mathsf{C}^{\vec{q}}\tau$ and since $t[u/x] = (\lambda y.t')[u/x] = \lambda y.t'[u/x]$ we conclude by applying an instance of the same rule.
- if the last rule is
$$\frac{\Gamma, x : \mathsf{s} \vdash^{X} t_1 : \mathscr{c}_1 \rightarrowtail \mathsf{C}^{\vec{q}}(\mathsf{t}' \Rightarrow \tau) \qquad \Gamma, x : \mathsf{s} \vdash^{X} t_2 : \mathscr{c}_2 \rightarrowtail \mathsf{t}' \qquad \mathscr{c} \vDash^{X} \mathscr{c}_1 \wedge \mathscr{c}_2}{\Gamma, x : \mathsf{s} \vdash^{X} t_1 t_2 : \mathscr{c} \rightarrowtail \mathsf{C}^{\vec{q}}\tau}\ (@)$$

Then, by IH, we deduce $\Gamma \vdash^X t_1[u/x] : \mathscr{c}_1 \wedge \mathscr{d} \rightarrowtail \mathrm{C}^{\vec{q}}(\mathrm{t}' \Rightarrow \tau)$ and $\Gamma \vdash^X t_2[u/x] : \mathscr{c}_2 \wedge \mathscr{d} \rightarrowtail \mathrm{t}'$, and since $(t_1 t_2)[u/x] = (t_1[u/x])(t_2[u/x])$ and $\mathscr{b} \vDash^X (\mathscr{c}_1 \wedge \mathscr{d}) \wedge (\mathscr{c}_2 \wedge \mathscr{d})$, we conclude by applying an instance of the same rule.

- if the last rule is

$$\frac{\Gamma, x : \mathfrak{s} \vdash^X t_1 : \mathscr{c}_1 \rightarrowtail \mathrm{C}^{\vec{q}}(\mathrm{t}' \Rightarrow \tau) \qquad \Gamma, x : \mathfrak{s} \vdash^X t_2 : \mathscr{c}_2 \rightarrowtail \mathrm{C}^{s}\mathrm{t}' \qquad \mathscr{c} \vDash^X \mathscr{c}_1 \wedge \mathscr{c}_2}{\Gamma, x : \mathfrak{s} \vdash^X \{t_1\}t_2 : \mathscr{c} \rightarrowtail \mathrm{C}^{s}\mathrm{C}^{\vec{q}}\tau} (\{\})$$

Then, by IH, we deduce $\Gamma \vdash^X t_1[u/x] : \mathscr{c}_1 \wedge \mathscr{d} \rightarrowtail \mathrm{C}^{\vec{q}}(\mathrm{t}' \Rightarrow \tau)$ and $\Gamma \vdash^X t_2[u/x] : \mathscr{c}_2 \wedge \mathscr{d} \rightarrowtail \mathrm{C}^{s}\mathrm{t}'$, and since $\{t_1\}t_2)[u/x] = \{t_1[u/x]\}(t_2[u/x])$ and $\mathscr{b} \vDash^X (\mathscr{c}_1 \wedge \mathscr{d}) \wedge (\mathscr{c}_2 \wedge \mathscr{d})$, we conclude by applying an instance of the same rule.

- if the last rule is

$$\frac{\Gamma, x : \mathfrak{s} \vdash^{X \cup \{a\}} t : \mathscr{c}' \wedge \mathscr{e} \rightarrowtail \mathrm{t} \qquad \mu(\mathscr{e}) \geq s \qquad \mathscr{c} \vDash^X \mathscr{c}'}{\Gamma, x : \mathfrak{s} \vdash^X \nu a.t : \mathscr{c} \rightarrowtail \mathrm{C}^{s}\mathrm{t}} (\mu)$$

Then, by IH, $\Gamma \vdash^{X \cup \{a\}} t[u/x] : (\mathscr{c}' \wedge \mathscr{d}) \wedge \mathscr{e}_i \rightarrowtail \mathrm{t}$. Hence, from the fact that $\mathscr{b} \vDash^X \mathscr{c} \wedge \mathscr{d}$ and that a cannot occur in $\mathscr{d}$, and since $(\nu a.t)[u/x] = \nu a.t[u/x]$, we can deduce the claim by applying an instance of the same rule.

□

We now have all ingredients to establish the subject reduction property of $\mathrm{C}\lambda^{\{\}}_{\rightarrow}$.

Proof of Proposition C.1. First observe that if the typing derivation D of t ends by a $(\vee)$-rule, it suffices to establish the property for the immediate sub-derivations of D and then apply an instance of $(\vee)$-rule to the resulting derivations. So we will always suppose that the typing derivation of D does not end by a $(\vee)$-rule.

For the case of β-reduction it suffices to check the claim when t is a redex $(\lambda x.t_1)t_2$ and u is $t_1[t_2/x]$. From $\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathfrak{s}$ we can suppose w.l.o.g. that the typing derivation is as below:

$$\frac{\dfrac{\left\{\dfrac{\Gamma, x : \mathrm{t} \vdash^X t_1 : \mathscr{c}_i \rightarrowtail \mathrm{C}^{\vec{q}}\sigma}{\Gamma \vdash^X \lambda x.t_1 : \mathscr{c}_i \rightarrowtail \mathrm{C}^{\vec{q}}(\mathrm{t} \Rightarrow \sigma)}\right\}_i \qquad \mathscr{b}_1 \vDash \bigvee_i \mathscr{c}_i}{\Gamma \vdash^X \lambda x.t_1 : \mathscr{b}_1 \rightarrowtail \mathrm{C}^{\vec{q}}(\mathrm{t} \Rightarrow \sigma)} (\vee) \qquad \dfrac{\left\{\Gamma \vdash^X t_2 : \mathscr{d}_j \rightarrowtail \mathrm{t}\right\}_j \qquad \mathscr{b}_2 \vDash \bigvee_j \mathscr{d}_j}{\Gamma \vdash^X t_2 : \mathscr{b}_2 \rightarrowtail \mathrm{t}} (\vee) \qquad \mathscr{b} \vDash \mathscr{b}_1 \wedge \mathscr{b}_2}{\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathrm{C}^{\vec{q}}\sigma}$$

From Lemma C.4 we deduce the existence of derivations of $\Gamma \vdash^X u : \mathscr{c}_i \wedge \mathscr{d}_j \rightarrowtail \mathrm{C}^{\vec{q}}\sigma$; from $\mathscr{b} \vDash^X \mathscr{b}_1 \wedge \mathscr{b}_2$ we deduce then $\mathscr{b} \vDash^X \bigvee_{i,j} \mathscr{c}_i \wedge \mathscr{d}_j$, using the fact that $\bigvee_i \mathscr{c}_i \wedge \bigvee_j \mathscr{d}_j \equiv \bigvee_{i,j} \mathscr{c}_i \wedge \mathscr{d}_j$. We can thus conclude as follows:

$$\frac{\left\{\Gamma \vdash^X u : \mathscr{c}_i \wedge \mathscr{d}_j \rightarrowtail \mathrm{C}^{\vec{q}}\sigma\right\}_{i,j} \qquad \mathscr{b} \vDash^X \bigvee_{i,j} \mathscr{c}_i \wedge \mathscr{d}_j}{\Gamma \vdash^X u : \mathscr{b} \rightarrowtail \mathrm{C}^{\vec{q}}\sigma} (\vee)$$

For the case of $\rightarrow_{\mathrm{p}\{\}}$ wee consider reduction rules one by one.

($t \oplus^i_a t \rightarrow_{\mathrm{p}\{\}} t$) The last rule of t is either

$$\frac{\Gamma \vdash^X t : \mathscr{b}' \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash^X \mathscr{x}^i_a \wedge \mathscr{b}'}{\Gamma \vdash^X t \oplus^a_i t : \mathscr{b} \rightarrowtail \mathfrak{s}} (\oplus l)$$

or

$$\frac{\Gamma \vdash^X t : \mathscr{b}' \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash^X \neg \mathscr{x}^i_a \wedge \mathscr{b}'}{\Gamma \vdash^X t \oplus^a_i t : \mathscr{b} \rightarrowtail \mathfrak{s}} (\oplus r)$$

Then, in either case, from $\Gamma \vdash^X t : \mathscr{b}' \rightarrowtail \mathfrak{s}$, $\mathscr{b} \vDash^X \mathscr{b}'$, using Lemma C.3 we deduce $\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathfrak{s}$.

($(t \oplus^i_a u) \oplus^i_a v \rightarrow_{\mathrm{p}\{\}} t \oplus^i_a v$) There are three possible sub-cases:

1. the type derivation is as follows:

$$\frac{\dfrac{\Gamma \vdash^X t : \mathscr{b}'' \rightarrowtail \mathfrak{s} \qquad \mathscr{b}' \vDash^X \mathscr{x}^i_a \wedge \mathscr{b}''}{\Gamma \vdash^X t \oplus^i_a u : \mathscr{b}' \rightarrowtail \mathfrak{s}} (\oplus l) \qquad \mathscr{b} \vDash^X \mathscr{x}^i_a \wedge \mathscr{b}'}{\Gamma \vdash^X (t \oplus^i_a u) \oplus^i_a : \mathscr{b} \rightarrowtail \mathfrak{s}} (\oplus l)$$

Then from $\Gamma \vdash^X t : \mathscr{b}'' \rightarrowtail \mathfrak{s}$ and since we have $\mathscr{b} \vdash^X \mathscr{x}^i_a \wedge \mathscr{b}''$ we deduce $\Gamma \vdash^X t \oplus^i_a v : \mathscr{b} \rightarrowtail \mathfrak{s}$.

2. the type derivation is as follows:

$$\dfrac{\dfrac{\Gamma \vdash^X u : \mathscr{b}'' \rightarrowtail \mathfrak{s} \qquad \mathscr{b}' \vDash^X \neg \mathscr{x}^i_a \wedge \mathscr{b}''}{\Gamma \vdash^X t \oplus^i_a u : \mathscr{b}' \rightarrowtail \mathfrak{s}}\,(\oplus r) \qquad \mathscr{b} \vDash^X \mathscr{x}^i_a \wedge \mathscr{b}'}{\Gamma \vdash^X (t \oplus^i_a u) \oplus^i_a v : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus l)$$

Then, from $\mathscr{b} \vDash^X \mathscr{x}^i_a \wedge \mathscr{b}'$ and $\mathscr{b}' \vDash^X \neg \mathscr{x}^i_a \wedge \mathscr{b}''$ we deduce $\mathscr{b} \vDash^X \bot$, so we conclude $\Gamma \vdash^X (t \oplus^i_a v) : \mathscr{b} \rightarrowtail \mathfrak{s}$ using one of the initial rules.

3. the type derivation is as follows:

$$\dfrac{\Gamma \vdash^X v : \mathscr{b}'' \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash^X \neg \mathscr{x}^i_a \wedge \mathscr{b}''}{\Gamma \vdash^X (t \oplus^i_a u) \oplus^i_a v : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus r)$$

Then from $\Gamma \vdash^X u : \mathscr{b}' \rightarrowtail \mathfrak{s}$ and $\mathscr{b} \vDash^X \neg \mathscr{x}^i_a \wedge \mathscr{b}''$ we deduce $\Gamma \vdash^X t \oplus^i_a v : \mathscr{b} \rightarrowtail \mathfrak{s}$.

$(t \oplus^i_a (u \oplus^i_a v) \mathrel{-\!\!\triangleright}_{\mathsf{p}\{\}} t \oplus^i_a v)$ Similar to the case above.

$(\lambda x.(t \oplus^i_a u) \mathrel{-\!\!\triangleright}_{\mathsf{p}\{\}} (\lambda x.t) \oplus^i_a (\lambda x.u))$ There are two possible sub-cases:

1. $$\dfrac{\dfrac{\Gamma, x : \mathfrak{s} \vdash^X t : \mathscr{b}' \rightarrowtail \mathbf{C}^{\vec{q}}\tau \qquad \mathscr{b} \vDash^X \mathscr{x}^i_a \wedge \mathscr{b}'}{\Gamma, x : \mathfrak{s} \vdash^X t \oplus^i_a u : \mathscr{b} \rightarrowtail \mathbf{C}^{\vec{q}}\tau}\,(\oplus l)}{\Gamma \vdash^X \lambda x.(t \oplus^i_a u) : \mathscr{b} \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \tau)}\,(\lambda)$$

   Then, we deduce

   $$\dfrac{\dfrac{\Gamma, x : \mathfrak{s} \vdash^X t : \mathscr{b}' \rightarrowtail \mathbf{C}^{\vec{q}}\tau}{\Gamma \vdash^X \lambda x.t : \mathscr{b}' \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \tau)}\,(\lambda) \qquad \mathscr{b} \vDash^X \mathscr{x}^i_a \wedge \mathscr{b}'}{\Gamma \vdash^X (\lambda x.t) \oplus^i_a (\lambda x.u) : \mathscr{b} \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \tau)}\,(\oplus l)$$

2. $$\dfrac{\dfrac{\Gamma, x : \mathfrak{s} \vdash^X u : \mathscr{b}' \rightarrowtail \mathbf{C}^{\vec{q}}\tau \qquad \mathscr{b} \vDash^X \neg \mathscr{x}^i_a \wedge \mathscr{b}'}{\Gamma, x : \mathfrak{s} \vdash^X t \oplus^i_a u : \mathscr{b} \rightarrowtail \mathbf{C}^{\vec{q}}\tau}\,(\oplus r)}{\Gamma \vdash^X \lambda x.(t \oplus^i_a u) : \mathscr{b} \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \tau)}\,(\lambda)$$

   Then, we can argue similarly to the previous case.

$((t \oplus^i_a u)v \mathrel{-\!\!\triangleright}_{\mathsf{p}\{\}} (tv) \oplus^i_a (uv))$ There are two possible sub-cases:

1. $$\dfrac{\dfrac{\Gamma \vdash^X t : \mathscr{b}'' \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \tau) \qquad \mathscr{b}' \vDash^X \mathscr{x}^i_a \wedge \mathscr{b}''}{\Gamma \vdash^X t \oplus^i_a u : \mathscr{b}' \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \tau)}\,(\oplus l) \qquad \Gamma \vdash^X v : \mathscr{c} \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash^X \mathscr{b}' \wedge \mathscr{c}}{\Gamma \vdash^X (t \oplus^i_a u)v : \mathscr{b} \rightarrowtail \mathbf{C}^{\vec{q}}\tau}\,(@)$$

   Then, we deduce

   $$\dfrac{\dfrac{\Gamma \vdash^X t : \mathscr{b}'' \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \tau) \qquad \Gamma \vdash^X v : \mathscr{c} \rightarrowtail \mathfrak{s} \qquad \mathscr{b}' \wedge \mathscr{c} \vDash^X \mathscr{b}'' \wedge \mathscr{c}}{\Gamma \vdash^X tv : \mathscr{b}' \wedge \mathscr{c} \rightarrowtail \mathbf{C}^{\vec{q}}\tau}\,(@) \qquad \mathscr{b} \vDash^X \mathscr{x}^i_a \wedge \mathscr{b}' \wedge \mathscr{c}}{\Gamma \vdash^X (tv) \oplus^i_a (uv) : \mathscr{b} \rightarrowtail \mathbf{C}^{\vec{q}}\tau}\,(\oplus l)$$

2. $$\dfrac{\dfrac{\Gamma \vdash^X u : \mathscr{b}'' \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \tau) \qquad \mathscr{b}' \vDash^X \neg \mathscr{x}^i_a \wedge \mathscr{b}''}{\Gamma \vdash^X t \oplus^i_a u : \mathscr{b}' \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \tau)}\,(\oplus r) \qquad \Gamma \vdash^X v : \mathscr{c} \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash^X \mathscr{b}' \wedge \mathscr{c}}{\Gamma \vdash^X (t \oplus^i_a u)v : \mathscr{b} \rightarrowtail \mathbf{C}^{\vec{q}}\tau}\,(@)$$

   Then, we can argue similarly to the previous case.

$(\{t \oplus^i_a u\}v \mathrel{-\!\!\triangleright}_{\mathsf{p}\{\}} (\{t\}v) \oplus^i_a (\{u\}v))$ Similar to the previous case.

$(t(u \oplus^i_a v) \mathrel{-\!\!\triangleright}_{\mathsf{p}\{\}} (tu) \oplus^i_a (tv))$ There are two sub-cases:

1. $$\dfrac{\Gamma \vdash^X t : \mathscr{b}' \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \tau) \qquad \dfrac{\Gamma \vdash^X u : \mathscr{c}' \rightarrowtail \mathfrak{s} \qquad \mathscr{c} \vDash^X \mathscr{x}^i_a \wedge \mathscr{c}'}{\Gamma \vdash^X u \oplus^i_a v : \mathscr{c} \rightarrowtail \mathfrak{s}}\,(\oplus l) \qquad \mathscr{b} \vDash^X \mathscr{b}' \wedge \mathscr{c}}{\Gamma \vdash^X t(u \oplus^i_a v) : \mathscr{b} \rightarrowtail \mathbf{C}^{\vec{q}}\tau}\,(@)$$

   Then, we deduce that

   $$\dfrac{\dfrac{\Gamma \vdash^X t : \mathscr{b}' \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \tau) \qquad \Gamma \vdash^X u : \mathscr{c}' \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash^X \mathscr{b}' \wedge \mathscr{c}'}{\Gamma \vdash^X tu : \mathscr{b} \rightarrowtail \mathbf{C}^{\vec{q}}\tau}\,(@) \qquad \mathscr{b} \vDash^X \mathscr{x}^i_a \wedge \mathscr{b}}{\Gamma \vdash^X (tu) \oplus^i_a (tv) : \mathscr{b} \rightarrowtail \mathbf{C}^{\vec{q}}\tau}\,(\oplus r)$$

2. $$\dfrac{\Gamma \vdash^X t : \mathscr{b}' \rightarrowtail \mathrm{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \tau) \qquad \dfrac{\Gamma \vdash^X v : \mathscr{c}' \rightarrowtail \mathfrak{s} \qquad \mathscr{c} \vDash^X \neg \mathscr{x}_a^i \wedge \mathscr{c}'}{\Gamma \vdash^X u \oplus_a^i v : \mathscr{c} \rightarrowtail \mathfrak{s}}\,(\oplus r) \qquad \mathscr{b} \vDash^X \mathscr{b}' \wedge \mathscr{c}}{\Gamma \vdash^X t(u \oplus_a^i v) : \mathscr{b} \rightarrowtail \mathrm{C}^{\vec{q}}\tau}\,(@)$$

Then, we can argue similarly to the previous case.

($\{t\}(u \oplus_a^i v) \rightarrow_{\mathrm{p}\{\}} (\{t\}u) \oplus_a^i (\{t\}v)$) Similar to the previous case.

($(t \oplus_a^i u) \oplus_b^j v \rightarrow_{\mathrm{p}\{\}} (t \oplus_b^j v) \oplus_a^i (u \oplus_b^j v)$) We suppose here $a \neq b$ or $i < j$. There are three sub-cases:

1. $$\dfrac{\dfrac{\Gamma \vdash^X t : \mathscr{b}'' \rightarrowtail \mathfrak{s} \qquad \mathscr{b}' \vDash^X \mathscr{x}_a^i \wedge \mathscr{b}''}{\Gamma \vdash^X t \oplus_a^i u : \mathscr{b}' \rightarrowtail \mathfrak{s}}\,(\oplus l) \qquad \mathscr{b} \vDash^X \mathscr{x}_b^j \wedge \mathscr{b}'}{\Gamma \vdash^X (t \oplus_a^i u) \oplus_b^j v : \mathscr{b}' \rightarrowtail \mathfrak{s}}\,(\oplus l)$$

Then, we deduce

$$\dfrac{\dfrac{\Gamma \vdash^X t : \mathscr{b}'' \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash^X \mathscr{x}_b^j \wedge \mathscr{b}''}{\Gamma \vdash^X t \oplus_b^j v : \mathscr{b}' \rightarrowtail \mathfrak{s}}\,(\oplus l) \qquad \mathscr{b} \vDash^X \mathscr{x}_a^i \wedge \mathscr{b}}{\Gamma \vdash^X (t \oplus_b^j u) \oplus_a^i (u \oplus_b^j v) : \mathscr{b}' \rightarrowtail \mathfrak{s}}\,(\oplus l)$$

2. $$\dfrac{\dfrac{\Gamma \vdash^X u : \mathscr{b}'' \rightarrowtail \mathfrak{s} \qquad \mathscr{b}' \vDash^X \neg\mathscr{x}_a^i \wedge \mathscr{b}''}{\Gamma \vdash^X t \oplus_a^i u : \mathscr{b}' \rightarrowtail \mathfrak{s}}\,(\oplus r) \qquad \mathscr{b} \vDash^X \mathscr{x}_b^j \wedge \mathscr{b}'}{\Gamma \vdash^X (t \oplus_a^i u) \oplus_b^j v : \mathscr{b}' \rightarrowtail \mathfrak{s}}\,(\oplus l)$$

Then, we can argue similarly to the previous case.

3. $$\dfrac{\Gamma \vdash^X v : \mathscr{b}' \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash^X \mathscr{x}_b^j \wedge \mathscr{b}'}{\Gamma \vdash^X (t \oplus_a^i u) \oplus_b^j v : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus r)$$

Then, we deduce (using the fact that $\mathscr{b} \equiv_X (\mathscr{x}_a^i \wedge \mathscr{b}) \vee (\neg\mathscr{x}_a^i \wedge \mathscr{b})$)

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash^X v : \mathscr{b}' \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash^X \mathscr{x}_b^j \wedge \mathscr{b}'}{\Gamma \vdash^X t \oplus_b^j v : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus r) \qquad \mathscr{x}_a^i \wedge \mathscr{b} \vDash^X \mathscr{x}_a^i \wedge \mathscr{b}}{\Gamma \vdash^X (t \oplus_b^j v) \oplus_a^i (u \oplus_b^j v) : \mathscr{x}_a^i \wedge \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus r) \qquad \dfrac{\dfrac{\Gamma \vdash^X v : \mathscr{b}' \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash^X \mathscr{x}_b^j \wedge \mathscr{b}'}{\Gamma \vdash^X u \oplus_b^j v : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus r) \qquad \neg\mathscr{x}_a^i \wedge \mathscr{b} \vDash^X \mathscr{x}_a^i \wedge \mathscr{b}}{\Gamma \vdash^X (t \oplus_b^j v) \oplus_a^i (u \oplus_b^j v) : \neg\mathscr{x}_a^i \wedge \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus r) \qquad \mathscr{b} \vDash^X (\mathscr{x}_a^i \wedge \mathscr{b}) \vee (\neg\mathscr{x}_a^i \wedge \mathscr{b})}{\Gamma \vdash^X (t \oplus_b^j v) \oplus_a^i (u \oplus_b^j v) : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\vee)$$

($t \oplus_b^j (u \oplus_a^i v) \rightarrow_{\mathrm{p}\{\}} (t \oplus_b^j u) \oplus_a^i (t \oplus_b^j v)$) Similar to the case above.

($\nu b.(t \oplus_a^i u) \rightarrow_{\mathrm{p}\{\}} (\nu b.t) \oplus_a^i (\nu b.u)$) We suppose $a \neq b$. There are two sub-cases:

1. $$\dfrac{\dfrac{\Gamma \vdash^{X \cup \{a,b\}} t : \mathscr{d} \rightarrowtail \mathfrak{s} \qquad \mathscr{b}' \wedge \mathscr{c} \vDash \mathscr{x}_a^i \wedge \mathscr{d}}{\Gamma \vdash^{X \cup \{a,b\}} t \oplus_a^i u : \mathscr{b}' \wedge \mathscr{c} \rightarrowtail \mathfrak{s}}\,(\oplus l) \qquad \mu(\mathscr{c}) \geq s \qquad \mathscr{b} \vDash \mathscr{b}'}{\Gamma \vdash^{X \cup \{a\}} \nu b.t \oplus_a^i u : \mathscr{b} \rightarrowtail \mathrm{C}^s\mathfrak{s}}\,(\mu)$$

From $\mathscr{b}' \wedge \mathscr{c}_i \vDash \mathscr{d}$, by Lemma C.3, we deduce the existence of a derivation of $\Gamma \vdash^{X \cup \{a,b\}} t : \mathscr{b}' \wedge \mathscr{c} \rightarrowtail \mathfrak{s}$. Moreover, from $\mathscr{b}' \wedge \mathscr{c}_i \vDash \mathscr{x}_a^i \wedge \mathscr{d}_i$ and the fact that $\mathrm{FN}(\mathscr{c}_i) \subseteq \{b\}$ it follows that $\mathscr{b}' \vDash \mathscr{x}_a^i$, so we can construct the following derivation:

$$\dfrac{\dfrac{\Gamma \vdash^{X \cup \{a,b\}} t : \mathscr{b}' \wedge \mathscr{c} \rightarrowtail \mathfrak{s} \qquad \mu(\mathscr{c}_i) \geq s}{\Gamma \vdash^{X \cup \{a\}} \nu b.t : \mathscr{b}' \rightarrowtail \mathrm{C}^s\mathfrak{s}}\,(\mu) \qquad \mathscr{b} \vDash \mathscr{x}_a^i \wedge \mathscr{b}'}{\Gamma \vdash^{X \cup \{a\}} (\nu b.t) \oplus_a^i (\nu b.u) : \mathscr{b} \rightarrowtail \mathrm{C}^s\mathfrak{s}}\,(\oplus l)$$

2. $$\dfrac{\dfrac{\Gamma \vdash^{X \cup \{a,b\}} u : \mathscr{d} \rightarrowtail \mathfrak{s} \qquad \mathscr{b}' \wedge \mathscr{c} \vDash \neg\mathscr{x}_a^i \wedge \mathscr{d}}{\Gamma \vdash^{X \cup \{a,b\}} t \oplus_a^i u : \mathscr{b}' \wedge \mathscr{c} \rightarrowtail \mathfrak{s}}\,(\oplus r) \qquad \mu(\mathscr{c}) \geq s \qquad \mathscr{b} \vDash \mathscr{b}'}{\Gamma \vdash^{X \cup \{a\}} \nu b.t \oplus_a^i u : \mathscr{b} \rightarrowtail \mathrm{C}^s\mathfrak{s}}\,(\mu)$$

The we can argue similarly to the previous case.

($\lambda x.\nu a.t \rightarrow_{\mathrm{p}\{\}} \nu a.\lambda x.t$) We have

$$\dfrac{\dfrac{\Gamma, x:\mathfrak{s} \vdash^{X\cup\{a\}} t : \mathscr{b}' \wedge \mathscr{c} \rightarrowtail \mathbf{C}^{\vec{q}}\sigma \qquad \mu(\mathscr{c}) \geq s \qquad \mathscr{b} \vDash^{X} \mathscr{b}'}{\Gamma, x:\mathfrak{s} \vdash^{X} \nu a.t : \mathscr{b} \rightarrowtail \mathbf{C}^{s}\mathbf{C}^{\vec{q}}\sigma}\,(\mu)}{\Gamma \vdash^{X} \lambda x.\nu a.t : \mathscr{b} \rightarrowtail \mathbf{C}^{s}\mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \sigma)}\,(\lambda)$$

from which we deduce:

$$\dfrac{\dfrac{\Gamma, x:\mathfrak{s} \vdash^{X\cup\{a\}} t : \mathscr{b}' \wedge \mathscr{c} \rightarrowtail \mathbf{C}^{\vec{q}}\sigma}{\Gamma \vdash^{X} \lambda x.t : \mathscr{b}' \wedge \mathscr{c} \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \sigma)}\,(\lambda) \qquad \mu(\mathscr{c}) \geq s \qquad \mathscr{b} \vDash^{X} \mathscr{b}'}{\Gamma \vdash^{X} \nu a.\lambda x.t : \mathscr{b} \rightarrowtail \mathbf{C}^{s}\mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \sigma)}\,(\mu)$$

($(\nu a.t)u \rightarrowtriangle_{\mathsf{p}\{\}} \nu a.(tu)$) We have

$$\dfrac{\dfrac{\Gamma \vdash^{X\cup\{a\},} t : \mathscr{b}' \wedge \mathscr{c} \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \sigma) \qquad \mu(\mathscr{c}) \geq s \qquad \mathscr{b}'' \vDash \mathscr{b}'}{\Gamma \vdash^{X} \nu a.t : \mathscr{b}'' \rightarrowtail \mathbf{C}^{s}\mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \sigma)}\,(\mu) \qquad \Gamma \vdash^{X} u : \mathscr{d} \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash \mathscr{b}'' \wedge \mathscr{d}}{\Gamma \vdash^{X} (\nu a.t)u : \mathscr{b} \rightarrowtail \mathbf{C}^{s}\mathbf{C}^{\vec{q}}\sigma}\,(@)$$

from which we deduce

$$\dfrac{\dfrac{\Gamma \vdash^{X\cup\{a\},} t : \mathscr{b}' \wedge \mathscr{c} \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \sigma) \qquad \Gamma \vdash^{X} u : \mathscr{d} \rightarrowtail \mathfrak{s} \qquad \mathscr{b}' \wedge \mathscr{c} \vDash (\mathscr{b}' \wedge \mathscr{c}) \wedge \mathscr{d}}{\Gamma \vdash^{X} tu : \mathscr{b}' \wedge \mathscr{c} \rightarrowtail \mathbf{C}^{\vec{q}}\sigma}\,(@) \qquad \mu(\mathscr{c}) \geq s \qquad \mathscr{b}'' \vDash \mathscr{b}'}{\Gamma \vdash^{X} \nu a.tu : \mathscr{b} \rightarrowtail \mathbf{C}^{s}\mathbf{C}^{\vec{q}}\sigma}\,(\mu)$$

($\{t\}\nu a.u \rightarrowtriangle_{\mathsf{p}\{\}} \nu a.(tu)$) We have

$$\dfrac{\Gamma \vdash^{X} t : \mathscr{b}' \wedge \mathscr{c} \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \sigma) \qquad \dfrac{\Gamma \vdash^{X\cup\{a\}} u : \mathscr{c}' \wedge \mathscr{d} \rightarrowtail \mathfrak{s} \qquad \mu(\mathscr{d}) \geq s \qquad \mathscr{c}' \vDash \mathscr{c}}{\Gamma \vdash^{X} \nu a.u : \mathscr{c} \rightarrowtail \mathbf{C}^{s}\mathfrak{s}}\,(\mu) \qquad \mathscr{b} \vDash \mathscr{b}' \wedge \mathscr{c}}{\Gamma \vdash^{X} \{t\}\nu a.u : \mathscr{b} \rightarrowtail \mathbf{C}^{s}\mathbf{C}^{\vec{q}}\sigma}\,(\{\})$$

from which we deduce (using Lemma C.2)

$$\dfrac{\dfrac{\Gamma \vdash^{X\cup\{a\}} t : \mathscr{b}' \wedge \mathscr{c} \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \sigma) \qquad \Gamma \vdash^{X\cup\{a\}} u : \mathscr{c}' \wedge \mathscr{d} \rightarrowtail \mathfrak{s} \qquad \mathscr{b}' \wedge \mathscr{d} \vDash \mathscr{b}' \wedge (\mathscr{c} \wedge \mathscr{d})}{\Gamma \vdash^{X} tu : \mathscr{b}' \wedge \mathscr{d} \rightarrowtail \mathbf{C}^{\vec{q}}\sigma}\,(@) \qquad \mu(\mathscr{d}) \geq s \qquad \mathscr{b} \vDash \mathscr{b}'}{\Gamma \vdash^{X} \nu a.tu : \mathscr{c} \rightarrowtail \mathbf{C}^{s}\mathbf{C}^{\vec{q}}\sigma}\,(\mu)$$

□

D Details about $\mathbf{C}\lambda_{\rightarrow,\cap}$.

D.1 Subject Reduction.

In this subsection we show that subject reduction holds for $\mathbf{C}\lambda_{\rightarrow,\cap}$ (and a fortiori for $\mathbf{C}\lambda_{\rightarrow}$).

Proposition D.1 (Subject Reduction). *If $\Gamma \vdash^{X} t : \mathscr{b} \rightarrowtail \mathfrak{s}$ and $t \rightarrowtriangle u$, then $\Gamma \vdash^{X} u : \mathscr{b} \rightarrowtail \mathfrak{s}$.*

As for $\mathbf{C}\lambda_{\rightarrow}^{\{\}}$ (see Section C), we replace the rule ($\oplus$) by the two rules ($\oplus l$) and ($\oplus r$). Moreover, we will ignore the rules (hn) and (n), as the result extends immediately to them.

To show the reduction property of $\mathbf{C}\lambda_{\rightarrow,\cap}$ we need to establish a few lemmas, some of which are analogous to results for $\mathbf{C}\lambda_{\rightarrow}^{\{\}}$, and proved in a similar way:

Lemma D.2. *If $\Gamma \vdash^{X} t : \mathscr{b} \rightarrowtail \mathfrak{s}$ and $X \subseteq Y$, then $\Gamma \vdash^{Y} t : \mathscr{b} \rightarrowtail \mathfrak{s}$.*

Lemma D.3. *If $\Gamma \vdash^{X} t : \mathscr{b} \rightarrowtail \mathfrak{s}$ holds and $\mathscr{c} \vDash^{X} \mathscr{b}$, then $\Gamma \vdash^{X} t : \mathscr{c} \rightarrowtail \mathfrak{s}$ is derivable by a derivation of the same length.*

The next lemmas are new:

Lemma D.4. *The following rule is admissible in $\mathbf{C}\lambda_{\rightarrow,\cap}$:*

$$\dfrac{\Gamma \vdash^{X} t : \mathscr{b} \rightarrowtail \mathfrak{s} \qquad \mathfrak{s} \preceq \mathfrak{t}}{\Gamma \vdash^{X} t : \mathscr{b} \rightarrowtail \mathfrak{t}}\,(\preceq)$$

Proof. We will show the admissibility of a more general rule, namely

$$\frac{\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathfrak{t} \qquad \Delta \preceq \Gamma, \mathfrak{s} \preceq \mathfrak{t}}{\Delta \vdash^X t : \mathscr{b} \rightarrowtail \mathfrak{t}} \ (\preceq^*)$$

where $\Delta \preceq \Gamma$ holds when $\Gamma = \{x_1 : \mathfrak{M}_1, \dots, x_n : \mathfrak{M}_n\}$, $\Delta = \{x_1 : \mathfrak{N}_1, \dots, x_n : \mathfrak{N}_n\}$ and $\mathfrak{M}_i \preceq \mathfrak{N}_i$.

We argue by induction on a typing derivation D of t:

- if D is
$$\frac{\mathsf{FN}(\mathscr{b}) \subseteq X \qquad \mathfrak{s}_i \preceq \mathfrak{t}}{\Gamma, x : [\mathfrak{s}_1, \dots, \mathfrak{s}_n] \vdash^X x : \mathscr{b} \rightarrowtail \mathfrak{t}}$$
then from $\Delta \preceq \Gamma, x : [\mathfrak{s}_1, \dots, \mathfrak{s}_n]$ we deduce that Δ contains $x : \mathfrak{M}$, with $\mathfrak{M} \preceq [\mathfrak{s}_1, \dots, \mathfrak{s}_n]$. This implies that $\mathfrak{M}$ contains $\mathfrak{u}_i$, where $\mathfrak{u}_i \preceq \mathfrak{s}_i$; by transitivity of $\preceq$, from $\mathfrak{u}_i \preceq \mathfrak{s}_i \preceq \mathfrak{t}$, we deduce $\mathfrak{u}_i \preceq \mathfrak{t}$, and thus we can construct the derivation below
$$\frac{\mathsf{FN}(\mathscr{b}) \subseteq X \qquad \mathfrak{u}_i \preceq \mathfrak{t}}{\Delta \vdash^X x : \mathscr{b} \rightarrowtail \mathfrak{t}}$$
- if D ends by any of the rules $(\vee)$, $(\oplus l)$, $(\oplus r)$ or (μ_Σ), then we can directly conclude by applying the I.H.
- if D is
$$\frac{\begin{array}{c}\vdots \\ \Gamma, x : \mathfrak{M} \vdash^X t : \mathscr{b} \rightarrowtail \mathbf{C}^q \sigma\end{array}}{\Gamma \vdash^X \lambda x.t : \mathscr{b} \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)} \ (\lambda)$$
then from $\mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma) \preceq \mathfrak{t}$ we deduce $\mathfrak{t} = \mathbf{C}^s(\mathfrak{M}' \Rightarrow \sigma')$, with $s \leq q$, $\mathfrak{M} \preceq \mathfrak{M}'$ and $\sigma \preceq \sigma''$, and from $\Delta \preceq \Gamma$, we deduce $\Delta, x : \mathfrak{M}' \preceq \Gamma, x : \mathfrak{M}$. So by the I.H. we deduce the existence of a derivation of $\Delta, x : \mathfrak{M}' \vdash^X t : \mathscr{b} \rightarrowtail \mathbf{C}^s \sigma'$ and we can conclude by applying an instance of (λ).
- if D is
$$\frac{\begin{array}{c}\vdots \\ \Gamma \vdash^X t : \mathscr{c} \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)\end{array} \qquad \left\{\begin{array}{c}\vdots \\ \Gamma \vdash^X u : \mathscr{d}_i \rightarrowtail \mathfrak{s}_i\end{array}\right\}_i \qquad \mathscr{b} \vDash \mathscr{c} \wedge \bigwedge_i \mathscr{d}_i}{\Gamma \vdash^X tu : \mathscr{b} \rightarrowtail \mathbf{C}^q \sigma} \ @_\cap$$
where $\mathfrak{M} = [\mathfrak{s}_1, \dots, \mathfrak{s}_n]$. Then from $\mathbf{C}^q \sigma \preceq \mathfrak{s}$ we deduce $\tau = \mathbf{C}^s \sigma'$, with $s \leq q$ and $\sigma \preceq \sigma'$, and thus $\mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma) \preceq \mathbf{C}^s(\mathfrak{M} \Rightarrow \sigma')$, so by the I.H. applied to the left-hand side sub-derivation we obtain a the existence of a derivation of $\Gamma \vdash^X t : \mathscr{c} \rightarrowtail \mathbf{C}^s(\mathfrak{M} \Rightarrow \sigma')$, and thus we can conclude by applying an instance of $(@_\cap)$.

□

Lemma D.5. *Let $\mathscr{b}, \mathscr{b}_1, \dots, \mathscr{b}_n$ and $\mathscr{c}, \mathscr{c}_1, \dots, \mathscr{c}_n$ be such that $\mathsf{FN}(\mathscr{b}), \mathsf{FN}(\mathscr{b}_i) \subseteq X$ and $\mathsf{FN}(\mathscr{c}), \mathsf{FN}(\mathscr{c}_i) \subseteq \{a\}$, where $a \notin X$. If $\mathscr{c}$ is satisfiable, then if $\mathscr{b} \wedge \mathscr{c} \vDash^{X \cup \{a\}} \bigvee_i^n \mathscr{b}_i \wedge \mathscr{c}_i$ holds, also $\mathscr{b} \vDash^X \bigvee_i^n \mathscr{b}_i$ holds.*

Proof. Let $v \in 2^X$ be a model of $\mathscr{b}$. Since $\mathscr{c}$ is satisfiable, v can be extended to a model $v' \in 2^{X \cup \{a\}}$ of $\mathscr{b} \wedge \mathscr{c}$. By hypothesis, then v' satisfies $\bigvee_i^n \mathscr{b}_i \wedge \mathscr{c}_i$, so for some $i_0 \leq n$, it satisfies $\mathscr{b}_{i_0} \wedge \mathscr{c}_{i_0}$. We deduce then that v satisfies $\mathscr{b}_{i_0}$, and thus v satisfies $\bigvee_i^n \mathscr{b}_i$. □

Lemma D.6. *If $\Gamma \vdash^{X \cup Y} t : \mathscr{b} \wedge \mathscr{c} \rightarrowtail \mathfrak{s}$ is derivable, where $X \cap Y = \emptyset$, $\mathsf{FN}(t) \subseteq X$, $\mathsf{FN}(\mathscr{b}) \subseteq X$, $\mathsf{FN}(\mathscr{c}) \subseteq Y$, and $\mathscr{c}$ is satisfiable, then $\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathfrak{s}$ is also derivable.*

Proof. By induction on a typing derivation of t:

- if the last rule is
$$\frac{\mathfrak{s}_i \preceq \mathfrak{t} \qquad \mathsf{FN}(\mathscr{b} \wedge \mathscr{c}) \subseteq X \cup Y}{\Gamma, x : [\mathfrak{s}_1, \dots, \mathfrak{s}_n] \vdash^{X \cup \{a\}} x : \mathscr{b} \wedge \mathscr{c} \rightarrowtail \mathfrak{t}} \ (\mathrm{id}_\cap)$$
then the claim can be deduced by an instance of the same rule.
- if the last rule is
$$\frac{\left\{\Gamma \vdash^{X \cup Y} t : \mathscr{b}_i \rightarrowtail \mathfrak{s}\right\}_i \qquad \mathscr{b} \wedge \mathscr{c} \vDash^{X \cup Y} \bigvee_i \mathscr{b}_i}{\Gamma \vdash^{X \cup Y} x : \mathscr{b} \wedge \mathscr{c} \rightarrowtail \mathfrak{s}} \ (\vee)$$
Let $\bigvee_j \mathscr{b}_{ij} \wedge \mathscr{d}_{ij}$ be weak Y-decompositions of the $\mathscr{b}_i$ (see [[3], p. 17]); since $\mathscr{b}_{ij} \wedge \mathscr{d}_{ij} \vDash \mathscr{b}_i$, by Lemma D.3 we deduce that $\Gamma \vdash^{X \cup Y} t : \mathscr{b}_{ij} \wedge \mathscr{d}_{ij} \rightarrowtail \mathfrak{s}$ is derivable for all i and j, by a derivation of same length as the corresponding derivation of $\Gamma \vdash^{X \cup Y} t : \mathscr{b}_i \rightarrowtail \mathfrak{s}$; hence we can apply the I.H. to such derivations, yielding derivations of $\Gamma \vdash^X t : \mathscr{b}_{ij} \rightarrowtail \mathfrak{s}$.
Using Lemma D.5, from $\mathscr{b} \wedge \mathscr{c} \vDash \bigvee_i \mathscr{b}_i$ and $\bigvee_i \mathscr{b}_i \equiv \bigvee_{ij} \mathscr{b}_{ij} \wedge \mathscr{c}_{ij}$, we deduce $\mathscr{b} \vDash \bigvee_{ij} \mathscr{b}_{ij}$, so we conclude

$$\dfrac{\left\{\Gamma \vdash^X t : \mathscr{b}_{ij} \rightarrowtail \mathfrak{s}\right\}_{ij} \qquad \mathscr{b} \vDash^X \bigvee_{ij} \mathscr{b}_{ij}}{\Gamma \vdash^X x : \mathscr{b} \rightarrowtail \mathfrak{s}}\ (\vee)$$

- if the last rule is

$$\dfrac{\Gamma \vdash^{X\cup Y} t_1 : \mathscr{b}' \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \wedge \mathscr{c} \vDash x_i^b \wedge \mathscr{b}'}{\Gamma \vdash^{X\cup Y} t_1 \oplus_b^i t_2 : \mathscr{b} \wedge \mathscr{c} \rightarrowtail \mathfrak{s}}\ (\oplus l)$$

  Then, let $\bigvee_j \mathscr{d}_j \wedge \mathscr{c}_j$ be a Y-decomposition of $\mathscr{b}'$; by Lemma D.3 there exist derivations of $\Gamma \vdash^{X\cup Y} t_1 : \mathscr{d}_j \wedge \mathscr{c}_j \rightarrowtail \mathfrak{s}$ of same length as the derivation $\Gamma \vdash^{X\cup Y} t_1 : \mathscr{b}' \rightarrowtail \mathfrak{s}$, so by I.H. we obtain derivations of $\Gamma \vdash^{X\cup Y} t_1 : \mathscr{d}_j \rightarrowtail \mathfrak{s}$.
  From $\mathscr{b} \wedge \mathscr{c} \vDash x_i^b \wedge \mathscr{b}'$ we deduce $\mathscr{b} \wedge \mathscr{c} \vDash \bigvee_j (x_i^b \wedge \mathscr{d}_j) \wedge \mathscr{c}_j$, and using Lemma D.5 we deduce $\mathscr{b} \vDash \bigvee_j x_i^b \wedge \mathscr{d}_j \equiv x_i^b \wedge \bigvee_j \mathscr{d}_j$.
  We can thus conclude as follows:

$$\dfrac{\dfrac{\left\{\Gamma \vdash^X t_1 : \mathscr{d}_j \rightarrowtail \mathfrak{s}\right\}_j}{\Gamma \vdash^X t_1 : \bigvee_j \mathscr{d}_j \rightarrowtail \mathfrak{s}}\ (\vee) \qquad \mathscr{b} \vDash^X x_i^b \wedge \bigvee_j \mathscr{d}_j}{\Gamma \vdash^X t_1 \oplus_a^i t_2 : \mathscr{b} \rightarrowtail \mathfrak{s}}\ (\oplus l)$$

- if the last rule is

$$\dfrac{\Gamma \vdash^{X\cup Y} t_2 : \mathscr{b}' \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \wedge \mathscr{c} \vDash \neg x_i^b \wedge \mathscr{b}'}{\Gamma \vdash^{X\cup Y} t_1 \oplus_b^i t_2 : \mathscr{b} \wedge \mathscr{c} \rightarrowtail \mathfrak{s}}\ (\oplus r)$$

  then we can argue similarly to the previous case.
- if the last rule is

$$\dfrac{\Gamma, y : \mathfrak{M} \vdash^{X\cup Y} t : \mathscr{b} \wedge \mathscr{c} \rightarrowtail \mathbf{C}^q\sigma}{\Gamma \vdash^{X\cup Y} \lambda y.t : \mathscr{b} \wedge \mathscr{c} \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)}\ (\lambda)$$

  Then, the claim follows from the I.H. by applying an instance of the same rule.
- if the last rule is

$$\dfrac{\Gamma \vdash^{X\cup Y} t_1 : \mathscr{b}_1 \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma) \qquad \left\{\Gamma \vdash^{X\cup Y} t_2 : \mathscr{b}_i \rightarrowtail \mathfrak{s}_i\right\}_i \qquad \mathscr{b} \wedge \mathscr{c} \vDash^{X\cup\{a\}} \mathscr{b}_1 \wedge (\bigwedge_i \mathscr{b}_i)}{\Gamma \vdash^{X\cup Y} t_1 t_2 : \mathscr{b} \wedge \mathscr{c} \rightarrowtail \mathbf{C}^q\sigma}\ (@_\cap)$$

  Then let $\bigvee_j \mathscr{c}'_j \wedge \mathscr{d}'_j$ and $\bigvee_k \mathscr{c}''_{ik} \wedge \mathscr{d}''_{ik}$ be weak Y-decompositions of $\mathscr{b}_1$ and $\mathscr{b}_i$. By Lemma D.3 there exist then derivations of $\Gamma \vdash^{X\cup Y} t_1 : \mathscr{c}'_j \wedge \mathscr{d}'_j \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)$ and $\Gamma \vdash^{X\cup Y} t_2 : \mathscr{c}''_{ik} \wedge \mathscr{d}''_{ik} \rightarrowtail \mathfrak{s}_i$ of same length as the corresponding derivations. By applying the I.H. to such derivations we obtain then derivations of $\Gamma \vdash^X t_1 : \mathscr{c}'_j \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)$ and $\Gamma \vdash^X t_2 : \mathscr{c}''_{ik} \rightarrowtail \mathfrak{s}_i$, respectively.
  Moreover, from $\mathscr{b} \wedge \mathscr{c} \vDash \mathscr{b}_1 \wedge \bigwedge_i \mathscr{b}_i$ we deduce $\mathscr{b} \wedge \mathscr{c} \vDash \bigvee_j \mathscr{c}'_j \wedge \mathscr{d}'_j$ and $\mathscr{b} \wedge \mathscr{c} \vDash \bigvee_{ik} \mathscr{c}''_{ik} \wedge \mathscr{d}''_{ik}$, so by Lemma D.5 we deduce $\mathscr{b} \vDash \bigvee_j \mathscr{c}'_j$ and $\mathscr{b} \vDash \bigwedge_i \bigvee_{ik} \mathscr{c}''_{ik}$.
  Thus, we can conclude as follows:

$$\dfrac{\dfrac{\left\{\Gamma \vdash^X t_1 : \mathscr{c}'_j \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)\right\}_j}{\Gamma \vdash^X t : \bigvee_j \mathscr{c}'_j \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)}\ (\vee) \qquad \left\{\dfrac{\left\{\Gamma \vdash^X t_2 : \mathscr{c}''_{ik} \rightarrowtail \mathfrak{s}_i\right\}_k}{\Gamma \vdash^X t_2 : \bigvee_{ik} \mathscr{c}''_{ik} \rightarrowtail \mathfrak{s}_i}\ (\vee)\right\}_i \qquad \mathscr{b} \vDash^X \left(\bigvee_j \mathscr{c}'_j\right) \wedge \left(\bigwedge_i \bigvee_{ik} \mathscr{c}''_{ik}\right)}{\Gamma \vdash^X t_1 t_2 : \mathscr{b} \rightarrowtail \mathbf{C}^q\sigma}\ (@)$$

- if the last rule is

$$\dfrac{\left\{\Gamma \vdash^{X\cup Y\cup\{b\}, q_u} t : \mathscr{b}' \wedge \mathscr{f}_u \rightarrowtail \mathbf{C}^{q_u}\sigma\right\}_u \qquad \mu(\mathscr{f}_u) \geq s_u \qquad \mathscr{b} \wedge \mathscr{c} \vDash \mathscr{b}'}{\Gamma \vdash^{X\cup Y} \nu b.t : \mathscr{b} \wedge \mathscr{c} \rightarrowtail \mathbf{C}^r\sigma}\ (\mu_\Sigma)$$

  where $r = \sum_u q_u s_u$, then let $\bigvee_i \mathscr{c}_i \wedge \mathscr{d}_i$, be a weak a- decomposition of $\mathscr{b}$. By Lemma C.3 there exist derivations of $\Gamma \vdash^{X\cup Y\cup\{b\}} t : \mathscr{c}_i \wedge \mathscr{d}_i \wedge \mathscr{f}_u \rightarrowtail \mathbf{C}^{q_u}\sigma$ of same length as the corresponding derivation of $\Gamma \vdash^{X\cup Y\cup\{b\}} t : \mathscr{b}' \wedge \mathscr{f}_u \rightarrowtail \mathbf{C}^{q_u}\sigma$. Hence, by the I.H. there exist derivations of $\Gamma \vdash^{X\cup\{b\}} t : \mathscr{c}_i \wedge \mathscr{f}_u \rightarrowtail \mathbf{C}^{q_u}\sigma$.
  From $\mathscr{b} \wedge \mathscr{c} \vDash^{X\cup\{a\}} \mathscr{b}'$ we deduce by Lemma D.5 $\mathscr{b} \vDash \bigvee_i \mathscr{c}_i$ and so we can conclude as follows:

$$\dfrac{\left\{\dfrac{\left\{\Gamma \vdash^{X\cup\{b\}} t : \mathscr{c}_i \wedge \mathscr{f}_u \rightarrowtail \mathbf{C}^{q_u}\sigma\right\}_i}{\Gamma \vdash^{X\cup\{b\}} t : \bigvee_i \mathscr{c}_i \wedge \mathscr{f}_u \rightarrowtail \mathbf{C}^{q_u}\sigma}\ (\vee)\right\}_u \qquad \mu(\mathscr{f}_u) \geq s_u \qquad \mathscr{b} \vDash \bigvee_i \mathscr{c}_i}{\Gamma \vdash^X \nu b.t : \mathscr{b} \rightarrowtail \mathbf{C}^r\sigma}\ (\mu_\Sigma)$$

$\square$

The proof of the substitution lemma below is analogous to the proof of Lemma C.4.

Lemma D.7 (substitution lemma)**.** *The following rule is derivable:*

$$\frac{\Gamma, x : [\mathfrak{s}_1, \dots, \mathfrak{s}_n] \vdash^X t : \mathscr{c} \rightarrowtail \mathsf{C}^q\tau \qquad \{\Gamma \vdash^X u : \mathscr{d}_i \rightarrowtail \mathfrak{s}_i\}_{i=1,\dots,n} \qquad \mathscr{b} \vDash \mathscr{c} \wedge (\bigwedge_i \mathscr{d}_i)}{\Gamma \vdash^X t[u/x] : \mathscr{b} \rightarrowtail \mathsf{C}^q\tau}\ (\mathit{subst}_\cap)$$

We now have all elements to establish the subject reduction property.

Proof of Proposition D.1. As in the proof of Prop. C.1, observe that if the typing derivation D of t ends by a $(\vee)$-rule, it suffices to establish the property for the immediate sub-derivations of D and then apply an instance of $(\vee)$-rule to the resulting derivations. So we will always suppose that the typing derivation of D does not end by a $(\vee)$-rule.

The argument for β-reduction works similarly to the one in the Prop. C.1, using Lemma D.7 in place of Lemma C.4. $(\mathrm{subst}_\cap)$

For the case of $\rightarrowtriangle_{\mathrm{p}}$ wee consider reduction rules one by one. Most cases are analogous to those from the proof of Prop. C.1. We limit ourselves to the case of the permutation rule $(\neg\nu)$ (not considered before), and to those permutations that involve the rule (μ_Σ):

$(\nu b.(t \oplus^i_a u) \rightarrowtriangle_{\mathrm{p}} (\nu b.t) \oplus^i_a (\nu b.u))$ We suppose $a \neq b$. As in the proof of Prop. C.1 there are two sub-cases, we only consider the first one (the second one being treated similarly):

$$\frac{\left\{\dfrac{\Gamma \vdash^{X \cup \{a,b\}} t : \mathscr{d}_i \rightarrowtail \mathsf{C}^{q_i}\sigma \qquad \mathscr{b}' \wedge \mathscr{c}_i \vDash \mathscr{x}^i_a \wedge \mathscr{d}_i}{\Gamma \vdash^{X \cup \{a,b\}} t \oplus^i_a u : \mathscr{b}' \wedge \mathscr{c}_i \rightarrowtail \mathsf{C}^{q_i}\sigma}\ (\oplus l)\right\}_i \qquad \mu(\mathscr{c}_i) \geq s_i \qquad \mathscr{b} \vDash \mathscr{b}'}{\Gamma \vdash^{X \cup \{a\}} \nu b.t \oplus^i_a u \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}\sigma}\ (\mu_\Sigma)$$

From $\mathscr{b}' \wedge \mathscr{c}_i \vDash \mathscr{d}_i$, by Lemma C.3, we deduce the existence of a derivation of $\Gamma \vdash^{X \cup \{a,b\}} t : \mathscr{b}' \wedge \mathscr{c}_i \rightarrowtail \mathsf{C}^{q_i}\sigma$. Moreover, from $\mathscr{b}' \wedge \mathscr{c}_i \vDash \mathscr{x}^i_a \wedge \mathscr{d}_i$ and the fact that $\mathrm{FN}(\mathscr{c}_i) \subseteq \{b\}$ it follows that $\mathscr{b}' \vDash \mathscr{x}^i_a$, so we can construct the following derivation:

$$\frac{\dfrac{\left\{\Gamma \vdash^{X \cup \{a,b\}} t : \mathscr{b}' \wedge \mathscr{c}_i \rightarrowtail \mathsf{C}^{q_i}\sigma\right\}_i \qquad \mu(\mathscr{c}_i) \geq s_i}{\Gamma \vdash^{X \cup \{a\}} \nu b.t : \mathscr{b}' \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}\sigma}\ (\mu_\Sigma) \qquad \mathscr{b} \vDash \mathscr{x}^i_a \wedge \mathscr{b}'}{\Gamma \vdash^{X \cup \{a\}} (\nu b.t) \oplus^i_a (\nu b.u) : \mathscr{b} \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}\sigma}\ (\oplus l)$$

$(\nu a.t \rightarrowtriangle_{\mathrm{p}} t)$ We have

$$\frac{\left\{\Gamma \vdash^{X \cup \{a\}} t : \mathscr{b}' \wedge \mathscr{c}_i \rightarrowtail \mathsf{C}^{q_i}\sigma\right\}_i \qquad \mu([\![\mathscr{c}_i]\!]_{\{a\}}) \geq s_i \qquad \mathscr{b} \vDash^X \mathscr{b}'}{\Gamma \vdash^X \nu a.t : \mathscr{b} \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}\sigma}\ (\mu_\Sigma)$$

Since the $\mathscr{c}_i$ are pairwise disjoint, from $\mu(\mathscr{c}_i) \geq s_i$ it follows that $\sum_i s_i \leq 1$. Hence, by letting $Q = \max_i\{q_i\}$, we have $\sum_i q_i s_i \leq \sum_i Q s_i = Q \cdot \sum_i s_i \leq Q$. Let i_0 be the index such that $Q = q_{i_0}$. Since $a \notin \mathrm{FN}(t)$, by applying Lemma D.6 to the derivation of $\Gamma \vdash^{X \cup \{a\}} t : \mathscr{b}' \wedge \mathscr{c}_{i_0} \rightarrowtail \mathsf{C}^Q\sigma$, we deduce $\Gamma \vdash^X t : \mathscr{b}' \rightarrowtail \mathsf{C}^Q\sigma$, and since $\mathsf{C}^Q\sigma \leq \mathsf{C}^{\sum_i q_i s_i}\sigma$ and, we deduce, by Lemma D.4, $\Gamma \vdash^X t : \mathscr{b}' \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}\sigma$, and by Lemma D.3, $\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}\sigma$.

$(\lambda x.\nu a.t \rightarrowtriangle_{\mathrm{p}} \nu a.\lambda x.t\,)$ We have

$$\frac{\dfrac{\left\{\Gamma, x : \mathfrak{M} \vdash^{X \cup \{a\}} t : \mathscr{b}' \wedge \mathscr{c}_i \rightarrowtail \mathsf{C}^{q_i}\sigma\right\}_i \qquad \mu(\mathscr{c}_i) \geq s_i \qquad \mathscr{b} \vDash^X \mathscr{b}'}{\Gamma, x : \mathfrak{M} \vdash^X \nu a.t : \mathscr{b} \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}\sigma}\ (\mu_\Sigma)}{\Gamma \vdash^X \lambda x.\nu a.t : \mathscr{b} \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}(\mathfrak{M} \Rightarrow \sigma)}\ (\lambda)$$

from which we deduce:

$$\frac{\left\{\dfrac{\Gamma, x : \mathfrak{M} \vdash^{X \cup \{a\}} t : \mathscr{b}' \wedge \mathscr{c}_i \rightarrowtail \mathsf{C}^{q_i}\sigma}{\Gamma \vdash^X \lambda x.t : \mathscr{b}' \wedge \mathscr{c}_i \rightarrowtail \mathsf{C}^{q_i}(\mathfrak{M} \Rightarrow \sigma)}\ (\lambda)\right\}_i \qquad \mu(\mathscr{c}_i) \geq s_i \qquad \mathscr{b} \vDash^X \mathscr{b}'}{\Gamma \vdash^X \nu a.\lambda x.t : \mathscr{b} \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}(\mathfrak{M} \Rightarrow \sigma)}\ (\mu_\Sigma)$$

$((\nu a.t)u \rightarrowtriangle_{\mathrm{p}} \nu a.(tu))$ We have

$$\frac{\dfrac{\left\{\Gamma \vdash^{X \cup \{a\}} t : \mathscr{c}' \wedge \mathscr{d}_i \rightarrowtail \mathsf{C}^{q_i}(\mathfrak{M} \Rightarrow \tau)\right\}_i \qquad \mu(\mathscr{d}_i) \geq s_i \qquad \mathscr{c} \vDash \mathscr{c}'}{\Gamma \vdash^X \nu a.t : \mathscr{c} \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}(\mathfrak{M} \Rightarrow \tau)}\ (\mu_\Sigma) \qquad \left\{\Gamma \vdash^X u : \mathscr{e}_j \rightarrowtail \mathfrak{s}_i\right\}_j \qquad \mathscr{b} \vDash \mathscr{c} \wedge (\bigwedge_j \mathscr{e}_j)}{\Gamma \vdash^X (\nu a.t)u : \mathscr{b} \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}\tau}\ (@_\cap)$$

where $\mathfrak{M} = [\mathfrak{s}_1, \dots, \mathfrak{s}_n]$, from which we deduce

$$\dfrac{\left\{\dfrac{\Gamma \vdash^{X\cup\{a\}} t : \mathcal{c}' \wedge \mathcal{d}_i \rightarrowtail \mathbf{C}^{q_i}(\mathfrak{M} \Rightarrow \tau) \qquad \left\{\Gamma \vdash^{X} u : \mathcal{e}_j \rightarrowtail \mathfrak{s}_i\right\}_j \qquad \mathcal{b} \wedge \mathcal{d}_i \vDash (\mathcal{c}' \wedge \mathcal{d}_i) \wedge (\bigwedge_j \mathcal{e}_j)}{\Gamma \vdash^{X} tu : \mathcal{b} \wedge \mathcal{d}_i \rightarrowtail \mathbf{C}^{q_i}\tau}\,(@_\cap)\right\}_i \qquad \mu(\mathcal{d}_i) \geq s_i}{\Gamma \vdash^{X} \nu a.tu : \mathcal{b} \rightarrowtail \mathbf{C}^{\sum_i q_i s_i}\tau}\,(\mu_\Sigma)$$

□

D.2 Subject Expansion.

The goal of this section is to establish the following result for $\mathbf{C}\lambda_{\rightarrow,\cap}$:

Proposition D.8 (subject expansion)**.** *If* $\Gamma \vdash^X t : \mathcal{b} \rightarrowtail \mathfrak{s}$ *and* $u \rightarrow t$*, then* $\Gamma \vdash^X t : \mathcal{b} \rightarrowtail \mathfrak{s}$.

As for subject reduction, we will ignore the rules (hn) and (n), as the result extends immediately to them.

We will consider first the (more laborious case) of β-reduction, and then the more direct case of permutative reduction.

D.2.1 Subject β-Expansion. To establish the subject expansion property we need to develop a finer analysis of typing derivations. In particular we will need to prove that derivations can be turned into a canonical form. The main idea is to reduce to a case where ($\vee$)-rules are always applied "as late as possible", that is, either at the end of the derivation or right before a counting rule. Moreover, we will require that if a ($\vee$)-rule occurs before a counting rule for some name a, then these two rule follow a specific pattern, defined below, which ensures that no information is loss about other names.

Definition D.9 ($\vee\mu$-pattern)**.** For any name $a \in \mathbb{N}$ and typing derivation D, a $\vee\mu$*-pattern on* a *in* D is given by the occurrence in D of (μ)-rule preceded by multiple occurrences of ($\vee$)-rules in a configuration as illustrated below:

$$\dfrac{\left\{\dfrac{\left\{\Gamma \vdash^{X\cup\{a\}} t : \mathcal{b} \wedge \mathcal{d}_{ij} \rightarrowtail \mathbf{C}^{q_i}\sigma\right\}_{j=1,\dots,L_i} \qquad \mathcal{b} \wedge \mathcal{d}_i \vDash^X \bigvee_l \mathcal{b} \wedge \mathcal{d}_{ij}}{\Gamma \vdash^{X\cup\{a\}} t : \mathcal{b} \wedge \mathcal{d}_i \rightarrowtail \mathbf{C}^{q_i}\sigma}\,(\vee)\right\}_i \qquad \mu(\mathcal{d}_i) \geq s_i}{\Gamma \vdash^{X} \nu a.t : \mathcal{b} \rightarrowtail \mathbf{C}^{\sum_i q_i s_i}\sigma}\,(\mu_\Sigma) \tag{1}$$

where $\mathrm{FN}(\mathcal{c}) \subseteq X$, $\mathrm{FN}(\mathcal{d}_{ij}) \subseteq \{a\}$, and $\mathcal{c}$ and the $\mathcal{d}_{ij}$ are conjunctions of literals.

Definition D.10 (canonical and pseudo-canonical derivation)**.** A typing derivation D is said *canonical* when the following hold:

a. all Boolean formulas occurring in D are conjunctions of literals, except for those which occur either in the conclusion of a ($\vee$)-rule or in the major premiss of a (μ_Σ)-rule;
b. any occurrence of the ($\vee$) rule in D occurs in a $\vee\mu$-pattern.

A typing derivation D is said *pseudo-canonical* when D is obtained by one application of the ($\vee$)-rule where all premises are conclusions of canonical sub-derivations.

The fundamental property of canonical derivations that we will exploit is the following:

Lemma D.11. *Suppose* D *is a canonical derivation of conclusion* $\Gamma \vdash^X t : \mathcal{b} \rightarrowtail \mathfrak{s}$*. If* $\mathcal{c}$ *is a Boolean formula* $\mathcal{c}$ *with* $\mathrm{FN}(\mathcal{c}) \subseteq X$ *occurring in an axiom of* D*, then* $\mathcal{b} \vDash^X \mathcal{c}$.

Proof. By induction on D:

- if D only consists in an axiom, the claim is trivial;
- if D ends by any rule of the form

$$\dfrac{\{\Delta \vdash ?? : \mathcal{e}_i \rightarrowtail ??\}_i \qquad \text{other conditions}}{\Delta \vdash ?? : \mathcal{b} \rightarrowtail ??}$$

  where $\mathcal{b} \vDash \mathcal{e}_i$ (this includes the rules for $\oplus^i_a$, λ and application), then from the induction hypothesis $\mathcal{e}_i \vDash \mathcal{c}$ holds, where i is the index of the derivation where $\mathcal{c}$ occurs, and we can thus conclude $\mathcal{b} \vDash \mathcal{c}$.
- if D ends by a $\vee\mu$-pattern as in Eq. (1), then by the induction hypothesis $\mathcal{b} \wedge \mathcal{d}_{ij} \vDash \mathcal{c}$, where i and j are chosen so that $\mathcal{c}$ occurs in the associated sub-derivation. Since $\mathrm{FN}(\mathcal{d}_{ij}) \cap \mathrm{FN}(\mathcal{c}) = \emptyset$ and the $\mathcal{d}_{ij}$ are all satisfiable, we can conclude $\mathcal{b} \vDash \mathcal{c}$ by Lemma D.5.

□

Remark 3. *It is not difficult to see that the property of Lemma D.11 fails for non-canonical derivations. For example, consider the non-canonical derivation below:*

$$\dfrac{\dfrac{\dfrac{x : \mathrm{C}^q\sigma \vdash^{\{a,b\}} x : x^1_a \rightarrowtail \mathrm{C}^q\sigma}{x : \mathrm{C}^q\sigma \vdash^{\{a,b\}} x \oplus^0_b x : x^0_b \wedge x^1_a \rightarrowtail \mathrm{C}^q\sigma}\,(\oplus l) \qquad \mu(x^0_b) \geq \frac{1}{2}}{x : \mathrm{C}^q\sigma \vdash^{\{a,b\}} \nu b.x \oplus^0_b x : x^1_a \rightarrowtail \mathrm{C}^{\frac{q}{2}}\sigma}\,(\mu) \qquad \dfrac{\dfrac{x : \mathrm{C}^q\sigma \vdash^{\{a,b\}} x : \neg x^1_a \rightarrowtail \mathrm{C}^q\sigma}{x : \mathrm{C}^q\sigma \vdash^{\{a,b\}} x \oplus^0_b x : \neg x^0_b \wedge \neg x^1_a \rightarrowtail \mathrm{C}^q\sigma}\,(\oplus r) \qquad \mu(\neg x^0_b) \geq \frac{1}{2}}{x : \mathrm{C}^q\sigma \vdash^{\{a,b\}} \nu b.x \oplus^0_b x : \neg x^1_a \rightarrowtail \mathrm{C}^{\frac{q}{2}}\sigma}\,(\mu) \qquad \top \vDash x^1_a \wedge \neg x^1_a}{x : \mathrm{C}^q\sigma \vdash^{\{a\}} \nu b.x \oplus^0_b x : \top \rightarrowtail \mathrm{C}^{\frac{q}{2}}\sigma}\,(\vee)$$

where $\mathscr{d} = (x^0_b \wedge x^1_a) \vee (\neg x^0_b \wedge \neg x^1_a)$. *While the literal* x^1_a *occurs in an axiom, it is not true that* $\top \vDash x^1_a$*.*

Observe that the derivation above is pseudo-canonical. Instead, the derivation below is neither canonical nor pseudo-canonical, and similarly violates the property of Lemma D.11.

$$\dfrac{\dfrac{\dfrac{x : \mathrm{C}^q\sigma \vdash^{\{a,b\}} x : x^1_a \rightarrowtail \mathrm{C}^q\sigma}{x : \mathrm{C}^q\sigma \vdash^{\{a,b\}} x \oplus^0_b x : x^0_b \wedge x^1_a \rightarrowtail \mathrm{C}^q\sigma}\,(\oplus l) \qquad \dfrac{x : \mathrm{C}^q\sigma \vdash^{\{a,b\}} x : \neg x^1_a \rightarrowtail \mathrm{C}^q\sigma}{x : \mathrm{C}^q\sigma \vdash^{\{a,b\}} x \oplus^0_b x : x^0_b \wedge \neg x^1_a \rightarrowtail \mathrm{C}^q\sigma}\,(\oplus l) \qquad x^0_b \vDash (x^1_a \wedge x^0_b) \vee (\neg x^1_a \wedge x^0_b)}{x : \mathrm{C}^q\sigma \vdash^{\{a,b\}} x \oplus^0_b x : x^0_b \rightarrowtail \mathrm{C}^q\sigma}\,(\vee) \qquad \mu(x^0_b) \geq \frac{1}{2}}{x : \mathrm{C}^q\sigma \vdash^{\{a\}} \nu b.x \oplus^0_b x : \top \rightarrowtail \mathrm{C}^{\frac{q}{2}}\sigma}\,(\mu)$$

The following is the fundamental structural result we need to establish subject β-expansion.

Theorem D.12. *Any typing derivation can be transformed into a pseudo-canonical derivation.*

We need two preliminary lemmas.

Lemma D.13. *Suppose D is pseudo-canonical and ends as follows:*

$$\dfrac{\left\{\Gamma \vdash^{X \cup \{a\}} t : \mathscr{b}_i \wedge \mathscr{e}_i \rightarrowtail \mathfrak{s}\right\}_{i=1,\dots,n} \qquad \mathscr{b} \vDash^{X \cup \{a\}} \bigvee_i \mathscr{b}_i \wedge \mathscr{e}_i}{\Gamma \vdash^{X \cup \{a\}} t : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\vee)$$

where $\mathrm{FN}(\mathscr{b}_i) \subseteq X$ *and* $\mathrm{FN}(\mathscr{e}_i) \subseteq \{a\}$*. Then there exists literals* $\mathscr{c}_{ij}$ *with free names in* X *such that*
(i.) if $i \neq i'$*, then for all* j, j' *either* $\mathscr{c}_{ij} \equiv \mathscr{c}_{i'j'}$ *or* $\mathscr{c}_{ij} \wedge \mathscr{c}_{i'j'} \vDash \bot$*;*
(ii.) $\mathscr{b}_i \equiv \bigvee_j \mathscr{c}_{ij}$*, and more generally* $\bigvee_{i,j} \mathscr{c}_{ij} \equiv \bigvee_{i=1}^n \mathscr{b}_i$*.*
Moreover, D can be turned into a pseudo-canonical derivation D' ending as follows:

$$\dfrac{\left\{\Gamma \vdash^{X \cup \{a\}} t : \mathscr{c}_{ij} \wedge \mathscr{e}_i \rightarrowtail \mathfrak{s}\right\}_{i,j} \qquad \mathscr{b} \vDash^{X \cup \{a\}} \bigvee_{i,j} \mathscr{c}_{ij} \wedge \mathscr{e}_i}{\Gamma \vdash^{X \cup \{a\}} t : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\vee)$$

Proof. For any of the conjunction of literals $\mathscr{b}_i$, let $W_i \subseteq \mathbb{N} \times \mathbb{N}$ contain all pairs (j, b) such that at least one of the literals $x^j_b, \neg x^j_b$ occurs in $\bigvee_i \mathscr{b}_i$ but neither of them occurs in $\mathscr{b}_i$. For any such $\mathscr{b}_i$, let $\mathscr{c}_{i1}, \dots, \mathscr{c}_{i2^{\sharp W_i}}$ be the conjunctions of literals obtained by adding to $\mathscr{b}_i$ all possible consistent conjunctions of literals chosen from W_i (i.e. all possible choices between x^j_b and $\neg x^j_b$, for $(j, b) \in W_i$). It is clear that $\mathscr{b}_i \equiv \bigvee_j \mathscr{c}_{ij}$ and that all $\mathscr{c}_{ij}$ are disjoint, so *ii*. holds.

Moreover, let $i \neq i'$ and suppose $\mathscr{c}_{ij} = \mathscr{b}_i \wedge \mathscr{d}$ and $\mathscr{c}_{i'j'} = \mathscr{b}_j \wedge \mathscr{e}$ are not equivalent; since $\mathscr{c}_{ij}$ and $\mathscr{c}_{i'j'}$ are conjunctions of literals, and each of them contains either x^j_b or $\neg x^j_b$ for $(j, b) \in \bigcup_i W_i$, the only possibility is that for some pair (j, b), $\mathscr{c}_{ij}$ contains either x^j_b or $\neg x^j_b$ and $\mathscr{c}_{i'j'}$ contains its negation, so $\mathscr{c}_{ij} \wedge \mathscr{c}_{i'j'} \vDash \bot$. Hence *i*. also holds.

Finally, the pseudo-canonical derivation D' is obtained as follows: for any canonical sub-derivation D_i of D of conclusion $\Gamma \vdash^X t : \mathscr{b}_i \wedge \mathscr{e}_i \rightarrowtail \mathfrak{s}$ (whose conclusion contains the Boolean formula $\mathscr{b}_i$) and for each choice of j, define a sub-derivation D_{ij} of conclusion $\Gamma \vdash^X t : \mathscr{c}_{ij} \wedge \mathscr{e}_i \rightarrowtail \mathfrak{s}$ by replacing each Boolean formula $\mathscr{f}$ occurring in D_i by $\mathscr{f} \wedge \mathscr{d}$, where $\mathscr{c}_{ij} = \mathscr{b}_i \wedge \mathscr{d}$. Since $\mathscr{d}$ is a conjunction of literals, one can check by induction on D_i that D_{ij} is also canonical. This is immediate for all rules except for the $(\vee)$-rule. However, since D_i is canonical, $(\vee)$-rules occur in D_i only in $\vee\mu$-patterns, and one can check that one obtains then a $\vee\mu$-pattern by replacing each Boolean formula $\mathscr{a}$ by $\mathscr{a} \wedge \mathscr{d}$.

We finally obtain D' as follows:

$$\dfrac{\left\{\begin{array}{c} D_{ij} \\ \Gamma \vdash^{X \cup \{a\}} t : \mathscr{c}_{ij} \wedge \mathscr{e}_i \rightarrowtail \mathfrak{s} \end{array}\right\}_{i,j} \qquad \mathscr{b} \vDash \bigvee_{i,j} \mathscr{c}_{ij} \wedge \mathscr{e}_i}{\Gamma \vdash^{X \cup \{a\}} t : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\vee)$$

∎

Lemma D.14. *If D is a derivation ending with a rule whose premises are conclusions of pseudo-canonical derivations, then D can be turned into a pseudo-canonical derivation D'.*

Proof. We consider all possible rules:

- if D is as follows:

$$\dfrac{\dfrac{\left\{\begin{array}{c} D_j \\ \Gamma \vdash^{X\cup\{a\}} t : \mathscr{c}_j \rightarrowtail \mathfrak{s} \end{array}\right\}_j \qquad \mathscr{b} \vDash \bigvee_i \mathscr{c}_j}{\Gamma \vdash^{X\cup\{a\}} t : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\vee)}{\Gamma \vdash^{X\cup\{a\}} t \oplus_a^i u : \mathscr{x}_a^i \wedge \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus l)$$

then D' is as follows:

$$\dfrac{\left\{\dfrac{\begin{array}{c} D_j \\ \Gamma \vdash^{X\cup\{a\}} t : \mathscr{c}_j \rightarrowtail \mathfrak{s} \end{array}}{\Gamma \vdash^{X\cup\{a\}} t \oplus_a^i u : \mathscr{x}_a^i \wedge \mathscr{c}_j \rightarrowtail \mathfrak{s}}\,(\oplus l)\right\}_j \qquad \mathscr{x}_a^i \wedge \mathscr{b} \vDash \bigvee_j \mathscr{x}_a^i \wedge \mathscr{c}_j}{\Gamma \vdash^{X\cup\{a\}} t \oplus_a^i u : \mathscr{x}_a^i \wedge \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\vee)$$

The symmetric rule for $\oplus_a^i$ is treated similarly.

- If D is as follows:

$$\dfrac{\dfrac{\left\{\begin{array}{c} D_j \\ \Gamma, x : \mathfrak{M} \vdash^{X} t : \mathscr{c}_j \rightarrowtail \mathbf{C}^q\sigma \end{array}\right\}_j \qquad \mathscr{b} \vDash \bigvee_j \mathscr{c}_j}{\Gamma, x : \mathfrak{M} \vdash^{X} t : \mathscr{b} \rightarrowtail \mathbf{C}^q\sigma}\,(\vee)}{\Gamma \vdash^{X} \lambda x.t : \mathscr{b} \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)}\,(\lambda)$$

then D' is as follows:

$$\dfrac{\left\{\dfrac{\begin{array}{c} D_j \\ \Gamma, x : \mathfrak{M} \vdash^{X} t : \mathscr{c}_j \rightarrowtail \mathbf{C}^q\sigma \end{array}}{\Gamma \vdash^{X} \lambda x.t : \mathscr{c}_j \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)}\,(\lambda)\right\}_j \qquad \mathscr{b} \vDash \bigvee_j \mathscr{c}_j}{\Gamma \vdash^{X} \lambda x.t : \mathscr{b} \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)}\,(\vee)$$

- if D is as follows:

$$\dfrac{\dfrac{\left\{\begin{array}{c} D_j \\ \Gamma \vdash^{X} t : \mathscr{c}_j \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma) \end{array}\right\}_j \qquad \mathscr{c} \vDash \bigvee_j \mathscr{c}_j}{\Gamma \vdash^{X} t : \mathscr{c} \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)}\,(\vee) \qquad \left\{\begin{array}{c} E_l \\ \Gamma \vdash^{X} u : \mathscr{d}_l \rightarrowtail \mathfrak{s}_l \end{array}\right\}_l \qquad \mathscr{b} \vDash \mathscr{c} \wedge \left(\bigwedge_l \mathscr{d}_l\right)}{\Gamma \vdash^{X} tu : \mathscr{b} \rightarrowtail \mathbf{C}^q\sigma}\,(@_\cap)$$

where $\mathfrak{M} = [\mathfrak{s}_1, \dots, \mathfrak{s}_n]$ and each derivation E_l is of the form

$$\dfrac{\left\{\begin{array}{c} E_{lk} \\ \Gamma \vdash^{X} t : \mathscr{e}_{lk} \rightarrowtail \mathfrak{s}_l \end{array}\right\}_k \qquad \mathscr{d}_l \vDash \bigvee_k \mathscr{e}_{lk}}{\Gamma \vdash^{X} t : \mathscr{d}_l \rightarrowtail \mathfrak{s}_l}\,(\vee)$$

Then D' is as follows:

$$\dfrac{\left\{\dfrac{\begin{array}{c} D_j \\ \Gamma \vdash^{X} t : \mathscr{c}_j \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma) \end{array} \qquad \left\{\begin{array}{c} E_{lg(l)} \\ \Gamma \vdash^{X} u : \mathscr{e}_{lg(l)} \rightarrowtail \mathfrak{s}_l \end{array}\right\}_l \qquad \mathscr{b}_{jg} \vDash \mathscr{c}_j \wedge \left(\bigwedge_l \mathscr{e}_{lg(l)}\right)}{\Gamma \vdash^{X} tu : \mathscr{b}_{jg} \rightarrowtail \mathbf{C}^q\sigma}\,(@_\cap)\right\}_{j,g} \qquad \mathscr{b} \vDash \bigvee_{j,g} \mathscr{b}_{jg}}{\Gamma \vdash^{X} tu : \mathscr{b} \rightarrowtail \mathbf{C}^q\sigma}\,(\vee)$$

where $\mathscr{b}_{jg} = \mathscr{c}_j \wedge \left(\bigwedge_l \mathscr{e}_{lg(l)}\right)$ and g ranges over all possible choice functions associating each l with a suitable value k such that $\mathscr{e}_{lk}$ exists, using the fact that

$$\mathscr{b} \vDash \mathscr{c} \wedge \left(\bigwedge_l \mathscr{d}_l\right) \equiv \left(\bigvee_j \mathscr{c}_j\right) \wedge \left(\bigwedge_l \bigvee_k \mathscr{e}_{lk}\right) \equiv \bigvee_j \left(\mathscr{c}_j \wedge \left(\bigwedge_l \bigvee_k \mathscr{e}_{lk}\right)\right) \equiv \bigvee_{jg} \mathscr{c}_j \wedge \left(\bigwedge_l \mathscr{e}_{lg(l)}\right)$$

- if D is as follows:

$$\dfrac{\left\{\dfrac{\left\{\begin{array}{c} D_{ij} \\ \Gamma \vdash^{X\cup\{a\}} t : \mathscr{d}_{ij} \rightarrowtail \mathbf{C}^{q_i}\sigma \end{array}\right\}_j \qquad \mathscr{b}' \wedge \mathscr{c}_i \vDash \bigvee_j \mathscr{d}_{ij}}{\Gamma \vdash^{X\cup\{a\}} t : \mathscr{b}' \wedge \mathscr{c}_i \rightarrowtail \mathbf{C}^{q_i}\sigma}(\vee)\right\}_i \qquad \mu(\mathscr{c}_i) \geq s_i \qquad \mathscr{b} \vDash \mathscr{b}'}{\Gamma \vdash^X \nu a.t : \mathscr{b} \rightarrowtail \mathbf{C}^{\sum_i q_i s_i}\sigma}(\mu)$$

where $\mathrm{FN}(\mathscr{b}, \mathscr{b}') \subseteq X$ and $\mathrm{FN}(\mathscr{c}_i) \subseteq \{a\}$, then, since the D_{ij} are canonical, each $\mathscr{d}_{ij}$ is a conjunction of literals and can thus be written as $\mathscr{e}_{ij} \wedge \mathscr{f}_{ij}$, where $\mathrm{FN}(\mathscr{e}_{ij}) \subseteq X$ and $\mathrm{FN}(\mathscr{f}_{ij}) \subseteq \{a\}$.
Observe that we can suppose w.l.o.g. that none of either the $\mathscr{e}_{ij}$ or the $\mathscr{f}_{ij}$ is equivalent to $\bot$. Moreover, we can also suppose w.l.o.g. that $\mathscr{b}$ and $\mathscr{b}'$ coincide and are satisfiable.
Now, from $\mathscr{b} \wedge \mathscr{c}_i \vDash \bigvee_j \mathscr{e}_{ij} \wedge \mathscr{f}_{ij}$ (using the fact that the $\mathscr{c}_i$ are satisfiable) we deduce by Lemma D.5 that $\mathscr{b} \vDash \bigvee_j \mathscr{e}_{ij}$ and $\mathscr{c}_i \vDash \bigvee_j \mathscr{f}_{ij}$.
For each i, by applying Lemma D.13 to the family $(\mathscr{e}_{ij})_j$, we obtain a family $\mathscr{e}_{ijk}$ of conjunctions of literals, all satisfiable, with $\mathrm{FN}(\mathscr{e}_{ijk}) \subseteq X$, and such that $\mathscr{e}_{ij} \equiv \bigvee_k \mathscr{e}_{ijk}$, and for $j \neq j'$, either $\mathscr{e}_{ijk} \equiv \mathscr{e}_{ij'k'}$ or $\mathscr{e}_{ijk} \wedge \mathscr{e}_{ij'k'} \vDash \bot$. Moreover, we also deduce the existence of a family $D^{\dagger}_{ijk}$ of canonical derivations of conclusion $\Gamma \vdash^{X\cup\{a\}, q_i} t : \mathscr{e}_{ijk} \wedge \mathscr{f}_{ij} \rightarrowtail \mathfrak{s}$.
For each i, j, k, let U^i_{jk} be the set of pairs (j', k') such that $\mathscr{e}_{ij'k'} \equiv \mathscr{e}_{ijk}$. Observe that for any pair $(j'', k'') \notin U^i_{jk}$, $\mathscr{e}_{ijk} \wedge \mathscr{e}_{ij''k''} \vDash \bot$.
Let $\approx^i$ be the equivalence relation on pairs (j, k) defined by $(j, k) \approx^i (j', k')$ iff $\mathscr{e}_{ijk} \equiv \mathscr{e}_{ij'k'}$ (i.e. iff $(j', k') \in U^i_{jk}$), and let W^i contain a chosen representative of $[(j, k)]_{\approx^i}$ for each equivalence class. Then we have that

$$\bigvee_j \mathscr{e}_{ij} \wedge \mathscr{f}_{ij} \equiv \bigvee_{jk} \mathscr{e}_{ijk} \wedge \mathscr{f}_{ij} \equiv \bigvee_{(j,k)\in W^i} \mathscr{e}_{ijk} \wedge \Big(\bigvee_{j'k' \in U^i_{jk}} \mathscr{f}_{ij'} \Big) \tag{2}$$

Let $\widetilde{\mathscr{f}_{ijk}} = \bigvee_{j'k' \in U^i_{jk}} \mathscr{f}_{ij'}$. Let $\mathscr{b} \equiv \bigvee_u \mathscr{b}_u$, where the $\mathscr{b}_u$ are pairwise disjoint conjunctions of literals, all being satisfiable. From $\mathscr{b} \wedge \mathscr{c}_i \vDash \bigvee_j \mathscr{d}_{ij} \equiv \bigvee_{(j,k)\in W^i} \mathscr{e}_{ijk} \wedge \widetilde{\mathscr{f}_{ijk}}$ we deduce by Lemma D.5 and the satisfiability of $\mathscr{c}_i$ that $\mathscr{b} \vDash \bigvee_{(j,k)\in W^i} \mathscr{e}_{ijk}$; since the $\mathscr{e}_{ijk}$, with (j, k) varying in W^i, are pairwise disjoint, this implies the existence of functions J, K such that $\mathscr{b}_u \vDash \mathscr{e}_{iJ(u)K(u)}$, and thus a fortiori such that $\mathscr{b}_u \wedge \mathscr{c}_i \vDash \mathscr{e}_{iJ(u)K(u)} \wedge \widetilde{\mathscr{f}_{iJ(u)K(u)}}$. Since the $\mathscr{b}_u$ are satisfiable, by Lemma D.5, this implies $\mathscr{c}_i \vDash \widetilde{\mathscr{f}_{iJ(u)K(u)}}$.
For each u, we can now construct a derivation D^*_u of $\Gamma \vdash^X \nu a.t : \mathscr{e}_{iJ(u)K(u)} \rightarrowtail \mathbf{C}^{\sum_i q_i s_i}\sigma$ as shown below:

$$\dfrac{\left\{\dfrac{\left\{\begin{array}{c} D^{\ddagger}_{ij'k'} \\ \Gamma \vdash^{X\cup\{a\}} t : \mathscr{b}_u \wedge \mathscr{f}_{ij'} \rightarrowtail \mathbf{C}^{q_i}\sigma \end{array}\right\}_{(j',k')\in U^i_{J(u)K(u)}} \qquad \mathscr{b}_u \wedge \mathscr{c}_i \vDash \bigvee_{j'k' \in U_{J(u)K(u)}} \mathscr{b}_u \wedge \mathscr{f}_{ij'}}{\Gamma \vdash^{X\cup\{a\}} t : \mathscr{b}_u \wedge \mathscr{c}_i \rightarrowtail \mathbf{C}^{q_i}\sigma}(\vee)\right\}_i \qquad \mu(\mathscr{c}_i) \geq s_i}{\Gamma \vdash^X \nu a.t : \mathscr{b}_u \rightarrowtail \mathbf{C}^{\sum_i q_i s_i}\sigma}(\mu)$$

where $D^{\ddagger}_{ij'k'}$ is obtained from $D_{ij'k'}$ by replacing everywhere $\mathscr{e}_{ij'k'}$ by $\mathscr{b}_u$, using the fact that $\mathscr{b}_u \vDash \mathscr{e}_{iJ(u)K(u)} \equiv \mathscr{e}_{ij'k'}$.
Observe that the one above is a $\vee\mu$-pattern, and thus D^*_u is canonical.
We can now conclude as follows:

$$\dfrac{\left\{\begin{array}{c} D^*_u \\ \Gamma \vdash^X \nu a.t : \mathscr{b}_u \rightarrowtail \mathbf{C}^{\sum_i q_i s_i}\sigma \end{array}\right\}_u \qquad \mathscr{b} \vDash \bigvee_u \mathscr{b}_u}{\Gamma \vdash^X \nu a.t : \mathscr{b} \rightarrowtail \mathbf{C}^{\sum_i q_i s_i}\sigma}(\vee)$$

□

We can now prove Theorem D.12.

Proof of Theorem D.12. First, for any axiom

$$\dfrac{\mathfrak{s}_i \preceq \mathfrak{t} \qquad \mathrm{FN}(\mathscr{b}) \subseteq X}{\Gamma, x : [\mathfrak{s}_1, \ldots, \mathfrak{s}_n] \vdash^X x : \mathscr{b} \rightarrowtail \mathfrak{t}}(\mathrm{id}_{\preceq})$$

transform $\mathscr{b}$ into an equivalent disjunctive normal form $\bigvee_j \mathscr{b}_j$ (with the $\mathscr{b}_j$ conjunctions of literals), and replace the axiom by the derivation below

$$\dfrac{\left\{\dfrac{\mathfrak{s}_i \preceq \mathfrak{t} \qquad \mathrm{FN}(\mathscr{b}) \subseteq X}{\Gamma, x : [\mathfrak{s}_1, \ldots, \mathfrak{s}_n] \vdash^X x : \mathscr{b}_j \rightarrowtail \mathfrak{t}}\,(\mathrm{id}_{\preceq})\right\}_j \qquad \mathscr{b} \vDash^X \bigvee_j \mathscr{b}_j}{\Gamma, x : [\mathfrak{s}_1, \ldots, \mathfrak{s}_n] \vdash^X x : \mathscr{b} \rightarrowtail \mathfrak{t}}\,(\vee)$$

Now, starting from the axioms, for each rule occurring in the construction of the derivation, use Lemma D.14 to progressively permute $(\vee)$-rules downwards, hence turning the derivation into a pseudo-canonical one. □

Using Theorem D.12 we can finally prove the subject β-expansion property. We will make use of the following property (which is an easy consequence of Lemma D.6 and Lemma D.3):

Lemma D.15. *If* $\Gamma, \Delta \vdash^{X \cup Y} t : \mathscr{b} \rightarrowtail \mathfrak{s}$ *is derivable, where* $\mathrm{FV}(t) \subseteq \Gamma$ *and* $\mathrm{FN}(t) \subseteq X$*, then there exists a Boolean formula* $\mathscr{b}'$ *with* $\mathrm{FN}(\mathscr{b}') \subseteq X$ *such that* $\mathscr{b} \vDash \mathscr{b}'$ *and* $\Gamma \vdash^X t : \mathscr{b}' \rightarrowtail \mathfrak{s}$ *is derivable.*

Proposition D.16 (subject β-expansion). *If* $\Gamma \vdash^X t[u/x] : \mathscr{b} \rightarrowtail \mathfrak{s}$ *is derivable then* $\Gamma \vdash^X (\lambda x.t)u : \mathscr{b} \rightarrowtail \mathfrak{s}$ *is also derivable.*

Proof. A typing derivation of $\Gamma \vdash^X t[u/x] : \mathscr{b} \rightarrowtail \mathfrak{s}$ can be depicted as follows:

$$\dfrac{\begin{matrix} D_1 \\ \Gamma, \Delta_1 \vdash^{X \cup Y_1} u : \mathscr{f}_1 \rightarrowtail \mathfrak{s}_1 \end{matrix} \qquad \ldots \qquad \begin{matrix} D_N \\ \Gamma, \Delta_N \vdash^{X \cup Y_N} u : \mathscr{f}_N \rightarrowtail \mathfrak{s}_N \end{matrix}}{\Gamma \vdash^X t[u/x] : \mathscr{b} \rightarrowtail \mathfrak{s}}\, D$$

Observe that neither the variables in Δ_i nor the names in Y_i can occur in u. Hence using Lemma D.15 we obtain derivations D_i^* of $\Gamma \vdash^X u : \mathscr{d}_i \rightarrowtail \mathfrak{s}_i$, with $\mathscr{f}_i \vDash \mathscr{d}_i$.

Moreover, from D we can deduce a derivation D^* of $\Gamma, x : [\mathfrak{s}_1, \ldots, \mathfrak{s}_N] \vdash^X t : \mathscr{b}' \rightarrowtail \mathfrak{s}$, where each of the D_i is replaced by an axiom

$$\dfrac{\mathfrak{s}_i \preceq \mathfrak{s}_i \qquad \mathrm{FN}(\mathscr{d}_i) \subseteq X}{\Gamma, \Delta_i, x : [\mathfrak{s}_1, \ldots, \mathfrak{s}_N] \vdash^X x : \mathscr{d}_i \rightarrowtail \mathfrak{s}_i}\,(\mathrm{id}_{\preceq})$$

and $\mathscr{b}'$ is such that $\mathscr{b} \vDash \mathscr{b}'$.

Using Theorem D.12 we can turn D^* into a pseudo-canonical derivation $D^{*\dagger}$, which is thus of the form

$$D^{*\dagger} \quad = \quad \dfrac{\left\{\begin{matrix} E_l \\ \Gamma, x : [\mathfrak{s}_1, \ldots, \mathfrak{s}_N] \vdash^X t : \mathscr{c}_l \rightarrowtail \mathfrak{s} \end{matrix}\right\}_l \qquad \mathscr{b}' \vDash \bigvee_l \mathscr{c}_l}{\Gamma, x : [\mathfrak{s}_1, \ldots, \mathfrak{s}_N] \vdash^X t : \mathscr{b}' \rightarrowtail \mathfrak{s}}\,(\vee)$$

where each E_l is canonical and contains all axioms of the form

$$\dfrac{\mathfrak{s}_i \preceq \mathfrak{s}_i \qquad \mathrm{FN}(\mathscr{d}_{ig_l(i)}) \subseteq X}{\Gamma, \Delta_i, x : [\mathfrak{s}_1, \ldots, \mathfrak{s}_N] \vdash^X x : \mathscr{d}_{ig_l(i)} \rightarrowtail \mathfrak{s}_i}\,(\mathrm{id}_{\preceq})$$

where $\bigvee_k \mathscr{d}_{ik} \equiv \mathscr{d}_i$ is a disjunctive normal form and g_l is some choice function choosing one value k for each i.

Since $\mathscr{d}_{ik} \vDash \bigvee_k \mathscr{d}_{ik} \equiv \mathscr{d}_i$ holds we deduce by Lemma C.3 the existence of derivations D_{ik} of $\Gamma \vdash^X u : \mathscr{d}_{ik} \rightarrowtail \mathfrak{s}_i$.

Moreover, since the E_l are canonical, by Lemma D.11 we deduce that $\mathscr{c}_l \vDash \bigwedge_i \mathscr{d}_{ig_l(i)}$.

Using $\mathscr{b} \vDash \mathscr{b}'$ and $\mathscr{b}' \vDash \bigvee_l \mathscr{c}_l$, we can thus obtain a derivation of $\Gamma \vdash^X (\lambda x.t)u : \mathscr{b} \rightarrowtail \mathfrak{s}$ as follows (where $\mathfrak{M} = [\mathfrak{s}_1, \ldots, \mathfrak{s}_N]$ and $\mathfrak{s} = \mathbf{C}^q\sigma$):

$$\dfrac{\left\{\dfrac{\dfrac{\begin{matrix} E_l \\ \Gamma, x : \mathfrak{M} \vdash^X t : \mathscr{c}_l \rightarrowtail \mathfrak{s} \end{matrix}}{\Gamma \vdash^X \lambda x.t : \mathscr{c}_l \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)} \qquad \left\{\begin{matrix} D_{ig_l(i)} \\ \Gamma \vdash^X u : \mathscr{d}_{ig_l(i)} \rightarrowtail \mathfrak{s}_i \end{matrix}\right\}_i \qquad \mathscr{c}_l \vDash \bigwedge_i \mathscr{d}_{ig_l(i)}}{\Gamma \vdash^X (\lambda x.t)u : \mathscr{c}_l \rightarrowtail \mathfrak{s}}\right\}_l \qquad \mathscr{b} \vDash \bigvee_l \mathscr{c}_l}{\Gamma \vdash^X (\lambda x.t)u : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\vee)$$

□

D.2.2 Subject Permutative Expansion. The subject expansion property with respect to permutative reduction can be established by a direct inspection of permutative rules.

Proposition D.17. *If* $\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathfrak{s}$ *and* $u \multimap_{\mathrm{p}} t$*, then* $\Gamma \vdash^X u : \mathscr{b} \rightarrowtail \mathfrak{s}$*.*

Proof. As in the proof of Proposition C.1, if the typing derivation D of t ends by a $(\vee)$-rule, it suffices to establish the property for the immediate sub-derivations of D and then apply an instance of $(\vee)$-rule to the resulting derivations. So we will always suppose that the typing derivation of t does not end by a $(\vee)$-rule.

$(t \oplus_a^i t \multimap_{\mathrm{p}} t)$ From $\Gamma \vdash^{X\cup\{a\}} t : \mathscr{b} \rightarrowtail \mathfrak{s}$ we deduce

$$\dfrac{\dfrac{\Gamma \vdash^{X\cup\{a\}} t : \mathscr{b} \rightarrowtail \mathfrak{s}}{\Gamma \vdash^{X\cup\{a\}} t \oplus_a^i t : \mathscr{x}_a^i \wedge \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus l) \qquad \dfrac{\Gamma \vdash^{X\cup\{a\}} t : \mathscr{b} \rightarrowtail \mathfrak{s}}{\Gamma \vdash^{X\cup\{a\}} t \oplus_a^i t : \neg\mathscr{x}_a^i \wedge \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus r) \qquad \mathscr{b} \vDash (\mathscr{x}_a^i \wedge \mathscr{b}) \vee (\neg\mathscr{x}_a^i \wedge \mathscr{b})}{\Gamma \vdash^{X\cup\{a\}} t \oplus_a^i t : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\vee)$$

$((t \oplus_a^i u) \oplus_a^i v \multimap_{\mathrm{p}} t \oplus_a^i v)$ There are two possible sub-cases:

1. the type derivation is as follows:

$$\dfrac{\Gamma \vdash^{X\cup\{a\}} t : \mathscr{b}' \rightarrowtail \mathfrak{s} \qquad \mathscr{b}' \vDash \mathscr{x}_a^i \wedge \mathscr{b}'}{\Gamma \vdash^{X\cup\{a\}} t \oplus_a^i v : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus l)$$

Then we deduce

$$\dfrac{\dfrac{\Gamma \vdash^{X\cup\{a\}} t : \mathscr{b}' \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash \mathscr{x}_a^i \wedge \mathscr{b}'}{\Gamma \vdash^{X\cup\{a\}} t \oplus_a^i u : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus l) \qquad \mathscr{b} \vDash \mathscr{x}_a^i \wedge \mathscr{b}}{\Gamma \vdash^{X\cup\{a\}} (t \oplus_a^i u) \oplus_a^i v : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus l)$$

2. the type derivation is as follows:

$$\dfrac{\Gamma \vdash^{X\cup\{a\}} v : \mathscr{b}' \rightarrowtail \mathfrak{s} \qquad \mathscr{b}' \vDash \neg\mathscr{x}_a^i \wedge \mathscr{b}'}{\Gamma \vdash^{X\cup\{a\}} t \oplus_a^i v : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus l)$$

Then we deduce

$$\dfrac{\Gamma \vdash^{X\cup\{a\}} v : \mathscr{b}' \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash \neg\mathscr{x}_a^i \wedge \mathscr{b}'}{\Gamma \vdash^{X\cup\{a\}} (t \oplus_a^i u) \oplus_a^i v : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus l)$$

$(t \oplus_a^i (u \oplus_a^i v) \multimap_{\mathrm{p}} t \oplus_a^i v)$ Similar to the case above.

$(\lambda x.(t \oplus_a^i u) \multimap_{\mathrm{p}} (\lambda x.t) \oplus_a^i (\lambda x.u))$ There are two possible sub-cases, treated similarly. We only consider the first one:

1.

$$\dfrac{\dfrac{\Gamma, x : \mathfrak{M} \vdash^X t : \mathscr{b}' \rightarrowtail \mathbf{C}^q\sigma}{\Gamma \vdash^X \lambda x.t : \mathscr{b}' \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)}\,(\lambda) \qquad \mathscr{b} \vDash \mathscr{x}_a^i \wedge \mathscr{b}'}{\Gamma \vdash^X (\lambda x.t) \oplus_a^i (\lambda x.u) : \mathscr{b} \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)}\,(\oplus l)$$

Then we deduce

$$\dfrac{\dfrac{\Gamma, x : \mathfrak{M} \vdash^X t : \mathscr{b}' \rightarrowtail \mathbf{C}^q\sigma \qquad \mathscr{b} \vDash \mathscr{x}_a^i \wedge \mathscr{b}'}{\Gamma, x : \mathfrak{M} \vdash^X t \oplus_a^i u : \mathscr{b} \rightarrowtail \mathbf{C}^q\sigma}\,(\oplus l)}{\Gamma \vdash^X \lambda x.(t \oplus_a^i u) : \mathscr{b} \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)}\,(\lambda)$$

$((t \oplus_a^i u)v \multimap_{\mathrm{p}} (tv) \oplus_a^i (uv))$ There are two possible sub-cases, treated similarly. We only consider the first one (with $\mathfrak{M} = [\mathfrak{s}_1, \dots, \mathfrak{s}_n]$):

1.

$$\dfrac{\dfrac{\Gamma \vdash^X t : \mathscr{b}'' \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma) \qquad \left\{\Gamma \vdash^X v : \mathscr{c}_i \rightarrowtail \mathfrak{s}_i\right\}_i \qquad \mathscr{b}' \vDash \mathscr{b}'' \wedge (\bigwedge_i \mathscr{c}_i)}{\Gamma \vdash^X tv : \mathscr{b}' \rightarrowtail \mathbf{C}^q\sigma}\,(@_\cap) \qquad \mathscr{b} \vDash \mathscr{x}_a^i \wedge \mathscr{b}'}{\Gamma \vdash^X (tv) \oplus_a^i (uv) : \mathscr{b} \rightarrowtail \mathbf{C}^q\sigma}\,(\oplus l)$$

Then we deduce

$$\dfrac{\dfrac{\Gamma \vdash^X t : \mathscr{b}'' \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma) \qquad \mathscr{x}_a^i \wedge \mathscr{b}'' \vDash \mathscr{x}_a^i \wedge \mathscr{b}''}{\Gamma \vdash^X t \oplus_a^i u : \mathscr{x}_a^i \wedge \mathscr{b}'' \rightarrowtail \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)}\,(\oplus l) \qquad \left\{\Gamma \vdash^X v : \mathscr{c}_i \rightarrowtail \mathfrak{s}_i\right\}_i \qquad \mathscr{b} \vDash (\mathscr{x}_a^i \wedge \mathscr{b}'') \wedge (\bigwedge_i \mathscr{c}_i)}{\Gamma \vdash^X (t \oplus_a^i u)v : \mathscr{b} \rightarrowtail \mathbf{C}^q\sigma}\,(@_\cap)$$

$(t(u \oplus^i_a v) \to_{\mathrm{p}} (tu) \oplus^i_a (tv))$ There are two sub-cases, treated similarly. We only consider the first one:

1.
$$\dfrac{\dfrac{\Gamma \vdash^X t : \mathscr{b}'' \rightarrowtail \mathsf{C}^q(\mathfrak{M} \Rightarrow \sigma) \qquad \Big\{\Gamma \vdash^X u : \mathscr{c}_i \rightarrowtail \mathfrak{s}_i\Big\}_i \qquad \mathscr{b}' \vDash \mathscr{b}'' \wedge (\bigwedge_i \mathscr{c}_i)}{\Gamma \vdash^X tu : \mathscr{b}' \rightarrowtail \mathsf{C}^q\sigma}\,(@_\cap) \qquad \mathscr{b} \vDash \mathscr{x}^i_a \wedge \mathscr{b}'}{\Gamma \vdash^X (tu) \oplus^i_a (tv) : \mathscr{b} \rightarrowtail \mathsf{C}^q\sigma}\,(\oplus l)$$

Then we deduce that

$$\dfrac{\Gamma \vdash^X t : \mathscr{b}'' \rightarrowtail \mathsf{C}^q(\mathfrak{M} \Rightarrow \sigma) \qquad \left\{\dfrac{\Gamma \vdash^X u : \mathscr{c}_i \rightarrowtail \mathfrak{s}_i \qquad \mathscr{x}^i_a \wedge \mathscr{c}_i \vDash \mathscr{x}^i_a \wedge \mathscr{c}_i}{\Gamma \vdash^X u \oplus^i_a v : \mathscr{x}^i_a \wedge \mathscr{c}_i \rightarrowtail \mathfrak{s}_i}\,(\oplus l)\right\}_i \qquad \mathscr{b} \vDash \mathscr{b}'' \wedge \left(\bigwedge_i (\mathscr{x}^i_a \wedge \mathscr{c}_i)\right)}{\Gamma \vdash^X t(u \oplus^i_a v) : \mathscr{b} \rightarrowtail \mathsf{C}^q\sigma}\,(@_\cap)$$

$((t \oplus^i_a u) \oplus^j_b v \to_{\mathrm{p}} (t \oplus^j_b v) \oplus^i_a (u \oplus^j_b v))$ We suppose here $a \neq b$ or $i < j$. There are four sub-cases, all treated similarly. We only consider the first one:

1.
$$\dfrac{\dfrac{\Gamma \vdash^X t : \mathscr{b}'' \rightarrowtail \mathfrak{s} \qquad \mathscr{b}' \vDash \mathscr{x}^j_b \wedge \mathscr{b}''}{\Gamma \vdash^X t \oplus^j_b v : \mathscr{b}' \rightarrowtail \mathfrak{s}}\,(\oplus l) \qquad \mathscr{b} \vDash \mathscr{x}^i_a \wedge \mathscr{b}'}{\Gamma \vdash^X (t \oplus^j_b u) \oplus^i_a (u \oplus^j_b v) : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus l)$$

Then we deduce

$$\dfrac{\dfrac{\Gamma \vdash^X t : \mathscr{b}'' \rightarrowtail \mathfrak{s} \qquad \mathscr{b} \vDash \mathscr{x}^i_a \wedge \mathscr{b}''}{\Gamma \vdash^X t \oplus^i_a u : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus l) \qquad \mathscr{b} \vDash \mathscr{x}^j_b \wedge \mathscr{b}}{\Gamma \vdash^X (t \oplus^i_a u) \oplus^j_b v : \mathscr{b} \rightarrowtail \mathfrak{s}}\,(\oplus l)$$

$(t \oplus^j_b (u \oplus^i_a v) \to_{\mathrm{p}} (t \oplus^j_b u) \oplus^i_a (t \oplus^j_b v))$ Similar to the case above.

$(\nu b.(t \oplus^i_a u) \to_{\mathrm{p}} (\nu b.t) \oplus^i_a (\nu b.u))$ We suppose $a \neq b$. There are two sub-cases, treated similarly. We only consider the first one:

1.
$$\dfrac{\dfrac{\{\Gamma \vdash^{X \cup \{a,b\}} t : \mathscr{b}'' \wedge \mathscr{c}_i \rightarrowtail \mathsf{C}^{q_i}\sigma\}_i \qquad \mu(\mathscr{c}_i) \geq s_i \qquad \mathscr{b}' \vDash \mathscr{b}''}{\Gamma \vdash^{X \cup \{a\}} \nu b.t : \mathscr{b}' \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}\sigma}\,(\mu_\Sigma) \qquad \mathscr{b} \vDash \mathscr{x}^i_a \wedge \mathscr{b}'}{\Gamma \vdash^{X \cup \{a\}} (\nu b.t) \oplus^i_a (\nu b.u) : \mathscr{b} \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}\sigma}\,(\oplus l)$$

From $\mathscr{b} \vDash \mathscr{b}''$ we deduce by Lemma C.3 the existence of derivations of $\Gamma \vdash^{X \cup \{b\}} t : \mathscr{b} \wedge \mathscr{c}_i \rightarrowtail \mathsf{C}^{q_i}\sigma$. Then we can construct the following type derivation for u:

$$\dfrac{\left\{\dfrac{\Gamma \vdash^{X \cup \{a,b\}} t : \mathscr{b} \wedge \mathscr{c}_i \rightarrowtail \mathsf{C}^{q_i}\sigma}{\Gamma \vdash^{X \cup \{a,b\}} t \oplus^i_a u : \mathscr{b} \wedge \mathscr{c}_i \rightarrowtail \mathsf{C}^{q_i}\sigma}\,(\oplus l)\right\}_i \qquad \mu(\mathscr{c}_i) \geq s_i}{\Gamma \vdash^{X \cup \{a\}} \nu b.t \oplus^i_a u : \mathscr{b} \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}\sigma}\,(\mu_\Sigma)$$

$(\nu a.t \to_{\mathrm{p}} t)$ We can suppose that $a \notin \mathrm{FN}(\mathscr{b})$, so from $\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathfrak{s}$ and $\mathscr{b} \wedge \top \vDash \mathscr{b}$ we deduce $\Gamma \vdash^{X \cup \{a\}} t : \mathscr{b} \wedge \top \rightarrowtail \mathfrak{s}$ and since $\mu(\top) \geq 1$, $\Gamma \vdash^X \nu a.t : \mathscr{b} \rightarrowtail \mathfrak{s}$, since $\mathfrak{s} = \mathsf{C}^q\sigma = \mathsf{C}^{q \cdot 1}\sigma$.

$(\lambda x.\nu a.t \to_{\mathrm{p}} \nu a.\lambda x.t\,)$

$$\dfrac{\left\{\dfrac{\Gamma, x : \mathfrak{M} \vdash^{X \cup \{a\}} t : \mathscr{b}' \wedge \mathscr{c}_i \rightarrowtail \mathsf{C}^{q_i}\sigma}{\Gamma \vdash^{X \cup \{a\}} \lambda x.t : \mathscr{b}' \wedge \mathscr{c}_i \rightarrowtail \mathsf{C}^{q_i}(\mathfrak{M} \Rightarrow \sigma)}\,(\lambda)\right\}_i \qquad \mu(\mathscr{c}_i) \geq s_i \qquad \mathscr{b} \vDash \mathscr{b}'}{\Gamma \vdash^X \nu a.\lambda x.t : \mathscr{b} \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}(\mathfrak{M} \Rightarrow \sigma)}\,(\mu_\Sigma)$$

from which we deduce

$$\dfrac{\dfrac{\{\Gamma, x : \mathfrak{M} \vdash^{X \cup \{a\}} t : \mathscr{b}' \wedge \mathscr{c}_i \rightarrowtail \mathsf{C}^{q_i}\sigma\}_i \qquad \mu(\mathscr{c}_i) \geq s_i \qquad \mathscr{b} \vDash \mathscr{b}'}{\Gamma, x : \mathfrak{M} \vdash^X \nu a.t : \mathscr{b} \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}\sigma}\,(\mu_\Sigma)}{\Gamma \vdash^X \lambda x.\nu a.t : \mathscr{b} \rightarrowtail \mathsf{C}^{\sum_i q_i s_i}(\mathfrak{M} \Rightarrow \sigma)}\,(\lambda)$$

$((\nu a.t)u \to_{\mathrm{p}} \nu a.(tu))$ We have

$$\dfrac{\left\{\dfrac{\Gamma \vdash^{X\cup\{a\}} t : \mathscr{c}_i \rightarrowtail \mathbf{C}^{q_i}(\mathfrak{M}_i \Rightarrow \sigma) \qquad \left\{\Gamma \vdash^{X\cup\{a\}} u : \mathscr{d}_{ij} \rightarrowtail \mathsf{t}_{ij}\right\}_j \qquad \mathscr{b}' \wedge \mathscr{e}_i \vDash \mathscr{c}_i \wedge \bigwedge_j \mathscr{d}_{ij}}{\Gamma \vdash^{X\cup\{a\}} tu : \mathscr{b}' \wedge \mathscr{e}_i \rightarrowtail \mathbf{C}^{q_i}\sigma}(@_\cap)\right\}_i \qquad \mu(\mathscr{e}_i) \geq s_i \qquad \mathscr{b} \vDash \mathscr{b}'}{\Gamma \vdash^{X} \nu a.tu : \mathscr{b} \rightarrowtail \mathbf{C}^{\sum_i q_i s_i}\sigma}(\mu_\Sigma)$$

where $\mathfrak{M}_i = [\mathsf{t}_{i1}, \dots, \mathsf{t}_{ip_i}]$. Let $\mathfrak{M}$ be the concatenation of $\mathfrak{M}_1, \dots, \mathfrak{M}_{k+1}$. Since $\mathfrak{M} \leq \mathfrak{M}_i$, we have $\mathbf{C}^q(\mathfrak{M}_i \Rightarrow \sigma) \leq \mathbf{C}^q(\mathfrak{M} \Rightarrow \sigma)$, so from the derivations of $\Gamma \vdash^{X\cup\{a\}} t : \mathscr{c}_i \rightarrowtail \mathbf{C}^{q_i}(\mathfrak{M}_i \Rightarrow \sigma)$, using Lemma D.4 we deduce derivations of $\Gamma \vdash^{X\cup\{a\}} t : \mathscr{c}_i \rightarrowtail \mathbf{C}^{q}(\mathfrak{M} \Rightarrow \sigma)$. Moreover, from $\mathscr{b} \wedge \mathscr{e}_i \vDash \mathscr{c}_i$, using Lemma D.3, we deduce the existence of derivations of $\Gamma \vdash^{X\cup\{a\}} t : \mathscr{b} \wedge \mathscr{e}_i \rightarrowtail \mathbf{C}^{q}(\mathfrak{M} \Rightarrow \sigma)$, and from $\mathscr{b} \wedge \mathscr{e}_i \vDash \bigwedge_j \mathscr{d}_{ij}$, since $\mathrm{FN}(\mathscr{e}_i) \cap \mathrm{FN}(\mathscr{d}_{ij}) = \emptyset$, we deduce $\mathscr{b} \vDash \bigwedge_j \mathscr{d}_{ij}$. We can thus construct a type derivation for $(\nu a.t)u$ as follows:

$$\dfrac{\dfrac{\left\{\Gamma \vdash^{X\cup\{a\}} t : \mathscr{b} \wedge \mathscr{e}_i \rightarrowtail \mathbf{C}^{q}(\mathfrak{M} \Rightarrow \sigma)\right\}_i \qquad \mu(\mathscr{e}_i) \geq s_i}{\Gamma \vdash^{X} \nu a.t : \mathscr{b} \rightarrowtail \mathbf{C}^{\sum_i q_i s_i}(\mathfrak{M} \Rightarrow \sigma)}(\mu_\Sigma) \qquad \left\{\Gamma \vdash^{X, r_i} u : \mathscr{d}_{ij} \rightarrowtail \mathsf{t}_{ij}\right\}_{i,j} \qquad \mathscr{b} \vDash \bigwedge_{ij} \mathscr{d}_{ij}}{\Gamma \vdash^{X} (\nu a.t)u : \mathscr{b} \rightarrowtail \mathbf{C}^{\sum_i q_i s_i}\sigma}(@_\cap)$$

Observe that this is the unique case in the proof in which we use intersection types in an essential way. □

D.3 Completeness.

D.3.1 Deterministic Completeness. The goal of this subsection is to establish Proposition 6.5.

Proposition 6.5 (Deterministic Completeness). *For any closed term t,*
(i.) t is head-normalizable iff $\vdash_{\neg\vee} t : \top \rightarrowtail \mathbf{C}^1\mathsf{hn}$;
(ii.) t is normalizable iff $\vdash_{\neg\vee} t : \top \rightarrowtail \mathbf{C}^1\mathsf{n}$.
(iii.) t is strongly normalizable iff $\vdash_{\neg\vee} t : \top \rightarrowtail \mathbf{C}^1\mathsf{n}$ and all types in the derivation are safe.

We need a few preliminary lemmas:

Lemma D.18. *For any closed term t:*
(i.) if t is a HNV, then $\vdash_{\neg\vee} t : \top \rightarrowtail \mathbf{C}^1\sigma$ holds for some type;
(ii.) if t is a HNV and is $\rightarrow_\triangleright$-normal, then $\vdash_{\neg\vee} t : \top \rightarrowtail \mathbf{C}^1\sigma$ holds for some safe type.

Proof. If $t = \lambda x_1 \dots x_n.x_i u_1 \dots u_p$, then we can let $\sigma = \mathfrak{s}_1 \Rightarrow \dots \Rightarrow \mathfrak{s}_n \Rightarrow o$, where for $j \neq i$, $\mathfrak{s}_j = [\ ]$, and $\mathfrak{s}_i = \mathbf{C}^1[\mathbf{C}^1[\ ] \Rightarrow \dots \Rightarrow \mathbf{C}^1[\ ] \Rightarrow o]$. The second claim is proved by induction on normal forms, using a similar construction. □

Lemma D.19. *If $\Gamma \vdash_{\neg\vee} t[u/x] : \mathscr{b} \rightarrowtail \mathfrak{s}$ is derivable without using $[]$, and t contains at least one occurrence of x, then there exists $\mathfrak{M} \neq []$ such that $\Gamma, x : \mathfrak{M} \vdash_{\neg\vee} t : \mathscr{b} \rightarrowtail \mathfrak{s}$ and $\Gamma \vdash_{\neg\vee} u : \mathscr{b} \rightarrowtail \mathfrak{M}$ are derivable without using $[]$.*

Proof. It suffices to look at the proof of Proposition D.16, and observe that (1) since u occurs at least once in t, $\mathfrak{M}$ cannot be $[]$, and (2) since the original derivation has no occurrence of $(\vee)$, it must be canonical, and thus to construct the derivation of $\Gamma \vdash (\lambda x.t)u : \mathscr{b} \rightarrowtail \mathfrak{s}$ we do not need to make use of any application of $(\vee)$. □

Using Lemma D.18 together with the subject expansion property we can prove this first half of Proposition 6.5. The second half comes from Corollary E.16.

Proof of Proposition 6.5. For (i.), if t is head normalizable, with head-normal form u, then by Lemma B.8 the tree $RBT(t) = RBT(u)$ is finite and its leaves are HNV $u_1, \dots, u_N$. By Lemma D.18 we can deduce $\vdash u_i : \top \rightarrowtail \mathbf{C}^1\mathsf{hn}$; we can then climb up $RBT(t)$ using the rules $(\oplus l)$, $(\oplus r)$ and (μ_Σ), hence proving $\vdash u : \top \rightarrowtail \mathbf{C}^1\mathsf{hn}$, and finally $\vdash t : \top \rightarrowtail \mathbf{C}^1\mathsf{hn}$ by subject expansion. The converse direction follows from Corollary E.16 (i.).

The argument for (ii.) is similar.

For (iii.), let t be strongly normalizable, we argue by induction on the maximum length N of a reduction of t. If $N = 0$, then t is in normal form, so by an argument similar to (i.) we can deduce $\vdash_{\not\vee} t : \top \rightarrowtail \mathbf{C}^1\mathsf{n}$. If $N > 0$ and $t \rightarrow_\triangleright u$, then by IH $\vdash_{\not\vee} u : \top \rightarrowtail \mathbf{C}^1\mathsf{n}$ holds with a $[]$-free derivation, and $t = \mathsf{C}[(\lambda x.t')t'']$, $u = \mathsf{C}[t'[t''/x]]$. If x occurs in t', then we deduce from Lemma D.19 that $\vdash_{\not\vee} t : \top \rightarrowtail \mathbf{C}^1\mathsf{n}$ holds with a $[]$-derivation. Otherwise, since t'' must be strong normalizable, by IH $\vdash_{\not\vee} t'' : \top \rightarrowtail \mathbf{C}^1\mathsf{n}$, and we deduce, again by IH, that $\lambda x.t'$ can be typed by declaring $x : \mathbf{C}^1\mathsf{n}$. We can then conclude $\vdash_{\not\vee} t : \top \rightarrowtail \mathbf{C}^1\mathsf{n}$ also in this case.

The converse claim follows from Corollary E.16 (iii.) □

D.3.2 Probabilistic Completeness. The goal of this subsection is to establish the completeness part of Theorem 6.6:

Theorem 6.6 (Probabilistic Completeness). *For any closed term t,*

$$\mathrm{HNV}_{\rightarrowtriangle}(t) = \sup\{q \mid \vdash t : \top \rightarrowtail \mathbf{C}^q \mathsf{hn}\}$$
$$\mathrm{NF}_{\rightarrowtriangle}(t) = \sup\{q \mid \vdash t : \top \rightarrowtail \mathbf{C}^q \mathsf{n}\}$$

More precisely, we will show the $\leq$-part of the equations above, while soundness (i.e. the $\geq$-part of the equations above) will follow from Theorem 6.1 and Theorem 6.3, proved in the next section.

First, we need to study the structure of the trees $RBT(t)$ (see Subsection B.1) further.

Definition D.20. A *randomized multi-context* C is a term constructed from the grammar below

$$\mathsf{C} ::= [\ ]_i \ (i \in \mathbb{N}) \mid \mathsf{C} \oplus_a^i \mathsf{C} \mid \nu a.\mathsf{C}$$

where for each two occurrences $[\ ]_i$ and $[\ ]_j$, $i \neq j$. The rank $\mathrm{r}(\mathsf{C})$ is the maximum k such that $[\ ]_k$ occurs in C.

For any randomized multi-context C, we let $\mathsf{C}^{\neg\nu}$ be the randomized multi-context obtained by deleting all occurrences of ν-binders.

If C is a randomized multi-context of rank k, for all terms $t_1, \dots, t_k$, we let $\mathsf{C}[t_1, \dots, t_k]$ indicate the term obtained by replacing each hole $[\ ]_i$ in C by t_i.

An *initial segment* T of $RBT(t)$, noted $T \sqsubset RBT(t)$ is any sub-tree of $RBT(t)$ which contains its root.

Observe that a finite initial segment $T \sqsubset RBT(t)$ is the same thing as a term of the form $\mathsf{C}_T[u_1, \dots, u_k]$ for some randomized multi-context C_T.

Lemma D.21. *For any term t and HNF $u_1, \dots, u_k$, if there exist randomized paths from t to $u_1, \dots, u_k$, then there exists $N \in \mathbb{N}$, a randomized multi-context C of rank $N + k$ and terms $t_1, \dots, t_N$ such that $t \rightarrowtriangle_{\mathsf{p}}^* \mathsf{C}[u_1, \dots, u_k, t_1, \dots, t_N]$ and $RBT(\mathsf{C}[u_1, \dots, u_k, t_1, \dots, t_N]) \sqsubseteq RBT(t)$.*

Proof. We argue by induction on the maximum length K of the randomized paths considered. If $K = 0$, $k = 1$ and $t = u$, so we just let $N = 0$ and $\mathsf{C}[\ ] = [\ ]$.

If $K \geq 1$, then in a finite number of reduction steps we get $t \rightarrowtriangle_{\mathsf{p}}^* \nu a.T$, where T is a $(\mathscr{T}, a)$-tree such that for each $i = 1, \dots, k$, there exists $v_i \in \mathrm{supp}(T)$ with a randomized path from v_i to u_i of length $\leq K - 1$; observe that we can write T as $\mathsf{D}[v_1, \dots, v_q]$ for some randomized multi-context C.

For any $j = 1, \dots, q$, let I_j be the set of values $i \leq k$ such that there is a randomized path from v_j to u_i. The I_j are disjoint sets with $\bigcup_j I_j = \{1, \dots, k\}$. By the I.H. we then deduce that there exist randomized multicontexts C_j or rank $N_j + |I_j|$ and terms $w_{j1}, \dots, w_{jN_j}$ such that $v_j \rightarrowtriangle_{\mathsf{p}}^* \mathsf{C}[u_{i_1}, \dots, u_{i_{M_j}}, w_{j1}, \dots, w_{jN_j}]$, where $I_j = \{i_1, \dots, i_{M_j}\}$. We now have that

$$t \rightarrowtriangle_{\mathsf{p}}^* \nu a.T = \nu a.\mathsf{C}[v_1, \dots, v_q] \rightarrowtriangle_{\mathsf{p}}^* \nu a.\mathsf{D}[\mathsf{C}_1[\vec{u}, \vec{w}], \dots, \mathsf{C}_q[\vec{u}, \vec{w}]]$$

so we can let $N = \sum_j N_j$, $\mathsf{C} = \nu a.\mathsf{D}[\mathsf{C}_1, \dots, \mathsf{C}_q]$ and $t_1, \dots, t_N = w_{11}, \dots, w_{qM_q}$. ◻

Lemma D.22. *For any randomized multi-context C with n bound names $a_1, \dots, a_n$,*

$$\nu a_1. \dots .\nu a_n.\mathsf{C}^{\neg\nu} \rightarrowtriangle_{\mathsf{p}}^* \mathsf{C} \tag{3}$$

Proof. We will prove the following fact: for all terms $\nu b.t, t_1, \dots, t_k$, it C has no ν-binders, then

$$\nu b.\mathsf{C}[t_1, t_2, \dots, t_k, t] \rightarrowtriangle_{\mathsf{p}}^* \mathsf{C}[t_1, \dots, t_k, \nu b.t] \tag{4}$$

From this fact we can deduce the main claim as follows: let b be a bound variable in C whose binder ν has no other binder in its scope; then C splits as $\mathsf{C}[t_0, t_1, \dots, t_k] = \mathsf{E}[t_{i_1}, \dots, t_{i_p}, \nu b.\mathsf{D}[t_{j_1}, \dots, t_{j_q}]]$, where $I = \{i_1, \dots, i_p\}, J = \{j_1, \dots, j_q\} \subseteq \{1, \dots, k+1\}$ satisfy $I \cap J = \emptyset$ and $p + q = \mathrm{r}(\mathsf{C})$ (we are here supposing that $r(\mathsf{C}) \geq 1$, because otherwise the claim is trivial).

Then, by (4) we deduce that $\nu b.\mathsf{E}^{\neg\nu}[t_{i_1}, \dots, t_{i_p}, \mathsf{D}[t_{j_1}, \dots, t_{j_q}]] \rightarrowtriangle_{\mathsf{p}} \mathsf{E}^{\neg\nu}[t_{i_1}, \dots, t_{i_p}, \nu b.\mathsf{D}[t_{j_1}, \dots, t_{j_q}]]$; by arguing in this way on all binders ν we finally obtain (3).

We prove (4) by induction on the construction of C:

- if $\mathsf{C} = [\ ]_i$, then the claim trivially holds;

- if $\mathsf{C} = \mathsf{C}_1 \oplus_a^j \mathsf{C}_2$, then $[\ ]_{k+1}$ occurs in either C_1 or C_2, and not both; let us say $[\ ]_{k+1}$ occurs in C_1; we then have that

$$
\begin{aligned}
\nu b.\mathsf{C}[t_1,\dots,t_k,t] &= \nu b.\mathsf{C}_1[t_{i_1},\dots,t_{i_p},t] \oplus_a^j \mathsf{C}_2[t_{j_1},\dots,t_{j_q}] \\
&\rightarrowtriangle_{\mathrm{p}} \left(\nu b.\mathsf{C}_1[t_{i_1},\dots,t_{i_p},t]\right) \oplus_a^i \left(\nu b.\mathsf{C}_2[t_{j_1},\dots,t_{j_q}]\right) \\
&\rightarrowtriangle_{\mathrm{p}} \left(\nu b.\mathsf{C}_1[t_{i_1},\dots,t_{i_p},t]\right) \oplus_a^i \mathsf{C}_2[t_{j_1},\dots,t_{j_q}] \\
&\overset{\text{I.H.}}{\rightarrowtriangle_{\mathrm{p}}^{*}} \mathsf{C}_1[t_{i_1},\dots,t_{i_p},\nu b.t] \oplus_a^i \mathsf{C}_2[t_{j_1},\dots,t_{j_q}] = \mathsf{C}[t_1,\dots,t_k,\nu b.t]
\end{aligned}
$$

□

Still one technical lemmas before proceeding to the proof of Theorem 6.6.

Lemma D.23. *Let $N, K \geq 1$, $X = \{a_1,\dots,a_K\}$ be distinct names, and for all $i \in N$ and $j \in K$, $\mathscr{b}_{ij}$ be a conjunction of literals of name a_j such that for all $i \neq i'$:*

a. for all $k \leq K$, if $\bigwedge_{j=1}^{k} \mathscr{b}_{ij} \neq \bigwedge_{j=1}^{k} \mathscr{b}_{i'j}$, then $\left(\bigwedge_{j=1}^{k} \mathscr{b}_{ij}\right) \wedge \left(\bigwedge_{j=1}^{k} \mathscr{b}_{ij}\right) \vDash \bot$;

b. $\left(\bigwedge_{j=1}^{K} \mathscr{b}_{ij}\right) \wedge \left(\bigwedge_{j=1}^{K} \mathscr{b}_{i'j}\right) \vDash \bot$.

Suppose $\Gamma \vdash^{X,q_i} t : \mathscr{c} \wedge \left(\bigwedge_{j=1}^{K} \mathscr{b}_{ij}\right) \rightarrowtail \sigma$ is derivable for all $i \in \{1,\dots,N\}$. Then there exists a derivation of $\Gamma \vdash^{\emptyset} \nu a_1.\dots.\nu a_K.t : \mathscr{c} \rightarrowtail \mathbf{C}^r\sigma$, where $r = \sum_{i=1}^{N}\left(q_i \cdot \prod_{j=1}^{K} \mu(\mathscr{b}_{ij})\right)$.

Proof. We argue by induction on K. If $K = 1$ then $i \neq i'$ implies $\mathscr{b}_i \wedge \mathscr{b}_{i'} \vDash \bot$, so we can conclude as follows:

$$
\dfrac{\left\{ \begin{array}{c} D_i \\ \Gamma \vdash^{\{a\}} t : \mathscr{c} \wedge \mathscr{b}_i \rightarrowtail \mathbf{C}^{q_i}\sigma \end{array} \right\}_{i=1,\dots,N}}{\Gamma \vdash^{\emptyset} \nu a.t : \mathscr{c} \rightarrowtail \mathbf{C}^{\sum_{i=1}^{N} q_i \cdot \mu(\mathscr{b}_i)}\sigma}\ (\mu_\Sigma)
$$

If $K > 1$, then for all $i, i' \in \{1,\dots,N\}$, let $i \sim i'$ if $\mathscr{b}_{iK} \wedge \mathscr{b}_{i'K} \vDash \bot$ and for all $j \in \{1,\dots K-1\}$, $\mathscr{b}_{ij} \equiv \mathscr{b}_{i'j}$. Let $\mathsf{Cl}(N_\sim)$ indicate the set of cliques of the relation $\sim$, and for each $U \in \mathsf{Cl}(N_\sim)$, fix some $i_U \in U$. Then we claim that the following equation holds:

$$
\sum_{i=1}^{N} q_i \cdot \prod_{j=1}^{K} \mu(\mathscr{b}_{ij}) = \sum_{U \in \mathsf{Cl}_{N_\sim}} \left(\prod_{j=1}^{K-1} \mu(\mathscr{b}_{i_U j})\right) \cdot \left(\sum_{i \in U} q_i \cdot \mu(\mathscr{b}_{iK})\right) \tag{5}
$$

Indeed we can compute

$$
\begin{aligned}
\sum_{i=1}^{N}\left(q_i \cdot \prod_{j=1}^{K} \mu(\mathscr{b}_{ij})\right) &= \sum_{U \in \mathsf{Cl}_{N_\sim}} \sum_{i \in U} \left(q_i \cdot \prod_{j=1}^{K} \mu(\mathscr{b}_{ij})\right) \\
&= \sum_{U \in \mathsf{Cl}_{N_\sim}} \sum_{i \in U} \left(\prod_{j=1}^{K-1} \mu(\mathscr{b}_{i_U j})\right) \cdot \left(q_i \cdot \mu(\mathscr{b}_{iK})\right) \\
&= \sum_{U \in \mathsf{Cl}_{N_\sim}} \left(\prod_{j=1}^{K-1} \mu(\mathscr{b}_{i_U j})\right) \cdot \left(\sum_{i \in U} q_i \cdot \mu(\mathscr{b}_{iK})\right)
\end{aligned}
$$

Let us now show how to construct, for all $U \in \mathsf{Cl}_{N_\sim}$, a derivation D_U of $\Gamma \vdash^{\{a_1,\dots,a_{K-1}\}} \nu a_K.t : \mathscr{c} \wedge \left(\prod_{j=1}^{K-1} \mathscr{b}_{i_U j}\right) \rightarrowtail \mathbf{C}^{q_U}\sigma$, where $q_U = \sum_{i \in U} q_i \cdot \mu(\mathscr{b}_{iK})$:

$$
D_U \quad = \quad \dfrac{\left\{ \begin{array}{c} D_{i_U} \\ \Gamma \vdash^{X} t : \mathscr{c} \wedge \left(\bigwedge_{j=1}^{K-1} \mathscr{b}_{i_U j}\right) \wedge \mathscr{b}_{i_U K} \rightarrowtail \mathbf{C}^{q_i}\sigma \end{array} \right\}_{j \in U}}{\Gamma \vdash^{X - \{a_K\}} \nu a_K.t : \mathscr{c} \wedge \left(\bigwedge_{j=1}^{K-1} \mathscr{b}_{i_U j}\right) \rightarrowtail \mathbf{C}^{q_U}\sigma}\ (\mu_\Sigma)
$$

Let $\mathscr{d}_U = \bigwedge_{j=1}^{K-1} \mathscr{b}_{i_U j}$, $t' = \nu a_K.t$ and observe that we now have derivations D_U of $\Gamma \vdash^{\{a_1,\dots,a_{K-1}\},q_U} t' : \mathscr{c} \wedge \mathscr{d}_U \rightarrowtail \sigma$, where for all $U \neq V$, $\mathscr{d}_U \wedge \mathscr{d}_V \vDash \bot$. Moreover, for all $U \neq V$, from $\mathscr{d}_U \not\equiv \mathscr{d}_V$ we deduce by condition (a.) that $\mathscr{d}_U \wedge \mathscr{d}_V \vDash \bot$; finally condition (b.) for the $\mathscr{d}_U$ follows immediately from condition (b.) for the $\bigwedge_{j=1}^{K} \mathscr{b}_{ij}$.

Hence, by applying the induction hypothesis, we deduce the existence of a derivation of $\Gamma \vdash^{\emptyset} \nu X.t : \mathscr{c} \rightarrowtail \mathsf{C}^{q}\sigma$, where

$$
\begin{aligned}
q = \sum_{U \in \mathsf{Cl}_{N_\sim}} q_U \cdot \mu(\mathscr{d}_U)) &= \sum_{U \in \mathsf{Cl}_{N_\sim}} \left(q_U \cdot \prod_{j=1}^{K-1} \mu(\mathscr{b}_{i_U j})) \right) \\
&= \sum_{U \in \mathsf{Cl}_{N_\sim}} \left(\sum_{i \in U} q_i \cdot \mu(\mathscr{b}_{iK}) \right) \cdot \left(\prod_{j=1}^{K-1} \mu(\mathscr{b}_{i_U j})) \right) \\
&\overset{(5)}{=} \sum_{i=1}^{N} \left(q_i \cdot \prod_{j=1}^{K} \mu(\mathscr{b}_{ij}) \right)
\end{aligned}
$$

so we are done. ◻

The technical lemma just proved can be used to establish Proposition D.24, i.e. the admissibility of the generalized counting rule (μ^*).

Proposition D.24. *The following generalized counting rule is derivable in* $\mathsf{C}\lambda_{\rightarrow,\cap}$*:*

$$
\frac{\Gamma \vdash^{\{a_1,\dots,a_K\}} t : \mathscr{b} \rightarrowtail \mathsf{C}^{q}\sigma}{\Gamma \vdash^{\emptyset} \nu a_1.\dots.\nu a_K.t : \top \rightarrowtail \mathsf{C}^{q\cdot\mu(\mathscr{b})}\sigma}\ (\mu^*)
$$

Proof. First, let us transform $\mathscr{b}$ into a disjunctive normal form of the form $\bigvee_{i=1}^{N} \bigwedge_{j=1}^{K} \mathscr{c}_{ij}$, where each $\mathscr{c}_{ij}$ is a conjunction of literals of name a_j. Now, as it was done in the proof of Lemma D.13, the $\mathscr{c}_{ij}$ can be turned into conjunctions of literals $\mathscr{e}_{ij}$, with $i \leq N'$, for some $N' \geq N$, and $j \leq K$, where for all $i \neq i'$ and j, j', either $\mathscr{e}_{ij} = \mathscr{e}_{i'j'}$ or $\mathscr{e}_{ij} \wedge \mathscr{e}_{i'j'} \vDash \bot$, and $\bigvee_{i=1}^{N'} \bigwedge_{j=1}^{K} \mathscr{e}_{ij} \equiv \bigvee_{i=1}^{N} \bigwedge_{j=1}^{K} \mathscr{c}_{ij}$.

Observe that the $\mathscr{e}_{ij}$ now satisfy conditions (a.) and (b.) of Lemma D.23. Moreover, by letting $\mathscr{f}_i = \bigwedge_{j=1}^{K} \mathscr{e}_{ij}$, from $\mathscr{f}_i \vDash \mathscr{b}$ we deduce that for all $i = 1, \dots, N'$, $\Gamma \vdash^{\{a_1,\dots,a_K\},q} t : \mathscr{f}_i \rightarrowtail \sigma$ is derivable. Hence, by applying the Lemma, we can conclude that $\Gamma \vdash^{\emptyset,r} \nu a_1.\dots.\nu a_K.t : \top \rightarrowtail \sigma$, where $r = \sum_{i=1}^{N'} q \cdot \mu(\mathscr{f}_i) = q \cdot \sum_{i=1}^{N'} \mu(\mathscr{f}_i) = q \cdot \mu(\mathscr{b})$. ◻

We can finally turn to the proof of the completeness part Theorem 6.6.

Proposition D.25. *For any closed term t,*

$$
\mathrm{HNV}_{\twoheadrightarrow}(t) \leq \sup\{q \mid \vdash t : \top \rightarrowtail \mathsf{C}^{q}\mathsf{hn}\}
$$
$$
\mathrm{NF}_{\twoheadrightarrow}(t) \leq \sup\{q \mid \vdash t : \top \rightarrowtail \mathsf{C}^{q}\mathsf{n}\}
$$

Proof. First observe that (with the notations from Subsection B.1)

$$
\begin{aligned}
\mathrm{HNV}_{\twoheadrightarrow}(t) &= \sup\left\{ \sum_{\pi : v \mapsto w \in \mathrm{HNV}} \mu(\mathscr{b}^{v}_{\pi}) \mid t \twoheadrightarrow^{*}_{\mathsf{h}} v \right\} \\
&= \sup\left\{ \sum_{i=1}^{N} \mu(\mathscr{b}^{u}_{\pi_i}) \mid N \in \mathbb{N}, u_1, \dots, u_N \in \mathrm{HNV},\ \pi_1 : v \mapsto u_1, \dots, \pi_N : t \mapsto u_N, t \twoheadrightarrow^{*}_{\mathsf{h}} v \right\}
\end{aligned}
$$

so it is enough to show that for any $n \in \mathbb{N}$, term v such that $t \twoheadrightarrow^{*}_{\mathsf{h}} v$ and terms $u_1, \dots, u_n \in \mathrm{HNV}$ together with randomized paths $\pi_i : v \mapsto u_i$, we can prove that $\vdash t : \top \rightarrowtail \mathsf{C}^{q}\mathsf{hn}$ is derivable, with $q = \sum_{i=1}^{n} \mu(\mathscr{b}_{\pi_i})$. Notice that, by subject expansion, it suffices to show $\vdash v : \top \rightarrowtail \mathsf{C}^{q}\mathsf{hn}$.

Let $\pi_1 : v \mapsto u_1, \dots, \pi_k : t \mapsto u_k$ be randomized paths, with $u_1, \dots, u_n \in \mathrm{HNV}$. By Lemma D.21 there exists N, terms $v_1, \dots, v_N$ and a randomized multi-context $\mathtt{C}$ of rank $k + N$ such that $v \twoheadrightarrow^{*}_{\mathsf{p}} \mathtt{C}[u_1, \dots, u_k, v_1, \dots, v_N]$. Moreover, by Lemma D.22 there exists a permutative reduction $\nu a_1 \dots \nu a_K.\mathtt{C}^{\neg\nu}[u_1, \dots, u_k, v_1, \dots, v_N] \twoheadrightarrow^{*}_{\mathsf{p}} \mathtt{C}[u_1, \dots, u_k, v_1, \dots, v_N]$.

Let v^* be the term $\mathtt{C}^{\neg\nu}[u_1, \dots, u_k, v_1, \dots, v_N]$. For each HNV u_i, by Lemma D.18, we have that $\vdash u_i : \top \rightarrowtail \mathsf{C}^{1}\mathsf{hn}$, so in particular $\vdash^{\{a_1,\dots,a_K\},1} u_i : \top \rightarrowtail \mathsf{C}^{1}\mathsf{hn}$ is derivable; using this fact one can construct by induction on $\mathtt{C}$ derivations D_i of $\vdash^{\{a_1,\dots,a_K\}} v^* : \mathscr{b}_{\pi_i} \rightarrowtail \mathsf{C}^{1}\mathsf{hn}$ (using the fact that $RBT(\mathtt{C}[u_1, \dots, u_k, v_1, \dots, v_N]) \sqsubseteq RBT(t)$) :

- if $\mathtt{C}^{\neg\nu} = [\ ]_i$, then $k = 1$, $u^* = u$, and $\mathscr{b}_{\pi_1} = \top$; then by the hypothesis we have $\vdash v^* : \top \rightarrowtail \mathsf{C}^{1}\mathsf{hn}$;
- if $\mathtt{C}^{\neg\nu} = \mathtt{C}_1 \oplus^{l}_{a} \mathtt{C}_2$, then $u^* = \mathtt{C}_1[u_{i_1}, \dots, u_{i_p}, v_{i'_1}, \dots, v_{i'_r}] \oplus^{l}_{a} \mathtt{C}_2[u_{j_1}, \dots, u_{j_q}, v_{j'_1}, \dots, v_{j'_s}]$, where $p + q = k$, $r + s = N$ and $I \cap J = \{i_1, \dots, i_p\} \cap \{j_1, \dots, j_q\} = \emptyset$, $I' \cap J' = \{i'_1, \dots, i'_r\} \cap \{j'_1, \dots, j'_s\} = \emptyset$. Moreover, we have that for all $m \in I$, $\mathscr{b}_{\pi_l} = \mathscr{x}^{l}_{a} \wedge \mathscr{b}_{\pi'_m}$, where π'_m is the path of u_m in $\mathtt{C}_1$; similarly, for all $m \in J$, $\mathscr{b}_{\pi_m} = \neg\mathscr{x}^{l}_{a} \wedge \mathscr{b}_{\pi'_m}$, where π'_m is the path of u_l in $\mathtt{C}_2$.

By the induction hypothesis we then have derivations of $\vdash^{\{a_1,\dots,a_K\}} \mathsf{C}_1[u_{i_1},\dots,u_{i_p},v_{i'_1},\dots,v_{i'_r}] : \mathscr{b}_{\pi'_m} \rightarrowtail \mathsf{C}^1\mathsf{hn}$ and $\vdash^{\{a_1,\dots,a_K\}} \mathsf{C}_2[u_{j_1},\dots,u_{j_q},v_{j'_1},\dots,v_{j'_s}] : \mathscr{b}_{\pi'_m} \rightarrowtail \mathsf{C}^1\mathsf{hn}$. Since $\mathscr{b}_{\pi_i}$ is either $x^l_a \wedge \mathscr{b}_{\pi'_i}$ or $\neg x^l_a \wedge \mathscr{b}_{\pi'_i}$, on i, we obtain then in any case a derivation of $\vdash^{\{a_1,\dots,a_K\}} v^* : \mathscr{b}_{\pi_i} \rightarrowtail \mathsf{C}^1\mathsf{hn}$ using either the $(\oplus l)$- or the $(\oplus r)$-rule.

Using the derivations D_i we thus obtain a derivation of $\vdash^{\emptyset} \nu a_1.\dots.\nu a_K.v^* : \top \rightarrowtail \mathsf{C}^{\sum_i \mu(\mathscr{b}_{\pi_i})}\mathsf{hn}$ by applying the derivable rule (μ^*) (Corollary D.24) as shown below:

$$\dfrac{\dfrac{\left\{\begin{array}{c} D_i \\ \vdash^{\{a_1,\dots,a_K\}} v^* : \mathscr{b}_{\pi_i} \rightarrowtail \mathsf{C}^1\mathsf{hn}\end{array}\right\}_{i=1,\dots,k} \qquad \bigvee_i \mathscr{b}_{\pi_i} \vDash \bigvee_i \mathscr{b}_{\pi_i}}{\vdash^{\{a_1,\dots,a_K\}} v^* : \bigvee_i \mathscr{b}_{\pi_i} \rightarrowtail \mathsf{C}^1\mathsf{hn}}(\vee)}{\vdash^{\emptyset} \nu a_1.\dots.\nu a_K.v^* : \top \rightarrowtail \mathsf{C}^{\mu(\bigvee_i \mathscr{b}_{\pi_i})}\mathsf{hn}}(\mu^*)$$

Since $\mu(\bigvee_i \mathscr{b}_{\pi_i}) = \sum_i \mu(\mathscr{b}_{\pi_i})$, using subject reduction and subject expansion we can finally deduce $\vdash^{\emptyset} v : \top \rightarrowtail \mathsf{C}^{\sum_i \mu(\mathscr{b}_{\pi_i})}\mathsf{hn}$.

For the case of $\mathrm{NF}_{\rightharpoonup}(t)$, we establish the result first for $\mathrm{NF}(t)$, arguing by induction on normal forms with an argument similar to the one above, exploiting the fact that if $q_i = \sup S_i$, then $\prod_{i=1}^k q_i = \sup\{\prod_{i=1}^k s_i \mid s_i \in S_i\}$. □

E Normalization Results

In this section we establish a few normalization results for $\mathsf{C}\lambda^{\{\}}_{\rightarrow}$ and $\mathsf{C}\lambda_{\rightarrow,\cap}$. All results are based on some variant of the standard reducibility predicates technique.

First, we prove Theorem 5.3 by adapting a result from [17] to the language $\Lambda^{\{\}}_{\mathsf{PE}}$. We then turn to probabilistic normalization. In Subsection E.2 we develop a theory of reducibility candidates for probabilistic head normal forms and probabilistic normal forms for Λ_{PE} and for a general family of types which comprises both those of $\mathsf{C}\lambda^{\{\}}_{\rightarrow}$ an $\mathsf{C}\lambda_{\rightarrow,\cap}$. This will allows us to establish the normalization theorem for $\mathsf{C}\lambda_{\rightarrow,\cap}$, but will also provide most of the steps needed to establish the normalization theorem for $\mathsf{C}\lambda^{\{\}}_{\rightarrow}$. In Subsection E.3 we discuss reducibility candidates for $\Lambda^{\{\}}_{\mathsf{PE}}$, by adapting most of results from the previous subsection, and we complete the proof of Theorem 6.1.

E.1 Deterministic Normalization.

E.1.1 Deterministic Normalization of $\mathsf{C}\lambda^{\{\}}_{\rightarrow}$. In this section we adapt the proof of [[17], Theorem 1] to $\mathsf{C}\lambda^{\{\}}_{\rightarrow}$, in order to prove Theorem 5.3:

Theorem 5.3 (Deterministic Normalization). *In both $\mathsf{C}\lambda_{\rightarrow}$ and $\mathsf{C}\lambda^{\{\}}_{\rightarrow}$, if $\Gamma \vdash_{\neg\vee} t : \mathscr{b} \rightarrowtail \mathfrak{s}$, then t is strongly normalizable.*

We introduce a simply typed λ-calculus $\lambda^{\{\}}_{\rightarrow}$ for $\Lambda^{\{\}}_{\mathsf{PE}}$ and we show that typable terms are strongly normalizable under $\rightharpoonup_{\{\}}$. Simple types are defined by the grammar below

$$\sigma ::= o \mid \sigma \Rightarrow \sigma$$

A judgement of $\lambda^{\{\}}_{\rightarrow}$ is of the form $\Gamma \vdash^{\theta} t : \sigma$, where θ is a *list* of names and $\mathsf{FN}(t) \subseteq \theta$ (observe that we consider here lists and not sets). The rules of $\lambda^{\{\}}_{\rightarrow}$ are illustrated in Fig. 17.

The following is easily established by inspecting all reduction rules:

Proposition E.1 (subject reduction). *If $\Gamma \vdash^{\theta} t : \sigma$ and $t \rightharpoonup_{\{\}} u$, then $\Gamma \vdash^{\theta} u : \sigma$.*

For any type $\mathfrak{s}$ of $\mathsf{C}\lambda^{\{\}}_{\rightarrow}$, let us define the corresponding simple type $|\mathfrak{s}|$ by letting $|o| = o$, $|\mathfrak{s} \Rightarrow \sigma| = |\mathfrak{s}| \Rightarrow |\sigma|$ and $|\mathsf{C}^q\mathfrak{s}| = |\mathfrak{s}|$. The following is easily checked by induction:

Proposition E.2. *If $\Gamma \vdash^X_{\neg\vee} t : \mathscr{b} \rightarrowtail \mathfrak{s}$ holds $\mathsf{C}\lambda^{\{\}}_{\rightarrow}$, then $|\Gamma| \vdash^{\theta} t : |\mathfrak{s}|$ is derivable in $\lambda^{\{\}}_{\rightarrow}$, for any suitable θ.*

Let $\Lambda_{\oplus,\nu,\{\}}$ indicate the sets of λ-terms of $\Lambda^{\{\}}_{\mathsf{PE}}$, and for all finite set of names X, let $\Lambda^X_{\oplus,\nu,\{\}}$ indicate the λ-terms with free names in X.

For any term t with $\mathsf{FN}(t) \subseteq \theta = a_1 \dots a_k$, we let $\nu^{\theta}t$ indicate the name-closed term $\nu a_1.\dots.\nu a_k.t$. For any list of names θ, let $\mathrm{SN}^{\theta} = \{t \in \Lambda^X_{\oplus,\nu,\{\}} \mid \nu^{\theta}t \text{ is strongly normalizing}\}$, and for any $t \in \mathrm{SN}^{\theta}$, let $\mathrm{sn}^{\theta}(t)$ be the maximum length of a reduction of $\nu^{\theta}t$.

We let a $\{\}$-*redex* be a redex for any of the tree reductions rules ($\{\}\nu$)-($\{\}\oplus_2$).

Lemma E.3. *(i.) if $tv_1 \dots v_n \in \mathrm{SN}^{\theta}$, $utv_1 \dots v_n \in \mathrm{SN}^{\theta}$, and a occurs in θ; then $(t \oplus^i_a u)v_1 \dots v_n \in \mathrm{SN}^{\theta}$;*
(ii.) if $tv_1 \dots v_n \in \mathrm{SN}^{a\cdot\theta}$, where $\mathsf{FN}(v_i) \subseteq X$, then $(\nu a.t)v_1 \dots v_n \in \mathrm{SN}^{\theta}$;

$$\frac{}{\Gamma, x : \sigma \vdash^{\theta} x : \sigma}$$

$$\frac{\Gamma, x : \sigma \vdash^{\theta} t : \tau}{\Gamma \vdash^{\theta} \lambda x.t : \sigma \Rightarrow \tau} \qquad \frac{\Gamma \vdash^{\theta} t : \sigma \Rightarrow \tau \qquad \Gamma \vdash^{\theta} u : \sigma}{\Gamma \vdash^{\theta} tu : \tau}$$

$$\frac{\Gamma \vdash^{\theta} t : \sigma \qquad \Gamma \vdash^{\theta} u : \sigma \qquad a \in \theta}{\Gamma \vdash^{\theta} t \oplus_a^i u : \sigma} \qquad \frac{\Gamma \vdash^{\theta} t : \sigma}{\Gamma \vdash^{\theta - \{a\}} \nu a.t : \sigma} \qquad \frac{\Gamma \vdash^{\theta} t : \sigma \Rightarrow \tau \qquad \Gamma \vdash^{\theta} u : \sigma}{\Gamma \vdash^{\theta} \{t\}u : \tau}$$

Figure 17. Typing rules for the simply typed λ-calculus $\lambda_{\rightarrow}^{\{\}}$.

(iii.) if $v_i \in \mathrm{SN}^{\theta}$, then $xv_1 \dots v_n \in \mathrm{SN}^{\theta}$;
(iv.) if $(t[u/x])v_1 \dots v_n \in \mathrm{SN}^{\theta}$ and $u \in \mathrm{SN}^{\theta}$, then $(\lambda x.t)uv_1 \dots v_n \in \mathrm{SN}^{\theta}$;

Proof. (i.) Same argument as [[17], Lemma 24], observing that a $\{\}$-redex in $(t \oplus_a^i u)v_1 \dots v_n$ can only occur inside one of $t, u, v_1, \dots, v_n$.
(ii.) Same argument as [[17], Lemma 25], again observing that a $\{\}$-redex in $(\nu a.t)v_1 \dots v_n$ can only occur inside one of $t, u, v_1, \dots, v_n$.
(iii.) Immediate.
(iv.) Same argument as [[17], Lemma 27], again observing that a $\{\}$-redex in $(\lambda x.t)uv_1 \dots v_n$ can only occur inside one of $t, u, v_1, \dots, v_n$. □

Lemma E.4. *Let $t^\bullet$ be obtained from t by deleting a finite number of ν-binders $\nu a_1, \dots, \nu a_k$. If $t \in \mathrm{SN}^{\theta}$, then $t^\bullet \in \mathrm{SN}^{\theta'}$, for any suitable θ'.*

Proof. We argue by induction on $|t| + \mathrm{sn}^{\theta}(t)$:

- if $t = x$ or $t = y$, the claim is immediate;
- if $t = \lambda y.t'$, the claim follows by IH;
- if $t = t_1 t_2$, then $t^\bullet = t_1^\bullet t_2^\bullet$; let us consider all possible reductions of $t^\bullet$:
  - a reduction of either $t_1^\bullet$ or $t_2^\bullet$, in which case we apply the IH;
  - a reduction $(\lambda y.u^\bullet)t_2^\bullet \rightarrowtriangle_\beta u^\bullet[t_2^\bullet/y]$, where $t_1 = \lambda y.u$, in which case we also apply the IH;
  - a reduction of the form $u^\bullet t_2^\bullet \rightarrowtriangle_{\{\}} v^\bullet$, where $t_1 = \nu a.u$, then $t_1 t_2 \rightarrowtriangle_{\mathsf{p}\{\}} \nu a.ut_2$, so we can apply the IH to ut_2.
- if $t = \{t_1\}t_2$, then $t^\bullet = \{t_1^\bullet\}t_2^\bullet$; then a reduction is either a reduction of either $t_1^\bullet$ or $t_2^\bullet$, in which case we apply the IH, either a $(\{\}\oplus_1)$, $(\{\}\oplus_2)$-reduction, in which case we also apply the IH, or a reduction of the form $\{t_1^\bullet\}\nu a.u^\bullet \rightarrowtriangle_{\mathsf{p}\{\}} \nu a.t_1^\bullet u^\bullet$, where $t_2 = \nu a.u$, in which case we also have $\{t_1\}t_2 \rightarrowtriangle_{\mathsf{p}\{\}} \nu a.t_1 u$, so we can apply the IH.
- if $t = t_1 \oplus_a^i t_2$, then a reduction of $t^\bullet$ is either a reduction of $t_1^\bullet$ or $t_2^\bullet$, in which case we apply the IH, or a reduction of the form $t_1^\bullet \oplus_a^i (u^\bullet \oplus_b^j v^\bullet) \rightarrowtriangle_{\mathsf{p}\{\}} (t_1^\bullet \oplus_a^i u^\bullet) \oplus_b^j (t_1^\bullet \oplus_a^i v^\bullet)$, where $t_2 = v \oplus_a^i w$, in which case $t \rightarrowtriangle_{\mathsf{p}\{\}} (t_1 \oplus_a^i u) \oplus_b^j (t_2 \oplus_a^i v)$, so we can apply the IH;
- if $t = \nu b.u$, then if $t^\bullet = u^\bullet$, we can argue by the IH; if $t^\bullet = \nu b.u^\bullet$, then a reduction of $t^\bullet$ is either a reduction of $u^\bullet$, in which case we can apply the IH, or a permutation $\nu b.u_1^\bullet \oplus_a^i u_2^\bullet \rightarrowtriangle_{\mathsf{p}\{\}} (\nu b.u_1^\bullet) \oplus_a^i (\nu b.u_2^\bullet)$, where $u = u_1 \oplus_a^i u_2$, in which case $t \rightarrowtriangle_{\mathsf{p}\{\}} (\nu b.u_1) \oplus_a^i (\nu b.u_2)$ so we can apply the IH.

□

Lemma E.5. *(i.) If $(\nu a.tu)v_1 \dots v_n \in \mathrm{SN}^{\theta}$, then $\{t\}(\nu a.u)v_1 \dots v_n \in \mathrm{SN}^{\theta}$.*
(ii.) If $t[\nu a.u/x]v_1 \dots v_n \in \mathrm{SN}^{\theta}$, then $t[u/x]v_1 \dots v_n \in \mathrm{SN}^{\theta'}$, for any suitable θ'.
(iii.) If $(tu)v_1 \dots v_n \in \mathrm{SN}^{\theta}$, then $(\{t\}u)v_1 \dots v_n \in \mathrm{SN}^{\theta}$.

Proof. (i.) We argue by induction on $|t| + \mathrm{sn}^{\theta}(t) + \mathrm{sn}^{\theta}(v) + \sum_i \mathrm{sn}^{\theta}(v_i)$.
A reduction of $\{t\}(\nu a.u)v_1 \dots v_n$ can only be a reduction in either of $t, u, v_1 \dots, v_n$, a reduction of the form $\{t_1 \oplus_a^i t_2\}(\nu a.u)v_1 \dots v_n \rightarrowtriangle_{\mathsf{p}\{\}} (\{t_1\}(\nu a.u)v_1 \dots v_n) \oplus_a^i (\{t_1\}(\nu a.u)v_1 \dots v_n)$, or a reduction $\{t\}(\nu a.u)v_1 \dots v_n \rightarrowtriangle_{\mathsf{p}\{\}} (\nu a.tu)v_1 \dots v_n$.
In the first case we can use the IH; in the second case by the IH we deduce $(\{t_i\}(\nu a.u)v_1 \dots v_n) \in \mathrm{SN}^{\theta}$ and we can conclude by Lemma E.3 (i.). In the third case we conclude by the hypothesis.
(ii.) Immediate application of Lemma E.4.
(iii.) We argue by induction on $n + |t| + |u| + \mathrm{sn}^{\theta}(t) + \mathrm{sn}^{\theta}(u) + \sum_i \mathrm{sn}^{\theta}(v_i)$.
A reduction of $(\{t\}u)v_1 \dots v_n$ is of one of the following forms:

- a reduction of t or of $u, v_1,\ldots, v_n$ in which case we argue by the IH;
- a reduction of the form $(\{t\}\nu a.u')v_1\ldots v_n \rightharpoonup_{\mathrm{p}\{\}} (\nu a.tu')v_1\ldots v_n$, where $u = \nu a.u'$; then by Lemma E.4, from $(tu)v_1\ldots v_n \in \mathrm{SN}^{\theta}$ we deduce $(tu')v_1\ldots v_n \in \mathrm{SN}^{X\cup\{a\}}$, and by Lemma E.3 (ii.), $(\nu a.tu')v_1\ldots v_n \in \mathrm{SN}^{\theta}$;
- a reduction of the form $(\{t_1 \oplus_a^i t_2\}u)v_1\ldots v_n \rightharpoonup_{\mathrm{p}\{\}} ((\{t_1\}u) \oplus_a^i (\{t_2\}u))v_1\ldots v_n$,where $t = t_1 \oplus_a^i t_2$; then by IH $(\{t_i\}u)v_1\ldots v_n \in \mathrm{SN}^{\theta}$, so we conclude $((\{t_1\}u) \oplus_a^i (\{t_2\}u))v_1\ldots v_n$ by Lemma E.3 (i.).
- a reduction of the form $(\{t\}(u_1 \oplus_a^i u_2))v_1\ldots v_n \rightharpoonup_{\mathrm{p}\{\}} ((\{t\}u_1) \oplus_a^i (\{t\}u_2))v_1\ldots v_n$; then by IH $(\{t\}u_i)v_1\ldots v_n \in \mathrm{SN}^{\theta}$, so we conclude $((\{t\}u_1) \oplus_a^i (\{t\}u_2))v_1\ldots v_n$ by Lemma E.3 (i.).
- a reduction of the form $((\{t\}u)v_1\ldots v_k)(v_{k+1}^1 \oplus_a^i v_{k+1}^2)v_{k+2}\ldots v_n \rightharpoonup_{\mathrm{p}\{\}} ((\{t\}u)v_1\ldots v_k v_{k+1}^1)\oplus_a^i((\{t\}u)v_1\ldots v_k v_{k+1}^2)v_{k+2}\ldots v_n$; then by IH $(\{t\}u)v_1\ldots v_k v_{k+1}^1 \in \mathrm{SN}^{\theta}$, $(\{t\}u)v_1\ldots v_k v_{k+1}^2 \in \mathrm{SN}^{\theta}$, and by Lemma E.3 (i.) we conclude $((\{t\}u)v_1\ldots v_k v_{k+1}^1)\oplus_a^i ((\{t\}u)v_1\ldots v_k v_{k+1}^2)v_{k+2}\ldots v_n \in \mathrm{SN}^{\theta}$.

□

The family of reducibility predicates $\mathrm{Red}_{\sigma}^{\theta}$ of $\lambda_{\to}^{\{\}}$ is defined as follows:

$$\mathrm{Red}_{o}^{\theta} := \mathrm{SN}^{\theta}$$

$$\mathrm{Red}_{\sigma\Rightarrow\tau}^{\theta} := \{t \in \Lambda_{\oplus,\nu,\{\}}^{X} \mid \forall u \in \mathrm{Red}_{\sigma}^{\theta},\ tu \in \mathrm{Red}_{\tau}^{\theta}\}$$

Lemma E.6. 1. *If $t \in \mathrm{Red}_{\sigma}^{\theta}$, then $t \in \mathrm{SN}^{\theta}$;*
2. *if $\Gamma \vdash xu_1\ldots u_n$ and $u_1,\ldots,u_n \in \mathrm{SN}^{\theta}$, then $xu_1\ldots u_n \in \mathrm{Red}_{\sigma}^{\theta}$;*
3. *If $t[u/x]v_1\ldots v_n \in \mathrm{Red}_{\sigma}^{\theta}$, and $u \in \mathrm{SN}^{\theta}$, then $(\lambda x.t)uv_1\ldots v_n \in \mathrm{Red}_{\sigma}^{\theta}$;*
4. *if $tv_1\ldots v_n \in \mathrm{Red}_{\sigma}^{\theta}$ and $uv_1\ldots v_n \in \mathrm{Red}_{\sigma}^{\theta}$, and $a \in \theta$, then $(t \oplus_a^i u)v_1\ldots v_n \in \mathrm{Red}_{\sigma}^{\theta}$*
5. *if $tv_1\ldots v_n \in \mathrm{Red}_{\sigma}^{a\cdot\theta}$ and $\mathsf{FN}(v_i) \subseteq X$, then $(\nu a.t)v_1\ldots v_n \in \mathrm{Red}_{\sigma}^{\theta}$.*

Proof. The proof is similar to the proof of [[17], Lemma 28]. □

Lemma E.7. *If $t \in \mathrm{Red}_{\sigma\Rightarrow\tau}^{\theta}$ and $u \in \mathrm{Red}_{\sigma}^{\theta}$, then $\{t\}u \in \mathrm{Red}_{\tau}^{\theta}$.*

Proof. Let $\tau = \sigma_1 \Rightarrow \ldots \Rightarrow \sigma_n \Rightarrow o$ and let $v_1 \in \mathrm{Red}_{\sigma_1}^{\theta},\ldots,v_n \in \mathrm{Red}_{\sigma_n}^{\theta}$. Then $(tu)v_1\ldots v_n \in \mathrm{SN}^{\theta}$ which, by Lemma E.5, implies $\{t\}uv_1\ldots v_n \in \mathrm{SN}^{\theta}$, from which we conclude $\{t\}u \in \mathrm{NRed}_{\tau}^{X}$. □

Proposition E.8. *If $\Gamma \vdash^{\theta} t : \sigma$ is derivable in $\lambda_{\to}^{\{\}}$, where $\Gamma = \{x_1 : \sigma_1,\ldots,x_n : \sigma_n\}$, then for all $u_i \in \mathrm{Red}_{\sigma_i}^{\theta}$, $t[u_1/x_1,\ldots,u_n/x_n] \in \mathrm{Red}_{\sigma}^{\theta}$.*

Proof. The argument is by induction on t. All inductive cases are treated as in the proof of [[17], Proposition 29], using the lemmas proved above; the only new case is the one below:

- if t is $\{t_1\}t_2$ then the last rule is

$$\frac{\Gamma \vdash^{\theta} t_1 : \sigma \Rightarrow \tau \qquad \Gamma \vdash^{\theta} t_2 : \sigma}{\Gamma \vdash^{\theta} t : \tau}$$

then by IH for all $u_i \in \mathrm{Red}_{\sigma_i}^{X}$, $t_1[u_1/x_1,\ldots,u_n/x_n] \in \mathrm{Red}_{\sigma\Rightarrow\tau}^{\theta}$ and $t_2[u_1/x_1,\ldots,u_n/x_n] \in \mathrm{Red}_{\sigma}$, so by Lemma E.7 $t[u_1/x_1,\ldots,u_n/x_n] = \{t_1[u_1/x_1,\ldots,u_n/x_n]\}(t_2[u_1/x_1,\ldots,u_n/x_n]) \in \mathrm{Red}_{\tau}^{\theta}$.

□

Theorem E.9. *If $\Gamma \vdash^{\theta} t : \sigma$ is derivable in $\lambda_{\to}^{\{\}}$, then $t \in \mathrm{SN}^{\theta}$.*

From this, using Proposition E.2 we immediately deduce:

Corollary E.10. *If $\Gamma \vdash^{X} t : \mathscr{b} \rightarrowtail \sigma$ is derivable in $\lambda_{\to}^{\{\}}$ with no 0-ary instance of the $(\vee)$-rule, then $t \in \mathrm{SN}^{\theta}$, for any suitable θ.*

E.1.2 Deterministic Normalization of $\mathrm{C}\lambda_{\to,\cap}$. We now adapt the proof of [[17], Theorem 1] to $\mathrm{C}\lambda_{\to,\cap}$.

For this we consider an intersection type system $\lambda_{\to,\cap}$ for Λ_{PE} with the following types

$$\sigma ::= [] \mid o \mid \mathsf{hn} \mid \mathsf{n} \mid \sigma \Rightarrow \sigma \mid \sigma \wedge \sigma$$

The rules of $\lambda_{\to,\cap}$ include all those of $\lambda_{\to}^{\{\}}$ except the one for $\{\}$, together with the rules in Fig. 18.

For any type $\mathfrak{s}$ of $\mathrm{C}\lambda_{\to,\cap}$, we define the corresponding type $|\mathfrak{s}|$ of $\lambda_{\to,\cap}$ by $|\sigma| = \sigma$, for $\sigma \in \{\mathsf{hn},\mathsf{n}\}$, $|\mathfrak{M} \Rightarrow \sigma| = |\mathfrak{M}| \Rightarrow |\sigma$, where $|[]| = []$, $|[\mathfrak{s}_1,\ldots,\mathfrak{s}_{n+1}]| = |\mathfrak{s}_1| \wedge \cdots \wedge |\mathfrak{s}_{n+1}|$, and $|\mathrm{C}^{q}\sigma| = |\sigma|$.

The following is easily checked by induction:

Proposition E.11. *If $\Gamma \vdash_{\neg\vee}^{X} t : \mathscr{b} \rightarrowtail \mathfrak{s}$ is derivable in $\mathrm{C}\lambda_{\to,\cap}$, then $|\Gamma| \vdash^{\theta} t : |\mathfrak{s}|$ is derivable in $\lambda_{\to,\cap}$, for any suitable θ.*

$$\frac{\Gamma \vdash^{\theta} t : \sigma}{\Gamma \vdash^{\theta} t : \mathsf{hn}} \qquad \frac{\Gamma \vdash^{\theta} t : \sigma}{\Gamma \vdash^{\theta} t : \mathsf{n}}\ (\sigma\ \{[], \mathsf{hn}\}\text{-free})$$

$$\frac{\mathsf{FV}(t) \subseteq \Gamma}{\Gamma \vdash^{\theta} t : []} \qquad \frac{\Gamma \vdash^{\theta} t : \sigma \qquad \Gamma \vdash^{\theta} t : \tau}{\Gamma \vdash^{\theta} t : \sigma \wedge \tau}$$

Figure 18. Typing rules for the simply typed λ-calculus $\lambda_{\rightarrow,\cap}$.

Let $\Lambda_{\oplus,\nu}$ indicate the sets of λ-terms of Λ_{PE}, and for all finite set of names X, let $\Lambda^{X}_{\oplus,\nu}$ indicate the λ-terms with free names in X. For any finite list of names θ contained in X, we let

$$\begin{aligned}
\mathrm{HN}^{\theta} &:= \{t \in \Lambda^{X}_{\oplus,\nu} \mid \nu^{\theta} t \text{ is head-normalizing}\}\\
\mathrm{NN}^{\theta} &:= \{t \in \Lambda^{X}_{\oplus,\nu} \mid \nu^{\theta} t \text{ is normalizing}\}\\
\mathrm{SN}^{\theta} &:= \{t \in \Lambda^{X}_{\oplus,\nu} \mid \nu^{\theta} t \text{ is strongly normalizing}\}
\end{aligned}$$

We define two families of reducibility predicates $\mathrm{NNRed}^{\theta}_{\sigma}, \mathrm{HNRed}^{\theta}_{\sigma} \subseteq \Lambda^{X}_{\oplus,\nu}$ as follows:

$$\begin{aligned}
\mathrm{HNRed}^{\theta}_{[]} &:= \Lambda^{X}_{\oplus,\nu} & \mathrm{NNRed}^{\theta}_{[]} &:= \mathrm{NNRed}_{\mathsf{hn}} = \Lambda^{X}_{\oplus,\nu}\\
\mathrm{HNRed}^{\theta}_{o} &:= \mathrm{HNRed}^{\theta}_{\mathsf{hn}} = \mathrm{HNRed}^{\theta}_{\mathsf{hn}} = \mathrm{HN}^{\theta} & \mathrm{NNRed}^{\theta}_{o} &:= \mathrm{NNRed}^{\theta}_{\mathsf{n}} = \mathbb{N}^{\theta}\\
\mathrm{HNRed}^{\theta}_{\sigma\Rightarrow\tau} &:= \{t \in \Lambda^{X}_{\oplus,\nu} \mid \forall u \in \mathrm{HNRed}^{\theta}_{\sigma},\ tu \in \mathrm{HNRed}^{\theta}_{\tau}\} & \mathrm{NNRed}^{\theta}_{\sigma\Rightarrow\tau} &:= \{t \in \Lambda^{X}_{\oplus,\nu} \mid \forall u \in \mathrm{NNRed}^{\theta}_{\sigma},\ tu \in \mathrm{NNRed}^{\theta}_{\tau}\}\\
\mathrm{HNRed}^{\theta}_{\sigma\wedge\tau} &:= \mathrm{HNRed}^{\theta}_{\sigma} \cap \mathrm{HNRed}^{\theta}_{\tau} & \mathrm{NNRed}^{\theta}_{\sigma\wedge\tau} &:= \mathrm{NNRed}^{\theta}_{\sigma} \cap \mathrm{NNRed}^{\theta}_{\tau}
\end{aligned}$$

Moreover, we define a third family of reducibility predicates $\mathrm{SNRed}^{\theta}_{\sigma} \subseteq \Lambda^{X}_{\oplus,\nu}$, indexed on all $\{[], \mathsf{hn}\}$-free types as follows:

$$\begin{aligned}
\mathrm{SNRed}^{\theta}_{o} &= \mathrm{SNRed}^{\theta}_{\mathsf{n}} = \mathrm{SN}^{\theta}\\
\mathrm{SNRed}^{\theta}_{\sigma\Rightarrow\tau} &:= \{t \in \Lambda^{X}_{\oplus,\nu} \mid \forall u \in \mathrm{NNRed}^{\theta}_{\sigma},\ tu \in \mathrm{NNRed}^{\theta}_{\tau}\}\\
\mathrm{SNRed}^{\theta}_{\sigma\wedge\tau} &:= \mathrm{NNRed}^{\theta}_{\sigma} \cap \mathrm{NNRed}^{\theta}_{\tau}
\end{aligned}$$

Definition E.12 (non-trivial types)**.** The set of *non-trivial types* is defined by induction as follows:

- $o, \mathsf{hn}, \mathsf{n}$ are non-trivial;
- if τ is non-trivial, $\sigma \Rightarrow \tau$ is non-trivial;
- if either σ or τ are non-trivial, $\sigma \wedge \tau$ is non-trivial.

Lemma E.13. *For any type σ,*
(i.) if σ is non-trivial then $\mathrm{HNRed}^{\theta}_{\sigma} \subseteq \mathrm{HN}^{\theta}$;
(ii.) if σ is $\{[], \mathsf{hn}\}$-free, then $\mathrm{NNRed}^{\theta}_{\sigma} \subseteq \mathrm{NN}^{\theta}$;
(iii.) if σ is $\{[], \mathsf{hn}\}$-free, then $\mathrm{SNRed}^{\theta}_{\sigma} \subseteq \mathrm{SN}^{\theta}$.

Proof. (i.) is proved in a standard way by induction on all types σ, also proving at each step that for any term t of the form $xt_1 \dots t_n$, $t \in \mathrm{HNRed}^{\theta}_{\sigma}$.

(ii.) is proved in a standard way by induction on all types σ, also proving at each step that for any term t of the form $xt_1 \dots t_n$, where $t_1, \dots, t_n \in \mathrm{NN}^{\theta}$, $t \in \mathrm{NNRed}^{\theta}_{\sigma}$.

(iii.) is proved in a standard way by induction on all $\{[], \mathsf{hn}\}$-free types σ, also proving at each step that for any term t of the form $xt_1 \dots t_n$, where $t_1, \dots, t_n \in \mathrm{SN}^{\theta}$, $t \in \mathrm{SNRed}^{\theta}_{\sigma}$. □

Proposition E.14. *If $\Gamma \vdash^{\theta} t : \sigma$ is derivable in $\lambda_{\rightarrow,\cap}$, where $\Gamma = \{x_1 : \sigma_1, \dots, x_n : \sigma_n\}$, and $\mathsf{FN}(t) \subseteq X$, then*
(i.) for all $u_i \in \mathrm{HNRed}^{\theta}_{\sigma_i}$, $t[u_1/x_1, \dots, u_n/x_n] \in \mathrm{HNRed}_{\sigma}$;
(ii.) for all $u_i \in \mathrm{NNRed}^{\theta}_{\sigma_i}$, $t[u_1/x_1, \dots, u_n/x_n] \in \mathrm{NNRed}_{\sigma}$;
(iii.) if the types $[]$ and hn never occurs in the derivation, then for all $u_i \in \mathrm{SNRed}^{\theta}_{\sigma_i}$, $t[u_1/x_1, \dots, u_n/x_n] \in \mathrm{SNRed}_{\sigma}$;

Proof. Both arguments are by induction on the typing derivation of t. All inductive cases are treated as in the proof of [[17], Proposition 29], except for those below (we consider here only case (ii.)):

- if the last rule is
$$\frac{\Gamma \vdash^{\theta} t : \sigma}{\Gamma \vdash^{\theta} t : \mathsf{hn}}$$
then for all $u_i \in \mathrm{NNRed}^{\theta}_{\sigma_i}$, $t[u_1/x_1, \ldots, u_n/x_n] \in \Lambda^{X}_{\oplus,\nu} = \mathrm{NNRed}^{\theta}_{\mathsf{hn}}$.
- if the last rule is
$$\frac{\Gamma \vdash^{\theta} t : \sigma}{\Gamma \vdash^{\theta} t : \mathsf{n}}$$
where σ is $\{[], \mathsf{hn}\}$-free, then for all $u_i \in \mathrm{NNRed}^{\theta}_{\sigma_i}$, $t[u_1/x_1, \ldots, u_n/x_n] \in \mathrm{NNRed}^{\theta}_{\sigma}$, and by Lemma E.13 $\mathrm{NNRed}^{\theta}_{\sigma} \subseteq \mathrm{NN}^{\theta} = \mathrm{NNRed}^{\theta}_{\mathsf{n}}$, so $t[u_1/x_1, \ldots, u_n/x_n] \in \mathrm{NNRed}^{\theta}_{\mathsf{n}}$.
- if the last rule is
$$\frac{\mathsf{FN}(t) \subseteq X}{\Gamma \vdash^{\theta} t : []}$$
then for all $u_i \in \mathrm{NNRed}^{\theta}_{\sigma_i}$, $t[u_1/x_1, \ldots, u_n/x_n] \in \Lambda^{X}_{\oplus,\nu} = \mathrm{NNRed}^{\theta}_{[]}$.
- if the last rule is
$$\frac{\Gamma \vdash^{\theta} t : \sigma \qquad \Gamma \vdash^{\theta} t : \tau}{\Gamma \vdash^{\theta} t : \sigma \wedge \tau}$$
then for all $u_i \in \mathrm{NNRed}^{\theta}_{\sigma_i}$, by IH $t[u_1/x_1, \ldots, u_n/x_n] \in \mathrm{NNRed}^{\theta}_{\sigma}$ and $t[u_1/x_1, \ldots, u_n/x_n] \in \mathrm{NNRed}^{\theta}_{\tau}$, whence $t[u_1/x_1, \ldots, u_n/x_n] \in \mathrm{NNRed}^{\theta}_{\sigma} \cap \mathrm{NNRed}^{\theta}_{\tau} = \mathrm{NNRed}^{\theta}_{\sigma\wedge\tau}$.

□

Proposition E.15. *If $\Gamma \vdash^{\theta} t : \sigma$ is derivable in $\lambda_{\rightarrow,\cap}$, then:*
(i.) if $[]$ is non-trivial, then $\nu^{\theta} t$ is head-normalizable;
(ii.) if Γ and σ are $\{[], \mathsf{hn}\}$-free, then $\nu^{\theta} t$ is normalizable;
(iii.) if $[]$ and hn never occur in the derivation, then $\nu^{\theta} t$ is strongly normalizable.

Using Proposition E.11

Corollary E.16. *If $\Gamma \vdash^{X}_{\neg\vee} t : \mathscr{b} \rightarrowtail \mathfrak{s}$ in $\mathrm{C}\lambda_{\rightarrow,\cap}$, then for all suitable θ:*
(i.) $\nu^{\theta} t$ is head-normalizable;
(ii.) if Γ and σ are $\{[], \mathsf{hn}\}$-free, then $\nu^{\theta} t$ is normalizable;
(iii.) if $[]$ and hn never occur in the derivation, then $\nu^{\theta} t$ is strongly normalizable.

E.2 Probabilistic Reducibility Candidates for Λ_{PE}.

In this subsection and the following one we work with a general class of types which comprises both those of $\mathrm{C}\lambda^{\{\}}_{\rightarrow}$ and those of $\mathrm{C}\lambda_{\rightarrow,\cap}$. These types are defined by the grammar below:
$$\sigma ::= [] \mid o \mid \mathsf{hn} \mid \mathsf{n} \mid \sigma \Rightarrow \sigma \mid \sigma \wedge \sigma \mid \mathbf{C}^{q}\sigma$$

It is clear that any type of $\mathrm{C}\lambda^{\{\}}_{\rightarrow}$ is a type of the grammar above. For the types of $\mathrm{C}\lambda_{\rightarrow,\cap}$ it suffices to see $\mathscr{M} = [\mathfrak{s}_1, \ldots, \mathfrak{s}_n]$ as $(\ldots(\mathfrak{s}_1 \wedge \mathfrak{s}_2) \wedge \cdots \wedge \mathfrak{s}_{n-1}) \wedge \mathfrak{s}_n$, where $n > 0$, and as $[]$ otherwise.

For any type σ, we let the positive real $\lceil\sigma\rceil \in [0, 1]$ be defined by $\lceil[]\rceil = 0$, $\lceil o\rceil = \lceil\mathsf{hn}\rceil = \lceil\mathsf{n}\rceil = 1$, $\lceil\sigma \Rightarrow \tau\rceil = \lceil\sigma\rceil \cdot \lceil\tau\rceil$, $\lceil\sigma \wedge \tau\rceil = \max\{\lceil\sigma\rceil, \lceil\tau\rceil\}$, and $\lceil\mathbf{C}^{q}\sigma\rceil = q \cdot \lceil\sigma\rceil$.

One can check that for all type σ coming from either $\mathrm{C}\lambda^{\{\}}_{\rightarrow}$ or $\mathrm{C}\lambda_{\rightarrow,\cap}$, $\lceil\sigma\rceil > 0$.

Let us establish a few preliminary lemmas.

Lemma E.17. *For all $t \in \mathscr{T}$ and variables x, y, with y not occurring in t,*
(i.) $\mathrm{HNV}(t) = \mathrm{HNV}(t[y/x])$;
(ii.) $\mathrm{NF}(t) = \mathrm{NF}(t[y/x])$.

Proof. We only prove case (ii.), the other one being proved in a similar way. Observe that if t is not a strong head normal form and $\mathscr{D}_t(u) > 0$, it must be that u is of size strictly smaller that t. So we can argue by induction on t:

- if $t = \lambda\vec{x}.zt_1 \ldots t_n$, then $\mathrm{NF}(t) = \prod_i \mathrm{NF}(t_i) \overset{\text{[I.H.]}}{=} \prod_i \mathrm{NF}(t_i[y/x]) = \mathrm{NF}(t[y/x])$;
- otherwise, since $\mathscr{D}_t(u) = \mathscr{D}_{t[y/x]}(u[y/x])$, we have that $\mathrm{NF}(t) = \sum_{u\in\mathrm{HNV}} \mathscr{D}_t(u) \cdot \mathrm{NF}(u) = \sum_{u\in\mathrm{HNV}} \mathscr{D}_{t[y/x]}(u[y/x]) \cdot \mathrm{NF}(u) \overset{\text{[I.H.]}}{=} \sum_{u\in\mathrm{HNV}} \mathscr{D}_{t[y/x]}(u[y/x]) \cdot \mathrm{NF}(u[y/x]) = \sum_{u\in\mathrm{HNV}} \mathscr{D}_{t[y/x]}(u) \cdot \mathrm{NF}(u) = \mathrm{NF}(t[y/x])$.

□

Lemma E.18. *For any name-closed term t,*
(i.) $\mathrm{HNV}(t) = \mathrm{HNV}(\lambda y.t)$*;*
(ii.) $\mathrm{NF}(t) = \mathrm{NF}(\lambda y.t)$*.*

Proof. We only prove case (ii.), the other one being proved in a similar way. We argue by induction on t:

- if $t \rightarrow^*_{\mathsf{p}\{\}} \lambda\vec{x}.zt_1 \dots t_n$, then $\lambda y.t \rightarrow^*_{\mathsf{p}\{\}} \lambda y\lambda\vec{x}.zt_1 \dots t_n$, so by definition $\mathrm{NF}(t) = \prod_i \mathrm{NF}(t_i) = \mathrm{NF}(\lambda y.t)$;
- otherwise, if $t \rightarrow^*_{\mathsf{p}\{\}} \nu a.t'$, then $\lambda y.t \rightarrow^*_{\mathsf{p}\{\}} \nu a.t^{\lambda y}$, and one can show by induction on a $\rightarrow_{\mathsf{p}}$-reduction path that $\mathscr{D}_{t'}(u) = \mathscr{D}_{t^{\lambda y}}(\lambda y.u)$. So we have that $\mathrm{NF}(t) = \sum_{u\in\mathrm{HNV}} \mathscr{D}_{t'}(u)\cdot\mathrm{NF}(u) = \sum_{u\in\mathrm{HNV}} \mathscr{D}_{t^{\lambda y}}(\lambda y.u)\cdot\mathrm{NF}(u) \overset{[\mathrm{I.H.}]}{=} \sum_{u\in\mathrm{HNV}} \mathscr{D}_{t^{\lambda y}}(\lambda y.u)\cdot\mathrm{NF}(\lambda y.u) = \sum_{u\in\mathrm{HNV}} \mathscr{D}_{t^{\lambda y}}(u)\cdot\mathrm{NF}(u) = \mathrm{NF}(\lambda y.t)$.

□

Lemma E.19. *For any name-closed term t,*
(i.) $\mathrm{HNV}(t) \geq \mathrm{HNV}(tx)$*;*
(ii.) $\mathrm{NF}(t) \geq \mathrm{NF}(tx)$*.*

Proof. We only prove case (ii.), the other one being proved in a similar way. Suppose $tx \rightarrow^* u$ and $\mathrm{NF}(u) \geq q$. We will show that $\mathrm{NF}(t) \geq q$. Two cases arise:

- $u = u'x$ where $t \rightarrow^* u'$. Then u' cannot start with either ν (otherwise it would not be a PNF, since $(\nu a.v)x \rightarrow_{\mathsf{p}\{\}} \nu a.vx$), $\oplus$ (since $(v_1 \oplus^i_a v_2)x \rightarrow_{\mathsf{p}\{\}} (v_1x) \oplus^i_a (v_2x)$), or λ (since $(\lambda y.v)x$ is a β-redex, which would imply $\mathrm{NF}(u) = 0$). Hence u' is of the form $yu_1 \dots u_n$, and thus $\mathrm{NF}(u) = \mathrm{NF}(u') \geq q$, which implies $\mathrm{NF}(t) \geq q$.
- $tx \rightarrow^*_{\{\}} (\lambda y.t')x$, and $t'[y \mapsto x] \rightarrow^*_{\{\}} u$. Then t reduces to the PNF $\lambda y.u[x \mapsto y]$, and we can conclude by Lemma E.17 and Lemma E.18.

□

In the following, we will sometimes write $t \in \mathrm{HNV}^r_{\rightarrow}$ (resp. $t \in \mathrm{NF}^r_{\rightarrow}$) for $\mathrm{HNV}_{\rightarrow}(t) \geq r$ (resp. $\mathrm{NF}_{\rightarrow}(t) \geq r$).

We will define two different classes of *reducibility predicates*, one related to probabilistic head normal forms and the other related to probabilistic normal forms. For any type σ, finite set X of names, $r \in [0,1]$, and $S \subseteq (2^{\mathbb{N}})^X$, we define the sets $\mathrm{HRed}^{X,r}_{\sigma}(S), \mathrm{NRed}^{X,r}_{\sigma}(S) \subseteq \Lambda^X_{\oplus,\nu}$ by induction as follows:

$$
\begin{aligned}
\mathrm{HRed}^{X,r}_{[]}(S) &= \Lambda^X_{\oplus,\nu}\\
\mathrm{HRed}^{X,r}_{o}(S) = \mathrm{HRed}^{X,r}_{\mathsf{hn}}(S) = \mathrm{HRed}^{X,r}_{\mathsf{n}}(S) &= \{t \in \Lambda^X_{\oplus,\nu} \mid \forall\omega \in S\ \pi^{\omega}_X(t) \in \mathrm{HNV}^r_{\rightarrow}\}\\
\mathrm{HRed}^{X,r}_{\sigma\Rightarrow\tau}(S) &= \{t \in \Lambda^X_{\oplus,\nu} \mid \forall S' \subseteq S, \forall s \in (0,1],\ \forall u \in \mathrm{HRed}^{X,1}_{\sigma}(S'),\ tu \in \mathrm{HRed}^{X,r}_{\tau}(S')\}\\
\mathrm{HRed}^{X,r}_{\sigma\wedge\tau}(S) &= \mathrm{HRed}^{X,r}_{\sigma}(S) \cap \mathrm{HRed}^{X,r}_{\tau}(S)\\
\mathrm{HRed}^{X,r}_{\mathsf{C}^q\sigma}(S) &= \mathrm{HRed}^{X,qr}_{\sigma}(S)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{NRed}^{X,r}_{[]}(S) = \mathrm{NRed}^{X,r}_{\mathsf{hn}}(S) &= \Lambda^X_{\oplus,\nu}\\
\mathrm{NRed}^{X,r}_{o}(S) = \mathrm{NRed}^{X,r}_{\mathsf{n}}(S) &= \{t \in \Lambda^X_{\oplus,\nu} \mid \forall\omega \in S\ \pi^{\omega}_X(t) \in \mathrm{NF}^r_{\rightarrow}\}\\
\mathrm{NRed}^{X,r}_{\sigma\Rightarrow\tau}(S) &= \{t \in \Lambda^X_{\oplus,\nu} \mid \forall S' \subseteq S, \forall s \in (0,1],\ \forall u \in \mathrm{NRed}^{X,1}_{\sigma}(S'),\ tu \in \mathrm{NRed}^{X,r}_{\tau}(S')\}\\
\mathrm{HRed}^{X,r}_{\sigma\wedge\tau}(S) &= \mathrm{NRed}^{X,r}_{\sigma}(S) \cap \mathrm{NRed}^{X,r}_{\tau}(S)\\
\mathrm{NRed}^{X,r}_{\mathsf{C}^q\sigma}(S) &= \mathrm{NRed}^{X,qr}_{\sigma}(S)
\end{aligned}
$$

We will now establish a few properties of the families $\mathrm{HRed}^{X,r}_{\sigma}(S)$ and $\mathrm{NRed}^{X,r}_{\sigma}(S)$. Since the proofs for the two cases are similar, we will provide details only for the second case.

Lemma E.20. *For any type σ and $q, r \in [0,1]$, if $q \geq r$, then*
(i.) $\mathrm{HRed}^q_{\sigma}(S) \subseteq \mathrm{HRed}^r_{\sigma}(S)$*.*
(ii.) $\mathrm{NRed}^q_{\sigma}(S) \subseteq \mathrm{NRed}^r_{\sigma}(S)$*.*

Proof. We only prove case (ii.), the other one being proved in a similar way. By induction on σ:

- if $\sigma \in \{[], \mathsf{hn}\}$, the claim is immediate;
- if $\sigma \in \{o, , \mathsf{n}\}$, then $t \in \mathrm{NRed}^q_{\sigma}(S)$ iff for all $\omega \in S$, $\pi^{\omega}_X(t) \in \mathrm{NF}^q_{\rightarrow} \subseteq \mathrm{NF}^r_{\rightarrow}$, so $t \in \mathrm{NRed}^r_{\sigma}(S)$;

- if $\sigma = (\tau \Rightarrow \rho)$ and $t \in \mathrm{NRed}_\tau^{X,q}(S)$ then for all $S' \subseteq S$, $u \in \mathrm{NRed}_\tau^{X,1}(S')$, $tu \in \mathrm{NRed}_\rho^{q}(S') \overset{[\text{I.H.}]}{\subseteq} \mathrm{NRed}_\rho^{r}(S')$; hence we can conclude $t \in \mathrm{NRed}_\sigma^{r}(S)$;
- if $\sigma = \tau \wedge \rho$ and $t \in \mathrm{NRed}_\sigma^{X,q}(S)$, then by IH $t \in \mathrm{NRed}_\tau^{X,r}(S)$ and $t \in \mathrm{NRed}_\rho^{X,r}(S)$, so $t \in \mathrm{NRed}_\sigma^{X,r}(S)$;
- if $\sigma = \mathrm{C}^s\tau$, then $\mathrm{NRed}_\sigma^{X,q}(S) = \mathrm{NRed}_\tau^{X,sq}(S) \overset{[\text{I.H.}]}{\subseteq} \mathrm{NRed}_\tau^{X,sr}(S) = \mathrm{NRed}_\sigma^{X,r}(S)$.

□

Lemma E.21. *(i.)* $\mathrm{HRed}_\sigma^{X,r}(\emptyset) = \{t \in \Lambda_{\oplus,\nu} \mid \mathsf{FN}(t) \subseteq X\}$;
(ii.) $\mathrm{NRed}_\sigma^{X,r}(\emptyset) = \{t \in \Lambda_{\oplus,\nu} \mid \mathsf{FN}(t) \subseteq X\}$.

Proof. We only prove (ii.) as (i.) is proved similarly. Let $\Lambda_{\oplus,\nu}^X = \{t \in \Lambda_{\oplus,\nu} \mid \mathsf{FN}(t) \subseteq X\}$. Observe that the inclusion $\mathrm{NRed}_\sigma^{X,r}(\emptyset) \subseteq \Lambda_{\oplus,\nu}^X$ is immediate. For the converse direction, we argue by induction on σ:

- if $\sigma \in \{[], \mathsf{hn}\}$, the claim is immediate;
- If $\sigma \in \{o, \mathsf{n}\}$, then trivially for all $\omega \in \emptyset$ and $t \in \Lambda_{\oplus,\nu}^X$, $\pi_X^\omega(t) \in \mathrm{NF}_{\twoheadrightarrow}^r$, so $\Lambda_{\oplus,\nu}^X \subseteq \mathrm{NRed}_\sigma^{X,r}(\emptyset)$.
- if $\sigma = \tau \Rightarrow \rho$, then for all $t \in \Lambda_{\oplus,\nu}^X$, and $u \in \mathrm{NRed}_\tau^{X,1}(\emptyset)$, $tu \in \Lambda_{\oplus,\nu}^X$, so by IH $tu \in \mathrm{NRed}_\rho^{X,r}(\emptyset)$. We can thus conclude that $t \in \mathrm{NRed}_\sigma^{X,r}(\emptyset)$.
- if $\sigma = \tau \wedge \rho$, then for all $t \in \Lambda_{\oplus,\nu}^X$, by IH $t \in \mathrm{NRed}_\tau^{X,r}(\emptyset)$ and $t \in \mathrm{NRed}_\rho^{X,r}(\emptyset)$, so $t \in \mathrm{NRed}_\sigma^{X,r}(\emptyset)$.
- If $\sigma = \mathrm{C}^q\tau$, then by the I.H. $\Lambda_{\oplus,\nu}^X \subseteq \mathrm{NRed}_\tau^{X,qr}(\emptyset) = \mathrm{NRed}_\sigma^{X,q}(\emptyset)$.

□

Lemma E.22. *(i.) $t \in \mathrm{HRed}_\sigma^{X,r}(S)$ iff for all $\omega \in S$, $\pi_X^\omega(t) \in \mathrm{HRed}_\sigma^{\emptyset,r}$;*
(ii.) $t \in \mathrm{NRed}_\sigma^{X,r}(S)$ iff for all $\omega \in S$, $\pi_X^\omega(t) \in \mathrm{NRed}_\sigma^{\emptyset,r}$.

Proof. We only prove (ii.) as (i.) is proved similarly. We argue by induction on σ:

- if $\sigma \in \{[], \mathsf{hn}\}$, the claim is immediate;
- If $\sigma \in \{o, \mathsf{n}\}$, the claim follows from the definition.
- If $\sigma = \tau \Rightarrow \rho$, suppose $t \in \mathrm{NRed}_\sigma^{X,r}(S)$, let $\omega \in S$ and $u \in \mathrm{NRed}_\tau^{\emptyset,1}$. Then u is name-closed hence for all $g \in S$, $\pi^g(u) = u$, which by the IH, implies $u \in \mathrm{NRed}_\tau^{X,1}(S)$; we deduce then that $tu \in \mathrm{NRed}_\rho^{X,r}(S)$, so by IH $\pi_X^\omega(tu) = \pi_X^\omega(t)u \in \mathrm{NRed}_\rho^{\emptyset,r}$. By a similar argument we can show that for all $v \in \mathrm{NRed}_\tau^{\emptyset,s}$, $\pi_X^\omega(\{t\}v) = \{\pi_X^\omega(t)\}v \in \mathrm{NRed}_\rho^{\emptyset,rs}$. We can thus conclude that $\pi_X^\omega(t) \in \mathrm{NRed}_\sigma^{\emptyset,r}$.
  Conversely, suppose that for all $\omega \in S$, $\pi_X^\omega(t) \in \mathrm{NRed}_\sigma^{\emptyset,r}$, let $S' \subseteq S$ and $u \in \mathrm{NRed}_\tau^{X,1}(S')$; if $\omega \in S'$, then by IH $\pi_X^\omega(u) \in \mathrm{NRed}_\tau^{\emptyset,1}$, so $\pi_X^\omega(tu) = \pi_X^\omega(t)\pi_X^\omega(u) \in \mathrm{NRed}_\rho^{\emptyset,r}$; We have thus proved that for all $\omega \in S'$, $\pi_X^\omega(tu) \in \mathrm{NRed}_\rho^{\emptyset,r}$, which by IH implies that $tu \in \mathrm{NRed}_\rho^{X,r}(S')$ We conclude then that $t \in \mathrm{NRed}_\sigma^{X,r}(S)$.
- if $\sigma = \tau \wedge \rho$ then $t \in \mathrm{NRed}_\sigma^{X,r}(S)$ iff $t \in \mathrm{NRed}_\tau^{X,r}(S)$ and $t \in \mathrm{NRed}_\rho^{X,r}(S)$ iff (by the IH) for all $\omega \in S$, $\pi_X^\omega(t) \in \mathrm{NRed}_\tau^{X,r}(\emptyset)$ and $\pi_X^\omega(t) \in \mathrm{NRed}_\rho^{X,r}(\emptyset)$, iff for all $\omega \in S$, $\pi_X^\omega(t) \in \mathrm{NRed}_\sigma^{X,r}(\emptyset)$.
- If $\sigma = \mathrm{C}^q\tau$, then $t \in \mathrm{NRed}_\sigma^{X,r}(S)$ iff $t \in \mathrm{NRed}_\tau^{X,rq}(S) \overset{[\text{I.H.}]}{\text{iff}} \forall \omega \in S\ \pi_X^\omega(t) \in \mathrm{NRed}_\tau^{X,rq} \overset{[\text{I.H.}]}{\text{iff}} \forall \omega \in S\ \pi_X^\omega(t) \in \mathrm{NRed}_\sigma^{X,r}$.

□

Lemma E.23. *(i.) If $t \in \mathrm{HRed}_\sigma^{X,r}(S)$ and $t' \twoheadrightarrow_{\mathsf{p}} t$, then $t' \in \mathrm{HRed}_\sigma^{X,r}(S)$.*
(ii.) If $t \in \mathrm{NRed}_\sigma^{X,r}(S)$ and $t' \twoheadrightarrow_{\mathsf{p}} t$, then $t' \in \mathrm{NRed}_\sigma^{X,r}(S)$.

Proof. By induction on σ. □

Lemma E.24. *For all terms $t, u_1, \dots, u_n$ with $\mathsf{FN}(t) \subseteq X \cup \{a\}$, $\mathsf{FN}(u_1), \dots, \mathsf{FN}(u_n) \subseteq X$, and measurable sets $S_1, \dots, S_{k+1} \subseteq (2^{\mathbb{N}})^{X \cup \{a\}}$ and $S' \subseteq (2^{\mathbb{N}})^X$, if*

1. *the S_i are pairwise disjoint;*
2. *for all $\omega \in S_i$, $\pi_X^\omega(tu_1 \dots u_n) \in \mathrm{HNV}_{\twoheadrightarrow}^{r_i}$ (resp. $\pi_X^\omega(tu_1 \dots u_n) \in \mathrm{NF}_{\twoheadrightarrow}^{r_i}$);*
3. *for all $\omega \in S'$, $\mu(\Pi^\omega(S_i)) \geq s_i$;*

then for all $\omega \in S'$, $\pi_X^\omega((\nu a.t)u_1 \dots u_n) \in \mathrm{HNV}_{\twoheadrightarrow}^{\sum_{i=1}^{k+1} r_i s_i}$ (resp. $\pi_X^\omega((\nu a.t)u_1 \dots u_n) \in \mathrm{NF}_{\twoheadrightarrow}^{\sum_{i=1}^{k+1} r_i s_i}$).

Proof. Let us first consider the case where $n = 0$. Let $\omega \in S'$. Since any term reduces to a (unique) PNF, we can suppose w.l.o.g. that the name-closed term $\pi_X^\omega(\nu a.t) = \nu a.t^*$ is in PNF. Then by Corollary B.5 t^* is a $(\mathcal{T}, a)$-tree of level $I_a(t^*)$. Observe that for all $\omega' \in (2^{\mathbb{N}})^{\{a\}}$, $\pi_{\{a\}}^{\omega'}(t^*) = \pi_{\{a\}}^{\omega'}(\pi_X^\omega(t)) = \pi_{X \cup \{a\}}^{\omega + \omega'}(t)$.

If N is the cardinality of $\mathrm{Supp}(t^*)$, by 3. and the fact that $s_i > 0$ we deduce that for all $i = 1, \dots, k+1$ there exists a finite number K_i of elements $w_{i1}, \dots, w_{iK_i}$ of $\mathrm{Supp}(t^*)$ such that $\pi^{\omega_{ij}}_{\{a\}}(t^*) = w_{ij} \in \mathrm{NF}^{r_i}_{\rightarrow}$ for some $\omega_{ij} \in (2^{\mathbb{N}})^{\{a\}}$ such that $\omega + \omega_{ij} \in S_i$, and using 3. we deduce that

$$\sum_{w \in \mathrm{NF}^{r_i}_{\rightarrow}} \mu\{\omega \mid \pi^{\omega}_{\{a\}}(t^*) = w\} \geq \sum_{j=1}^{K_i} \mu\{\omega \mid \pi^{\omega}_{\{a\}}(t^*) = w_{ij}\} \geq s_i$$

Now, by reducing, for all $i \leq k+1$, each such term w_{ij} inside $\nu a.t$, to some PNF $w'_{ij} \in \mathrm{NF}^{r_i}$, we obtain a new PNF $\nu a.t^{\sharp}$ and we can compute (using the fact that $\sum_{u \in \mathrm{HNV}} \mathscr{D}_{w'_{ij}}(u) \cdot \mathrm{NF}(u) \geq r_i$):

$$\begin{aligned}
\mathrm{NF}(\nu a.t^{\sharp}) = \sum_{u \in \mathrm{HNV}} \mathcal{D}_{\nu a.t^{\sharp}}(u) \cdot \mathrm{NF}(u) &= \sum_{u \in \mathrm{HNV}} \left(\sum_{t' \in \mathrm{Supp}(t^{\sharp})} \mathcal{D}_{t'}(u) \cdot \mu\{\omega \mid \pi^{\omega}_{\{a\}}(t^{\sharp}) = t'\} \right) \cdot \mathrm{NF}(u) \\
&= \sum_{u \in \mathrm{HNV}} \left(\sum_{t' \in \mathrm{Supp}(t^{\sharp})} \mathcal{D}_{t'}(u) \cdot \mathrm{NF}(u) \cdot \mu\{\omega \mid \pi^{\omega}_{\{a\}}(t^{\sharp}) = t'\} \right) \\
&= \sum_{t' \in \mathrm{Supp}(t^{\sharp})} \left(\sum_{u \in \mathrm{HNV}} \mathcal{D}_{t'}(u) \cdot \mathrm{NF}(u) \cdot \mu\{\omega \mid \pi^{\omega}_{\{a\}}(t^{\sharp}) = t'\} \right) \\
&= \sum_{t' \in \mathrm{Supp}(t^{\sharp})} \left(\sum_{u \in \mathrm{HNV}} \mathcal{D}_{t'}(u) \cdot \mathrm{NF}(u) \right) \cdot \mu\{\omega \mid \pi^{\omega}_{\{a\}}(t^{\sharp}) = t'\} \\
&\geq \sum_{i=1}^{k+1} \sum_{j=1}^{K_i} \left(\sum_{u \in \mathrm{HNV}} \mathcal{D}_{w'_{ij}}(u) \cdot \mathrm{NF}(u) \right) \cdot \mu\{\omega \mid \pi^{\omega}_{\{a\}}(t^{\sharp}) = w'_{ij}\} \\
&\geq \sum_{i=1}^{k+1} \sum_{j=1}^{K_i} r_i \cdot \mu\{\omega \mid \pi^{\omega}_{\{a\}}(t^{\sharp}) = w'_{ij}\} \\
&= \sum_{i=1}^{k+1} r_i \cdot \left(\sum_{j=1}^{K_i} \mu\{\omega \mid \pi^{\omega}_{\{a\}}(t^{\sharp}) = w'_{ij}\} \right) \\
&= \sum_{i=1}^{k+1} r_i \cdot \left(\sum_{j=1}^{K_i} \mu\{\omega \mid \pi^{\omega}_{\{a\}}(t^{*}) = w'_{ij}\} \right) \\
&\geq \sum_{i=1}^{k+1} r_i \cdot s_i
\end{aligned}$$

Now, from $\nu a.t \rightarrow^*_{\{\}} \nu a.t^{\sharp} \in \mathrm{NF}^{\sum_i r_i s_i}$, we conclude $\nu a.t \in \mathrm{NF}^{\sum_i r_i s_i}_{\rightarrow}$.

For the case in which $n > 0$, we argue as follows: from the hypotheses we deduce by the first point that $\nu a.(tu_1u_2 \dots u_n) \in \mathrm{NF}^{\sum_i r_i s_i}_{\rightarrow}$. We can then conclude by observing that $(\nu a.t)u_1u_2 \dots u_n \rightarrow_{\mathrm{p}\{\}} \nu a.(tu_1u_2 \dots u_n)$. ◻

Lemma E.25. *For all types σ, terms $t, u_1, \dots, u_n$ with $\mathsf{FN}(t) \subseteq X \cup \{a\}$, $\mathsf{FN}(u_i) \subseteq X$, and measurable sets $S_1, \dots, S_{k+1} \subseteq (2^{\mathbb{N}})^{X \cup \{a\}}$ and $S' \subseteq (2^{\mathbb{N}})^X$, if*

1. *the S_i are pairwise disjoint;*
2. *$tu_1 \dots u_n \in \mathrm{HRed}^{X \cup \{a\}, r_i}_{\sigma}(S_i)$ (resp. $tu_1 \dots u_n \in \mathrm{NRed}^{X \cup \{a\}, r_i}_{\sigma}(S_i)$), for all $i = 1, \dots, k+1$;*
3. *for all $\omega \in S'$, $\mu(\Pi^{\omega}(S_i)) \geq s_i$;*

then $(\nu a.t)u_1 \dots u_n \in \mathrm{HRed}^{X, \sum_{i=1}^{k+1} r_i \cdot s_i}_{\sigma}(S')$ (resp. $(\nu a.t)u_1 \dots u_n \in \mathrm{NRed}^{X, \sum_{i=1}^{k+1} r_i \cdot s_i}_{\sigma}(S')$).

Lemma E.26. *(i.) $t[u/x]u_1 \dots u_n \in \mathrm{HRed}^{X,r}_{\sigma}(S) \Rightarrow (\lambda x.t)uu_1 \dots u_n \in \mathrm{HRed}^{X,r}_{\sigma}(S)$.*
(ii.) $t[u/x]u_1 \dots u_n \in \mathrm{NRed}^{X,r}_{\sigma}(S) \Rightarrow (\lambda x.t)uu_1 \dots u_n \in \mathrm{NRed}^{X,r}_{\sigma}(S)$.

Proof. We only prove claim (ii.) as claim (i.) is proved similarly. By induction on σ:

- if $\sigma \in \{[], \mathrm{hn}\}$, the claim is immediate;

- If $\sigma \in \{o, \mathsf{n}\}$, then if $t[u/x]u_1 \dots u_n \in \mathrm{NRed}^{X,r}_{\sigma}(S)$, then for all $\omega \in S$, $\pi^{\omega}_X(t[u/x]u_1 \dots u_n) = \pi^{\omega}_X(t)[\pi^{\omega}_X(u)/x]\pi^{\omega}_X(u_1)\dots\pi^{\omega}_X(u_n) \in \mathrm{NF}^{r}_{\rightarrow}$. Since the reduction tree of $\pi^{\omega}_X(t'')$, where $t'' = (\lambda x.t)uu_1 \dots u_n$, is contained in that of $\pi^{\omega}_X(t[u/x]u_1 \dots u_n)$, we deduce $\pi^{\omega}_X(t'') \in \mathrm{NF}^{r}$, and we conclude that $t'' \in \mathrm{NRed}^{X,r}_{\sigma}(S)$.
- If $\sigma = \tau \Rightarrow \rho$, then let $S' \subseteq S$ and $v \in \mathrm{NRed}^{X,1}_{\tau}(S')$; then $t[u/x]u_1 \dots u_n v \in \mathrm{NRed}^{X,r}_{\rho}(S')$ so by IH $(\lambda x.t)uu_1 \dots u_n v \in \mathrm{NRed}^{X,r}_{\rho}(S')$. This proves in particular that $(\lambda x.t)uu_1 \dots u_n \in \mathrm{NRed}^{X,r}_{\sigma}(S)$.
- If $\sigma = \tau \wedge \rho$ and $t[u/x]u_1 \dots u_n \in \mathrm{NRed}^{X,r}_{\sigma}(S)$, then $t[u/x]u_1 \dots u_n \in \mathrm{NRed}^{X,r}_{\tau}(S)$ and $t[u/x]u_1 \dots u_n \in \mathrm{NRed}^{X,r}_{\tau}(S)$, so by IH $(\lambda x.t)uu_1 \dots u_n \in \mathrm{NRed}^{X,r}_{\tau}(S)$ and $(\lambda x.t)uu_1 \dots u_n \in \mathrm{NRed}^{X,r}_{\rho}(S)$, which implies $(\lambda x.t)uu_1 \dots u_n \in \mathrm{NRed}^{X,r}_{\sigma}(S)$.
- If $\sigma = \mathbf{C}^q\tau$, then if $t[u/x]u_1 \dots u_n \in \mathrm{NRed}^{X,r}_{\sigma}(S) = \mathrm{NRed}^{X,rq}_{\tau}(S)$, then by I.H. $(\lambda x.t)uu_1 \dots u_n \in \mathrm{NRed}^{X,rq}_{\tau}(S) = \mathrm{NRed}^{X,r}_{\sigma}(S)$.

□

Lemma E.27. *For all t such that $\mathsf{FN}(t) \subseteq X$ and measurable sets $S, S' \subseteq (2^{\mathbb{N}})^X$,*
(i.) If for all $S'' \subseteq S'$ and $u \in \mathrm{HRed}^{X,s}_{\sigma}(S'')$, $t[u/x] \in \mathrm{HRed}^{X,rs}_{\tau}(S \cap S'')$, then $\lambda x.t \in \mathrm{HRed}^{X,r}_{\sigma \Rightarrow \tau}(S \cap S')$.
(ii.) If for all $S'' \subseteq S'$ and $u \in \mathrm{NRed}^{X,s}_{\sigma}(S'')$, $t[u/x] \in \mathrm{NRed}^{X,rs}_{\tau}(S \cap S'')$, then $\lambda x.t \in \mathrm{NRed}^{X,r}_{\sigma \Rightarrow \tau}(S \cap S')$.

Proof. As usual, we only prove claim (ii.). Let $T \subseteq S \cap S'$ and $u \in \mathrm{NRed}^{X,s}_{\sigma}(T)$. Then, by hypothesis, $t[u/x] \in \mathrm{NRed}^{X,rs}_{\tau}(S \cap T) = \mathrm{NRed}^{X,rs}_{\tau}(S \cap S')$ and we conclude by Lemma E.26 that $(\lambda x.t)u \in \mathrm{NRed}^{X,rs}_{\tau}(S \cap S')$. This proves that $\lambda x.t \in \mathrm{NRed}^{X,r}_{\sigma \Rightarrow \tau}(S \cap S')$.

□

Lemma E.28. *For any two types $\mathfrak{s}, \mathfrak{t}$ of $\mathbf{C}\lambda_{\rightarrow,\cap}$, if $\mathfrak{s} \preceq \mathfrak{t}$, then for all $S \subseteq (2^{\mathbb{N}})^X$,*
(i.) $\mathrm{HRed}^{X,r}_{\sigma}(S) \subseteq \mathrm{HRed}^{X,r}_{\tau}(S)$;
(ii.) $\mathrm{NRed}^{X,r}_{\sigma}(S) \subseteq \mathrm{NRed}^{X,r}_{\tau}(S)$.

Proof. As usual, we only prove claim (ii.). We argue by induction on $\mathfrak{s}$:
- if $\mathfrak{s} \in \{[], o, \mathsf{hn}, \mathsf{n}\}$, then it must be $\mathfrak{s} = \mathfrak{t}$, so the claim is immediate.
- If $\mathfrak{s} = [\mathfrak{M}_1, \dots, \mathfrak{M}_n] \Rightarrow \mathfrak{s}'$, then $\mathfrak{t} = [\mathfrak{N}_1, \dots, \mathfrak{N}_m] \Rightarrow \mathfrak{t}'$, where $\mathfrak{s}' \preceq \mathfrak{t}'$, and $\mathfrak{s}_i \succeq \mathfrak{t}_{f(i)}$ holds for some injective function $f : \{1, \dots, n\} \to \{1, \dots, m\}$. By the I.H. we have then that $\mathrm{NRed}^{X,r}_{\mathfrak{s}'}(S) \subseteq \mathrm{NRed}^{X,r}_{\mathfrak{t}'}(S)$ and $\mathrm{NRed}^{X,1}_{\mathfrak{t}_{f(i)}}(S') \subseteq \mathrm{NRed}^{X,1}_{\mathfrak{s}_i}(S')$; by the definition of $\mathrm{NRed}^{X,r}_{\mathfrak{s}}(S)$ this implies then $\mathrm{NRed}^{X,r}_{\mathfrak{s}}(S) \subseteq \mathrm{NRed}^{X,r}_{\mathfrak{t}}(S)$.
- if $\mathfrak{s} = \mathbf{C}^q\mathfrak{s}'$, then it must be $\mathfrak{t} = \mathbf{C}^s\mathfrak{t}'$, with $\mathfrak{s}' \preceq \mathfrak{t}'$ and $q \geq s$; then by IH and Lemma E.20 we have $\mathrm{NRed}^{X,r}_{\mathfrak{s}}(S) = \mathrm{NRed}^{X,qr}_{\mathfrak{s}'}(S) \subseteq \mathrm{NRed}^{X,sr}_{\mathfrak{s}'}(S) \subseteq \mathrm{NRed}^{X,sr}_{\mathfrak{t}'}(S) = \mathrm{NRed}^{X,r}_{\mathfrak{t}}(S)$.

□

We now show how the two families of reducibility candidates differ on the kind of normalization properties they warrant. Let us introduce a (first) notion of *neutral term* for Λ_{PE}:

Definition E.29 (neutral terms, version 1)**.** For any measurable set $S \subseteq (2^{\mathbb{N}})^X$, the set $\mathrm{Neut}(S)$ is defined by induction as follows:
- for any variable x, $x \in \mathrm{Neut}(S)$;
- if $t \in \mathrm{Neut}(S)$ then for all $u \in \Lambda^X_{\oplus,\nu}$, $tu \in \mathrm{Neut}(S)$.

It is easily checked that for all $t \in \mathrm{Neut}(S)$ and $\omega \in S$, $\mathrm{HNV}(\pi^{\omega}_X(t)) = 1$.
Probabilistic neutral terms can be used to show that reducible terms reduce to a probabilistic head normal form.

Definition E.30 (non-trivial type)**.** The set of *non-trivial types* is defined by induction as follows:
- $o, \mathsf{hn}, \mathsf{n}$ are non-trivial;
- if τ is non-trivial, $\sigma \Rightarrow \tau$ is non-trivial;
- if either σ or τ are non-trivial, $\sigma \wedge \tau$ is non-trivial;
- if σ is non-trivial, $\mathbf{C}^q\sigma$ is non-trivial.

Lemma E.31. *For any type σ:*

1. *if σ is non-trivial and $t \in \mathrm{HRed}^{X,r}_{\sigma}(S)$ and $\omega \in S$, then $\pi^{\omega}_X(t) \in \mathrm{HNV}^{\lceil \sigma \rceil \cdot r}_{\rightarrow}$;*
2. *$\mathrm{Neut}(S) \subseteq \mathrm{HRed}^{X,1}_{\sigma}(S)$.*

Proof. We argue by induction on σ:
- if $\sigma = []$, then both claims are immediate;
- if $\sigma \in \{o, \mathsf{hn}, \mathsf{n}\}$ then by definition $t \in \mathrm{HRed}^{X,r}_{\sigma}(S)$ iff for any $\omega \in S$, $\pi^{\omega}_X(t) \in \mathrm{HNV}^{r}_{\rightarrow} = \mathrm{HNV}^{\lceil \sigma \rceil r}_{\rightarrow}$. Moreover, if $t \in \mathrm{Neut}(S)$, then $\mathrm{HNV}(t) = 1$, so $t \in \mathrm{HRed}^{X,1}_{\sigma}(S)$.

- if $\sigma = \tau \Rightarrow \rho$ and $t \in \mathrm{HRed}^{X,r}_{\sigma}(S)$, then by IH $x \in \mathrm{HRed}^{X,1}_{\tau}(S)$, hence $tx \in \mathrm{HRed}^{X,r}_{\tau}(S)$, and again by IH for all $\omega \in S$ $\pi^{\omega}_{X}(tx) = \pi^{\omega}_{X}(t)x \in \mathrm{HNV}^{\lceil\tau\rceil\cdot r}_{\rightarrow} = \mathrm{HNV}^{\lceil\sigma\rceil\cdot r}_{\rightarrow}$, and thus $\pi^{\omega}_{X}(t) \in \mathrm{HNV}^{\lceil\sigma\rceil\cdot r}_{\rightarrow}$.
  Moreover, if $t \in \mathrm{Neut}(S)$, let $S' \subseteq S$ and $u \in \mathrm{HRed}^{X,1}_{\tau}(S)$, then by IH for all $\omega \in S'$, $\pi^{\omega}(u) \in \mathrm{HNV}^{\lceil\tau\rceil}_{\rightarrow}$ and thus $tu \in \mathrm{Neut}(S')$, so by IH $tu \in \mathrm{HRed}^{X,1}_{\rho}(S') \subseteq \mathrm{HRed}^{X,r}_{\rho}(S')$; we can thus conclude that $t \in \mathrm{HRed}^{X,1}_{\sigma}(S)$.
- if $\sigma = \tau \wedge \rho$ and $t \in \mathrm{HRed}^{X,r}_{\sigma}(S)$, then $t \in \mathrm{HRed}^{X,r}_{\tau}(S)$ and $t \in \mathrm{HRed}^{X,r}_{\rho}(S)$, so by IH for all $\omega \in S$, $\pi^{\omega}_{X}(t) \in \mathrm{HNV}^{\lceil\tau\rceil r}_{\rightarrow}$ and $\pi^{\omega}_{X}(t) \in \mathrm{HNV}^{\lceil\rho\rceil r}_{\rightarrow}$; hence we deduce that for all $\omega \in S$ $\pi^{\omega}_{X}(t) \in \mathrm{HNV}^{\max\{\lceil\tau\rceil,\lceil\rho\rceil\}\cdot r}_{\rightarrow} = \mathrm{HNV}^{\lceil\sigma\rceil\cdot r}_{\rightarrow}$.
  Moreover, if $t \in \mathrm{Neut}(S)$, then by IH $t \in \mathrm{HRed}^{X,1}_{\tau}(S)$ and $t \in \mathrm{HRed}^{X,1}_{\rho}(S)$, so $t \in \mathrm{HRed}^{X,1}_{\sigma}(S)$.
- if $\sigma = \mathbf{C}^{s}\tau$ and $t \in \mathrm{HRed}^{X,q}_{\sigma}(S) = \mathrm{HRed}^{X,qs}_{\tau}(S)$, then by IH for all $\omega \in S$, $\pi^{\omega}_{X}(t) \in \mathrm{HNV}^{\lceil\tau\rceil qs}_{\rightarrow} = \mathrm{HNV}^{\lceil\sigma\rceil q}_{\rightarrow}$. Moreover, if $t \in \mathrm{Neut}(S)$, then by IH $t \in \mathrm{HRed}^{X,1}_{\tau}(S) \subseteq \mathrm{HRed}^{X,s}_{\tau}(S) = \mathrm{HRed}^{X,1}_{\sigma}(S)$.

□

To establish a result like Lemma E.31 for the family $\mathrm{NRed}^{X,r}_{\sigma}(S)$, as discussed in Section 6, we need to restrict to *balanced* types. For our general class of types, these are defined as follows:

Definition E.32. The set Bal of *balanced types* is defined inductively as follows:

- $\mathbf{C}^{\vec{q}}\sigma$, for $\sigma \in \{o, \mathsf{hn}, \mathsf{n}\}$;
- if $\sigma, \tau \in \mathrm{Bal}$, $\sigma \wedge \tau \in \mathrm{Bal}$;
- if $\sigma_1, \dots, \sigma_n \in \mathrm{Bal}$ and $\prod_{i=1}^{k} q_i \leq \prod_{j=1}^{n}\lceil\sigma_j\rceil$, then $\mathbf{C}^{q_1}\dots\mathbf{C}^{q_n}(\sigma_1 \Rightarrow \dots \Rightarrow \sigma_n \Rightarrow o) \in \mathrm{Bal}$.

For instance, the type $\mathbf{C}^{\frac{1}{2}}(\mathbf{C}^{\frac{1}{2}}o \Rightarrow o)$ is balanced, while $\mathbf{C}^{1}(\mathbf{C}^{\frac{1}{2}}o \Rightarrow o)$ is not.

One can check that any balanced type is also non-trivial.

To study normalization for balanced types we need a more refined notion of neutral term.

Definition E.33 (neutral terms, version 2)**.** For any $q \in (0,1]$, and measurable set S, the set $\mathrm{Neut}^{q}(S)$ is defined by induction as follows:

- for any variable x, $x \in \mathrm{Neut}^{q}(S)$;
- if $t \in \mathrm{Neut}^{q}(S)$ and for all $\omega \in S$, $\mathrm{NF}_{\rightarrow}(\pi^{\omega}_{X}(u)) \geq s$, then $tu \in \mathrm{Neut}^{qs}(S)$.

It is easily checked that for all $t \in \mathrm{Neut}^{q}(S)$ and $\omega \in S$, $\mathrm{NF}_{\rightarrow}(\pi^{\omega}_{X}(t)) \geq q$.

Lemma E.34. *For any balanced and $\{[], \mathsf{hn}\}$-free type σ:*

1. *if $t \in \mathrm{NRed}^{X,r}_{\sigma}(S)$ and $\omega \in S$, then $\pi^{\omega}_{X}(S) \in \mathrm{NF}^{\lceil\sigma\rceil\cdot r}_{\rightarrow}$;*
2. $\mathrm{Neut}^{q}(S) \subseteq \mathrm{NRed}^{X,q}_{\sigma}(S)$.

Proof. We argue by induction on $\mathfrak{s}$:

- if $\sigma = \mathbf{C}^{q_1}\dots\mathbf{C}^{q_k}\tau$, with $\tau \in \{o, \mathsf{n}\}$, then by definition $t \in \mathrm{NRed}^{X,r}_{\sigma}(S)$ iff for all $\omega \in S$, $\pi^{\omega}_{X}(t) \in \mathrm{NF}^{q_1\cdots q_k r}_{\rightarrow} = \mathrm{NF}^{\lceil o\rceil\cdot r}$.
  Moreover if $t \in \mathrm{Neut}^{q}(S)$, then for all $\omega \in S$, $\mathrm{NF}_{\rightarrow}(\pi^{\omega}_{X}(t)) \geq q$, hence $t \in \mathrm{NRed}^{X,q}_{o}(S) \subseteq \mathrm{NRed}^{X,q\lceil\sigma\rceil}_{\sigma}(S)$.
- if $\sigma = \tau \wedge \rho$, then we can argue as in the corresponding point in the proof of Lemma E.31.
- if $\sigma = \mathbf{C}^{q_1}\dots\mathbf{C}^{q_k}(\sigma_1 \Rightarrow \dots \Rightarrow \sigma_n \Rightarrow o)$, where $\prod_i q_i \leq \prod_j\lceil\sigma_j\rceil$, and $t \in \mathrm{NRed}^{X,r}_{\sigma}(S)$, then by IH $x_j \in \mathrm{Neut}^{1}(S) \subseteq \mathrm{NRed}^{X,1}_{\sigma_i}(S)$, and thus $tx_1\dots x_n \in \mathrm{NRed}^{rq_1\cdots q_k}_{o}(S)$, so for all $\omega \in S$, $\pi^{\omega}_{X}(tx_1\dots x_n) = \pi^{\omega}_{X}(t)x_1\dots x_n \in \mathrm{NF}^{rq_1\cdots q_k}_{\rightarrow} \subseteq \mathrm{NF}^{rq_1\cdots q_n\prod_j\lceil\sigma_j\rceil}_{\rightarrow} = \mathrm{NF}^{\lceil\sigma\rceil r}_{\rightarrow}$. We deduce then $\pi^{\omega}_{X}(t) \in \mathrm{NF}^{\lceil\sigma\rceil r}_{\rightarrow}$ by applying Lemma E.19 a finite number of times.
  Moreover, if $t \in \mathrm{Neut}^{q}(S)$, $S' \subseteq S$, $\omega \in S'$, and $u_j \in \mathrm{NRed}^{X,1}_{\sigma_j}(S')$, then by the IH for all $\omega \in S'$ $\pi^{\omega}_{X}(u_j) \in \mathrm{NF}^{\lceil\sigma_j\rceil}_{\rightarrow}$, so $tu_1\dots u_n \in \mathrm{Neut}^{q\prod_k\lceil\sigma_j\rceil}(S') \subseteq \mathrm{Neut}^{qq_1\cdots q_k}(S')$; by the IH it follows then $tu_1\dots u_n \in \mathrm{NRed}^{X,q_1\cdots q_k s}_{o}(S')$. We can thus conclude $t \in \mathrm{NRed}^{X,qq_1\dots q_k}_{\sigma_1\Rightarrow\dots\Rightarrow\sigma_n\Rightarrow o}(S) = \mathrm{NRed}^{X,q}_{\sigma}(S)$.

□

The last ingredient of our reducibility argument is the following:

Proposition E.35. *If $\Gamma \vdash^{X} t : \mathscr{b} \rightarrowtail \mathfrak{s}$ is derivable in $\mathsf{C}\lambda_{\rightarrow,\cap}$, where $\Gamma = \{x_1 : \mathfrak{M}_1, \dots, x_m : \mathfrak{M}_n\}$, then for all $S \subseteq [\![\mathscr{b}]\!]_X$, the following two claims hold:*

(i.) for all $u_i \in \mathrm{HRed}^{X,1}_{\mathfrak{M}_i}(S)$, $t[u_1/x_1, \dots, u_m/x_m] \in \mathrm{HRed}^{X,1}_{\mathfrak{s}}(S)$;

(ii.) for all $u_i \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_i}(S)$, $t[u_1/x_1, \dots, u_m/x_m] \in \mathrm{NRed}^{X,1}_{\mathfrak{s}}(S)$.

Proof. We only prove claim (ii.). The argument is by induction on a type derivation of t:

- if $t = x$ and the last rule is

$$\frac{\mathfrak{s}_i \preceq \mathfrak{t} \qquad \mathsf{FN}(\mathscr{b}) \subseteq X}{\Gamma, x : [\mathfrak{s}_1, \dots, \mathfrak{s}_n] \vdash^X x : \mathscr{b} \rightarrowtail \mathfrak{t}} \ (\mathrm{id}_\cap)$$

Then for all $S \subseteq [\![\mathscr{b}]\!]_X$, and for all $u_1 \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_1}(S), \dots u_n \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_n}(S)$, and $u \in \mathrm{NRed}^{X,1}_{\mathfrak{M}}(S)$, using Lemma E.28 we deduce $u \in \mathrm{NRed}^{X,1}_{\mathfrak{s}_i}(S) \subseteq \mathrm{NRed}^{X,1}_{\mathfrak{t}}(S)$, whence $t[u_1/x_1, \dots, u_n/x_n, u/x] = u \in \mathrm{NRed}^{X,1}_{\mathfrak{t}}([\![\mathscr{b}]\!]_X \cap S) = \mathrm{NRed}^{X,1}_{\mathfrak{s}}(S)$.

- if the last rule is

$$\frac{\left\{\Gamma \vdash^X t : \mathscr{b}_j \rightarrowtail \mathfrak{s}\right\}_{j=1,\dots,k} \qquad \mathscr{b} \vDash^X \bigvee_j \mathscr{b}_j}{\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathfrak{s}} \ (\vee)$$

then by IH, for all $j = 1, \dots, k$, $S \subseteq [\![\mathscr{b}_j]\!]_X$, for all $u_1 \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_1}(S), \dots, u_n \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_n}(S)$ and for all $\omega \in S$, $\pi^\omega_X(t[u_1/x_1, \dots, u_n/x_n]) \in \mathrm{NRed}^{\emptyset,1}_\rho$.
If $k = 0$, then $\mathscr{b} \vdash \bigvee_j \mathscr{b}_j = \bot$, so the claim trivially holds. Let then $k > 0$. Let $S \subseteq [\![\mathscr{b}]\!]_X \subseteq [\![\bigvee_j \mathscr{b}_j]\!]_X = \bigcup_i [\![\mathscr{b}_j]\!]_X$ and $\omega \in S$. Then $f \in [\![\mathscr{b}_j]\!]_X$, for some $j \leq k$. Then first observe that from $u_i \in \mathrm{NRed}^{X,q_i}_{\mathfrak{M}_i}(S)$ it follows in particular that $u_i \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_i}(S \cap [\![\mathscr{b}_j]\!]_X)$: for all $\omega \in S \cap [\![\mathscr{b}_j]\!]_X$, since $\omega \in S$, $\pi^\omega_X(u_i) \in \mathrm{NRed}^{\emptyset,1}_{\mathfrak{M}_i}$, so by Lemma E.22 we conclude that $u_i \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_i}(S \cap [\![\mathscr{b}_j]\!]_X)$. Hence we deduce that $\pi^\omega_X(t[u_1/x_1, \dots, u_n/x_n]) \in \mathrm{NRed}^{\emptyset, q_1 \dots q_n}_{\mathfrak{s}}$, and again by Lemma E.22 we conclude $t[u_1/x_1, \dots, u_n/x_n] \in \mathrm{NRed}^{X,1}_{\mathfrak{s}}(S)$.

- if $t = t_1 \oplus^i_a t_2$, and the last rule is

$$\frac{\Gamma \vdash^X t_1 : \mathscr{b} \rightarrowtail \mathfrak{s} \qquad \mathscr{c} \vDash (\neg \mathscr{x}^i_a \wedge \mathscr{b})}{\Gamma \vdash^X t_1 \oplus^i_a t_2 : \mathscr{c} \rightarrowtail \mathfrak{s}} \ (\oplus l)$$

By IH and Lemma E.22, for all $S \subseteq [\![\mathscr{b}]\!]_X$, $\omega \in S$, $u_1 \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_1}(S), \dots u_n \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_n}(S)$, $\pi^\omega_X(t_1[u_1/x_1, \dots, u_n/x_n]) \in \mathrm{NRed}^{\emptyset,1}_{\mathfrak{s}}$.
Let now $S \subseteq [\![\mathscr{c}]\!]_X \subseteq [\![\mathscr{x}^i_a]\!]_X \wedge [\![\mathscr{b}]\!]_X$ and $u_1 \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_1}(S), \dots, u_n \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_n}(S)$. If now $\omega \in S \subseteq [\![\mathscr{c}]\!]_{X \cup \{a\}}$, then $\omega(a)(i) = 0$ so in particular $\pi^\omega_X(t[u_1/x_1, \dots, u_n/x_n]) = \pi^\omega_X(t_1[u_1/x_1, \dots, u_n/x_n]) \in \mathrm{NRed}^{\emptyset,1}_{\mathfrak{s}}$. Hence, using Lemma E.22 we conclude that $t[u_1/x_1, \dots, u_n/x_n] \in \mathrm{NRed}^{X,1}_{\mathfrak{s}}(S)$.

- if $t = t_1 \oplus^i_a t_2$, and the last rule is

$$\frac{\Gamma \vdash^X t_2 : \mathscr{b} \rightarrowtail \mathfrak{s} \qquad \mathscr{c} \vDash (\neg \mathscr{x}^i_a \wedge \mathscr{b})}{\Gamma \vdash^X t_1 \oplus^i_a t_2 : \mathscr{c} \rightarrowtail \mathfrak{s}} \ (\oplus r)$$

then we can argue similarly to the previous case.

- if the last rule is

$$\frac{\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathbf{C}^q \sigma}{\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathbf{C}^q \mathsf{hn}} \ (\mathsf{hn})$$

then by IH for all $S \subseteq [\![\mathscr{b}]\!]_X$, for all $u_1 \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_1}(S), \dots, u_n \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_n}(S)$, $\pi^\omega_X(t[u_1/x_1, \dots, t_n/x_n]) \in \mathrm{NRed}^{\emptyset,q}_\sigma$, so by Lemma E.31, $\mathrm{HNV}(\pi^\omega_X(t[u_1/x_1, \dots, t_n/x_n])) \geq q \cdot \lceil \sigma \rceil = q$, which implies $t[u_1/x_1, \dots, t_n/x_n] \in \mathrm{NRed}^{X,q}_{\mathsf{hn}}(S)$.

- if the last rule is

$$\frac{\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathbf{C}^q \sigma \qquad \sigma \text{ safe}}{\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathbf{C}^q \mathsf{n}} \ (\mathsf{n})$$

then σ is $\{[], \mathsf{hn}\}$-free and balanced; by IH for all $S \subseteq [\![\mathscr{b}]\!]_X$, for all $u_1 \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_1}(S), \dots, u_n \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_n}(S)$, $\pi^\omega_X(t[u_1/x_1, \dots, t_n/x_n]) \in \mathrm{NRed}^{\emptyset,q}_\sigma$; furthermore, by Lemma E.34, $\mathrm{NF}_{\rightarrow}(\pi^\omega_X(t[u_1/x_1, \dots, t_n/x_n])) \geq q \cdot \lceil \sigma \rceil = q$, which implies $t[u_1/x_1, \dots, t_n/x_n] \in \mathrm{NRed}^{X,q}_{\mathsf{n}}(S)$.

- if $t = \lambda x.u$ and the last rule is

$$\frac{\Gamma, x : \mathfrak{M} \vdash^X u : \mathscr{b} \rightarrowtail \mathbf{C}^{\vec{s}} \tau}{\Gamma \vdash^X \lambda x.u : \mathscr{b} \rightarrowtail \mathbf{C}^{\vec{s}}(\mathfrak{M} \Rightarrow \tau)} \ (\lambda)$$

where $\vec{s} = s_1, \dots, s_l$; then, by IH, for all $S \subseteq [\![\mathscr{b}]\!]_X$, for all $u_1 \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_1}(S), \dots, u_n \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_n}(S)$, and $v \in \mathrm{NRed}^{X,1}_{\mathfrak{M}}(S)$, $u[u_1/x_1, \dots, u_n/x_n, v/x] \in \mathrm{NRed}^{X,1}_{\mathbf{C}^{\vec{s}}\tau}(S) = \mathrm{NRed}^{X, s_1 \dots s_k}_\tau(S)$. In particular, for all choice of $u_1, \dots, u_n$ and variable y not occurring free in $u_1, \dots, u_n$, by letting $t' = u[u_1/x_1, \dots, u_n/x_n, y/x]$, we have that for all $v \in \mathrm{NRed}^{X,1}_{\mathfrak{M}}(S \cap [\![\mathscr{b}]\!]_X)$, $t'[v/y] = u[u_1/x_1, \dots, u_n/x_n, v/x] \in \mathrm{NRed}^{X, s_1 \dots s_k}_\tau(S)$. By Lemma E.27, then, we can conclude that $\lambda y.t' = (\lambda y.t[y/x])[u_1/x_1, \dots, u_n/x_n] = (\lambda x.t)[u_1/x_1, \dots, u_n/x_n] \in \mathrm{NRed}^{X, s_1 \dots s_k}_{\mathfrak{M} \Rightarrow \sigma}(S) = \mathrm{NRed}^{X,1}_{\mathbf{C}^{\vec{s}}(\mathfrak{M} \Rightarrow \sigma)}(S)$.

- $t = uv$ and the last rule is

$$\frac{\Gamma \vdash^X u : \mathscr{c} \rightarrowtail \mathsf{C}^{\vec{s}}(\mathfrak{M} \Rightarrow \sigma) \qquad \left\{\Gamma \vdash^X v : \mathscr{d}_i \rightarrowtail \mathfrak{s}_i\right\}_i \qquad \mathscr{b} \vDash^X \mathscr{c} \wedge (\bigwedge_i \mathscr{d}_i)}{\Gamma \vdash^X uv : \mathscr{b} \rightarrowtail \mathsf{C}^{\vec{s}}\sigma}\,(@)$$

where $\vec{s} = s_1, \dots, s_k$ and $\mathscr{M} = [\mathfrak{s}_1, \dots, \mathfrak{s}_n]$, then by IH for all $S \subseteq [\![\mathscr{b}]\!]_X \subseteq [\![\mathscr{c}]\!]_X \cap \bigcap_i [\![\mathscr{d}_i]\!]_X$ and $u_1 \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_1}(S), \dots u_n \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_n}(S)$, $u[u_1/x, \dots, u_n/x] \in \mathrm{NRed}^{X,s_1\dots s_k}_{\mathfrak{M}\Rightarrow\sigma}(S)$ and $v[u_1/x, \dots, u_n/x] \in \bigcap_i \mathrm{NRed}^{X,1}_{\mathfrak{s}_i}(S) = \mathrm{NRed}^{X,1}_{\mathfrak{M}}(S)$ (where, if $n = 0$, $v[u_1/x, \dots, u_n/x] \in \mathrm{NRed}^{X,1}_{\mathfrak{M}}(S)$ does not follow from the induction hypothesis but from the fact that $\mathrm{NRed}^{X,1}_{[]}(S) = \Lambda^X_{\oplus,\nu}$) so we deduce $u[u_1/x, \dots, u_n/x]v[u_1/x, \dots, u_n/x] = (uv)[u_1/x, \dots, u_n/x] \in \mathrm{NRed}^{X,s_1\dots s_k}_{\sigma}(S) = \mathrm{NRed}^{X,1}_{\mathsf{C}^{\vec{s}}\sigma}(S)$.

- If $t = \nu a.u$ and the last rule is

$$\frac{\left\{\Gamma \vdash^{X\cup\{a\}} u : \mathscr{c} \wedge \mathscr{d}_i \rightarrowtail \mathsf{C}^{q_j}\sigma\right\}_{j=1,\dots,n+1} \qquad \vDash \mu(\mathscr{d}_j) \geq r_j \qquad \mathscr{b} \vDash \mathscr{c}}{\Gamma \vdash^X \nu a.u : \mathscr{b} \rightarrowtail \mathsf{C}^{\sum_j r_j q_j}\sigma}\,(\mu)$$

  where $a \notin FV(\Gamma) \cup FV(\mathscr{c})$, then let $S \subseteq [\![\mathscr{b}]\!]_X \subseteq [\![\mathscr{c}]\!]_X$ and $u_1 \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_1}(S), \dots u_n \in \mathrm{NRed}^{X,1}_{\mathfrak{M}_n}(S)$. Let $T = \{\omega + \omega' \mid \omega \in S\} \subseteq (2^{\mathbb{N}})^{X\cup\{a\}}$, which is measurable since counter-image of a measurable set through the projection function; then since $\mathsf{FN}(u_i) \subseteq X$, we deduce using Lemma E.22 that $u_i \in \mathrm{NRed}^{X\cup\{a\},1}_{\mathfrak{M}_i}(T \cap [\![\mathscr{d}_j]\!]_{X\cup\{a\}})$. By IH and the hypothesis we deduce then that
  - the sets $[\![\mathscr{d}_j]\!]_{X\cup\{a\}}$ are pairwise disjoint;
  - $u[u_1/x_1, \dots, u_n/x_n] \in \mathrm{NRed}^{X\cup\{a\},1}_{\mathfrak{s}}(T \cap [\![\mathscr{d}_j]\!]_{X\cup\{a\}})$;
  - for all $\omega \in S$, $\mu(\Pi^\omega(T \cap [\![\mathscr{d}_j]\!]_{X\cup\{a\}})) \geq r_j$;

  Hence, by Lemma E.25 we conclude that $\nu a.u[u_1/x_1, \dots, u_n/x_n] = (\nu a.u)[u_1/x_1, \dots, u_n/x_n] \in \mathrm{NRed}^{X,\sum_j r_j q_j}_{\mathfrak{s}}(S) = \mathrm{NRed}^{X,1}_{\mathsf{C}^{\sum_j r_j q_j}\mathfrak{s}}(S)$. □

From Proposition E.35 (i.) and Lemma E.31, by observing that any type of $\mathsf{C}\lambda_{\to,\cap}$ is non-trivial, we deduce the following, which generalizes the first part of Theorem 6.1.

Theorem E.36. *If $\Gamma \vdash t : \mathscr{b} \rightarrowtail \mathsf{C}^q\sigma$ holds in $\mathsf{C}\lambda_{\to,\cap}$, then for all $\omega \in [\![\mathscr{b}]\!]$, $\mathrm{HNV}_{\rightharpoonup}(\pi^\omega_X(t)) \geq q$.*

From Proposition E.35 (i.) and Lemma E.34 we similarly deduce a generalization of Theorem 6.3:

Theorem E.37. *If $\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathsf{C}^q\sigma$ holds in $\mathsf{C}\lambda_{\to,\cap}$, where Γ and $\mathfrak{s}$ are formed of balanced types, then for all $\omega \in [\![\mathscr{b}]\!]$, $\mathrm{NF}_{\rightharpoonup}(\pi^\omega_X(t)) \geq q$.*

From the two results above we can finally deduce the soundness part of Theorem 6.6:

Corollary E.38. *For any closed term t,*

$$\mathrm{HNV}_{\rightharpoonup}(t) \geq \sup\{q \mid \vdash t : \top \rightarrowtail \mathsf{C}^q\mathsf{hn}\}$$
$$\mathrm{NF}_{\rightharpoonup}(t) \geq \sup\{q \mid \vdash t : \top \rightarrowtail \mathsf{C}^q\mathsf{n}\}$$

E.3 Probabilistic Reducibility Candidates for $\Lambda^{\{\}}_{\mathsf{PE}}$.

We now show how to adapt the family of reducibility candidates $\mathrm{HRed}^{X,1}_\sigma(S)$ to $\Lambda^{\{\}}_{\mathsf{PE}}$, in order to prove the second part of Theorem 6.1. We will still work with the general family of types from the previous section (hence suggesting that intersection types could be added to $\mathsf{C}\lambda^{\{\}}_{\to}$, too).

For any $t \in \mathcal{T}$, let $\mathscr{D}^1_t : \mathcal{T} \to [0,1]$ be δ_t if $t \in \mathrm{HNV}$, and if $t = \nu a.t'$, let

$$\mathscr{D}^1_t(u) = \mu\{\omega \in 2^{\mathbb{N}} \mid \pi^\omega_{\{a\}}(t') = u\}$$

By recalling the definition of head-reduction in $\Lambda^{\{\}}_{\mathsf{PE}}$ from Subsection B.2, one can easily check the following useful properties:

Lemma E.39. *(i.) If $t \rightharpoonup^*_{\{\}h} u$, then $\pi^\omega_X(t) \rightharpoonup^*_{\{\}h} \pi^\omega_X(u)$.*
(ii.) if $t \rightharpoonup^*_{\{\}h} u$, then $\mathrm{HNV}(t) \geq \mathrm{HNV}(u)$.

Lemma E.40. *For all $t, u, w \in \mathcal{T}$, $\mathscr{D}^1_t(u) = \mathscr{D}^1_{tw}(uw)$.*

Proof. If t is a HNV, tw is a pseudo-value, so $\mathscr{D}^1_t = \delta_t$ and $\mathscr{D}^1_{tw} = \delta_{tw}$, which implies $\mathscr{D}^1_t(u) = 1$ iff $t = u$ iff $tw = uw$ iff $\mathscr{D}^1_{tw} = tw$. If $t = \nu a.t'$, since we can suppose $a \notin \mathsf{FN}(w)$, from $\pi^\omega_{\{a\}}(t'w) = \pi^\omega_{\{a\}}(t')w$ we deduce $\{\omega \mid \pi^\omega_{\{a\}}(t'w) = uw\} = \{\omega \mid \pi^\omega_{\{a\}}(t') = u\}$, from which the claim follows. □

The following is a simple consequence of Lemma E.39:

Lemma E.41. *Let W be a set of terms such that $t \in W$ and $u \rightarrow\!\!\triangleright^*_{\{\}h} t$ implies $u \in W$; then for any $t, u \in \mathcal{T}$,] if $t \rightarrow\!\!\triangleright^*_{\{\}h} u$, then $\sum_{w \in W} \mathscr{D}^1_t(w) \geq \sum_{w \in W} \mathscr{D}^1_u(w)$.*

To adapt reducibility predicates to $\Lambda^{\{\}}_{\mathsf{PE}}$, we need to adapt the definition of reducibility for counting quantified types. We define yet a new family of reducibility predicates $\mathrm{H^*Red}^{X,r}_{o}(S) \subseteq \Lambda^X_{\oplus,\nu,\{\}}$ as follows:

$$
\begin{aligned}
\mathrm{H^*Red}^{X,r}_{[]}(S) &= \Lambda^X_{\oplus,\nu,\{\}} \\
\mathrm{H^*Red}^{X,r}_{o}(S) = \mathrm{H^*Red}^{X,r}_{\mathsf{hn}}(S) = \mathrm{H^*Red}^{X,r}_{\mathsf{n}}(S) &= \{t \mid \forall \omega \in S\ \mathrm{HNV}_{\rightarrow\!\triangleright}(\pi^\omega_X(t)) \geq r\} \\
\mathrm{H^*Red}^{X,r}_{\sigma \Rightarrow \tau}(S) &= \{t \mid \forall S' \subseteq S,\ \forall u \in \mathrm{HRed}^{X,1}_{\sigma}(S'),\ tu \in \mathrm{HRed}^{X,r}_{\tau}(S')\} \\
\mathrm{H^*Red}^{X,r}_{\sigma \wedge \tau}(S) &= \mathrm{H^*Red}^{X,r}_{\sigma}(S) \cap \mathrm{H^*Red}^{X,r}_{\tau}(S) \\
\mathrm{H^*Red}^{X,r}_{\mathbf{C}^q\sigma}(S) &= \left\{ t \mid \forall \omega \in S \sum_{u \in \mathrm{H^*Red}^{\emptyset,r}_{\sigma}} \mathscr{D}^1_{\pi^\omega_X(t)}(u) \geq q \right\}
\end{aligned}
$$

Most properties of $\mathrm{H^*Red}_\sigma$ are proved as in the corresponding cases of HRed_σ. We here only consider the cases in which the proofs are different.

Lemma E.42. *If $t \in \mathrm{H^*Red}^{X,r}_{\sigma}(S)$ and $u \rightarrow\!\!\triangleright^*_{\{\}h} t$, then $u \in \mathrm{H^*Red}^{X,r}_{\sigma}(S)$.*

Proof. We argue by induction on σ:

- if $\sigma \in \{o, \mathsf{hn}, \mathsf{n}\}$, then $t \in \mathrm{H^*Red}^{X,r}_{\sigma}(S)$ iff for all $\omega \in S$, $\mathrm{HNV}_{\rightarrow\!\triangleright}(\pi^\omega_X(t)) \geq r$; the claim then follows from Lemma E.39 (i.).
- if $\sigma = \tau \Rightarrow \rho$ and $t \in \mathrm{H^*Red}^{X,r}_{\sigma}(S)$, then for all $S' \subseteq S$ and $v \in \mathrm{H^*Red}^{X,1}_{\tau}(S')$, $tv \in \mathrm{H^*Red}^{X,r}_{\rho}(S')$; since $uv \rightarrow\!\!\triangleright^*_{\{\}h} tv$, by IH we deduce then $uv \in \mathrm{H^*Red}^{X,r}_{\rho}(S')$; we can thus conclude $v \in \mathrm{H^*Red}^{X,r}_{\sigma}(S)$;
- if $\sigma = \tau \wedge \rho$, and $t \in \mathrm{H^*Red}^{X,r}_{\sigma}(S)$, then by IH $u \in \mathrm{H^*Red}^{X,r}_{\tau}(S)$ and $u \in \mathrm{H^*Red}^{X,r}_{\rho}(S)$, so $u \in \mathrm{H^*Red}^{X,r}_{\sigma}(S)$.
- if $\sigma = \mathbf{C}^q\tau$ and $t \in \mathrm{H^*Red}^{X,r}_{\sigma}(S)$, then $\sum_{v \in \mathrm{H^*Red}^{X,r}_{\tau}(S)} \mathscr{D}^1_{\pi^\omega_X(t)}(v) \geq q$, and we can conclude $\sum_{v \in \mathrm{H^*Red}^{X,r}_{\tau}(S)} \mathscr{D}^1_{\pi^\omega_X(u)}(v) \geq q$ from the IH and Lemma E.41 (i).

$\square$

Using Lemma E.24 (i.), whose proof is adapted without difficulties to $\Lambda^{\{\}}_{\mathsf{PE}}$, Lemma E.25 is adapted as follows:

Lemma E.43. *For all types σ, terms $t, u_1, \dots, u_n$ with $\mathsf{FN}(t) \subseteq X \cup \{a\}$, $\mathsf{FN}(u_i) \subseteq X$, and measurable set $S \subseteq (2^{\mathbb{N}})^{X \cup \{a\}}$ and $S' \subseteq (2^{\mathbb{N}})^X$, if*

1. *$tu_1 \dots u_n \in \mathrm{H^*Red}^{X \cup \{a\}, r}_{\sigma}(S)$;*
2. *for all $\omega \in S'$, $\mu(\Pi^\omega(S)) \geq s$;*

then $(\nu a.t)u_1 \dots u_n \in \mathrm{H^*Red}^{X,r}_{\mathbf{C}^s\sigma}(S')$.

The following adaptation of Lemma E.26 immediately follows from Lemma E.42:

Lemma E.44. $t[u/x]u_1 \dots u_n \in \mathrm{H^*Red}^{X,r}_{\sigma}(S) \Rightarrow (\lambda x.t)uu_1 \dots u_n \in \mathrm{H^*Red}^{X,r}_{\sigma}(S)$.

Lemma E.45. *For all types σ and τ, $\mathrm{H^*Red}^{X,r}_{\mathbf{C}^{q_1} \dots \mathbf{C}^{q_k}(\sigma \Rightarrow \tau)}(S) \subseteq \mathrm{H^*Red}^{X,r}_{\sigma \Rightarrow \mathbf{C}^{q_1} \dots \mathbf{C}^{q_k}\tau}(S)$.*

Proof. For any $j \leq k$, let $U_j := \mathrm{H^*Red}^{X,r}_{\mathbf{C}^{q_{k-j+1}} \dots \mathbf{C}^{q_k}(\sigma \Rightarrow \tau)}(S)$ and $V_j := \mathrm{H^*Red}^{X,r}_{\sigma \Rightarrow \mathbf{C}^{q_{k-j+1}} \dots \mathbf{C}^{q_k}\tau}(S)$.

We argue, by induction on $j \leq k$, that $U_j \subseteq V_j$. If $j = 0$ then $U_j = \sigma \Rightarrow \tau = V_j$.

Let then $j > 0$ and $\omega \in S$. Suppose $t \in U_j$ and let $w \in \mathrm{H^*Red}^{X,1}_{\sigma}(S)$.

By IH we have $U_{j-1} \subseteq V_{j-1}$, from which we deduce $q_1 \leq \sum_{u \in U_{j-1}} \mathscr{D}^1_{\pi^\omega_X(t)}(u) \leq \sum_{u \in V_{j-1}} \mathscr{D}^1_{\pi^\omega_X(t)}(u)$ $\stackrel{\text{[Lemma E.40]}}{=} \sum_{u \in V_{j-1}} \mathscr{D}^1_{\pi^\omega_X(t)\pi^\omega_X(w)}(u\pi^\omega_X(w)) = \sum_{u \in V_{j-1}} \mathscr{D}^1_{\pi^\omega_X(tw)}(u\pi^\omega_X(w))$ which implies $t \in V_j$.

$\square$

We call a term t *ν-safe* if t never reduces to a term of the form $\nu a.t'$.

Definition E.46 (neutral terms for $\Lambda^{\{\}}_{\mathsf{PE}}$.)**.** For any measurable set $S \subseteq (2^{\mathbb{N}})^X$, the set $\mathrm{Neut}_{\{\}}(S)$ is defined by induction as follows:

- for any variable x, $x \in \mathrm{Neut}_{\{\}}(S)$;
- if $t \in \mathrm{Neut}_{\{\}}(S)$ and for all $\omega \in S$ $\pi_X^\omega(u)$ is ν-safe and $\mathrm{HNV}_{\twoheadrightarrow}(u) = 1$, $\{t\}u \in \mathrm{Neut}_{\{\}}(S)$.
- if $t \in \mathrm{Neut}_{\{\}}(S)$, then for all $u \in \Lambda^X_{\oplus,\nu,\{\}}$, $tu \in \mathrm{Neut}_{\{\}}(S)$.

It is easily checked that for all $t \in \mathrm{Neut}_{\{\}}(S)$ and $\omega \in S$, $\mathrm{HNV}(\pi_X^\omega(t)) = 1$.

Lemma E.47. *For any type σ,*
(i.) if σ is non-trivial, then $t \in \mathrm{H}^*\mathrm{Red}^{X,r}_\sigma(S)$ and $\omega \in S$, then $\mathrm{HNV}(\pi_X^\omega(t)) \geq \lceil\sigma\rceil \cdot r$;
(ii.) $\mathrm{Neut}_{\{\}}(S) \subseteq \mathrm{H}^*\mathrm{Red}^{X,1}_\sigma(S)$.

Proof. By induction on σ:

- if $\sigma = []$, both claims are immediate.
- if $\sigma \in \{o, \mathsf{hn}, \mathsf{n}\}$, claim (i.) holds by definition, and claim (ii.) follows from the fact that for all $t \in \mathrm{Neut}_{\{\}}(S)$ and for all $\omega \in S$, $\mathrm{HNV}(\pi_X^\omega(t)) = 1$, so $t \in \mathrm{H}^*\mathrm{Red}^{X,1}_\sigma(S)$.
- if $\sigma = \tau \Rightarrow \rho$ and $t \in \mathrm{H}^*\mathrm{Red}^{X,r}_\sigma(S)$, then, since by IH $x \in \mathrm{H}^*\mathrm{Red}^{X,1}_\tau(S)$, $tx \in \mathrm{H}^*\mathrm{Red}^{X,1}_\tau(S)$, hence for all $\omega \in S$, $\mathrm{HNV}(\pi_X^\omega(t)) \geq \mathrm{HNV}(\pi_X^\omega(tx)) \geq \lceil\rho\rceil \cdot r = \lceil\sigma\rceil \cdot r$.
  Moreover, if $t \in \mathrm{Neut}_{\{\}}(S)$, then for all $S' \subseteq S$, $u \in \mathrm{H}^*\mathrm{Red}^{X,1}_\tau(S')$ and $\omega \in S'$, $tu \in \mathrm{Neut}_{\{\}}(S)$, so by IH $tu \in \mathrm{H}^*\mathrm{Red}^{X,r}_\rho(S')$, and we conclude then $t \in \mathrm{H}^*\mathrm{Red}^{X,r}_\sigma(S)$.
- if $\sigma = \tau \wedge \rho$ and $t \in \mathrm{H}^*\mathrm{Red}^{X,r}_\sigma(S)$, then $t \in \mathrm{H}^*\mathrm{Red}^{X,r}_\tau(S)$ and $t \in \mathrm{H}^*\mathrm{Red}^{X,r}_\rho(S)$, so by IH for all $\omega \in S$, $\mathrm{HNV}(\pi_X^\omega(t)) \geq \max\{r \cdot \lceil\tau\rceil, r \cdot \lceil\rho\rceil\} = r \cdot \max\{\lceil\tau\rceil, \lceil\rho\rceil\} = r \cdot \lceil\sigma\rceil$.
  Moreover, it $t \in \mathrm{Neut}_{\{\}}(S)$, by IH $t \in \mathrm{H}^*\mathrm{Red}^{X,1}_\tau(S)$ and $t \in \mathrm{H}^*\mathrm{Red}^{X,1}_\rho(S)$, whence $t \in \mathrm{H}^*\mathrm{Red}^{X,1}_\sigma(S)$.
- if $\sigma = \mathbf{C}^q\tau$, and $t \in \mathrm{H}^*\mathrm{Red}^{X,r}_{\mathbf{C}^q\tau}(S)$, let $\omega \in S$; then we consider two cases:
  - if $\pi_X^\omega(t)$ (that we can suppose being in $\mathscr{T}$) is a pseudo-value, then $\sum_{w \in \mathrm{H}^*\mathrm{Red}^{X,r}_\tau} \mathscr{D}^1_t(w) \geq q > 0$ implies $\pi_X^\omega(t) \in \mathrm{H}^*\mathrm{Red}^{X,r}_\tau$, so by IH we deduce $\mathrm{HNV}(\pi_X^\omega(t)) \geq r \cdot \lceil\tau\rceil \geq r \cdot q \cdot \lceil\tau\rceil = r \cdot \lceil\sigma\rceil$.
  - if $\pi_X^\omega(t) = \nu a.t'$, then $\sum_{w \in \mathrm{H}^*\mathrm{Red}^{X,r}_\tau} \mathscr{D}^1_t(w) \geq q > 0$ implies, by IH, that $\mu(\{\omega' \in 2^{\mathbb{N}} \mid \pi^{\omega'}(t') \in \mathrm{H}^*\mathrm{Red}^{\emptyset,r}_\tau\}) \geq \mu(\{\omega' \in 2^{\mathbb{N}} \mid \mathrm{HNV}(\pi^{\omega'}(t)) \geq \lceil\tau\rceil r\}) \geq q$, and this implies $\mathrm{HNV}(\pi_X^\omega(t)) \geq q \cdot r \cdot \lceil\tau\rceil = r \cdot \lceil\sigma\rceil$.

  Moreover, if $t \in \mathrm{Neut}_{\{\}}(S)$, then for all $\omega \in S$, $\mathscr{D}^1_{\pi_X^\omega(t)} = \delta_{\pi_X^\omega(t)}$, and since by the IH $t \in \mathrm{H}^*\mathrm{Red}^{X,r}_\tau(S)$, we deduce $\sum_{u \in \mathrm{H}^*\mathrm{Red}^{X,r}_\tau} \mathscr{D}^1_{\pi_X^\omega(t)}(u) = \delta_{\pi_X^\omega(t)}(\pi_X^\omega(t)) = 1 \geq q$, so $t \in \mathrm{H}^*\mathrm{Red}^{X,r}_{\mathbf{C}^q\tau}(S)$.

□

Lemma E.48. *For any non-trivial type σ, and term t name-closed and ν-safe, if $t \in \mathrm{H}^*\mathrm{Red}^{\emptyset,r}_\sigma$, then $\mathrm{HNV}_{\twoheadrightarrow}(t) = 1$.*

Proof. By induction on σ:

- if $\sigma \in \{o, \mathsf{hn}, \mathsf{n}\}$, from $t \in \mathrm{H}^*\mathrm{Red}^{\emptyset,r}_\sigma$ we deduce $\mathrm{HNV}_{\twoheadrightarrow}(t) \geq r > 0$, and since for all u such that $t \twoheadrightarrow^*_{\{\}h} u$, $\mathscr{D}_u = \delta_u$, we deduce that $\sup\{\mathrm{HNV}(u) \mid t \twoheadrightarrow^*_{\{\}h} u\} = \sup\{\sum_{v \in \mathrm{HNV}^{\{\}}} \delta_u(v) \mid t \twoheadrightarrow^*_{\{\}h} u\} > 0$, this implies that t reduces to a HNV, whence $\mathrm{HNV}_{\twoheadrightarrow}(t) = 1$.
- if $\sigma = \tau \Rightarrow \rho$ and $t \in \mathrm{H}^*\mathrm{Red}^{\emptyset,r}_\sigma$ then since by Lemma E.47 $x \in \mathrm{H}^*\mathrm{Red}^{\emptyset,1}_\tau$, $tx \in \mathrm{H}^*\mathrm{Red}^{\emptyset,r}_\rho$; suppose $tx \twoheadrightarrow^*_{\{\}h} \nu a.t'$; then since t is ν-safe, the only possibility is that $t \twoheadrightarrow^*_{\{\}h} (\lambda x.u)$ and $u \twoheadrightarrow^*_{\{\}h} \nu a.t'$; but then $t \twoheadrightarrow^*_{\{\}h} \lambda x.\nu a.t' \twoheadrightarrow_{\{\}h} \nu a.\lambda x.t'$, against the hypothesis. We conclude that also tx must be ν-safe, so by IH $\mathrm{HNV}_{\twoheadrightarrow}(tx) = 1$, whence $\mathrm{HNV}_{\twoheadrightarrow}(t) = 1$.
- if $\sigma = \tau \wedge \rho$ and $t \in \mathrm{H}^*\mathrm{Red}^{\emptyset,r}_\sigma$, then $t \in \mathrm{H}^*\mathrm{Red}^{\emptyset,r}_\tau$, so by IH, $\mathrm{HNV}_{\twoheadrightarrow}(t) = 1$.
- if $\sigma = \mathbf{C}^s\tau$ and $t \in \mathrm{H}^*\mathrm{Red}^{\emptyset,r}_\sigma$, then since $\mathscr{D}_t = \delta_t$, from $\sum_{w \in \mathrm{H}^*\mathrm{Red}^{\emptyset,r}_\tau} \delta_t(w) \geq s > 0$, it follows that $t \in \mathrm{H}^*\mathrm{Red}^{\emptyset,r}_\tau$, so by IH $\mathrm{HNV}_{\twoheadrightarrow}(t) = 1$

□

Lemma E.49. *If $t \in \mathrm{H}^*\mathrm{Red}^{X,q}_{\sigma\Rightarrow\tau}(S)$ and $u \in \mathrm{H}^*\mathrm{Red}^{X,1}_{\mathbf{C}^s\sigma}(S)$, where σ is non-trivial, then $\{t\}u \in \mathrm{H}^*\mathrm{Red}^{X,q}_{\mathbf{C}^s\tau}(S)$.*

Proof. Let $\omega \in S$. We distinguish two cases:

- $\pi_X^\omega(u) \twoheadrightarrow^*_{\{\}h} \nu a.u'$; then, letting $W_\sigma := \mathrm{H}^*\mathrm{Red}^{X,1}_\sigma(S)$ and $W_\tau := \mathrm{H}^*\mathrm{Red}^{X,q}_\tau(S)$, we have that $\sum_{w \in W} \mathscr{D}^1_{\pi_X^\omega(\nu a.tu')}(w) = \sum_{w \in W} \mu(\{\omega' \mid \pi^{\omega+\omega'}_{X\cup\{a\}}(tu') = w\} = \mu\{\omega' \mid \pi^{\omega+\omega'}_{X\cup\{a\}}(tu') \in W\} = \mu\{\omega' \mid t\pi^{\omega+\omega'}_{X\cup\{a\}}(u') \in W\} \geq \mu\{\omega' \mid \pi^{\omega+\omega'}_{X\cup\{a\}}(u') \in U\} = \sum_{w \in U} \mathscr{D}^1_{\nu a.u}(w) \geq s$, which proves that $\nu a.tu' \in W_\tau$. Since $\{t\}u \twoheadrightarrow^*_{\{\}h} \nu a.tu'$, we conclude then $\{t\}u \in W_\tau$ from Lemma E.42.
- $\pi_X^\omega(u)$ is ν-safe; then from $\pi_X^\omega(u) \in \mathrm{H}^*\mathrm{Red}^{\emptyset,q}_\sigma$ we deduce, by Lemma E.48, that $\mathrm{HNV}_{\twoheadrightarrow}(\pi_X^\omega(u)) = 1$. We deduce then that $\{t\}u \in \mathrm{Neut}_{\{\}}(S)$, and by Lemma E.47 we conclude $\{t\}u \in \mathrm{H}^*\mathrm{Red}^{X,q}_{\mathbf{C}^s\sigma}(S)$.

□

Proposition E.50. *If* $\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathfrak{s}$ *is derivable in* $\mathsf{C}\lambda^{\{\}}_{\rightarrow}$, *where* $\Gamma = \{x_1 : \mathfrak{s}_1, \dots, x_m : \mathfrak{s}_n\}$, *then for all* $S \subseteq [\![\mathscr{b}]\!]_X$, *for all* $u_i \in \mathrm{H}^*\mathrm{Red}^{X,1}_{\mathfrak{s}_i}(S)$, $t[u_1/x_1, \dots, u_m/x_m] \in \mathrm{H}^*\mathrm{Red}^{X,1}_{\mathfrak{s}}(S)$.

Proof. We can argue as in the proof of Proposition E.35, with the following two new cases:

- if $t = \{t_1\}t_2$ and the last rule is

$$\frac{\Gamma \vdash^X t_1 : \mathscr{c} \rightarrowtail \mathbf{C}^{\vec{q}}(\mathfrak{s} \Rightarrow \sigma) \qquad \Gamma \vdash^X t_2 : \mathscr{d} \rightarrowtail \mathbf{C}^s\mathfrak{s} \qquad \mathscr{b} \vDash \mathscr{c} \wedge \mathscr{d}}{\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathbf{C}^s\mathbf{C}^{\vec{q}}\sigma} (\{\})$$

  then by IH for all $u_i \in \mathrm{H}^*\mathrm{Red}^{X,1}_{\mathfrak{s}_i}(S)$, $t_1[u_1/x_1, \dots, u_n/x_n] \in \mathrm{H}^*\mathrm{Red}^{X,1}_{\mathrm{C}^{\vec{q}}(\mathfrak{s}\Rightarrow\sigma)}(S)$ and $t_2[u_1/x_1, \dots, u_n/x_n] \in \mathrm{H}^*\mathrm{Red}^{X,1}_{\mathrm{C}^s\mathfrak{s}}(S)$; then by Lemma E.45 and Lemma E.49 (observing that $\mathfrak{s}$ is non-trivial) we deduce
  $(\{t_1\}t_2)[u_1/x_1, \dots, u_n/x_n] = \{t_1[u_1/x_1, \dots, u_n/x_n]\}(t_2[u_1/x_1, \dots, u_n/x_n]) \in \mathrm{H}^*\mathrm{Red}^{X,1}_{\mathrm{C}^{\vec{q}}\sigma}(S)$.
- If $t = \nu a.u$ and the last rule is

$$\frac{\Gamma \vdash^{X\cup\{a\}} u : \mathscr{c} \wedge \mathscr{d} \rightarrowtail \mathbf{C}^{q_j}\sigma \qquad \vDash \mu(\mathscr{d}) \geq r \qquad \mathscr{b} \vDash \mathscr{c}}{\Gamma \vdash^X \nu a.u : \mathscr{b} \rightarrowtail \mathbf{C}^r\sigma} (\mu)$$

  where $a \notin FV(\Gamma) \cup FV(\mathscr{c})$, then let $S \subseteq [\![\mathscr{b}]\!]_X \subseteq [\![\mathscr{c}]\!]_X$ and $u_1 \in \mathrm{H}^*\mathrm{Red}^{X,1}_{\mathfrak{s}_1}(S), \dots u_n \in \mathrm{H}^*\mathrm{Red}^{X,1}_{\mathfrak{s}_n}(S)$. Let $T = \{g + f \mid \omega \in S\} \subseteq (2^{\mathbb{N}})^{X\cup\{a\}}$, which is measurable since counter-image of a measurable set through the projection function; then since $\mathsf{FN}(u_i) \subseteq X$, we deduce using Lemma E.22 that $u_i \in \mathrm{H}^*\mathrm{Red}^{X\cup\{a\},1}_{\mathfrak{s}_i}(T \cap [\![\mathscr{d}_j]\!]_{X\cup\{a\}})$. By IH and the hypothesis we deduce then that
  - $u[u_1/x_1, \dots, u_n/x_n] \in \mathrm{H}^*\mathrm{Red}^{X\cup\{a\},1}_{\mathfrak{s}}(T \cap [\![\mathscr{d}_j]\!]_{X\cup\{a\}})$;
  - for all $\omega \in S$, $\mu(\Pi^\omega(T \cap [\![\mathscr{d}_j]\!]_{X\cup\{a\}})) \geq r$;

  Hence, by Lemma E.43 we conclude that $\nu a.u[u_1/x_1, \dots, u_n/x_n] = (\nu a.u)[u_1/x_1, \dots, u_n/x_n] \in \mathrm{H}^*\mathrm{Red}^{X,\sum_j r_j q_j}_{\mathfrak{s}}(S) = \mathrm{H}^*\mathrm{Red}^{X,1}_{\mathrm{C}^r\mathfrak{s}}(S)$.

□

We can now deduce Theorem 6.1 from Proposition E.50 as in the case of $\mathsf{C}\lambda_{\rightarrow,\cap}$.

Theorem E.51. *If* $\Gamma \vdash^X t : \mathscr{b} \rightarrowtail \mathfrak{s}$, *then for all* $\omega \in [\![\mathscr{b}]\!]$, $\mathrm{HNV}_{\rightarrow}(\pi^\omega_X(t)) \geq \lceil\sigma\rceil$.