Towards Logical Foundations for Probabilistic Computation

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Abstract

The overall purpose of the present work is to lay the foundations for a new approach to bridge logic with probabilistic computation. To this aim we introduce extensions of classical and intuitionistic propositional logic with *counting quantifiers*, that is, quantifiers that measure *to which extent* a formula is true. The resulting systems, called CPL and iCPL, respectively, admit a natural semantics, based on the Borel σ-algebra of the Cantor space, together with a sound and complete proof system. Our main results relate CPL and iCPL with some central concepts in the study of probabilistic computation. On the one hand, the validity of CPL-formulas in prenex form characterizes the corresponding level of the counting hierarchy, a hierarchy of counting complexity classes closely related to probabilistic complexity. On the other hand, proofs in iCPL correspond, in the sense of Curry and Howard, to typing derivations for a randomized extension of the λ-calculus, so that counting quantifiers reveal the probability of termination of the underlying probabilistic programs.

Keywords: Probabilistic Computation, Propositional Logic, Counting Quantifiers, Counting Complexity, Typed λ-Calculi

Contents

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1. Introduction

Among the many intriguing relationships existing between logic and computer science, we must certainly mention those between classical propositional logic, on the one hand, and computational complexity, the theory of programming languages, and many other branches of computer science on the other. For instance, it is well-known that classical propositional logic provided the first example of an NP-complete problem [15], while the *Curry-Howard correspondence* unveiled a fundamental correspondence between type systems for abstract functional languages and proof calculi for numerous constructive logics, e.g. intuitionistic and linear logic [33, 67].

These lines of research evolved in various directions, resulting in active subareas of computer science. In particular, variations of propositional logic have been put in relation with complexity classes other than P and NP and with type systems other than simple types. For example, the complexity of deciding *quantified* propositional logic formulas is known to correspond to the polynomial hierarchy [47, 48, 69, 80, 13]. On the other hand, proof systems for propositional *linear* logic or *bunched* logic have inspired resource-conscious type systems in which duplication and sharing are taken into account and appropriately dealt with through type systems [54, 73].

Nevertheless, there is at least one crucial aspect of the theory of computation which has only been marginally touched by these fruitful interactions, namely *probabilistic* computation. Indeed, in spite of the appearance in the literature of several and well-studied abstract models for probabilistic computations (e.g. probabilistic automata [60], Turing machines [63, 30], and λ-calculi [62]), these seem not to have yet found a precise logical counterpart. Such a missing connection looks even more striking nowadays, due to the increasing pervasiveness of probabilistic models in several fast-growing areas of computer science like statistical learning or approximate computing.

When considering probabilistic algorithms, behavioral properties like termination or equivalence have an inherently *quantitative* nature: any computation terminates *with a given probability*, and a program might simulate a desired function *up to* some probability of error (e.g. think of probabilistic primality tests or learning algorithms). Can such quantitative properties be studied within a logical system?

Recent work by the authors [4, 2] suggests that a fruitful strategy to express properties of probabilistic programs consists in enriching logical languages with suitable *measure-quantifiers* (see [51]). A somehow compelling example comes from [4], where a generalization to the probabilistic setting of Gödel's well-known representation theorem for (deterministic) recursive functions is obtained by extending the usual language $\mathscr{L}$ of first-order arithmetics with this kind of quantifiers. Let $\mathscr{L}_{\mathsf{FLIP}}$ indicate the extension of $\mathscr{L}$ with a unary predicate symbol $\mathsf{FLIP}(\cdot)$, intuitively standing for a *random sample* from the Cantor space $\{0,1\}^{\mathbb{N}}$, and a quantifier $\mathbf{C}^{t/u}A$, intuitively meaning: "A is true with probability greater than t/u". The function computed by a probabilistic Turing Machine can be described as some $f : \mathbb{N}^k \to \mathbb{D}(\mathbb{N})$, where $\mathbb{D}(\mathbb{N})$ indicates the set of *sub-distributions on* $\mathbb{N}$, i.e. of functions $\mu : \mathbb{N} \to [0,1]$ such that $\sum_{n\in\mathbb{N}} \mu(n) \leq 1$. This leads then to the following:

Theorem 1 ([4])**.** *If $f : \mathbb{N}^k \to \mathbb{D}(\mathbb{N})$ is the function computed by some probabilistic Turing Machine, then there exists a Σ^0_1-formula A_f of $\mathscr{L}_{\mathsf{FLIP}}$ such that for all $n_1, \dots, n_k, m, p, q \in \mathbb{N}$,*

$$f(n_1, \dots, n_k)(m) \geq p/q \qquad \Leftrightarrow \qquad \mathbb{N} \vDash \mathbf{C}^{\overline{p}/\overline{q}} A_f(\overline{n_1}, \dots, \overline{n_k}, \overline{m}).$$

What happens if we try to introduce similar quantifications in standard *propositional* logic? When A is a formula of classical logic (say $A = \boldsymbol{x} \vee (\neg \boldsymbol{y})$), measuring the probability that A is valid actually amounts to *counting*, among the finitely many valuations of the variables of the formula, those which satisfy the formula (in our example, three out of four), and thus to a purely recursive operation. This suggests that measure quantifiers for classical propositional logic may lead to express *counting problems*, i.e. problems in which one does not ask *if* a machine will accept some input, but *how many times* it will do so. As recalled below, such problems are deeply related to the study of probabilistic complexity classes such as e.g. PP. Moreover, just as a proof of A in intuitionistic propositional logic can be seen, in the light of the Curry-Howard correspondence, as some program of type A, measure quantifiers for intuitionistic propositional logic can be used to express that some program involving probabilistic choice has type A *with a certain probability*.

In this paper these ideas will be developed by demonstrating that the extension of classical and intuitionistic propositional logic with counting quantifiers provides ways to relate logic with probabilistic computation. More precisely, to bridge classical logic with *counting complexity*, and intuitionistic logic with *probabilistic functional programming*. Our contribution is threefold:

- We introduce *classical* and *intuitionistic counting propositional logic* (CPL and iCPL, for short), and for each of these logics we provide a semantics based on the Borel σ-algebra of the Cantor space, as well as a sound and complete proof system.
- We show that the validity of CPL-formulas in (a special kind of) prenex normal form characterizes the corresponding level of the *counting hierarchy* [75, 77, 76], a hierarchy of complexity classes related to counting problems, as $\sharp$`SAT`, `MajSAT` and `MajMajSAT`, and tightly related to probabilistic complexity classes like PP [31, 72, 64].
- We show that derivations in iCPL can be decorated with terms of a probabilistic extension of the λ-calculus (inspired from [20]). We describe in this way an extension of the usual Curry-Howard correspondence between intuitionistic logic and typed λ-calculi, in which counting quantifiers *reveal* the actual probability of termination, and proof normalization describes well-known evaluation strategies of probabilistic functional programs.

2. From Logic to Counting and Probability: a Roadmap

This section provides an overview of the different ideas developed in this paper. In particular, we provide a first sketch of how counting quantifiers can be used to capture ideas from computational complexity and programming language theory from a logical perspective.

2.1. From Counting Quantifiers to the Counting Hierarchy

As is well-known, checking the satisfiability of a formula of classical propositional logic, **PL**, represents the paradigmatic NP-complete problem, while determining whether any such formula is a tautology is, dually, coNP-complete. Furthermore, these two classes can be captured uniformly by switching to *quantified* propositional logic [47, 48, 69, 80, 13]. Indeed, satisfiability corresponds to the truth of closed, existentially quantified formulas in the form $\exists x_1 \ldots \exists x_n A$ (where A is quantifier-free), while validity stands for the truth of closed, universally quantified formulas in the form $\forall x_1 \ldots \forall x_n A$ (where, again, A is quantifier-free). For example, the formula,

$$\exists x_1 \exists x_2 \exists x_3 \big((x_1 \wedge \neg x_2) \vee (x_2 \wedge \neg x_3) \vee (x_3 \wedge \neg x_1)\big)$$

expresses that $(x_1 \wedge \neg x_2) \vee (x_2 \wedge \neg x_3) \vee (x_3 \wedge \neg x_1)$ is satisfied by *at least* one model. In this way, checking the validity of quantified formulas provides complete problems for the whole *polynomial hierarchy* (PH, for short), so that each level of the hierarchy is characterized by the number of alternations in the given quantified propositional formulas (in prenex normal form). In such correspondence, universal and existential quantifications basically play the role of the *acceptance condition* in the (non-deterministic) machines defining the corresponding complexity class, i.e. they correspond to requiring that either *at least one* computation path or *all* paths of the machine is/are accepting one/ones.

A somehow natural question which may arise in this context is: what if other kinds over quantifications of computation paths replace universal and existential quantifications? For instance, the complexity class PP concerns problems which are computable by a polytime probabilistic Turing Machine such that the produced answer is correct for *at least half* of its accepting paths (and similarly for the non-accepting states). Interestingly, a complete problem for PP can still be expressed in terms of **PL**-formulas: this is the problem `MajSAT`, i.e. the problem of checking if a formula of **PL** is true in *at least half* of its models. Moreover, PP related to $\mathsf{P}^{\sharp\mathtt{SAT}}$, the class of counting problems associated with the decision problems in NP, since $\mathsf{P}^{\mathsf{PP}} = \mathsf{P}^{\sharp\mathtt{SAT}}$ (and both these classes contain the whole polynomial hierarchy, by Toda's Theorem [70, 71]).

Starting from these classes and in analogy with PH, in 1984/86 Wagner defined the so-called *counting hierarchy* (CH, for short) [75, 77, 76], given by

$$\mathsf{CH}_0 = \mathsf{P} \quad \text{and} \quad \mathsf{CH}_{n+1} = \mathsf{PP}^{\mathsf{CH}_n}.$$

A typical problem belonging to this hierarchy (in fact, one which is complete for $\mathsf{CH}_2 = \mathsf{PP}^{\mathsf{PP}}$) is `MajMajSAT`, the problem of determining, given a **PL**-formula A containing two disjoint sets $\mathbf{x}$ and $\mathbf{y}$ of variables, whether it holds that for the majority of the valuations of $\mathbf{x}$, the majority of the valuations of $\mathbf{y}$ makes A true.

Going back to logic, one may naturally wonder whether it is possible to capture complete problems for the levels of this hierarchy by means of suitable measure quantifiers on propositional formulas. At a first glance, this question could even appear as meaningless as the variables one would like to quantify over only take *two* possible values. The perspective changes when allowing to simultaneously quantify over all the propositional variables which are free in the argument formula or, otherwise said, when the interpretation of a formula is no more interpreted as the single truth-value that it gets in a given valuation, but as the set of *all* valuations making it true.

Indeed, letting $[\![A]\!]$ indicate such a set, we can interpret a quantified formula of the form $\mathbf{C}^r A$ as saying that A is true in *at least a fraction* $r \cdot 2^n$ of the possible 2^n assignments of its n variables. For instance, we can assert that the formula below,

$$\mathbf{C}^{1/2}\big((x_1 \wedge \neg x_2) \vee (x_2 \wedge \neg x_3) \vee (x_3 \wedge \neg x_1)\big)$$

is true, as $(x_1 \wedge \neg x_2) \vee (x_2 \wedge \neg x_3) \vee (x_3 \wedge \neg x_1)$ is satisfied in at least 4 (actually, in 6) of its $2^3 = 8$ models. By the way, the aforementioned counting quantifier is very reminiscent of operators on classes of languages which Wagner introduced in his seminal works on the counting hierarchy [75, 77].

Following these insights, in Section 3 we introduce an extension of classical propositional logic with counting quantifiers, called CPL. In the formulas of CPL propositional variables are grouped into disjoint sets using *names*, $a, b, c, \dots$, and counting quantified formulas are of the form $\mathbf{C}^q_a A$, with the intuitive meaning that "A is true in at least $q \cdot n$ of the n valuations of the variables of name a". In this way, a problem in `MajMajSAT` can be captured by formulas of CPL of the form $\mathbf{C}^q_a \mathbf{C}^r_b A$. Then, in Section 4, we provide a characterization of the full counting hierarchy using CPL by showing that: (1) any formula of CPL is equivalent to one in the special *prenex normal form* $\mathbf{C}^{q_1}_{a_1} \dots \mathbf{C}^{q_k}_{a_k} A$, where A is quantifier-free, and (2) prenex formulas with k nested counting quantifiers characterize the level k of Wagner's counting hierarchy.

2.2. Curry and Howard Meet Borel

Among the many ways in which mathematical logic influenced programming language theory, the so-called Curry-Howard correspondence is certainly among the most intriguing and meaningful ones. Traditionally, the correspondence, identified by Curry [17] and formalized by Howard [40], (CHC in the following) relates propositional intuitionistic logic and the simply-typed λ-calculus. It is well-known, though, that the correspondence holds in other contexts, too. Indeed, in the last fifty years more sophisticated type systems have been put in relation with logical formalisms: from polymorphism [32, 34] to various forms of session typing [74, 14], from control operators [56] to dependent types [46, 66].

Nevertheless, for languages with probabilistic effects, a similar CHC has not yet been defined. Type-theoretic accounts for such languages have recently been put forward in various ways, e.g. type systems based on sized types [19], intersection types [11] or type systems in the style of so-called amortized analysis [78]. Nevertheless, in all the aforementioned cases, a type system was built by modifying well-known type systems for deterministic languages *without* being guided by logic, and, instead, incepting quantitative concepts directly from probability theory. Then, is there any logical system behind all this? What kind of logic could possibly play the role of **PL** in suggesting meaningful and expressive type systems for a λ-calculus endowed with probabilistic choice effects?

Before giving an answer to this question, we should first ask ourselves what kind of programs we are actually dealing with. By a probabilistic functional program we mean here a functional program with the additional ability of sampling from some distributions, or to or of performing some form of discrete probabilistic choice.[4] This has a couple of crucial consequences: program evaluation becomes an essentially stochastic process, while programs satisfy a given specification *up to* a certain probability. As an example, consider the λ-term,

$$\Xi_{\mathsf{half}} := \lambda x.\lambda y.x \oplus y$$

[4]Here we are not concerned with sampling from continuous distributions, nor with capturing any form of conditioning.

where $\oplus$ is a binary infix operator for *fair* probabilistic choice. When applied to two arguments t and u, the evaluation of Ξ_{half} results in either t, with probability one half, or in u, again with probability one half. Now, if we try to take Ξ_{half} as a proof of, say, a propositional logic formula, we see that this system is simply not rich enough to capture the behavior above. Indeed, given that Ξ_{half} is a function of two arguments, it is natural to see it as a proof of an implication $A \to B \to C$, namely (following the BHK-interpretation) as a function turning a proof of A and a proof of B into a proof of C. What is C, then? Should it be A or should it be B? Actually, it could be both, with some degree of uncertainty, but propositional logic is not able to express all this. What seems to be lacking in intuitionistic logic is precisely a way to express that C could be $\mathbf{C}^{1/2}A$, namely that it should be A *with probability at least* $1/2$ and, similarly, that C could be $\mathbf{C}^{1/2}B$, which is that it is B still with probability greater or equal than $1/2$.

In Section 5 we introduce an intuitionistic version of CPL, called iCPL, which enriches intuitionistic logic with Boolean variables as well as the counting quantifier $\mathbf{C}^q$. Intuitively, if a proof of a formula A can be seen as a deterministic program satisfying the specification A, a proof of $\mathbf{C}^q A$ corresponds to a *probabilistic* program that satisfies the specification A with probability q. Then, in Section 6, we show that proofs in iCPL correspond to (families of) functional probabilistic programs and, most importantly, that proof normalization describes the evaluation of such programs.

2.3. Preliminaries on the Cantor Space

Before proceeding to the actual technical content of this article, we recall some basic facts about the Borel σ-algebra of the Cantor space, and we introduce the class of *named Boolean formulas*, that will be used throughout the text as a language for *finitary* Borel sets.

We consider a countably infinite set $\mathscr{A}$ of *names*, noted $a, b, c, \dots$. For any finite subset $X \subseteq \mathscr{A}$, we let $\mathbf{B}_X$ (resp. $\mathbf{B}_{\mathscr{A}}$) indicate the *Borel σ-algebra* on the X-th product of the Cantor space $(2^{\mathbb{N}})^X$ $\big($resp. on the $\mathscr{A}$-th product $(2^{\mathbb{N}})^{\mathscr{A}}\big)$, that is the smallest σ-algebra containing all open sets under the product topology. There exists a unique measure μ of $\mathbf{B}_{\mathscr{A}}$ such that $\mu(\text{Cylinder}(a,i)) = 1/2$ for all *cylinders* $\text{Cylinder}(a,i) = \{\omega \mid \omega(a)(i) = 1\}$. μ restricts to a measure μ_X on $\mathbf{B}_X$ by letting $\mu_X(S) = \mu\big(S \times (2^{\mathbb{N}})^{\mathscr{A}-X}\big)$. For any Borel set $S \in \mathbf{B}_{X\cup Y}$ and $\omega \in (2^{\mathbb{N}})^X$, let

$$\Pi_\omega(S) = \{\omega' \in (2^{\mathbb{N}})^Y \mid \omega + \omega' \in S\} \subseteq (2^{\mathbb{N}})^X.$$

Notice that the set $\Pi_\omega(S)$ needs not be Borel. However, since it is an *analytic* set (see [43]) one can show that its Lebesgue measure is always well-defined. Moreover, we will make use in several places of the following fact:

Lemma 1 ([43], Theorem 14.11 + Theorem 29.26)**.** *For any $S \in \mathbf{B}_{X\cup Y}$, with $X \cap Y = \emptyset$, and $r \in [0,1]$, $\{\omega \in (2^{\mathbb{N}})^X \mid \mu(\Pi_\omega(S)) \geq r\} \in \mathbf{B}_X$.*

Boolean formulas with names in $\mathscr{A}$ are defined by the grammar below:

$$\mathscr{b} ::= \top \mid \bot \mid \boldsymbol{x}_a^i \mid \neg\mathscr{b} \mid \mathscr{b} \wedge \mathscr{b} \mid \mathscr{b} \vee \mathscr{b}$$

where $a \in \mathscr{A}$ and $i \in \mathbb{N}$. We let $\mathrm{FN}(\mathscr{b}) \subseteq \mathscr{A}$ be the set of names occurring in $\mathscr{b}$. For all Boolean formula $\mathscr{b}$ and $X \supseteq \mathrm{FN}(\mathscr{b})$, we let $[\![\mathscr{b}]\!]_X$ indicate the Borel set $\{\omega \in (2^{\mathbb{N}})^X \mid \omega \vDash \mathscr{b}\}$. The value $\mu_X([\![\mathscr{b}]\!]_X) \in [0,1] \cap \mathbb{Q}$ is independent from the choice of $X \supseteq \mathrm{FN}(\mathscr{b})$, and we will denote it simply as $\mu(\mathscr{b})$.

3. Classical Counting Propositional Logic

In this section we introduce the logic CPL together with its semantics and proof-theory. Most of the material in this section has appeard in [2, 3]. The main novelty presented here is a sound and complete natural deduction system for CPL.

3.1. On Counting Quantifiers and Their Semantics

In standard propositional logic, the interpretation of a formula A containing propositional atoms $p_1, \dots, p_n$ is a truth-value $[\![A]\!]_v$ depending on a valuation $v : \{p_1, \dots, p_n\} \to 2$ (where $2 = \{0, 1\}$). Yet, what if one defined the semantics of A as consisting of *all* valuations making A true? Since propositional formulas can have an arbitrary number of propositional values, valuations can be described more generally as functions $\omega \in 2^{\mathbb{N}}$. Hence, a formula, A, may be interpreted as the set $[\![A]\!] \subseteq 2^{\mathbb{N}}$ made of all $\omega \in 2^{\mathbb{N}}$ "making A true".

In fact, it is not difficult to see that such sets are *measurable sets* of the standard Borel σ-algebra $\mathcal{B}(2^{\mathbb{N}})$. An atomic proposition p_i is interpreted as the *cylinder set* [9], $\{\omega \in 2^{\mathbb{N}} \mid \omega(i) = 1\}$. The interpretation of non-atomic propositions is provided by the standard σ-algebra operations of complementation, finite intersection and finite union. The main novelty concerns counting-quantified formulas. First, we define two counting quantifiers, as inspired by Wagner's notion of counting operator over languages [75, 77, 76]. Observe that, as discussed in Section 2, the counting problems coming from **PL**, such as `MajMajSAT`, involve the relation between valuations of *different* groups of variables. To this end, we consider *named* logical atoms of the form $\boldsymbol{x}_a^i$, as well as named quantifiers of the forms $\mathbf{C}_a^q A$, $\mathbf{D}_a^q A$. Intuitively, named quantifiers count models relative to the variables of name a. So, given a formula A and a group of variables in it (that we can suppose to be those of name a), it becomes possible to express that A is satisfied by, e.g. *at least half* or by *strictly less than half* of the valuations of the variables of this group.

Definition 1 (CPL-Formulas)**.** *The* formulas *of* CPL *are defined by the grammar below:*

$$A, B := \top \;\Big|\; \bot \;\Big|\; \boldsymbol{x}_a^i \;\Big|\; \neg A \;\Big|\; A \wedge B \;\Big|\; A \vee B \;\Big|\; \mathbf{C}_a^q A \;\Big|\; \mathbf{D}_a^q A$$

where $a \in \mathscr{A}$*,* $i \in \mathbb{N}$ *and* $q \in [0,1] \cap \mathbb{Q}$*.*

The name a is considered bound in the formulas $\mathbf{C}^q_a A$ and $\mathbf{D}^q_a A$. We denote the set of names which occur *free* in a formula A, as $\mathrm{FN}(A)$.

While for any formula A of classical propositional logic, the Borel set $[\![A]\!]$ can be defined in a straightforward way, when A is a formula of the form $\mathbf{C}^q_a A$ or $\mathbf{D}^q_a A$, the definition of the Borel set $[\![A]\!]$ is slightly subtler and relies on the construction of Lemma 1 from Section 2.

Definition 2. *For any formula A of* CPL *and finite set of names $X \supseteq \mathrm{FN}(A)$, the set $[\![A]\!]_X$ is inductively defined as follows:*

$$\begin{aligned}
[\![\top]\!]_X &= (2^{\mathbb{N}})^X \\
[\![\bot]\!]_X &= \emptyset \\
[\![\boldsymbol{x}^i_a]\!]_X &= \mathrm{Cylinder}(a, i) \\
[\![\neg A]\!]_X &= (2^{\mathbb{N}})^X \ - \ [\![A]\!] \\
[\![A \wedge B]\!]_X &= [\![A]\!]_X \cap [\![B]\!]_X \\
[\![A \vee B]\!]_X &= [\![A]\!]_X \cup [\![B]\!]_X \\
[\![\mathbf{C}^q_a A]\!]_X &= \{\omega \mid \mu(\Pi_\omega([\![A]\!]_{X \cup \{a\}})) \geq q\} \\
[\![\mathbf{D}^q_a A]\!]_X &= \{\omega \mid \mu(\Pi_\omega([\![A]\!]_{X \cup \{a\}})) < q\}.
\end{aligned}$$

For any formula A, the set $[\![A]\!]_X \in \mathcal{B}\big((2^{\mathbb{N}})^X\big)$ is Borel: the definition above uses basic operations of the Borel σ-algebra, and exploits Lemma 1 for the case of quantifiers. Lemma 3 below provides a more direct characterization of the sets $[\![\mathbf{C}^q_a A]\!]_X$ and $[\![\mathbf{D}^q_a A]\!]_X$, from which the fact that they are Borel can be deduced without relying on Lemma 1.

To have a grasp of the semantics of named quantifiers, observe that if all variables of A have name a, then $[\![\mathbf{C}^q_a A]\!]_X$ will be the full set $(2^{\mathbb{N}})^X$ if $\mu([\![A]\!]_{\{a\}}) \geq q$, and the empty set otherwise. In case the variables of A are grouped by different names, things are slightly subtler. For instance, consider the formula:

$$A := (\boldsymbol{x}^0_a \wedge (\neg \boldsymbol{x}^0_b \wedge \boldsymbol{x}^1_b)) \vee (\neg \boldsymbol{x}^0_a \wedge \boldsymbol{x}^0_b \wedge \neg \boldsymbol{x}^1_b) \vee ((\neg \boldsymbol{x}^0_a \vee \boldsymbol{x}^1_a) \wedge \boldsymbol{x}^1_b).$$

The valuations $\omega \in (2^{\mathbb{N}})^{\{b\}}$ belonging to $[\![\mathbf{C}^{1/2}_a A]\!]_{\{b\}}$ are those which can be extended to models of A in at least $1/2$ of the cases. In this simple case we can enumerate them exhaustively:

1. if $\omega(b)(0) = \omega(b)(1) = 1$, then A has $1/4$ chances of being true (since both $\neg \boldsymbol{x}^0_a$ and $\boldsymbol{x}^1_a$ must be true)
2. if $\omega(b)(0) = 1, \omega(b)(1) = 0$, then A has $1/2$ chances of being true (since $\neg \boldsymbol{x}^0_a$ must be true)
3. if $\omega(b)(0) = 0, \omega(b)(1) = 1$, then A has $3/4$ chances of being true (since either $\boldsymbol{x}^0_a$ or both $\neg \boldsymbol{x}^0_a$ and $\boldsymbol{x}^1_a$ must be true)
4. if $\omega(b)(0) = \omega(b)(1) = 0$, then A has 0 chances of being true.

Thus, $[\![\mathbf{C}^{1/2}_a A]\!]_{\{b\}}$ only contains the valuations which agree with cases 2. and 3. We conclude that $[\![\mathbf{C}^{1/2}_b \mathbf{C}^{1/2}_a A]\!]_\emptyset = 2^{\mathbb{N}}$, i.e. that $\mathbf{C}^{1/2}_b \mathbf{C}^{1/2}_a A$ is valid, since half of the valuations of b has at least $1/2$ chances of being extendable to a model of A.

Notice that $\mathbf{C}$ and $\mathbf{D}$, contrarily to $\forall$ and $\exists$, are *not* dual operators in the same sense in which, say, $\Box$ is dual to $\Diamond$ in modal logic. Indeed, $\neg\mathbf{D}^q_a A$ is not equivalent to $\mathbf{C}^q_a \neg A$. In fact, the following equivalences hold:

$$\neg\mathbf{D}^q_a A \equiv \mathbf{C}^q_a A \qquad\qquad \neg\mathbf{C}^q_a A \equiv \mathbf{D}^q_a A$$

In order to prove an alternative characterization of the sets $[\![\mathbf{C}^q_a A]\!]_X$ we introduce *a-decompositions* of Boolean formulas:

Definition 3. *Let b be a named Boolean formula with free names in $X \cup \{a\}$ (with $a \notin X$). An* a-decomposition *of b is a Boolean formula $c = \bigvee_{i=0}^{k-1} d_i \wedge e_i$ such that:*

- $[\![c]\!]_{X\cup\{a\}} = [\![b]\!]_{X\cup\{a\}}$
- $\mathrm{FN}(d_i) \subseteq \{a\}$ *and* $\mathrm{FN}(e_i) \subseteq X$
- *if $i \neq j$, then $[\![e_i]\!]_X \cap [\![e_j]\!]_X = \emptyset$.*

The proof of Lemma 2 below is postponed to Appendix A.1.

Lemma 2. *Any named Boolean formula b with $\mathrm{FN}(b) \subseteq X \cup \{a\}$ admits an a-decomposition in X.*

It is worth observing that while an a-decomposition of b always exists, finding it is not necessarily easy, since this formula can be of exponential length with respect to b. Yet, a-decompositions can be used to show that the interpretation of a quantified formula is a finite union of measurable sets:

Lemma 3 (Fundamental Lemma)**.** *Let b be a named Boolean formula with $\mathrm{FN}(b) \subseteq X \cup \{a\}$ and $c = \bigvee_{i=0}^{k-1} d_i \wedge e_i$ be an a-decomposition of b. Then, for all $q \in [0,1]$,*

$$\begin{aligned}
\{\omega \in (2^{\mathbb{N}})^X \mid \mu(\Pi_\omega([\![b]\!]_{X\cup\{a\}})) \geq q\} &= \bigcup\{[\![e_i]\!]_X \mid \mu([\![d_i]\!]_{\{a\}}) \geq q\} \\
\{\omega \in (2^{\mathbb{N}})^X \mid \mu(\Pi_\omega([\![b]\!]_{X\cup\{a\}})) < q\} &= \bigcup\{[\![e_i]\!]_X \mid \mu([\![d_i]\!]_{\{a\}}) < q\}.
\end{aligned}$$

Proof. We only prove the first equality, the second one being proved in a similar way. First note that if $r = 0$, then both sets are equal to $(2^{\mathbb{N}})^X$, so we can suppose $r > 0$.

($\subseteq$) Suppose $\mu(\Pi_\omega([\![b]\!]_{X\cup\{a\}})) \geq r$. Then $\Pi_\omega([\![b]\!]_{X\cup\{a\}})$ is non-empty and from $b \equiv \bigvee_i^k d_i \wedge e_i$ we deduce that there exists $i \leq k$ such that $\omega \in [\![e_i]\!]_X$ and for all $\omega' \in [\![d_i]\!]_{\{a\}}$, $\omega + \omega' \in [\![d_i \wedge e_i]\!]_{X\cup\{a\}}$. This implies then that $[\![d_i]\!]_{\{a\}} \subseteq \Pi_\omega([\![b]\!]_{X\cup\{a\}})$. Moreover, since the sets $[\![e_i]\!]_X$ are pairwise disjoint, for all $j \neq i$, $\omega \notin [\![e_j]\!]_X$, which implies that $\Pi_\omega([\![b]\!]_{X\cup\{a\}}) \subseteq [\![d_i]\!]_{\{a\}}$. Hence $\Pi_\omega([\![b]\!]_{X\cup\{a\}}) = [\![d_i]\!]_{\{a\}}$, which implies $\mu([\![d_i]\!]_{\{a\}}) \geq r$.

($\supseteq$) If $\omega \in [\![e_i]\!]_X$, where $\mu([\![d_i]\!]_{\{a\}}) \geq r$, then since $d_i \wedge e_i \vDash^{X\cup\{a\}} b$, $\mu(\Pi_\omega([\![b]\!]_{X\cup\{a\}})) \geq \mu(\Pi_\omega([\![d_i \wedge e_i]\!]_{X\cup\{a\}})) = \mu([\![d_i]\!]_{\{a\}}) \geq r$.

$\Box$

Corollary 1. *For any formula A (with $\mathrm{FN}(A) \subseteq X$), there exists a Boolean formula $\mathscr{b}_A$, such that $[\![A]\!]_X = [\![\mathscr{b}_A]\!]_X$.*

Proof. By induction on A. The crucial steps, concerning $[\![\mathbf{C}^q_a A]\!]_X$ and $[\![\mathbf{D}^q_a A]\!]_X$, are handled using the fact that, thanks to Lemma 3, these sets are finite unions of sets of the form $[\![\mathscr{e}_i]\!]_X$, where $\bigvee_{i=0}^{k-1} \mathscr{d}_i \wedge \mathscr{e}_i$ is an a-decomposition of $\mathscr{b}_A$. $\square$

Then, from Corollary 1 one immediately deduces that the sets $[\![A]\!]_X$ are Borel (hence without relying on Lemma 1).

3.2. The Proof-Theory of CPL

We now introduce a natural deduction system $\mathbf{G}_{\mathsf{CPL}}$ for CPL. For simplicity, and since this is enough to obtain completeness of provability with respect to semantics, we consider a calculus based on judgements containing *exactly one* (labeled) formula. Nevertheless, it is possible to extend $\mathbf{G}_{\mathsf{CPL}}$ to a calculus $\mathbf{G}^{\mathrm{Hyp}}_{\mathsf{CPL}}$ with hypotheses, i.e. with judgements of the form $\Gamma \vdash L$, without altering the main properties of the rule system (see Remark 2 below).

More precisely, the judgements of $\mathbf{G}_{\mathsf{CPL}}$ are of the form $\vdash L$, where L is of one of the two forms $\mathscr{b} \rightarrowtail A$ and $\mathscr{b} \leftarrowtail A$, with A a formula of CPL and $\mathscr{b}$ a Boolean formula. A judgement $\vdash \mathscr{b} \rightarrowtail A$ (resp. $\vdash \mathscr{b} \leftarrowtail A$) is *valid*, noted $\vDash \mathscr{b} \rightarrowtail A$ (resp. $\vDash \mathscr{b} \leftarrowtail A$) when for all $X \supseteq \mathrm{FN}(\mathscr{b})$, $[\![\mathscr{b}]\!]_X \subseteq [\![A]\!]_X$ (resp. $[\![\mathscr{b}]\!]_X \supseteq [\![A]\!]_X$). In other words, the judgement $\vdash \mathscr{b} \rightarrowtail A$ is valid when the set of valuations making $\mathscr{b}$ (seen as a Borel set) true is included in $[\![A]\!]$, and similarly, the judgement $\mathscr{b} \leftarrowtail A$ is valid when for all $X \supseteq \mathrm{FN}(\mathscr{b})$, $[\![A]\!]_X$ is included in the set of valuations making $[\![\mathscr{b}]\!]_X$ true.

The rules of the calculus $\mathbf{G}_{\mathsf{CPL}}$ include semantic conditions, called *external hypotheses*: these are expressions of the form $\mathscr{b} \vDash \mathscr{c}$, $\mu(\mathscr{b}) \geq q$ or $\mu(\mathscr{b}) \leq q$, where $q \in [0,1] \cap \mathbb{Q}$. Otherwise said, proofs in $\mathbf{G}_{\mathsf{CPL}}$ may rely on an oracle to express logical consequence between Boolean formulas as well as to measure Boolean formulas. Notice that the latter is a purely recursive operation, indeed one which is performed in $\mathsf{P}^{\sharp\mathtt{SAT}}$, i.e. it is in the complexity class of problems which are solved in polynomial time with an oracle on the problem $\sharp\mathtt{SAT}(\mathscr{b}) = \sharp\{\text{valuations satisfying } \mathscr{b}\}$:

Lemma 4. *For each Boolean formula $\mathscr{b}$ with n distinct propositional variables,*

$$\mu(\mathscr{b}) = \sharp\mathtt{SAT}(\mathscr{b}) \cdot 2^{-n}.$$

Proof. To any valuation v of the Boolean variables of $\mathscr{b}$ we can associate a measurable set $X(v) \subseteq (2^{\mathbb{N}})^{\mathrm{FN}(\mathscr{b})}$ by letting $X(v) = \{\omega \mid \forall a, i \text{ occurring in } \mathscr{b}, \omega(a)(i) = v(\mathscr{x}^i_a)\} = \bigcap_{i=0}^{n-1} \mathrm{Cylinder}(a,i)^{v(\mathscr{x}^i_a)}$, where $\mathrm{Cylinder}(a,i)^{v(\mathscr{x}^i_a)}$ is $\mathrm{Cylinder}(a,i)$ if $v(\mathscr{x}^i_a) = 1$ and $\overline{\mathrm{Cylinder}(a,i)}$ otherwise. One can check by induction that for any Boolean formula $\mathscr{b}$, $[\![\mathscr{b}]\!] = \bigcup_{v \vDash \mathscr{b}} X(v)$. Then, since for all distinct v, v', $X(v) \cap X(v') = \emptyset$, we deduce $\sharp\mathtt{SAT}(\mathscr{b}) \cdot 2^{-n} = \sum_{v \vDash \mathscr{b}} 2^{-n} = \sum_{v \vDash \mathscr{b}} \mu(X(v)) = \mu(\bigcup_{v \vDash \mathscr{b}} X(v)) = \mu([\![\mathscr{b}]\!])$. $\square$

The rules of $\mathbf{G}_{\mathsf{CPL}}$ are illustrated in Fig. 1. We call *introduction rules* for a n-ary connective c those whose conclusion is of the form $\vdash \mathbb{b} \rightarrowtail \mathsf{c}(A_1, \dots, A_n)$, and *elimination rules* those whose conclusion is of the form $\vdash \mathbb{b} \leftarrowtail \mathsf{c}(A_1, \dots, A_n)$. In the rules for counting quantifiers it is assumed that $\bigvee_i \mathbb{e}_i \wedge \mathbb{d}_i$ is an a-decomposition of $\mathbb{c}$. The logical rules of $\mathbf{G}_{\mathsf{CPL}}$ are straightforward extensions of usual natural deduction rules (with an empty context). In the rules ($\mathbf{C}$I) and ($\mathbf{D}$E) it is assumed that a does not occur free in $\mathbb{b}$ and is the only free name in $\mathbb{d}$. In the rules ($\mathbf{C}$E) and ($\mathbf{D}$I) it is assumed that $\bigvee_i \mathbb{e}_i \wedge \mathbb{d}_i$ is an a-decomposition of $\mathbb{c}$.

Remark 1. *The rules (*$\mathbf{C}$I*) and (*$\mathbf{D}$E*) look significantly simpler than the corresponding rules (*$\mathbf{C}$E*) and (*$\mathbf{D}$I*), which make reference to an a-decomposition of $\mathbb{c}$. Actually, one can show that the following variants (*$\mathbf{C}$I**) and (*$\mathbf{D}$E**), which are more similar to the corresponding E- and I-rules, are admissible in* $\mathbf{G}_{\mathsf{CPL}}$*:*

$$\frac{\vdash \mathbb{c} \rightarrowtail A \qquad \mathbb{b} \vDash \bigvee_i \{\mathbb{e}_i \mid \mu([\![\mathbb{d}_i]\!]_{\{a\}}) \geq q\}}{\vdash \mathbb{b} \rightarrowtail \mathbf{C}^q_a A}\,(\mathbf{C}\mathrm{I}^*) \qquad \frac{\vdash \mathbb{c} \rightarrowtail A \qquad \neg\mathbb{b} \vDash \bigvee_i \{\mathbb{e}_i \mid \mu([\![\mathbb{d}_i]\!]_{\{a\}}) \geq q\}}{\vdash \mathbb{b} \leftarrowtail \mathbf{D}^q_a A}\,(\mathbf{D}\mathrm{E}^*)$$

The result below, easily shown by induction, provides a good exercise to practice with the rules:

Lemma 5. *For any formula A, the judgements $\vdash \mathbb{b}_A \rightarrowtail A$ and $\vdash \mathbb{b}_A \leftarrowtail A$ (cf. Corollary 1) are derivable in* $\mathbf{G}_{\mathsf{CPL}}$*.*

Observe that, since judgements have only one formula, there is no axiom of the form $\vdash A, \neg A$. In order to prove the valid judgement $\vdash \top \rightarrowtail (A \vee \neg A)$, one can use Lemma 5 to show that $\vdash \mathbb{b}_A \rightarrowtail A$ and $\vdash \mathbb{b}_{\neg A} \rightarrowtail \neg A$, and conclude using rules ($\vee$I$_1$), ($\vee$I$_2$) followed by the rule ($\vDash$I).

The soundness of $\mathbf{G}_{\mathsf{CPL}}$ is easily established by induction on derivations:

Theorem 2 (Soundness of $\mathbf{G}_{\mathsf{CPL}}$)**.** *If $\vdash L$, then $\vDash L$.*

Completeness requires a more sophisticated argument. We introduce a *decomposition relation* $\rightsquigarrow$ between finite sets of labelled formulas. We will show that the following three properties hold for all finite sets of judgements Φ, Ψ:

i. If Φ is $\rightsquigarrow$-normal, then $\vDash \Phi$ if and only if $\vdash \Phi$
ii. if $\vDash \Phi$, then there is a Ψ such that $\Phi \rightsquigarrow \Psi$ and $\vDash \Psi$
iii. if $\vdash \Psi$ and $\Phi \rightsquigarrow \Psi$, then $\vdash \Phi$.

Definition 4. *The relation $L \rightsquigarrow_0 \Phi$ between labelled formulas and finite sets of*

Initial Rules

$$
\frac{\boldsymbol{b} \vDash \boldsymbol{x}_i^a}{\vdash \boldsymbol{b} \rightarrowtail \boldsymbol{x}_a^i}\ (\boldsymbol{x}\mathrm{I}) \qquad \frac{\boldsymbol{x}_i^a \vDash \boldsymbol{b}}{\vdash \boldsymbol{b} \leftarrowtail \boldsymbol{x}_a^i}\ (\boldsymbol{x}\mathrm{E})
$$

Structural Rules

$$
\frac{\boldsymbol{b} \vDash \bot}{\vdash \boldsymbol{b} \rightarrowtail A}\ (\bot\mathrm{E}) \qquad \frac{\top \vDash \boldsymbol{b}}{\vdash \boldsymbol{b} \leftarrowtail A}\ (\top\mathrm{I})
$$

$$
\frac{\vdash \boldsymbol{c} \rightarrowtail A \qquad \vdash \boldsymbol{d} \rightarrowtail A \qquad \boldsymbol{b} \vDash \boldsymbol{c} \vee \boldsymbol{d}}{\vdash \boldsymbol{b} \rightarrowtail A}\ (\vDash\mathrm{I}) \qquad \frac{\vdash \boldsymbol{c} \leftarrowtail A \qquad \vdash \boldsymbol{d} \leftarrowtail A \qquad \boldsymbol{c} \wedge \boldsymbol{d} \vDash \boldsymbol{b}}{\vdash \boldsymbol{b} \leftarrowtail A}\ (\vDash\mathrm{E})
$$

Logical Rules

$$
\frac{\vdash \boldsymbol{c} \leftarrowtail A \qquad \boldsymbol{b} \vDash \neg \boldsymbol{c}}{\vdash \boldsymbol{b} \rightarrowtail \neg A}\ (\neg\mathrm{E}) \qquad \frac{\vdash \boldsymbol{c} \rightarrowtail A \qquad \neg \boldsymbol{c} \vDash \boldsymbol{b}}{\vdash \boldsymbol{b} \leftarrowtail \neg A}\ (\neg\mathrm{I})
$$

$$
\frac{\vdash \boldsymbol{b} \rightarrowtail A}{\vdash \boldsymbol{b} \rightarrowtail A \vee B}\ (\vee\mathrm{I}_1) \qquad \frac{\vdash \boldsymbol{b} \rightarrowtail B}{\vdash \boldsymbol{b} \rightarrowtail A \vee B}\ (\vee\mathrm{I}_2) \qquad \frac{\vdash \boldsymbol{b} \leftarrowtail A \qquad \vdash \boldsymbol{b} \leftarrowtail B}{\vdash \boldsymbol{b} \leftarrowtail A \vee B}\ (\vee\mathrm{E})
$$

$$
\frac{\vdash \boldsymbol{b} \rightarrowtail A \qquad \vdash \boldsymbol{b} \rightarrowtail B}{\vdash \boldsymbol{b} \rightarrowtail A \wedge B}\ (\wedge\mathrm{I}) \qquad \frac{\vdash \boldsymbol{b} \leftarrowtail A}{\vdash \boldsymbol{b} \leftarrowtail A \wedge B}\ (\wedge\mathrm{E}_1) \qquad \frac{\vdash \boldsymbol{b} \leftarrowtail B}{\vdash \boldsymbol{b} \leftarrowtail A \wedge B}\ (\wedge\mathrm{E}_2)
$$

Counting Rules

$$
\frac{\vdash \boldsymbol{b} \wedge \boldsymbol{d} \rightarrowtail A \qquad \mu(\boldsymbol{d}) \geq q}{\vdash \boldsymbol{b} \rightarrowtail \mathbf{C}_a^q A}\ (\mathbf{C}\mathrm{I}) \qquad \frac{\vdash \boldsymbol{c} \leftarrowtail A \qquad \bigvee_i \{\boldsymbol{e}_i \mid \mu(\llbracket \boldsymbol{d}_i \rrbracket_{\{a\}}) \geq q\} \vDash \boldsymbol{b}}{\vdash \boldsymbol{b} \leftarrowtail \mathbf{C}_a^q A}\ (\mathbf{C}\mathrm{E})
$$

$$
\frac{\vdash \boldsymbol{c} \leftarrowtail A \qquad \boldsymbol{b} \vDash \bigvee_i \{\boldsymbol{e}_i \mid \mu(\llbracket \boldsymbol{d}_i \rrbracket_{\{a\}}) < q\}}{\vdash \boldsymbol{b} \rightarrowtail \mathbf{D}_a^q A}\ (\mathbf{D}\mathrm{I}) \qquad \frac{\vdash \neg \boldsymbol{b} \wedge \boldsymbol{d} \rightarrowtail A \qquad \mu(\boldsymbol{d}) \geq q}{\vdash \boldsymbol{b} \leftarrowtail \mathbf{D}_a^q A}\ (\mathbf{D}\mathrm{E})
$$

Figure 1: Rules of $\mathbf{G}_{\mathsf{CPL}}$.

labelled formulas is generated by the following clauses:

$$
\begin{array}{rcllr}
\boldsymbol{b} \rightarrowtail \neg A & \rightsquigarrow_0 & \{\boldsymbol{c} \leftarrowtail A\} & (\boldsymbol{b} \vDash \neg \boldsymbol{c}) & (\rightsquigarrow_0 1) \\
\boldsymbol{b} \leftarrowtail \neg A & \rightsquigarrow_0 & \{\boldsymbol{c} \rightarrowtail A\} & (\neg \boldsymbol{c} \vDash \boldsymbol{b}) & (\rightsquigarrow_0 2) \\
\boldsymbol{b} \rightarrowtail A \vee B & \rightsquigarrow_0 & \{\boldsymbol{c} \rightarrowtail A,\ \boldsymbol{d} \rightarrowtail B\} & (\boldsymbol{b} \vDash \boldsymbol{c} \vee \boldsymbol{d}) & (\rightsquigarrow_0 3) \\
\boldsymbol{b} \leftarrowtail A \vee B & \rightsquigarrow_0 & \{\boldsymbol{b} \leftarrowtail A,\ \boldsymbol{b} \leftarrowtail B\} & & (\rightsquigarrow_0 4) \\
\boldsymbol{b} \rightarrowtail A \wedge B & \rightsquigarrow_0 & \{\boldsymbol{b} \rightarrowtail A,\ \boldsymbol{b} \rightarrowtail B\} & & (\rightsquigarrow_0 5) \\
\boldsymbol{b} \leftarrowtail A \wedge B & \rightsquigarrow_0 & \{\boldsymbol{c} \leftarrowtail A,\ \boldsymbol{d} \leftarrowtail B\} & (\boldsymbol{c} \wedge \boldsymbol{d} \vDash \boldsymbol{b}) & (\rightsquigarrow_0 6) \\
\boldsymbol{b} \rightarrowtail \mathbf{C}_a^q A & \rightsquigarrow_0 & \{\boldsymbol{c} \rightarrowtail A\} & & (\rightsquigarrow_0 7) \\
\boldsymbol{b} \leftarrowtail \mathbf{C}_a^q A & \rightsquigarrow_0 & \{\boldsymbol{c} \leftarrowtail A\} & & (\rightsquigarrow_0 8) \\
\boldsymbol{b} \rightarrowtail \mathbf{D}_a^q A & \rightsquigarrow_0 & \{\boldsymbol{c} \leftarrowtail A\} & & (\rightsquigarrow_0 9) \\
\boldsymbol{b} \leftarrowtail \mathbf{D}_a^q A & \rightsquigarrow_0 & \{\boldsymbol{c} \rightarrowtail A\} & & (\rightsquigarrow_0 10)
\end{array}
$$

where in ($\rightsquigarrow_0$ 7)-($\rightsquigarrow_0$ 10) *it is assumed that* $\boldsymbol{c} = \bigvee \boldsymbol{e}_i \wedge \boldsymbol{d}_i$ *is an a-decomposition* *of $\mathbb{c}$ and moreover:*

- *in ($\rightsquigarrow_0$ 7) it is assumed that $\mathbb{b} \vDash \bigvee\{\mathbb{e}_i \mid \mu(\llbracket \mathbb{d}_i \rrbracket_{\{a\}}) \geq q\}$*
- *in ($\rightsquigarrow_0$ 8) it is assumed that $\bigvee\{\mathbb{e}_i \mid \mu(\llbracket \mathbb{d}_i \rrbracket_{\{a\}}) \geq q\} \vDash \mathbb{b}$*
- *in ($\rightsquigarrow_0$ 9) it is assumed that $\mathbb{b} \vDash \bigvee\{\mathbb{e}_i \mid \mu(\llbracket \mathbb{d}_i \rrbracket_{\{a\}}) < q\}$*
- *in ($\rightsquigarrow_0$ 10) it is assumed that $\neg\mathbb{b} \vDash \bigvee\{\mathbb{e}_i \mid \mu(\llbracket \mathbb{d}_i \rrbracket_{\{a\}}) \geq q\}$.*

*The relation $\Phi \rightsquigarrow \Psi$ between finite sets of labelled formulas is defined by letting $\{L_1, \dots, L_n\} \rightsquigarrow \bigcup_{i=1}^{n} \Phi_i$, whenever $L_i \rightsquigarrow_0 \Phi_i$. Φ is said $\rightsquigarrow$-*normal *when for no Ψ, $\Phi \rightsquigarrow \Psi$ holds.*

Lemma 6. *If L is $\rightsquigarrow_0$-normal, then $\vdash L$ if and only if $\vDash L$.*

Proof. It is easily checked that L is $\rightsquigarrow_0$-normal if and only if the formula A occurring in L is atomic, and from this it follows by inspection that $\vdash L$ if and only if $\vDash L$. □

Lemma 7. *$\rightsquigarrow$ is strongly normalizing.*

Proof. For any set of labelled formulas Φ, let $\mathsf{ms}(\Phi) \in \mathbb{N}$ be defined by letting $\mathsf{ms}(\emptyset) = 0$, and $\mathsf{ms}(\{L_1, \dots, L_n\}) = 3^{\sum_{i=1}^{n} \mathsf{cn}(L_i)}$, where $\mathsf{cn}(\mathbb{b} \rightarrowtail A) = \mathsf{cn}(\mathbb{b} \leftarrowtail A)$ indicates the number of connectives occurring in A. One can then check that whenever $\Phi \rightsquigarrow \Psi$, $\mathsf{ms}(\Psi) < \mathsf{ms}(\Phi)$, from which the claim descends. □

Finally, the two lemmas below, correspond to the properties (i.) and (ii.) anticipated above, and are proved by inspecting all possible cases:

Lemma 8. *Let $\Phi \rightsquigarrow \Psi$, where $\Phi = \{L_1, \dots, L_n\}$ and $\Psi = \{J_1, \dots, J_m\}$. If $\vDash J_i$ holds for all $i = 1, \dots, m$, then $\vDash L_i$ holds too, for all $i = 1, \dots, n$.*

Lemma 9. *Let $\Phi \rightsquigarrow \Psi$, where $\Phi = \{L_1, \dots, L_n\}$ and $\Psi = \{J_1, \dots, J_m\}$. If $\vdash L_i$ holds for all $i = 1, \dots, n$, then $\vdash J_i$ holds too, for all $i = 1, \dots, m$.*

The lemmas above allows us to conclude the proof.

Theorem 3 (Completeness of $\mathbf{G}_{\mathsf{CPL}}$.)**.** *If $\vDash L$, then $\vdash L$.*

Proof. By combining Lemma 8, Lemma 6, and Lemma 9. □

Remark 2. *It is possible to consider a more general natural deduction system $\mathbf{G}_{\mathsf{CPL}}^{\mathrm{Hyp}}$ given by judgements of the form $\Gamma \vdash \mathbb{b} \rightarrowtail A$ or $\Gamma \vdash \mathbb{b} \leftarrowtail A$, where Γ is a finite set of (non-labeled) formulas. The intended meaning of such judgements is that $\llbracket \mathbb{b} \rrbracket \cap \llbracket \Gamma \rrbracket \subseteq \llbracket A \rrbracket$ (resp. $\llbracket A \rrbracket \cap \llbracket \Gamma \rrbracket \subseteq \llbracket \mathbb{b} \rrbracket$). The rules of $\mathbf{G}_{\mathsf{CPL}}^{\mathrm{Hyp}}$ include those in Fig. 1 (with the addition of the hypotheses Γ, and where in the rules for counting quantifiers one must require that the quantified name does not occur free in the context Γ), together with a new axiom of conclusion $\Gamma, A \vdash \mathbb{b} \rightarrowtail A$ and two structural intro-elim rules for $\rightarrowtail$ as follows:*

$$\frac{\Gamma, \mathbb{c} \vdash \mathbb{d} \rightarrowtail A \qquad \mathbb{b} \vDash \mathbb{c} \wedge \mathbb{d}}{\Gamma \vdash \mathbb{b} \rightarrowtail A}\ (\rightarrowtail I) \qquad \frac{\Gamma \vdash \mathbb{d} \rightarrowtail A \qquad \Gamma \vdash \mathbb{b} \rightarrowtail \mathbb{d}}{\Gamma \vdash \mathbb{b} \rightarrowtail A}\ (\rightarrowtail E)$$

One can check then that $A_1, \dots, A_n \vdash \mathbb{b} \rightarrowtail A$ is derivable in $\mathbf{G}_{\mathsf{CPL}}^{\mathrm{Hyp}}$ iff $\vdash \mathbb{b} \rightarrowtail (\neg A_1 \vee \dots \vee \neg A_n \vee A)$ is derivable in $\mathbf{G}_{\mathsf{CPL}}$, so one can transport the soundness and completeness result from $\mathbf{G}_{\mathsf{CPL}}$ to $\mathbf{G}_{\mathsf{CPL}}^{\mathrm{Hyp}}$.

4. Characterization of the Counting Hierarchy

In his seminal work on counting complexity [75, 77, 76], Wagner defined canonical complete problems for each level of the counting hierarchy by iteratively applying a "counting" operator to the language consisting of satisfiable formulas, thereby obtaining results similar to classical ones holding for the polynomial hierarchy [47, 48, 69, 80, 13]. In this context, the counting quantifiers of CPL provide a way to internalize Wagner's construction inside a proper logical system. Nevertheless, contrarily to Wagner's operator over classes of languages, counting quantifiers, can be used deep inside formulas, rather than just at top-level. Hence, in order to show that validity in CPL characterizes the counting hierarchy, we first need to show that any formula in CPL can be put in a prenex form of a special kind, that we call a *positive prenex normal form*, essentially corresponding to Wagner's construction. Using this fact, to conclude it will then suffice to show that Wagner's complete problem can be expressed in the language of CPL.

4.1. Prenex Normal Forms

The first step is to show that formulas can be put in a (standardly defined) prenex normal form.

Definition 5 (Prenex Normal Form, PNF)**.** *A formula is an n-ary* prenex normal form *(or simply a* prenex normal form*, PNF for short) if it can be written as $\triangle_1\triangle_2\ldots\triangle_n A$ where, for every $i \in \{1,\ldots,n\}$, the operator $\triangle_i$ is either in the form $\mathbf{C}^q_a$ or $\mathbf{D}^q_a$, and A does not contain any counting operator. In this case, the formula A is said to be the* matrix *of the PNF.*

To convert a formula into an equivalent prenex normal form, we need some preliminary lemmas. First, it is worth noticing that for every CPL-formula A, name a, and finite set X, with $\mathrm{FN}(A) \subseteq X$ and $a \notin X$, if $q = 0$, then $[\![\mathbf{C}^q_a A]\!]_X = (2^{\mathbb{N}})^X$ and $[\![\mathbf{D}^q_a A]\!]_X = \emptyset$.[5]

Moreover, it is possible to get rid of negated quantifiers using the equivalences $\neg\mathbf{D}^q_a A \equiv \mathbf{C}^q_a A$ and $\neg\mathbf{C}^q_a A \equiv \mathbf{D}^q_a A$. Finally, the following Lemma 10 states that counting quantifiers occurring inside any conjunction or disjunction can somehow be extruded from it.[6]

Lemma 10. *Let $a \notin \mathrm{FN}(A)$ and $q > 0$. Then the following equivalences hold:*

$$A \wedge \mathbf{C}^q_a B \equiv \mathbf{C}^q_a(A \wedge B) \qquad A \vee \mathbf{C}^q_a B \equiv \mathbf{C}^q_a(A \vee B)$$
$$A \wedge \mathbf{D}^q_a B \equiv \mathbf{D}^q_a(\neg A \vee B) \qquad A \vee \mathbf{D}^q_a B \equiv \mathbf{D}^q_a(\neg A \wedge B).$$

As the equivalences from Lemma 10 can be oriented, we conclude that each formula of CPL can be converted into an equivalent term in PNF.

[5] So, in particular, for any A, B and name a, $A \wedge \mathbf{C}^0_a B \equiv A$, $A \wedge \mathbf{D}^0_a B \equiv \emptyset^X$, $A \vee \mathbf{C}^0_a B \equiv (2^{\mathbb{N}})^X$, and $A \vee \mathbf{D}^0_a B \equiv A$.

[6] For the proof, see Appendix A.2

Proposition 1. *For every* CPL*-formula* A *there is a PNF* B *such that* $A \equiv B$*. Moreover,* B *can be computed in polynomial time from* A*.*

4.2. Positive Prenex Normal Form

Prenex normal forms of CPL are already very close to what we need. There is one last step to be made, namely getting rid of the $\mathbf{D}^q_a$ operator, which does not have any counterpart in Wagner's construction. In other words, we need prenex normal forms *of a special kind*.

Definition 6 (Positive Prenex Normal Forms, PPNF)**.** *A formula is said to be in* positive prenex normal form *(PPNF) if it is a PNF in which the operator* $\mathbf{D}$ *does not occur.*

In order to get rid of $\mathbf{D}$, the idea is to turn any instance of $\mathbf{D}$ into $\mathbf{C}$ using $\mathbf{D}^q_a A \equiv \neg\mathbf{C}^q_a A$, and, then, to use the fact that $\mathbf{C}$ enjoys a weak form of self duality, expressed by the lemma below.

Lemma 11. *For every formula* A *and* $q \in [0,1] \cap \mathbb{Q}$ *there is a* $p \in [0,1] \cap \mathbb{Q}$ *such that, for every* X*, with* $\mathrm{FN}(A) \subseteq X$ *and* $a \notin X$*,* $\neg\mathbf{C}^q_a A \equiv \mathbf{C}^p_a \neg A$*. Moreover,* p *can be computed from* q *in polynomial time.*

Proof. For all $k \in \mathbb{N}$, let $[0,1]_k$ be the set of rationals of the form $q = \sum_{i=0}^k b_i \cdot 2^{-i}$, where $b_i \in \{0,1\}$. Observe that for all $p \in \mathbb{N}$, if $p \leq 2^k$, $p/2^k \in [0,1]_k$. Indeed, let $b_0 \dots b_k = (p)_2$ the base 2 writing of p (with possibly all 0s at the end so that the length is precisely k), then $p = \sum_{i=0}^k b_i \cdot 2^i$, so $p/2^k = \dfrac{\sum_{i=0}^k b_i \cdot 2^i}{2^k} = \sum_{i=0}^k b_i \cdot 2^{-k+i} = \sum_{i=0}^k b_{k-i} \cdot 2^{-i}$. The fundamental observation is that, for any Boolean formula $\mathscr{b}$ with $\mathrm{FN}(\mathscr{b}) \subseteq \{a\}$, $\mu(\mathscr{b}) \in [0,1]_k$, where k is the maximum natural number such that $\boldsymbol{x}^a_k$ occurs in $\mathscr{b}$. Indeed, by Lemma 4, we have that,

$$\mu(\mathscr{b}) = \sharp\mathscr{b} = (\sharp\{m : \{\boldsymbol{x}^a_0, \dots, \boldsymbol{x}^a_k\} \to \{0,1\} \mid m \vDash \mathscr{b}\}) \cdot 2^{-k} \in [0,1]_k.$$

Let now $\mathscr{b}_A$ (cf. Corollary 1) be a-decomposable as $\bigvee_i^n \mathscr{d}_i \wedge \mathscr{e}_i$ and let k be maximum such that $\boldsymbol{x}^a_k$ occurs in $\mathscr{b}_A$. Since for all $i = 0, \dots, n$, $\mu([\![\mathscr{d}_i]\!]_{\{a\}}) \in [0,1]_k$, for all $\omega \in (2^{\mathbb{N}})^X$, $\mu(\Pi_\omega([\![A]\!]_{X\cup\{a\}})) \in [0,1]_k$, since $\Pi_\omega([\![A]\!]_{X\cup\{a\}}))$ coincides with the unique $[\![\mathscr{d}_i]\!]_{\{a\}}$ such that $\omega \in [\![\mathscr{e}_i]\!]_X$ (by Lemma 3). Now, if $q \notin [0,1]_k$, let $\epsilon = 0$; and if $q \in [0,1]_k$, then if $q = 1$, let $\epsilon = -2^{-(k+1)}$, and if $q \neq 1$, let $\epsilon = 2^{-(k+1)}$. In all cases $p := 1 - (q + \epsilon) \notin [0,1]_k$, so we deduce:

$$\begin{aligned}
[\![\neg\mathbf{C}^q_a A]\!]_X &= \{\omega \in (2^{\mathbb{N}})^X \mid \mu(\Pi_\omega([\![A]\!]_{X\cup\{a\}})) < q\} \\
&= \{\omega \in (2^{\mathbb{N}})^X \mid \mu(\Pi_\omega([\![A]\!]_{X\cup\{a\}})) \leq q + \epsilon\} \\
&\overset{(1)}{=} \{\omega \in (2^{\mathbb{N}})^X \mid \mu(\overline{\Pi_\omega([\![A]\!]_{X\cup\{a\}})}) \geq 1 - (q + \epsilon)\} \\
&\overset{(2)}{=} \{\omega \in (2^{\mathbb{N}})^X \mid \mu(\Pi_\omega(\overline{[\![A]\!]_{X\cup\{a\}}})) \geq 1 - (q + \epsilon)\} \\
&= \{\omega \in (2^{\mathbb{N}})^X \mid \mu(\Pi_\omega([\![\neg A]\!]_{X\cup\{a\}})) \geq 1 - (q + \epsilon)\} \\
&= [\![\mathbf{C}^p_a \neg A]\!]_X,
\end{aligned}$$

where we used the easily established facts (1) $\mu(S) \leq r$ iff $\mu(\overline{S}) \geq 1-r$ and (2) $\overline{\Pi_\omega(S)} = \Pi_\omega(\overline{S})$. $\Box$

Actually, the value of p is very close to $1-q$, the difference between the two being easily computable from the formula A. Hence, any negation occurring in the counting prefix of a PNF can be pushed back into the formula. Summing up:

Proposition 2. *For every formula A of* CPL *there is a formula in PPNF B, such that $A \equiv B$. Moreover, A can be computed from A in polynomial time.*

4.3. CPL *and the Counting Hierarchy*

Below, we state a slightly weaker version of Wagner's characterization of the counting hierarchy, which however perfectly fits our needs. Let S be a set and $\mathcal{L} \subseteq S^n$, let $1 \leq m < n$, and $b \in \mathbb{N}$. We define $\mathbf{C}^b_m \mathcal{L}$ as the following subset of S^{n-m}:

$$\{(a_n, \dots, a_{m+1}) \mid \#(\{(a_m, \dots, a_1) \mid (a_n, \dots, a_1) \in \mathcal{L}\}) \geq b\}\,.$$

Now, take PL as a set comprising the language of propositional logic formulas, including the propositional constants $\mathbf{T}$ and $\mathbf{F}$. For any natural number $n \in \mathbb{N}$, let $\mathcal{TF}^n$ be the subset of PL^{n+1} containing all tuples in the form $(A, t_1, \dots, t_n)$, where A is a propositional formula in CNF with at most n free variables, and $t_1, \dots, t_n \in \{\mathbf{T}, \mathbf{F}\}$, which render A true. Finally, for every $k \in \mathbb{N}$, we indicate as W^k the language consisting of all (binary encodings of) tuples in the form $(A, m_1, \dots, m_k, b_1, \dots, b_k)$ and such that $A \in \mathbf{C}^{b_1}_{m_1} \cdots \mathbf{C}^{b_k}_{m_k} \mathcal{TF}^{\sum m_i}$.

Theorem 4 (Wagner)**.** *For every k, the language W^k is complete for* CH_k*.*

Please notice that elements of W^k can be seen as nothing else than alternative representations for CPL's PPNFs, once any m_i is replaced by $\min\{1, \frac{m_i}{2^{b_i}}\}$. As a consequence:

Corollary 2. *The closed and valid k-ary PPNFs, whose matrix is in CNF, define a complete set for* CH_k*.*

5. Intuitionistic Counting Propositional Logic

In this section we introduce a variant of CPL extending standard intuitionistic logic with Boolean variables and counting quantifiers. The logic iCPL combines constructive reasoning, corresponding, under the CHC, to functional programming, with semantic reasoning on Boolean formulas and their models, corresponding, as we have seen, to discrete probabilistic reasoning.

Intuitionistic logic is often presented as the system obtained from classical logic when the excluded middle principle is suppressed. This is not the road we take here: we maintain Boolean formulas as providing a way to manipulate finitary Borel spaces of the form $[\![\mathscr{b}]\!]$; instead we add a *new* countable set of propositional variables $\mathcal{p}, \mathcal{q}, \dots$, which are not subject to the excluded middle and thus can be used to formalize constructive reasoning (this is in analogy with what happens in so-called *ecumenical* logics, see [57]). Nevertheless, we do not retain *all* the expressive power of CPL: to obtain a constructive interpretation we are forced to suppress the dual quantifier $\mathbf{D}$, as well as the dual judgements of the form $\mathcal{b} \leftarrow A$ (see Remark 6 in Section 6).

After presenting the syntax of iCPL, we introduce a Kripke-style semantics for this logic, which adapts ideas from the semantics of CPL discussed in the previous sections. Then we present a natural deduction system $\mathbf{G}_{\mathsf{iCPL}}$ for iCPL obtained by suitably adapting the system $\mathbf{G}^{\mathrm{Hyp}}_{\mathsf{CPL}}$ from Remark 2. We establish the soundness and completeness of $\mathbf{G}_{\mathsf{iCPL}}$ with respect to Kripke semantics, by adapting standard arguments for intuitionistic modal logic. We finally introduce a natural deduction system $\mathbf{G}_{\mathsf{iCPL}_0}$ for *purely intuitionistic* formulas, i.e. not containing any Boolean variable, and we relate derivations in $\mathbf{G}_{\mathsf{iCPL}}$ with families of derivations in $\mathbf{G}_{\mathsf{iCPL}_0}$. It is for this last calculus that we will obtain, in the next section, the desired connection with probabilistic functional programs.

5.1. iCPL *and its Kripke Semantics*

The formulas of iCPL comprise both a countable set $\mathscr{P} = \{\mathcal{p}, \mathcal{q}, \dots\}$ of intuitionistic propositional variables, and the Boolean propositional variables of CPL.

Definition 7. *The* formulas of iCPL *are defined by the grammar below:*

$$A ::= \top \;\big|\; \bot \;\big|\; \boldsymbol{x}^i_a \;\big|\; \mathcal{p} \;\big|\; A \wedge A \;\big|\; A \vee A \;\big|\; A \to A \;\big|\; \mathbf{C}^q_a A$$

where $q \in (0,1] \cap \mathbb{Q}$*. A formula A of* iCPL *is said* purely intuitionistic *if it contains no occurrence of Boolean variables. We let* IntForm *indicate the set of such formulas.*

As before, we take $\neg A$ as an abbreviation of $A \to \bot$. Notice that we no more consider counting quantified formulas of the form $\mathbf{C}^0_a A$. Indeed, in CPL we have that $\mathbf{C}^0_a A \equiv \top$, and since we no more consider formulas of the form $\mathbf{D}^q_a A$, we can thus simply suppress such formulas.

For example, consider the formula $A = \mathbf{C}^{1/2}_a B$, where $B = \mathcal{p} \to \big((\boldsymbol{x}^0_a \wedge \mathcal{p}) \vee (\neg \boldsymbol{x}^0_a \wedge \mathcal{q})\big)$. Intuitively, proving B amounts to showing that, whenever $\mathcal{p}$ holds, given that either $\boldsymbol{x}^0_a$ or $\neg \boldsymbol{x}^0_a$ holds, then $\mathcal{p}$ holds in the first case, while $\mathcal{q}$ in the second case. For a randomly chosen valuation of $\boldsymbol{x}^0_a$, such provability conditions can be met only if $\boldsymbol{x}^0_a$ turns out true, so we can conclude that B holds with probability $\frac{1}{2}$, and thus A is valid.

An alternative way of looking at the formula B is as a *distribution* of purely intutionistic formulas. Indeed, if we randomly choose a valuation v of the Boolean variables of A, then: (i) we have half chances that $\boldsymbol{x}^0_a$ turns out true (and $\neg \boldsymbol{x}^0_a$ false), so that A is equivalent, under v, to the valid intuitionistic formula $\mathcal{p} \to \mathcal{p}$, (ii) half chances that $\boldsymbol{x}^0_a$ turns out false (and $\neg \boldsymbol{x}^0_a$ true), so that A is equivalent, under v, to the invalid intuitionistic formula $\mathcal{p} \to \mathcal{q}$, and (iii) no chance that A produces any other formula.

These intuitive ideas will be made precise using the semantics of iCPL-formulas. This is based on the following class of Kripke-style structures.

Definition 8 (iCPL-Structure)**.** *A* iCPL-structure *is a triple* $\mathscr{M} = (W, \preceq, \mathbf{i})$ *where* W *is a countable set,* $\preceq$ *is a preorder on* W*, and* $\mathbf{i} : \mathscr{P} \to W^{\uparrow}$*, where* $W^{\uparrow}$ *is the set of upper subsets of* W*.*

The interpretation of iCPL-formulas in Kripke structures combines a set W of worlds (for the interpretation of intuitionistic propositional variables) with the choice of an element of the Cantor space (for the interpretation of Boolean variables).

Definition 9 (Semantics of iCPL)**.** *For any* iCPL*-structure* $\mathscr{M} = (W, \preceq, \mathbf{i})$ *and finite set* X*, we define the relation* $w, \omega \Vdash^X_{\mathscr{M}} A$ *(where* $w \in W$*,* $\omega \in (2^{\mathbb{N}})^X$ *and* $\mathrm{FN}(A) \subseteq X$*) by induction:*

- $w, \omega \nVdash^X_{\mathscr{M}} \bot$
- $w, \omega \Vdash^X_{\mathscr{M}} \top$
- $w, \omega \Vdash^X_{\mathscr{M}} \boldsymbol{x}^i_a$ *if and only if* $\omega(a)(i) = 1$
- $w, \omega \Vdash^X_{\mathscr{M}} \mathscr{p}$ *if and only if* $w \in \mathbf{i}(\mathscr{p})$
- $w, \omega \Vdash^X_{\mathscr{M}} A \wedge B$ *if and only if* $w, \omega \Vdash^X_{\mathscr{M}} A$ *and* $w, \omega \Vdash^X_{\mathscr{M}} B$
- $w, \omega \Vdash^X_{\mathscr{M}} A \vee B$ *if and only if* $w, \omega \Vdash^X_{\mathscr{M}} A$ *or* $w, \omega \Vdash^X_{\mathscr{M}} B$
- $w, \omega \Vdash^X_{\mathscr{M}} A \to B$ *if and only if for all* $w' \succeq w$*,* $w', \omega \Vdash^X_{\mathscr{M}} A$ *implies* $w', \omega \Vdash^X_{\mathscr{M}} B$
- $w, \omega \Vdash^X_{\mathscr{M}} \mathbf{C}^q_a A$ *if and only if*

$$\mu\left(\left\{\omega' \in 2^{\mathbb{N}} \mid w, \omega + \omega' \Vdash^{X \cup \{a\}}_{\mathscr{M}} A\right\}\right) \geq q.$$

We write $\Gamma \Vdash^X_{\mathscr{M}} \mathscr{b} \rightarrowtail A$ when for all $w \in W$ and $\omega \in [\![\mathscr{b}]\!]_X$, whenever $w, \omega \Vdash^X_{\mathscr{M}} \Gamma$ holds, also $w, \omega \Vdash^X_{\mathscr{M}} A$ holds. We write $\Gamma \vDash_I \mathscr{b} \rightarrowtail A$ when for any iCPL-structure $\mathscr{M}$ and $X \supseteq \mathrm{FN}(A)$, $\Gamma \Vdash^X_{\mathscr{M}} \mathscr{b} \rightarrowtail A$ holds. We use $\Gamma \vDash_I A$ as a shorthand for $\Gamma \vDash \top \rightarrowtail A$. We write $A \equiv_I B$ as a shorthand for $A \vDash_I B$ and $B \vDash_I A$.

Notice that, for the definition of $w, \omega \Vdash^X_{\mathscr{M}} A$ to be well-defined, we need to check that all considered subsets of $2^{\mathbb{N}}$ are measurable. This is ensured by the following lemma:

Lemma 12. *Given a* iCPL*-structure* $\mathscr{M} = (W, \preceq, \mathbf{i})$*, for any finite set* X*,* $w \in W$ *and formula* A *with* $\mathrm{FN}(A) \subseteq X$*, the set*

$$\mathrm{mod}_{\mathscr{M}}(A, X, w) = \{\omega \in (2^{\mathbb{N}})^X \mid w, \omega \Vdash^X_{\mathscr{M}} A\}$$

is Borel.

Proof. We argue by induction on A. All cases follow almost immediately from the induction hypothesis and properties of Borel sets, except the following two:

- if $A = B \to C$, then $\mathrm{mod}_{\mathscr{M}}(A, X, w) = \bigcap_{w' \succeq w} \overline{\mathrm{mod}_{\mathscr{M}}(B, X, w')} \cup \mathrm{mod}_{\mathscr{M}}(C, X, w')$ is, by the IH, a countable intersection of Borel sets, so it is Borel

- if $A = \mathbf{C}^q_a B$, then by the IH , for all $w' \in W$, $\mathrm{mod}_{\mathscr{M}}(B, X \cup \{a\}, w')$ is Borel; using Lemma 1 we then have that the set

$$\mathrm{mod}_{\mathscr{M}}(A, X, w) = \bigcap_{w' \succeq w} \left\{\omega \in (2^{\mathbb{N}})^X \mid \mu\left(\Pi^{\omega}\left(\mathrm{mod}_{\mathscr{M}}(B, X \cup \{a\}, w')\right)\right) \geq q\right\}$$

  is Borel.

$\square$

For any formula A and $\omega \in (2^{\mathbb{N}})^X$, with $\mathrm{FN}(A) \subseteq X$, one can check by induction on A that the set of worlds w such that $w, \omega \vDash^X_{\mathscr{M}} A$, is up-closed:

Lemma 13 (Monotonicity)**.** *If $w, \omega \Vdash^X_{\mathscr{M}} A$ and $w \preceq w'$, then $w', \omega \Vdash^X_{\mathscr{M}} A$.*

The Kripke-semantics of iCPL agrees with the semantics of CPL in the following sense:

Proposition 3. *For any $\mathbf{D}$-free formula A of CPL, $\vDash_I \mathscr{b} \rightarrowtail A$ iff $\vDash \mathscr{b} \rightarrowtail A$.*

Proof. Let A be $\mathbf{D}$-free formula A of CPL. It is easily checked by induction on A that for any iCPL-structure $\mathscr{M} = (W, \preceq, \mathbf{i})$ and for all $w \in W$, $w, \omega \Vdash^X_{\mathscr{M}} A$ iff $\omega \in [\![A]\!]_X$. $\square$

The following lemma shows that a few commutations (cf. Lemma 10) scale from CPL to iCPL:

Lemma 14. *Let $a \notin \mathrm{FN}(A)$ and $q > 0$. Then the following equivalences hold:*

$$A \wedge \mathbf{C}^q_a B \equiv_I \mathbf{C}^q_a(A \wedge B) \qquad\qquad A \vee \mathbf{C}^q_a B \equiv_I \mathbf{C}^q_a(A \vee B).$$

Remark 3. *A similar commutation for implication, e.g. $A \to \mathbf{C}^q_a B \equiv_I \mathbf{C}^q_a(A \to B)$, fails to hold. Indeed, while $\mathbf{C}^q_a(A \to B) \vDash_I A \to \mathbf{C}^q_a B$ is easily established, one can construct a counter-model to $A \to \mathbf{C}^q_a B \vDash_I \mathbf{C}^q_a(A \to B)$ by letting $A = \mathscr{p}$, $B = (\mathscr{x}^i_a \wedge \mathscr{q}) \vee (\neg \mathscr{x}^i_a \wedge \mathscr{r})$, $W = \{0 \preceq 1, 2\}$, and $\mathbf{i}(0) = \emptyset$, $\mathbf{i}(1) = \{\mathscr{p}, \mathscr{q}\}$, $\mathbf{i}(2) = \{\mathscr{p}, \mathscr{r}\}$.*

Remark 4. iCPL-*structures can be related to standard Kripke structures for intuitionistic modal logic [58, 65]. These are based on a set W with* two *pre-order relations $\leq$ and R enjoying a suitable "diamond" property $R; \leq \subseteq \leq; R$. We obtain a similar structure by taking worlds to be pairs w, ω made of a world and an outcome from the Cantor space, with $(w, \omega) \preceq (w', \omega)$ whenever $w \leq w'$, and $(w, \omega) R (w, \omega + \omega')$. The clause for $\mathbf{C}^q A$ can then be seen as a quantitative variant of the corresponding clause for $\Diamond A$. This is not very surprising, given the similarity between the introduction and elimination rules for $\mathbf{C}^q$ and those for $\Diamond$ (see e.g. [8, 1]).*

Using Kripke semantics we can make the informal interpretations of formulas provided at the beginning of this section more precise. First, for any formula A and $\omega \in (2^{\mathbb{N}})^{\mathrm{FN}(A)}$, let $|\omega|_A$ indicate the unique valuation of the Boolean formulas of A which is in accordance with ω.

Definition 10. *For any formula A of* iCPL *and valuation v of its Boolean variables, the purely intuitionistic formula A_v is defined inductively as follows:*

- *if $A = \top$, $A = \bot$, or $A = \mathcal{p}$, then $A_v = A$;*
- *if $A = \boldsymbol{x}_a^i$, then $A_v = \top$ if $v(a)(i) = 1$, and $A_v = \bot$ if $v(a)(i) = 0$;*
- *if $A = B\mathsf{c}C$, where $c \in \{\wedge, \vee, \to\}$, then $A_v = B_v\mathsf{c}C_v$;*
- *if $A = \mathbf{C}_a^q B$ and a does not occur in B, then $A_v := B_v$;*
- *if $A = \mathbf{C}_a^q B$ and a does occur in B, then $A_v = \mathbf{C}_a^q \bigvee_w B_{v+w}$.*

The distribution $\mathscr{D}_A : \mathrm{IntForm} \to [0,1]$ is defined by

$$\mathscr{D}_A(C) = \mu\Big(\big\{\omega \in (2^{\mathbb{N}})^{\mathrm{FN}(A)} \mid A_\omega = C\big\}\Big)$$

where, for any $\omega \in (2^{\mathbb{N}})^{\mathrm{FN}(A)}$, $A_\omega := A_{|\omega|_A}$.

For any formula C and $b \in \{0,1\}$, let $\neg^b C$ indicate the formula C if $b = 0$ and $\neg C$ if $b = 1$.

Definition 11 (Theory of ω)**.** *For any $\omega \in (2^{\mathbb{N}})^{\mathscr{A}}$, let the* theory of ω *be the set of Boolean formulas $\mathsf{Th}(\omega) = \{\neg^{(1-\omega(a)(i))}\boldsymbol{x}_a^i \mid a, i \in \mathbb{N}\}$. For all $K \in \mathbb{N}$ and finite set X, let $\mathsf{Th}_X^K(\omega)$ be the Boolean formula $\mathsf{Th}_X^K(\omega) = \bigwedge_{a \in A, i \le K} \neg^{(1-\omega(a)(i))}\boldsymbol{x}_a^i$.*

Observe that the formula $\mathsf{Th}_X^K(\omega)$ only depends on a finite amount of information of ω, namely, on the unique $v \in (2^{\{1,\dots,K\}})^X$ such that $v(a)(i) = \omega(a)(i)$. When v is clear from the context, we will indicate $\mathsf{Th}_X^K(\omega)$ simply as $\mathsf{Th}(v)$.

The following two lemmas show that any formula A of iCPL (resp. any formula of the form $\mathbf{C}_a^q B$) can be equivalently seen as a *family* (resp. as a *distribution*) of purely intuitionistic formulas.

Lemma 15. *For any formula A of* iCPL*, $A \equiv_I \bigvee_v \mathsf{Th}(v) \wedge A_v$.*

Proof. It can be shown, by induction on A, that for any valuation v of its Boolean variables, $\mathsf{Th}(v) \vDash_I A \leftrightarrow A_v$. Using $\mathsf{Th}(v) \vDash_I A \leftrightarrow A_v$ and the fact that $\vDash_I \bigvee_v \mathsf{Th}(v)$, we can conclude $A \equiv_I \bigvee_v \mathsf{Th}(v) \wedge A_v$. □

Lemma 16. *For any formula A of* iCPL*, $\vDash_I \mathbf{C}_a^q A$ iff $\sum_{C \in \mathrm{IntVal}} \mathscr{D}_A(C) \ge q$. where* IntVal *indicates the set of valid purely intuitionistic formulas of* iCPL.

Proof. Let $\mathscr{M} = (W, \preceq, \mathbf{i})$ be a iCPL-structure. Since, as shown in the proof of Lemma 15, for all $w \in W$, $\omega \in (2^{\mathbb{N}})^X$ and $\omega' \in (2^{\mathbb{N}})^{\{a\}}$, $\mathsf{Th}(\omega + \omega') \vDash_I A \leftrightarrow A_{\omega+\omega'}$, we have that $w, \omega + \omega' \Vdash_{\mathscr{M}}^{X \cup \{a\}} A$ iff $w \Vdash_{\mathscr{M}}^{\emptyset} A_{\omega+\omega'}$. From this, we can deduce the claim by observing that $\mathscr{D}_A(C) > 0$ iff C is of the form $A_{\omega+\omega'}$ for some $\omega \in (2^{\mathbb{N}})^X$ and $\omega' \in (2^{\mathbb{N}})^{\{a\}}$. □

5.2. The Proof-Theory of iCPL

We now introduce a natural deduction system $\mathbf{G}_{\mathsf{iCPL}}$ for iCPL. Similarly to the proof system $\mathbf{G}_{\mathsf{CPL}}^{\mathrm{Hyp}}$, the judgements of $\mathbf{G}_{\mathsf{iCPL}}$ are of the form $\Gamma \vdash \mathscr{b} \rightarrowtail A$, where Γ is a finite set of formulas. We let $\Gamma \vdash A$ be a shorthand of $\Gamma \vdash \top \rightarrowtail A$. Observe that we no more consider labeled formulas of the form $\mathscr{b} \leftarrowtail A$.

The rules of $\mathbf{G}_{\mathsf{iCPL}}$ comprise those illustrated in Fig. 2, where in the rule ($\mathbf{C}$I) it is assumed that $\mathrm{FN}(d) \subseteq \{a\}$ and in the rules ($\mathbf{C}$I) and ($\mathbf{C}\mathrm{E}_i$) a does not occur in either Γ or $\mathscr{b}$, together with all instances of the two axiom schema below:

$$\mathbf{C}^q_a(A \vee B) \to (A \vee (\mathbf{C}^q_a A)) \qquad (a \notin \mathrm{FN}(A)) \qquad (\mathbf{C}\vee)$$

$$\neg \mathbf{C}^q_a \mathscr{b} \qquad (\mathrm{FN}(\mathscr{b}) \subseteq \{a\}, \mu(\mathscr{b}) < q) \qquad (\mathbf{C}\bot)$$

We let $\mathbf{G}^-_{\mathsf{iCPL}}$ indicate $\mathbf{G}_{\mathsf{iCPL}}$ without ($\mathbf{C}\vee$) and ($\mathbf{C}\bot$).

The rule ($\mathbf{C}$I) is analogous to the corresponding intro-rule from $\mathbf{G}_{\mathsf{CPL}}$. Instead, the elimination rule ($\mathbf{C}$E) of $\mathbf{G}_{\mathsf{CPL}}$ (which relied on the suppressed judgements $\mathscr{b} \leftarrowtail A$) is replaced in $\mathbf{G}_{\mathsf{iCPL}}$ by two distinct elimination rules, which make no reference to a-decompositions.

A corollary of Proposition 3 and the completeness of $\mathbf{G}_{\mathsf{iCPL}}$ (Theorem 5) is that for any $\mathbf{D}$-free formula of system $\mathbf{G}_{\mathsf{CPL}}$, $\vdash \mathscr{b} \rightarrowtail A$ is derivable in $\mathbf{G}_{\mathsf{CPL}}$ iff $\vdash \mathscr{b} \rightarrowtail A$ is derivable in $\mathbf{G}_{\mathsf{iCPL}}$ (see Corollary 3 below).

Remark 5. *The reasons for distinguishing between the systems $\mathbf{G}_{\mathsf{iCPL}}$ and $\mathbf{G}^-_{\mathsf{iCPL}}$ are somehow analogous to the ones concerning intuitionistic modal logic (IML). Indeed, standard axiomatizations of IML include two axioms:*

$$\Diamond(A \vee B) \to \Diamond A \vee \Diamond B \qquad (\Diamond\vee)$$

$$\neg\Diamond\bot \qquad (\Diamond\bot)$$

which do not have a clear computational interpretation. Instead, a Curry-Howard correspondence can be defined for an axiomatization of IML (usually referred to as constructive modal logic*), which does not include these two axioms. In a similar way, the computational interpretation described in the next section will be limited to $\mathbf{G}^-_{\mathsf{iCPL}}$.*

Soundness and Completeness for $\mathbf{G}_{\mathsf{iCPL}}$. To establish the soundness of $\mathbf{G}_{\mathsf{iCPL}}$ with respect to iCPL-structures we need a few preliminary lemmas:

Lemma 17. *Let $\mathscr{M} = \langle W, \preceq, \alpha\rangle$ be an iCPL-structure. For any finite set X, $a \notin X$, $u \in W$, $\mathscr{b} \in (2^{\mathbb{N}})^X$, and formula A with $\mathrm{FN}(A) \subseteq X$, if $u, \omega \Vdash^X_{\mathscr{M}} A$, then $u, \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} A$ holds for any $\omega' \in (\{\omega\})^{\Uparrow X\cup\{a\}}$.*

Proof. By induction on the definition of the rules. □

Lemma 18. *Let $\mathscr{M} = \langle W, \preceq, \mathbf{i}\rangle$ be an iCPL-structure. For any finite set X, $a \notin X$, formula A with $\mathrm{FN}(A) \subseteq X$, $q \in (0, 1]$ and $\omega \in (2^{\mathbb{N}})^X$,*

$$w, \omega \Vdash^X_{\mathscr{M}} \mathbf{C}^q_a A \quad \Rightarrow \quad w, \omega \Vdash^X_{\mathscr{M}} A$$

Proof. $w', \omega \Vdash^X_{\mathscr{M}} \mathbf{C}^q_a A$ implies that $\mu(\{\omega' \in 2^{\mathbb{N}} \mid u, \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} A\}) \geq q > 0$. From $u, \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} A$ one can deduce by induction on A that $u, \omega \Vdash^X_{\mathscr{M}} A$. Since there exists at least one such ω', we conclude that $u, \omega \Vdash^X_{\mathscr{M}} A$ holds.

□

Initial Sequents

$$\frac{\boldsymbol{b} \vDash \boldsymbol{x}_i^a}{\Gamma \vdash \boldsymbol{b} \rightarrowtail x_a^i}\ \mathsf{id}_{C1} \qquad \frac{\boldsymbol{b} \vDash \neg \boldsymbol{x}_i^a}{\Gamma \vdash \boldsymbol{b} \rightarrowtail \neg x_a^i}\ \mathsf{id}_{C2} \qquad \frac{}{\Gamma, A \vdash \boldsymbol{b} \rightarrowtail A}\ \mathsf{id}_I$$

Structural Rules

$$\frac{\boldsymbol{b} \vDash \bot}{\Gamma \vdash \boldsymbol{b} \rightarrowtail A}\ \bot\mathrm{E} \qquad \frac{\Gamma \vdash \boldsymbol{c} \rightarrowtail A \qquad \Gamma \vdash \boldsymbol{d} \rightarrowtail A \qquad \boldsymbol{b} \vDash \boldsymbol{c} \vee \boldsymbol{d}}{\Gamma \vdash \boldsymbol{b} \rightarrowtail A}\ \vDash$$

$$\frac{\Gamma, \boldsymbol{c} \vdash \boldsymbol{d} \rightarrowtail A \qquad \boldsymbol{b} \vDash \boldsymbol{c} \wedge \boldsymbol{d}}{\Gamma \vdash \boldsymbol{b} \rightarrowtail A}\ (\rightarrowtail\mathrm{I}) \qquad \frac{\Gamma \vdash \boldsymbol{d} \rightarrowtail A \qquad \Gamma \vdash \boldsymbol{b} \rightarrowtail \boldsymbol{d}}{\Gamma \vdash \boldsymbol{b} \rightarrowtail A}\ (\rightarrowtail\mathrm{E})$$

Logical Rules

$$\frac{\Gamma \vdash \boldsymbol{b} \rightarrowtail A}{\Gamma \vdash \boldsymbol{b} \rightarrowtail A \vee B}\ (\vee\mathrm{I}_1) \qquad \frac{\Gamma \vdash \boldsymbol{b} \rightarrowtail B}{\Gamma \vdash \boldsymbol{b} \rightarrowtail A \vee B}\ (\vee\mathrm{I}_2)$$

$$\frac{\Gamma \vdash \boldsymbol{b} \rightarrowtail A \vee B \qquad \Gamma, A \vdash \boldsymbol{b} \rightarrowtail C \qquad \Gamma, B \vdash \boldsymbol{b} \rightarrowtail C}{\Gamma \vdash \boldsymbol{b} \rightarrowtail C}\ (\vee\mathrm{E})$$

$$\frac{\Gamma \vdash \boldsymbol{b} \rightarrowtail A \qquad \Gamma \vdash \boldsymbol{b} \rightarrowtail B}{\Gamma \vdash \boldsymbol{b} \rightarrowtail A \wedge B}\ (\wedge\mathrm{I}) \qquad \frac{\Gamma \vdash \boldsymbol{b} \rightarrowtail A}{\Gamma \vdash \boldsymbol{b} \rightarrowtail A \wedge B}\ (\wedge\mathrm{E}_1) \qquad \frac{\Gamma \vdash \boldsymbol{b} \rightarrowtail B}{\Gamma \vdash \boldsymbol{b} \rightarrowtail A \wedge B}\ (\wedge\mathrm{E}_2)$$

Counting Rules

$$\frac{\Gamma \vdash \boldsymbol{b} \wedge \boldsymbol{c} \rightarrowtail A \qquad \mu(\boldsymbol{c}) \geq q}{\Gamma \vdash \boldsymbol{b} \rightarrowtail \mathbf{C}_a^q A}\ (\mathbf{C}\mathrm{I})$$

$$\frac{\Gamma \vdash \boldsymbol{c} \rightarrowtail \mathbf{C}_a^q A}{\Gamma \vdash \boldsymbol{b} \rightarrowtail A}\ (\mathbf{C}\mathrm{E}_1)\ a \notin A \qquad \frac{\Gamma \vdash \boldsymbol{b} \rightarrowtail \mathbf{C}_a^q A \qquad \Gamma, A \vdash \boldsymbol{b} \rightarrowtail B}{\Gamma \vdash \boldsymbol{b} \rightarrowtail \mathbf{C}_a^{qs} B}\ (\mathbf{C}\mathrm{E}_2)$$

Figure 2: Rules of $\mathbf{G}_{\mathsf{iCPL}}$.

Proposition 4 (Soundness of $\mathbf{G}_{\mathsf{iCPL}}$). *If $\Gamma \vdash \boldsymbol{b} \rightarrowtail A$, then $\Gamma \vDash_I \boldsymbol{b} \rightarrowtail A$.*

Proof. Let $\mathcal{M} = \langle W, \preceq, \mathbf{i} \rangle$ be a iCPL-structure. We argue by induction on the rules of $\mathbf{G}_{\mathsf{iCPL}}$ (selected cases):

- if the last rule is of the form

$$\frac{\Gamma \vdash^{X \cup \{a\}} \boldsymbol{b} \wedge \boldsymbol{d} \rightarrowtail A \qquad \mu(\boldsymbol{d}) \geq q}{\Gamma \vdash \boldsymbol{b} \rightarrowtail \mathbf{C}_a^q A}\ (\mathbf{C}\mathrm{I})$$

  where $\mathrm{FN}(\boldsymbol{b}) \subseteq X$ and $\mathrm{FN}(\boldsymbol{d}) \subseteq \{a\}$ (with $a \notin X$), then by the IH together with Lemma 13, for all $w \in W$, for all $\omega \in [\![\boldsymbol{b}]\!]_X$ and $\omega' \in [\![\boldsymbol{d}]\!]_{\{a\}}$, if $w, \omega \Vdash_{\mathcal{M}}^X \Gamma$, then for all $w' \succeq w$, we have $w', \omega + \omega' \Vdash_{\mathcal{M}}^{X \cup \{a\}} A$. Since $\mu(\boldsymbol{d}) \geq q$, we deduce that for all $w' \succeq w$, the set $\{\omega' \mid w', \omega + \omega' \Vdash_{\mathcal{M}}^{X \cup \{a\}} A\}$ has measure greater than q, and we conclude then that $w, \omega \Vdash_{\mathcal{M}}^X \mathbf{C}_a^q A$.
- if the last rule is of the form

$$\frac{\Gamma \vdash \boldsymbol{b} \rightarrowtail \mathbf{C}_a^q A}{\Gamma \vdash \boldsymbol{b} \rightarrowtail A}\ (\mathbf{C}\mathrm{E}_1) \quad a \notin \mathrm{FN}(A)$$

let $w \in W$, $\omega \in [\![\mathscr{b}]\!]_X$ and $w, \omega \Vdash^X_{\mathscr{M}} \Gamma$. By IH we deduce that the set

$$S := \left\{\omega' \mid w, \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} A\right\}$$

has measure greater than $q > 0$, and it is thus non-empty. Let ω' be an element of S. Again, by the IH we deduce that $w, \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} A$. Since $a \notin \mathrm{FN}(C)$, by Lemma 18, we conclude then that $w, \omega \Vdash^X_{\mathscr{M}} A$.

- if the last rule is of the form

$$\frac{\Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^q_a A \qquad \Gamma, A \vdash^{X\cup\{a\}} \mathscr{b} \rightarrowtail C}{\Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^{qs}_a C} \ (\mathbf{C}\mathrm{E}_2) \quad a \notin \mathrm{FN}(\Gamma)$$

let $w \in W$, $\omega \in [\![\mathscr{b}]\!]_X$ and $w, \omega \Vdash^X_{\mathscr{M}} \Gamma$. By IH we deduce that the set

$$S := \left\{\omega' \mid w, \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} A\right\}$$

has measure greater than $q \geq qs$ and is contained in $\{\omega' \mid w, \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} C\}$. We can conclude then that $w, \omega \Vdash^X_{\mathscr{M}} \mathbf{C}^{qs}_a C$.

- Axiom ($\mathbf{C}\vee$) is valid: suppose $w, \omega \Vdash^X_{\mathscr{M}} \mathbf{C}^q_a(A \vee B)$, where $a \notin \mathrm{FN}(A)$. Then, the set $\{\omega' \mid w, \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} A \vee B\}$ has measure greater than q. Observe that $w, \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} A$ holds iff $w, \omega \Vdash^X_{\mathscr{M}} A$. So, suppose $w, \omega \Vdash^X_{\mathscr{M}} A$ does not hold; then for any ω', $w, \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} A$ does not hold, and thus the set $\{\omega' \mid w, \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathscr{M}} B\}$ must have measure greater than q. We have thus proved that either $w, \omega \Vdash^X_{\mathscr{M}} A$ holds or $w, \omega \Vdash^X_{\mathscr{M}} \mathbf{C}^q_a B$ holds, and thus that $w, \omega \Vdash^X A \vee \mathbf{C}^q_a B$.
- Axiom ($\mathbf{C}\bot$) is valid: suppose $\mathrm{FN}(\mathscr{b}) \subseteq \{a\}$ and $\mu(\mathscr{b}) < q$. For any $w \in W$, the set $\{\omega' \mid w, \omega' \vdash^{\{a\}} \mathscr{b}\}$ coincides with $[\![\mathscr{b}]\!]$, and has thus measure $< q$ by hypothesis. We deduce that for any $w' \geq w$, $w', * \Vdash^{\emptyset}_{\mathscr{M}} \mathbf{C}^q_a \mathscr{b}$ does not hold, and we conclude that $w, * \Vdash^{\emptyset}_{\mathscr{M}} \neg\mathbf{C}^q_a \mathscr{b}$.

□

The completeness of $\mathbf{G}_{\mathsf{iCPL}}$ is proved by adapting standard completeness arguments for intuitionistic modal logic (see e.g. [65]). Before proceeding to the proof, we need to study provability in iCPL a bit further.

Definition 12. *For any formula A, finite set X, and $\omega \in (2^{\mathbb{N}})^{\mathscr{A}}$, we let A^ω be the formula $\mathsf{Th}_{\mathrm{FN}(A)}(|\omega|_A) \to A$.*

We will exploit the following proof-theoretic variant of Lemma 15, which can be proved in a similar way:

Lemma 19. *Let A be a formula with $\mathrm{FN}(A) \subseteq X \cup Y$, where $X \cap Y = \emptyset$. Then, there exist formulas A^X_w, with $\mathrm{FN}(A^X_w) \subseteq X$, where w ranges over the valuations of the variables of name in Y in A, such that $\vdash A \leftrightarrow (\bigvee_w \mathsf{Th}(w) \wedge A^X_w)$.*

Definition 13. *For any formula A with* $\mathrm{FN}(A) \subseteq \{a\}$*, let $\mathscr{b}_A$ be the Boolean formula $\bigvee_v \{\mathsf{Th}(v) \mid A_v^X \nvdash \bot\}$, where v ranges over all valuations of the Boolean variables of A.*

For any formula A with $\mathrm{FN}(A) \subseteq X \cup \{a\}$*, with $a \notin X$, and for all $\omega \in (2^{\mathbb{N}})^{\mathscr{A}}$, let $\mathscr{b}_A^{X \mapsto \omega}$ be the Boolean formula $\bigvee_v \{\mathsf{Th}(v) \mid A_{|\omega|_A + v} \nvdash \bot\}$, where v ranges over all valuations of the Boolean variables of name a in A.*

The following result shows that measuring the formula $\mathscr{b}_A^{X \mapsto \omega}$ provides a test to know if $\mathbf{C}_a^q A$ is consistent under $\mathsf{Th}(\omega)$:

Lemma 20. *Let A be a formula with* $\mathrm{FN}(A) \subseteq X \cup \{a\}$ *with $a \notin X$. Then for all $\omega \in (2^{\mathbb{N}})^{\mathscr{A}}$ and $q \in (0, 1]$, if $\mu(\mathscr{b}_A^{X \mapsto \omega}) < q$ then $\mathsf{Th}(\omega) \cup \{\mathbf{C}_a^q A\} \vdash \bot$.*

Proof. The claim follows from the observation that $\mathsf{Th}(v) \cup \{A\} \vdash \mathscr{b}_A^{X \mapsto \omega}$, where v is the valuation of the variables of name X in A induced by ω, can be deduced by observing that $\mathsf{Th}(v) \cup \{A\} \vdash \bigvee_w A_{|\omega|+w} \wedge \mathsf{Th}(w) \vdash \mathscr{b}_A^{X \mapsto \omega}$, and by applying the rule ($\mathbf{C}\mathrm{E}_2$) and Axiom ($\mathbf{C}\bot$). $\square$

The proof of the completeness theorem relies on the construction of a suitable "canonical model" based on sets of formulas. Let us first introduce some terminology. Let Γ be a (possibly infinite) set of formulas Γ. We say that Γ is:

- *A-consistent* if $\Gamma \nvdash A$;
- *consistent* if it is $\bot$-consistent;
- *super-consistent* if for all $\omega \in (2^{\mathbb{N}})^{\mathscr{A}}$, $\mathsf{Th}(\omega) \cup \Gamma \nvdash \bot$;
- *closed* if $A_1, \ldots, A_n \in \Gamma$ and $A_1, \ldots, A_n \vdash A$ implies $A \in \Gamma$, and if $(\mathscr{d} \to A)^\omega \in \Gamma$ (where $\mathrm{FN}(\mathscr{d}) \subseteq \{a\}$) and $\mu(\mathscr{d}) \geq q$ then $(\mathbf{C}_a^q A)^\omega \in \Gamma$.

For any set Γ we indicate as $\overline{\Gamma}$ the smallest closed set containing Γ. Let Θ be the set of all super-consistent and closed sets Γ satisfying the following conditions, for all $\omega \in (2^{\mathbb{N}})^{\mathscr{A}}$:

$$(A \vee B)^\omega \in \Gamma \quad \Rightarrow \quad A^\omega \in \Gamma \text{ or } B^\omega \in \Gamma \qquad (\vee\text{-closure})$$
$$(\mathbf{C}_a^q A)^\omega \in \Gamma \quad \Rightarrow \quad \exists \mathscr{d} \text{ s.t. } \mu(\mathscr{d}) \geq q \text{ and } (\mathscr{d} \to A)^\omega \in \Gamma \qquad (\mathbf{C}\text{-closure})$$

The fundamental ingredients of the completeness proof are the two lemmas below (proved in Appendix 5), which will be used to lift any consistent set of formulas to an element of Θ.

Lemma 21 (Saturation Lemma)**.** *Let Γ be a super-consistent set and let $\omega_0 \in (2^{\mathbb{N}})^{\mathscr{A}}$ be such that $\mathsf{Th}(\omega_0) \cup \Gamma$ is A-consistent. Then there exists a set $\Gamma^* \supseteq \Gamma$ such that $\Gamma^* \in \Theta$ and $\mathsf{Th}_X(\omega_0) \cup \Gamma^*$ is A-consistent.*

Lemma 22. *Let Γ be A-consistent. Then there exists a super-consistent set Δ and $\omega \in (2^{\mathbb{N}})^{\mathscr{A}}$ such that $\mathsf{Th}(\omega) \cup \Delta$ is A-consistent and $\Gamma \subseteq \overline{\mathsf{Th}(\omega) \cup \Delta}$.*

Theorem 5 (Completeness of $\mathbf{G}_{\mathsf{iCPL}}$)**.** *If $\Gamma \vDash_I \mathscr{b} \rightarrowtail A$, then $\Gamma \vdash \mathscr{b} \rightarrowtail A$.*

Proof. Let $\mathscr{T} = \langle \Theta, \subseteq, \mathbf{i} \rangle$, where $\mathbf{i}(\mathscr{p}) = \{\Gamma \mid \mathscr{p} \in \Gamma\}$. We will prove that for all $\Gamma \in \Theta$ and $\omega \in (2^{\mathbb{N}})^X$, $\Gamma, \omega \Vdash_{\mathscr{T}}^X A$ iff $A^\omega \in \Gamma$. From this claim the theorem is proved as follows: suppose $\Gamma \not\vdash \mathscr{b} \rightarrowtail A$; by Lemma 22 we obtain a super-consistent set Δ and $\omega_0 \in (2^{\mathbb{N}})^{\mathscr{A}}$ such that $\mathsf{Th}(\omega_0) \cup \Delta$ is A-consistent and $\Gamma \cup \{\mathscr{b}\} \subseteq \overline{\mathsf{Th}(\omega_0) \cup \Delta}$. By Lemma 21 Δ extends to $\Gamma^* \in \Theta$ such that $\Gamma^* \supseteq \Gamma \cup \{\mathscr{b}\}$ and $\mathsf{Th}(\omega_0) \cup \Gamma^*$ is A-consistent. Then using the claim, from $\Gamma \subseteq \Gamma^*$, we deduce that $\Gamma^*, |\omega_0|_X \Vdash^X_{\mathscr{G}} \Gamma$; moreover, since $\mathscr{b} \in \Gamma^*$ and $\mathsf{Th}(\omega_0) \cup \Gamma^*$ is A-consistent, both $|\omega_0|_X \in [\![\mathscr{b}]\!]_X$, and $\Gamma^*, |\omega_0|_X \not\Vdash^X_{\mathscr{G}} A$ hold; we can conclude then that $\Gamma \not\Vdash^X_{\mathscr{G}} \mathscr{b} \rightarrowtail A$, and thus that $\Gamma \not\models_I \mathscr{b} \rightarrowtail A$.

Let us now prove the claim, by arguing by induction on A (selected cases):

- if $A = B \vee C$, and $\Gamma, \omega \Vdash^X_{\mathscr{G}} A$, then either $\Gamma, \omega \Vdash^X_{\mathscr{G}} B$ or $\Gamma, \omega \Vdash^X_{\mathscr{G}} C$ holds. If the first holds, then by the IH $B^\omega \in \Gamma$, which implies $(B \vee C)^\omega \in \Gamma$, and one can argue similarly if the second holds. Conversely, if $(B \vee C)^\omega \in \Gamma$, by $\vee$-closure either $B^\omega \in \Gamma$ or $C^\omega \in \Gamma$, so by the IH in each case we deduce $\Gamma, \omega \Vdash^X_{\mathscr{G}} A$.
- if $A = B \to C$, then suppose that $\Gamma, \omega \Vdash^X_{\mathscr{G}} A$; then for all $\Gamma' \in \Theta$ such that $\Gamma' \supseteq \Gamma$ and $\Gamma', \omega \Vdash^X_{\mathscr{G}} B$, also $\Gamma', \omega \Vdash^X_{\mathscr{G}} C$ holds. Suppose $\Gamma \cup \{B^\omega\}$ is super-consistent. Furthermore, suppose $\mathsf{Th}(\omega) \cup \Gamma \cup \{B^\omega\} \not\vdash C^\omega$. Then, by Lemma 21 (with $\omega_0 = \omega$ and $A = C$) there exists $\Gamma' \in \Theta$ with $\Gamma' \supseteq \Gamma \cup \{B^\omega\}$ such that $C^\omega \notin \Gamma'$. Using the IH we deduce then that $\Gamma', \omega \Vdash^X_{\mathscr{G}} B$ but $\Gamma', \omega \not\Vdash^X_{\mathscr{G}} C$, against the assumption. We conclude then that $\mathsf{Th}(\omega) \cup \Gamma \cup \{B^\omega\} \vdash C^\omega$, and thus that $\Gamma \vdash (B \to C)^\omega$, which implies by closure $(B \to C)^\omega \in \Gamma$.
  Suppose now that $\Gamma \cup \{B^\omega\}$ is not super-consistent. There exists ω' such that $\mathsf{Th}(\omega') \cup \Gamma \cup \{B^\omega\} \vdash \bot$. Since $\mathsf{Th}(\omega') \cup \Gamma$ is consistent the only possibility is that $\mathsf{Th}(\omega') \vdash \mathsf{Th}^{|B|}_{\mathrm{FN}(B)}(\omega)$ (which forces $\omega|_{|B|,\mathrm{FN}(B)}$ to coincide with $\omega'|_{|B|,\mathrm{FN}(B)}$) and $\mathsf{Th}(\omega') \cup \Gamma \cup \{B\} \vdash \bot$; but then we also have that $\mathsf{Th}(\omega) \cup \Gamma \cup \{B\} \vdash \bot$, which implies $\mathsf{Th}(\omega) \cup \Gamma \vdash B \to C$ and thus $\Gamma \vdash (B \to C)^\omega$; by closure this implies then $(B \to C)^\omega \in \Gamma$. Conversely, suppose $(B \to C)^\omega \in \Gamma$ and let $\Gamma' \in \Theta$ be such that $\Gamma' \supseteq \Gamma$ and $\Gamma', \omega \Vdash^X_{\mathscr{G}} B$; by the IH this implies $B^\omega \in \Gamma'$, and since $(B \to C)^\omega \in \Gamma \subseteq \Gamma'$ we also deduce by closure $C^\omega \in \Gamma'$, which by the IH implies $\Gamma', \omega \Vdash^X_{\mathscr{G}} C$.
- if $A = \mathbf{C}^q_a B$, then suppose that $\Gamma, \omega \Vdash^X_{\mathscr{G}} A$; then by the IH the Borel set

  $$S = \Big\{\omega' \mid B^{\omega + \omega'} \in \Gamma\Big\} \supseteq \Big\{\omega' \mid \Gamma, \omega + \omega' \Vdash^{X \cup \{a\}}_{\mathscr{G}} B\Big\}$$

  has measure greater than q. Observe that for all $\omega' \in 2^{\mathbb{N}}$, $B^{\omega+\omega'}$ is equivalent to

  $$\mathsf{Th}^K_X(|\omega|_X) \to \Big(\mathsf{Th}^H_{\{a\}}(\omega') \to B\Big).$$

  where K is the maximum i such that $\boldsymbol{x}^i_b$ occurs in B, for some $b \in X$, and H is the maximum i such that $\boldsymbol{x}^i_a$ occurs in B. Let $\mathscr{d} := \bigvee\{\mathsf{Th}^H_{\{a\}}(\omega') \mid \omega' \in S\}$ (observe that the disjunction is necessarily finite); since Γ is closed, we deduce then that $(\mathscr{d} \to B)^\omega \in \Gamma$ and that $\mu(\mathscr{d}) \geq q$; again, by closure, this implies $(\mathbf{C}^q_a B)^\omega \in \Gamma$.
  For the converse direction, suppose $(\mathbf{C}^q_a B)^\omega \in \Gamma$ and let $\Gamma' \supseteq \Gamma$; since Γ is $\mathbf{C}$-closed, there exists a Boolean formula $\mathscr{d}$ with $\mu(\mathscr{d}) \geq q$ such that

$(\mathscr{d} \to B)^\omega \in \Gamma \subseteq \Gamma'$. This implies that for all $\omega' \in [\![\mathscr{d}]\!]$, $B^{\omega+\omega'} \in \Gamma'$. Hence, by the IH for all $\omega' \in [\![\mathscr{d}]\!]$, $\Gamma', \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathcal{T}} B$. Since $\mu(\mathscr{d}) \geq q$, the set of ω' such that $\Gamma', \omega + \omega' \Vdash^{X\cup\{a\}}_{\mathcal{T}} B$ has measure greater than q, and we can conclude $\Gamma, \omega \Vdash^{X}_{\mathcal{T}} A$.

$\square$

Using Proposition 3, together with the completeness theorems of $\mathbf{G}_{\mathsf{CPL}}$ (with respect to the semantics of CPL) and $\mathbf{G}_{\mathsf{iCPL}}$ (with respect to the Kripke semantics of iCPL) one deduces the following:

Corollary 3. *For any* $\mathbf{D}$*-free formula* A *of* CPL*,* $\vdash \mathscr{b} \rightarrowtail A$ *is derivable in* $\mathbf{G}_{\mathsf{CPL}}$ *iff* $\vdash \mathscr{b} \rightarrowtail A$ *is derivable in* $\mathbf{G}_{\mathsf{iCPL}}$*.*

5.3. A Calculus for Purely Intuitionistic Formulas

As shown by Lemma 15, formulas of iCPL can be seen as families of purely intuitionistic formulas. A natural question is thus whether a proof of an iCPL-formula can similarly be seen as a family of proofs of purely intuitionistic formulas. In this subsection we show that this is indeed possible: derivations in $\mathbf{G}^-_{\mathsf{iCPL}}$ can be associated with families of derivations in a proof system $\mathbf{G}_{\mathsf{iCPL}_0}$ designed for purely intuitionistic formulas.

Let iCPL_0 indicate the fragment of iCPL formed by purely intutionistic formulas, i.e. formulas defined by the grammar below:

$$A ::= \bot \;\Big|\; \top \;\Big|\; \mathscr{p} \;\Big|\; A \to A \;\Big|\; A \wedge A \;\Big|\; A \vee A \;\Big|\; \mathbf{C}^q A$$

where $q \in (0,1] \cap \mathbb{Q}$. Notice that, since formulas do not contain Boolean variables, counting quantifiers in iCPL_0 are not named. We use $\mathbf{C}^{q_1 * \dots * q_n} A$ as an abbreviation for $\mathbf{C}^{q_1} \dots \mathbf{C}^{q_n} A$.

The natural deduction system $\mathbf{G}_{\mathsf{iCPL}_0}$ for iCPL_0 is defined as follows: as in $\mathbf{G}_{\mathsf{iCPL}}$ judgements are of the form $\Gamma \vdash \mathscr{b} \rightarrowtail A$, where now A and the formulas in Γ are iCPL_0-formulas, and $\mathscr{b}$ is a Boolean formula. The presence of the formula $\mathscr{b}$ is due to the fact that we still want proofs to manipulate finitary Borel sets, as in $\mathbf{G}_{\mathsf{iCPL}}$, by means of suitable structural rules. The rules of $\mathbf{G}_{\mathsf{iCPL}_0}$ are illustrated in Fig. 3. For simplicity, and since this will be enough for the CHC, we omit introduction and elimination rules for $\top, \bot, \wedge, \vee$, but these can defined as in $\mathbf{G}_{\mathsf{iCPL}}$.

The following lemma is easily checked by induction on the rules of $\mathbf{G}_{\mathsf{iCPL}_0}$:

Lemma 23. *In* $\mathbf{G}_{\mathsf{iCPL}_0}$*, if* $\Gamma \vdash \mathscr{b} \wedge \mathscr{c} \rightarrowtail A$*, where* $\mathscr{b} \wedge \mathscr{c}$ *is satisfiable, then* $\Gamma \vdash \mathscr{b} \rightarrowtail A$*.*

Our goal is to establish a precise correspondence between $\mathbf{G}^-_{\mathsf{iCPL}}$ and $\mathbf{G}_{\mathsf{iCPL}_0}$. For any purely intuitionistic formula A of iCPL, let $|A|$ indicate the corresponding formula of iCPL_0 obtained by deleting names from counting quantifiers, i.e. by replacing $\mathbf{C}^q_a$ by $\mathbf{C}^q$. By Lemma 15, any judgement $\Gamma \vdash \mathscr{b} \rightarrowtail A$ of iCPL is naturally associated with the family of iCPL_0-judgements of the form $|\Gamma_v| \vdash \mathsf{Th}(v) \rightarrowtail A_v$ where v ranges over all valuations of the Boolean variables of Γ

Identity Rule

$$\frac{}{\Gamma, A \vdash \mathbb{b} \rightarrowtail A}\ (\mathrm{id})$$

Structural Rules

$$\frac{\mathbb{b} \vDash \bot}{\Gamma \vdash \mathbb{b} \rightarrowtail A}\ (\bot) \qquad \frac{\Gamma \vdash \mathbb{c} \rightarrowtail A \qquad \Gamma \vdash \mathbb{d} \rightarrowtail A \qquad \mathbb{b} \vDash (\mathbb{c} \wedge \boldsymbol{x}_a^i) \vee (\mathbb{d} \wedge \neg \boldsymbol{x}_a^i)}{\Gamma \vdash \mathbb{b} \rightarrowtail A}\ (\mathsf{m})$$

Logical Rules

$$\frac{\Gamma, A \vdash \mathbb{b} \rightarrowtail B}{\Gamma \vdash \mathbb{b} \rightarrowtail (A \to B)}\ (\to\mathrm{I}) \qquad \frac{\Gamma \vdash \mathbb{b} \rightarrowtail (A \to B) \qquad \Gamma \vdash \mathbb{b} \rightarrowtail A}{\Gamma \vdash \mathbb{b} \rightarrowtail B}\ (\to\mathrm{E})$$

Counting Rules

$$\frac{\Gamma \vdash \mathbb{b} \wedge \mathbb{c} \rightarrowtail A \qquad \mu(\mathbb{c}) \geq q}{\Gamma \vdash \mathbb{b} \rightarrowtail \mathbf{C}^q A}\ (\mathbf{C}\mathrm{I}) \qquad \frac{\Gamma \vdash \mathbb{b} \rightarrowtail \mathbf{C}^q A \qquad \Gamma, A \vdash \mathbb{b} \rightarrowtail B}{\Gamma \vdash \mathbb{b} \rightarrowtail \mathbf{C}^{qs} B}\ (\mathbf{C}\mathrm{E})$$

Figure 3: Rules of $\mathbf{G}_{\mathsf{iCPL}_0}$

and A that satisfy $\mathbb{b}$. The following proposition extends this correspondence to derivations:

Proposition 5. *$\Gamma \vdash \mathbb{b} \rightarrowtail A$ is derivable in $\mathbf{G}^{-}_{\mathsf{iCPL}}$ if and only if for all valuations $v \vDash \mathbb{b}$, $|\Gamma_v| \vdash \mathsf{Th}(v) \rightarrowtail |A_v|$ is derivable in $\mathbf{G}_{\mathsf{iCPL}_0}$.*

Proof. ($\Leftarrow$) The claim follows from the decomposition lemma (Lemma 15).

($\Rightarrow$) We argue by induction on the rules of $\mathbf{G}_{\mathsf{iCPL}}$. All propositional cases are straightforward, so we only focus on the rules for counting quantifiers.

- If the derivation of $\Gamma \vdash A$ ends with a rule-instance of the following form:

$$\frac{\Gamma \vdash \mathbb{b} \wedge \mathbb{d} \rightarrowtail A \qquad \mu(\mathbb{d}) \geq q}{\Gamma \vdash \mathbb{b} \rightarrowtail \mathbf{C}^q_a A}$$

  then by the IH, for all valuations $v \vDash \mathbb{b}$ and $w \vDash \mathbb{d}$, there is a derivation of $|\Gamma_v| \vdash \mathsf{Th}(v) \rightarrowtail |A_{v+w}|$, and we can construct the desired derivation as follows:

$$\frac{\dfrac{\left\{ \dfrac{|\Gamma_v| \vdash \mathsf{Th}(v) \wedge \mathsf{Th}(w) \rightarrowtail |A_{v+w}|}{|\Gamma_v| \vdash \mathsf{Th}(v) \wedge \mathsf{Th}(w) \rightarrowtail \bigvee_w |A_{v+w}|}\ (\vee\mathrm{I}) \right\}_{w \vDash \mathbb{d}}}{|\Gamma_v| \vdash \mathsf{Th}(v) \wedge \mathbb{d} \rightarrowtail \bigvee_w |A_{v+w}|}\ (\mathsf{m}) \qquad \mu(\mathbb{d}) \geq q}{|\Gamma_v| \vdash \mathsf{Th}(v) \rightarrowtail |(\mathbf{C}^q_a A)_v|}\ (\mathbf{C}\mathrm{I})$$

- If the derivation of $\Gamma \vdash A$ ends with a rule-instance of the following form:

$$\frac{\Gamma \vdash \mathbb{b} \rightarrowtail \mathbf{C}^q_a A}{\Gamma \vdash \mathbb{b} \rightarrowtail A}\ (\mathbf{C}\mathrm{E}_2)$$

then, since $a \notin \mathrm{FN}(A)$, it follows that $(\mathbf{C}^q_a A)_v = A_v$. So, we can conclude by the IH.

- If the derivation of $\Gamma \vdash A$ ends with a rule-instance of the following form:

$$\frac{\Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^q_a A \qquad \Gamma, A \vdash \mathscr{b} \rightarrowtail B}{\Gamma \vdash \mathscr{b} \rightarrowtail \mathbf{C}^{qs}_a B}$$

  then by the IH for all $v \vDash \mathscr{b}$ and for all valuation w of the variables of name a, there exist derivations of $|\Gamma_v| \vdash \mathsf{Th}(v) \rightarrowtail |(\mathbf{C}^q_a A)_v|$ and $|\Gamma_v|, |A_{v+w}| \vdash \mathsf{Th}(v) \wedge \mathsf{Th}(w) \rightarrowtail |B_{v+w}|$, so we can construct the desired derivation as below

$$\dfrac{|\Gamma_v| \vdash \mathsf{Th}(v) \rightarrowtail |(\mathbf{C}^q_a A)_v| \qquad \dfrac{\left\{ \dfrac{\dfrac{|\Gamma_v|, |A_{v+w}| \vdash \mathsf{Th}(v) \wedge \mathsf{Th}(w) \rightarrowtail |B_{v+w}|}{|\Gamma_v|, |A_{v+w}| \vdash \mathsf{Th}(v) \rightarrowtail |B_{v+w}|}\,\text{[Lemma 23]}}{|\Gamma_v|, |A_{v+w}| \vdash \mathsf{Th}(v) \rightarrowtail (\bigvee_w |B_{v+w}|)}\,(\vee\mathrm{I}) \right\}_w}{|\Gamma_v|, \bigvee_w |A_{v+w}| \vdash \mathsf{Th}(v) \rightarrowtail (\bigvee_w |B_{v+w}|)}\,(\vee\mathrm{E})}{|\Gamma_v| \vdash \mathsf{Th}(v) \rightarrowtail |(\mathbf{C}^{qs}_a B)_v|}\,(\mathbf{C}\mathrm{E})$$

$\square$

6. The Curry-Howard Correspondence

The judgements $\Gamma \vdash \mathscr{b} \rightarrowtail A$ of iCPL and iCPL_0 have a natural computational interpretation: they express program specifications of the form "Π yields a proof of A from Γ, whenever its sampled function satisfies $\mathscr{b}$". Following this intuition, in this section we provide a computational interpretation of iCPL_0.

First we introduce a variant $\Lambda^{\{\}}_{\mathsf{PE}}$ of the *probabilistic event λ-calculus* from [20], and we define a probabilistic notion of (head-)normalization for the terms of this calculus. Then we show that any derivation Π of $\mathbf{G}_{\mathsf{iCPL}_0}$ can be decorated with a term t^Π from $\Lambda^{\{\}}_{\mathsf{PE}}$ in such a way that normalization of Π results in the evaluation of t^Π. Using Proposition 5, this yields then a computational interpretation of iCPL, by associating each derivation of $\mathbf{G}^-_{\mathsf{iCPL}}$ with a family of functional programs. Finally, we show that counting-quantified formulas reveal the probability of termination of their terms: if t decorates a proof of $\mathbf{C}^q A$, then t (head-)normalizes with probability q.

6.1. The Probabilistic Event λ-Calculus

In extensions of the standard λ-calculus with probabilistic choice operators, two evaluation strategies are generally considered: the *call-by-name* (CbN) strategy might duplicate occurrences of a choice operator before evaluating it, while the *call-by-value* (CbV) strategy evaluates choice operators before possibly duplicating them. Crucially, the probability of termination of a program might differ depending on the chosen strategy. For example, let us consider the application of the term $\mathbf{2} := \lambda yx.y(yx)$ (i.e. the second *Church numeral*) to $I \oplus \Omega$, where $I = \lambda x.x$ and Ω is the diverging term $(\lambda x.xx)\lambda x.xx$. Under CbN, the redex $\mathbf{2}(I \oplus \Omega)$ first produces $\lambda x.(I \oplus \Omega)\big((I \oplus \Omega)x\big)$, then reduces to any of the terms $\lambda x.u(vx)$, with u, v chosen from $\{I, \Omega\}$, each with probability $\frac{1}{4}$. Since $\lambda x.u(vx)$ converges only when $u = v = I$, the probability of convergence is $\frac{1}{4}$. Under CbV, in $\mathbf{2}(I \oplus \Omega)$ one first evaluates $I \oplus \Omega$, and then passes the result to $\mathbf{2}$, returning either the converging term $I(Ix)$ or the diverging term $\Omega(\Omega x)$, each with probability $\frac{1}{2}$.

If we now think of the Church numeral $\mathbf{2}$ as a proof of some counting quantified formula, we see that, depending on the reduction strategy we have in mind, it should prove a *different* formula. Indeed, given that $I \oplus \Omega$ proves $\mathbf{C}^{1/2}(A \to A)$, in the CbN case, $\mathbf{2}$ proves $\mathbf{C}^{1/2}(A \to A) \to A \to \mathbf{C}^{1/4}A$, since only in one over four cases it yields a proof of A, while in the CbV case, $\mathbf{2}$ proves the formula $\mathbf{C}^{1/2}(A \to A) \to A \to \mathbf{C}^{1/2}A$, as it yields a proof of A in one case over two.

In the literature on probabilistic λ-calculi, the apparent incompatibility of CbN and CbV evaluation is usually resolved by restricting to calculi admitting only one chosen strategy. Nevertheless, the observation above suggests that, if functional programs are typed using counting quantifiers, it should become possible to make the two evaluation strategies coexist, by assigning them different types.

Actually, a few recent approaches [18, 23, 25, 20] already suggest ways to make CbN and CbV evaluation coexist. In particular, in the *probabilistic event λ-calculus* [20] the choice operator $\oplus$ is decomposed into two different operators, yielding a confluent calculus: a *choice operator* $t \oplus_a u$, depending on some probabilistic event $a \in \{0, 1\}$, and a *probabilistic event generator* $\nu a.t$, which actually "flips the coin". In this language, the CbN and CbV applications of $\mathbf{2}$ to $I \oplus \Omega$ are encoded by two *distinct* terms $\mathbf{2}(\nu a.I \oplus_a \Omega)$ and $\nu a.\mathbf{2}(I \oplus_a \Omega)$, crucially distinguishing between duplicating probabilistic choice and duplicating its outcome.

This calculus constitutes then an ideal candidate for our CHC. Indeed, the rules for the counting quantifier $\mathbf{C}^q$ naturally give rise to typing rules for the event generator νa. We now introduce a slight variant of this calculus.

Definition 14. *The* terms of $\Lambda^{\{\}}_{\mathsf{PE}}$ *are defined by the grammar below:*

$$t ::= x \;\Big|\; \lambda x.t \;\Big|\; tt \;\Big|\; \{t\}t \;\Big|\; t \oplus^i_a u \;\Big|\; \nu a.t$$

with $a \in \mathscr{A}$, and $i \in \mathbb{N}$.

The intuition is that $\nu a.$ samples some function ω from the Cantor space, and $t \oplus^i_a u$ yields either t or u depending on the value $\omega(a)(i) \in \{0, 1\}$. In the following we let $t \oplus^i u$ be an abbreviation for $\nu a.t \oplus^i_a u$ (supposing a does not occur free in either t or u).

The two applications tu and $\{t\}u$ correspond to the CbN and CbV cases. For example, while the usual CbN application of $\mathbf{2}$ to $I \oplus^i \Omega$ is just $\mathbf{2}(I \oplus^i \Omega)$, the CbN application is given by $\{\mathbf{2}\}(I \oplus^i \Omega)$.

In usual randomized λ-calculi program execution is defined so as to be inherently probabilistic. For instance, a term $t \oplus u$ can reduce to either t or u, with probability $\frac{1}{2}$. In this way, chains of reductions can be described as *stochastic Markovian sequences* [59], leading to the formalization of the idea of *normalization with probability* $r \in [0,1]$ (see [10]). By contrast, reduction in Λ_{PE} is fully deterministic. Given a term t, names a, b bound in t and $i, j \in \mathbb{N}$ let $(a,i) <_t (b,j)$ if either νb occurs in the scope of νa in t, or $a = b$ and $i < j$.

Definition 15. β-reduction $\rightarrowtriangle_\beta$ *is the relation on terms of* $\Lambda^{\{\}}_{\mathsf{PE}}$ *defined as the context closure of the (unresticred)* β*-rule* $(\lambda x.t)u \rightarrowtriangle_\beta t[u/x]$*.* Permutative reduction $\rightarrowtriangle_{\mathsf{p}}$ *is the relation on terms of* $\Lambda^{\{\}}_{\mathsf{PE}}$ *defined as the context closure of the relation defined by the rules in Fig. 4.* Full reduction *is defined by* $\rightarrowtriangle := \rightarrowtriangle_\beta \cup \rightarrowtriangle_{\mathsf{p}}$.

Intuitively, permutative reductions implement probabilistic choices by computing the full tree of possible choices. For example, by combining β and permutative reduction steps, the CbN redex $\mathbf{2}(I \oplus^i \Omega)$ reduces to the term below:

$$\nu a.\nu a'.((\lambda x.x) \oplus^i_{a'} (\lambda x.\Omega x)) \oplus_a ((\lambda x.\Omega x) \oplus^i_{a'} (\lambda x.\Omega(\Omega x)))$$

which enumerates the four possible outcomes of the CbN application of $\mathbf{2}$ to $I \oplus^i \Omega$. The permutative reduction rule $(\{\}\nu)$ encodes CbN evaluation using β and permutative reduction steps, e.g. one has that $\{\mathbf{2}\}(I \oplus^i \Omega)$ reduces to the term below:

$$\nu a.(\lambda x.x) \oplus^i_a (\lambda x.\Omega(\Omega x))$$

which enumerates the two possible outcomes of the CbV application of $\mathbf{2}$ to $I \oplus^i \Omega$.

The fundamental properties of $\Lambda^{\{\}}_{\mathsf{PE}}$ are the following ones, the proofs of which are obtained by adapting those in [20]:

Theorem 6. $\rightarrowtriangle_{\mathsf{p}}$ *is confluent and strongly normalizing.* $\rightarrowtriangle$ *is confluent.*

The existence and unicity of normal forms for $\rightarrowtriangle_{\mathsf{p}}$ (that we call *permutative normal forms*, **PNF**s for short) naturally raises the question of what these normal forms represent. Let $\mathscr{T}$ indicate the set of **PNF**s containing no free name occurrence.

Definition 16 (Values)**.** *The sets* $\mathscr{V} \subseteq \mathscr{T}$ *of* values *is defined inductively as follows:*
(i.) $x \in \mathscr{V}$
(ii.) if $t \in \mathscr{V}$ *and* $u \in \mathscr{T}$, $tu \in \mathscr{V}$
(iii.) if $t \in \mathscr{V}$, $\lambda x.t \in \mathscr{V}$
(iv.) if $t \in \mathscr{T}$ *and* $u \in \mathscr{V}$, $\{t\}u \in \mathscr{V}$.

The following lemma provides a useful characterization of **PNF**s:

Lemma 24. *For all* $t \in \mathscr{T}$*, either* $t \in \mathscr{V}$ *or* $t = \nu a.t'$*, where* t' *is a tree of* a*-labeled choices* $\oplus^i_a$ *whose leaves form a finite set of* $\mathscr{T}$ *(the* support *of* t', $\mathrm{supp}(t)$*).*

$$
\begin{array}{rcll}
t \oplus_a^i t & \rightarrow_{\mathsf{p}} & t & & (\mathsf{i}) \\
(t \oplus_a^i u) \oplus_a^i v & \rightarrow_{\mathsf{p}} & t \oplus_a^i v & & (\mathsf{c}_1) \\
t \oplus_a^i (u \oplus_a^i v) & \rightarrow_{\mathsf{p}} & t \oplus_a^i v & & (\mathsf{c}_2) \\
\lambda x.(t \oplus_a^i u) & \rightarrow_{\mathsf{p}} & (\lambda x.t) \oplus_a^i (\lambda x.u) & & (\oplus\lambda) \\
(t \oplus_a^i u)v & \rightarrow_{\mathsf{p}} & (tu) \oplus_a^i (uv) & & (\oplus\mathsf{f}) \\
t(u \oplus_a^i v) & \rightarrow_{\mathsf{p}} & (tu) \oplus_a^i (tv) & & (\oplus\mathsf{a}) \\
(t \oplus_a^i u) \oplus_b^j v & \rightarrow_{\mathsf{p}} & (t \oplus_b^j v) \oplus_a^i (u \oplus_b^j v) & ((a,i) < (b,j)) & (\oplus\oplus_1) \\
\{t \oplus_a^i u\}v & \rightarrow_{\mathsf{p}} & \{t\}v \oplus_a^i \{t\}v & & (\{\}\oplus_1) \\
\{t\}(u \oplus_a^i v) & \rightarrow_{\mathsf{p}} & \{t\}u \oplus_a^i \{t\}v & & (\{\}\oplus_2) \\
t \oplus_b^j (u \oplus_a^i v) & \rightarrow_{\mathsf{p}} & (t \oplus_b^j u) \oplus_a^i (t \oplus_b^j v) & ((a,i) < (b,j)) & (\oplus\oplus_2) \\
\nu b.(t \oplus_a^i u) & \rightarrow_{\mathsf{p}} & (\nu b.t) \oplus_a^i (\nu b.u) & (a \neq b) & (\oplus\nu) \\
\lambda x.\nu a.t & \rightarrow_{\mathsf{p}} & \nu a.\lambda x.t & & (\nu\lambda) \\
(\nu a.t)u & \rightarrow_{\mathsf{p}} & \nu a.(tu) & & (\nu\mathsf{f}) \\
\{t\}\nu a.u & \rightarrow_{\mathsf{p}} & \nu a.tu & & (\{\}\nu)
\end{array}
$$

Figure 4: Permutative reductions.

Proof. We argue, by induction on the structure of t, that if t does not start with ν, then $t \in \mathscr{V}$. If t does not start with ν, then, as not being in the form $t_1 \oplus_a^i t_2$, it is of one of the forms $\lambda x.t'$, $t_1 t_2$, $\{t_1\}t_2$. In the first case by the IH $t' \in \mathscr{V}$, which implies $t \in \mathscr{V}$. In the second case, it cannot be $t_1 = \nu a.t_1'$, otherwise t would not be $\rightarrow_{\mathsf{p}}$-normal, so by IH $t_1 \in \mathscr{V}$, which implies $t \in \mathscr{V}$. In the third case, it cannot be $t_2 = \nu a.t_2'$, otherwise t would not be $\rightarrow_{\mathsf{p}}$-normal, so by IH $t_2 \in \mathscr{V}$, which implies $t \in \mathscr{V}$. Now, from $t = \nu a.t'$, we can deduce by induction on t', that t' is a finite tree of a-labeled choices, since any other operator commutes with $\oplus_a^i$ by permutative rules.

□

Using Lemma 24, any $t \in \mathscr{T}$ yields a *(sub-)distribution* of values $\mathscr{D}_t : \mathscr{V} \to [0,1]$ as follows. First, for any term t, finite set X and $\omega \in (2^{\mathbb{N}})^X$, let $\pi_X^\omega(t)$, the "application of ω to t through X", be defined as:

$$
\begin{aligned}
\pi_X^\omega(x) &= x \\
\pi_X^\omega(\lambda x.t) &= \lambda x.\pi_X^\omega(t) \\
\pi_X^\omega(tu) &= \pi_X^\omega(t)\pi_X^\omega(u) \\
\pi_X^\omega(\{t\}u) &= \{\pi_X^\omega(t)\}\pi_X^\omega(u) \\
\pi_X^\omega(t \oplus_a^i u) &= \begin{cases} \pi_X^\omega(t) & \text{if } a \in X \text{ and } \omega(a)(i) = 1 \\ \pi_X^\omega(u) & \text{if } a \in X \text{ and } \omega(a)(i) = 0 \\ \pi_X^\omega(t) \oplus_a^i \pi_X^\omega(u) & \text{otherwise} \end{cases} \\
\pi_X^\omega(\nu b.t) &= \nu b.\pi_X^\omega(t).
\end{aligned}
$$

Now, $\mathscr{D}_t$ is defined by letting $\mathscr{D}_t(v) = \delta_t$ (i.e. the *Dirac* distribution given by $\delta_t(t) = 1$ and $\delta_t(u) = 0$, for $t \neq u$) when $t \in \mathscr{V}$, and

$$\mathscr{D}_t(v) = \sum_{u \in \mathrm{supp}(t')} \mathscr{D}_u(v) \cdot \mu(\{\omega \in 2^{\mathbb{N}} \mid \pi^{\omega}_{\{a\}}(t') = u\})$$

if $t = \nu a.t'$. Intuitively, $\mathscr{D}_t(v)$ measures the probability of finding v by iteratively applying to t random choices of events from the Cantor space any time a ν is found.

While Theorem 6 assures the existence of **PNF**s, it certainly does not assure the existence of normal forms under the full reduction $\rightarrowtriangle$. Given a term $t \in \mathscr{T}$, the questions "is t in normal form?" and "does t reduce to a normal form?" have univocal yes/no answers, because $\rightarrowtriangle$ is deterministic. Nevertheless, focusing on normalizable terms excludes the most interesting part of the calculus, made of terms for which normalization is inherently probabilistic (indeed all terms considered so far, which contained the diverging term Ω). By the way, if we think of t rather as the distribution $\mathscr{D}_t$, the relevant questions are rather probabilistic variants of those considered above, e.g. "with what probability is t in normal form?" and "with what probability does t reduce to normal form?".

For simplicity, instead of considering reduction to normal forms, we will focus here on reduction to *head normal form*, defined below. A complete analysis of normal forms in $\Lambda^{\{\}}_{\mathsf{PE}}$ will be developed in a future paper. Let a *randomized context* be any context $\mathtt{R}[\ ]$ defined by the grammar:

$$\mathtt{R}[\ ] ::= [\ ] \mid \mathtt{R}[\ ] \oplus^i_a u \mid t \oplus^i_a \mathtt{R}[\ ] \mid \nu a.\mathtt{R}[\ ]$$

and a *head-context* be any context $\mathtt{H}[\ ]$ defined by the grammar:

$$\mathtt{H}[\ ] := [\ ] \mid \lambda x.\mathtt{H}[\ ] \mid \mathtt{H}[\ ]u \mid \{\mathtt{H}[\ ]\}u.$$

Head-reduction is formally defined inductively as follows:

Definition 17 (Head Reduction)**.** *A reduction $t \rightarrowtriangle u$ is a* head-reduction *(noted $t \rightarrowtriangle_{\mathsf{h}} u$) if it is of one of the following forms:*
- *a permutative reduction;*
- *a β-reduction of the form $\mathtt{R}[\mathtt{H}[(\lambda y.t)u]] \rightarrowtriangle_\beta \mathtt{R}[\mathtt{H}[t[u/y]]]$;*
- *a head reduction of the form $\mathtt{R}[\mathtt{H}[\{t\}u]] \rightarrowtriangle_{\mathsf{h}} \mathtt{R}[\mathtt{H}[\{t\}u']]$.*

A head normal value *is a value $t \in \mathscr{V}$, which is also $\rightarrowtriangle_{\mathsf{h}}$-normal. We let $\mathrm{HNV} \subseteq \mathscr{V}$ indicate the set of head normal values.*

The following characterization of head normal values is easily established by induction on the definition of $\mathscr{V}$:

Lemma 25. *For any $t \in \mathscr{V}$, $t \in \mathrm{HNV}$ iff one of the following holds:*
(i.) t is a variable
(ii.) $t = t_1t_2$, where $t_1 \in \mathrm{HNV}$ and does not start with λ, ν
(iii.) $t = \lambda x.t'$, where $t' \in \mathrm{HNV}$
(iv.) $t = \{t_1\}t_2$, where $t_2 \in \mathrm{HNV}$.

We now introduce a function $\mathrm{HNV}(t)$ measuring the probability that t reduces to a head normal form.

Definition 18. *For any* $t \in \mathcal{T}$*,* $\mathrm{HNV}(t) := \sum_{v\in\mathrm{HNV}} \mathscr{D}_t(v)$ *and* $\mathrm{HNV}_{\twoheadrightarrow}(t) := \sup\{\mathrm{HNV}(u) \mid t \twoheadrightarrow_{\mathsf{h}}^{*} u\}$*. When* $\mathrm{HNV}_{\twoheadrightarrow}(t) \geq q$*, we say that* t yields a head-normal value with probability at least q.

For example, if $t = \nu a.(\lambda x\lambda y.(y \oplus_a^i I)x)u$, where $u = I \oplus^j \Omega$, then $\mathrm{HNV}_{\twoheadrightarrow}(t) = \frac{3}{4}$. Indeed, we have $t \twoheadrightarrow_{\mathsf{h}}^{*} \nu a.(\lambda y.y(\nu b.I \oplus_b^j \Omega)) \oplus_a^i (\nu b'.I \oplus_{b'}^j \Omega)$ and three over the four possible choices (corresponding to choosing between either left or right for both νa and $\nu b'$) yield a HNV. Observe that the choice about νb does not matter, since $\lambda y.yu$ is already a HNV.

6.2. The Correspondence, Statically and Dynamically

We now present the heart of the CHC. First, we show that any derivation Π in iCPL_0 can be decorated with a term t^{Π} from $\Lambda_{\mathsf{PE}}^{\{\}}$; second, we show that whenever Π reduces to Π', $t^{\Pi} \twoheadrightarrow_{\mathsf{h}}^{*} t^{\Pi'}$; finally, we show that if a term t decorates a proof of, say, $\mathbf{C}^q A$, then t yields a head normal value with probability q.

The decoration of iCPL_0-derivations with terms from $\Lambda_{\mathsf{PE}}^{\{\}}$ is illustrated in Fig. 5. To any derivation of $\Gamma \vdash \mathscr{b} \rightarrowtail A$ is associated a finite set of variables $\vec{x}$, corresponding to the formulas in Γ, and a term t depending on such variables, noted $\vec{x} : \Gamma \vdash t : \mathscr{b} \rightarrowtail A$.

Observe that the rule ($\mathbf{C}$I) provides a natural way of typing the constructor ν: if a program t, depending on some name a, yields a proof of A whenever a satisfies a Boolean constraint $\mathscr{d}$, then, if $\mu(\mathscr{d}) \geq q$ and a is sampled in a random way, the program $\nu a.t$ has probability q of yielding a proof of A.

Remark 6 (Why $\mathbf{D}$ is not constructive)**.** *What makes the rule (*$\mathbf{C}$*I) a natural candidate to type the operator* ν *is that the fact that a program* t *enjoys a given property* P *with probability* at least q *can be approximated "from below" by looking at* finitely many *paths in the tree of possible outcomes of* t *(recall that any* finite set *of* finite *paths of labeled choices can be captured by means of Boolean formulas). By contrast, the fact that* t *enjoys* P *with probability* strictly less than q *cannot be similarly approximated "from below": one needs to look at* all *possible outcomes of random choices in (terms obtained by evaluating)* t*. The latter need not form a finite collection and thus cannot in general be captured by a Boolean formula.*

If the quantifier $\mathbf{D}$ *were to be added to the language of* iCPL*, the intuitive computational meaning of* $t : \mathbf{D}^q A$ *should be something like "*t *has type* A *with probability* less than q*", i.e. a property that needs not be approximated in a finite (nor computable) way. In a similar way, the intuitive computational meaning of a judgment of the form* $t : \mathscr{b} \hookleftarrow A$ *would be something like "*t *has type* A only when $\omega \vDash \mathscr{b}$*", and checking this would similarly require to inspect the full tree of randomized choices of* t*.*

Let us now look at normalization for $\mathbf{G}_{\mathsf{iCPL}_0}$, and to what happens to decorating terms. The two fundamental cases of normalization are that of a peak

Identity Rule

$$\frac{}{\vec{x}:\Gamma, y:A \vdash y:\mathbb{b} \rightarrowtail A}\ (\mathrm{id})$$

Structural Rules

$$\frac{\mathbb{b} \vDash \bot}{\vec{x}:\Gamma \vdash \Omega:\mathbb{b} \rightarrowtail A}\ (\bot)$$

$$\frac{\vec{x}:\Gamma \vdash t:\mathbb{c} \rightarrowtail A \qquad \vec{x}:\Gamma \vdash u:\mathbb{d} \rightarrowtail A \qquad \mathbb{b} \vDash (\mathbb{c} \wedge \boldsymbol{x}_a^i) \vee (\mathbb{d} \wedge \neg \boldsymbol{x}_a^i)}{\vec{x}:\Gamma \vdash t \oplus_a^i u:\mathbb{b} \rightarrowtail A}\ (\mathsf{m})$$

Logical Rules

$$\frac{\vec{x}:\Gamma, y:A \vdash t:\mathbb{b} \rightarrowtail B}{\vec{x}:\Gamma \vdash \lambda y.t:\mathbb{b} \rightarrowtail (A \to B)}\ (\to\mathrm{I})$$

$$\frac{\vec{x}:\Gamma \vdash t:\mathbb{b} \rightarrowtail (A \to B) \qquad \vec{x}:\Gamma \vdash u:\mathbb{b} \rightarrowtail A}{\vec{x}:\Gamma \vdash tu:\mathbb{b} \rightarrowtail B}\ (\to\mathrm{E})$$

Counting Rules

$$\frac{\vec{x}:\Gamma \vdash t:\mathbb{b} \wedge \mathbb{d} \rightarrowtail A \qquad \mu(\mathbb{d}) \geq q}{\vec{x}:\Gamma \vdash \nu\vec{a}.t:\mathbb{b} \rightarrowtail \mathbf{C}^q A}\ (\mathbf{C}\mathrm{I})$$

$$\frac{\vec{x}:\Gamma \vdash t:\mathbb{b} \rightarrowtail \mathbf{C}^q A \qquad \vec{x}:\Gamma, y:A \vdash u:\mathbb{b} \rightarrowtail B}{\vec{x}:\Gamma \vdash \{\lambda y.u\}t:\mathbb{b} \rightarrowtail \mathbf{C}^{qs} B}\ (\mathbf{C}\mathrm{E})$$

Figure 5: Decoration of the rules of $\mathbf{G}_{\mathsf{iCPL}_0}$ with terms from $\Lambda_{\mathsf{PE}}^{\{\}}$.

($\to$I/$\to$E) and that of a peak ($\mathbf{C}$I/$\mathbf{C}$E). The first case is treated as in usual intuitionistic natural deduction: the peak

$$\frac{\dfrac{\overset{\Pi}{\vec{x}:\Gamma, y:A \vdash t:\mathbb{b} \rightarrowtail B}}{\vec{x}:\Gamma \vdash \lambda y.t:\mathbb{b} \rightarrowtail (A \to B)}\ (\to\mathrm{I}) \qquad \overset{\Sigma}{\vec{x}:\Gamma \vdash u:\mathbb{b} \rightarrowtail A}}{\vec{x}:\Gamma \vdash (\lambda y.t)u:\mathbb{b} \rightarrowtail B}\ (\to\mathrm{E})$$

is replaced by

$$\frac{\overset{\Pi}{\vec{x}:\Gamma, y:A \vdash t:\mathbb{b} \rightarrowtail B} \qquad \overset{\Sigma}{\vec{x}:\Gamma \vdash u:\mathbb{b} \rightarrowtail A}}{\vec{x}:\Gamma \vdash t[u/x]:\mathbb{b} \rightarrowtail B}\ (\mathrm{subst})$$

The admissibility of the rule (subst) in $\mathbf{G}_{\mathsf{iCPL}_0}$ (as well as its correspondence with term substitution) is easily checked by induction on the rules of $\mathbf{G}_{\mathsf{iCPL}_0}$. In the second case a peak of the form

$$\dfrac{\dfrac{\dfrac{\Sigma}{\vec{x}:\Gamma\vdash t:\mathcal{b}\wedge\mathcal{d}\rightarrowtail A}\quad \mu(\mathcal{d})\geq q}{\vec{x}:\Gamma\vdash \nu a.t:\mathcal{b}\rightarrowtail \mathbf{C}^{q}A}\,(\mathbf{C}\mathrm{I})\qquad \dfrac{\Pi}{\vec{x}:\Gamma,y:A\vdash u:\mathcal{b}\rightarrowtail B}}{\vec{x}:\Gamma\vdash\{\lambda y.u\}\nu a.t:\mathcal{b}\rightarrowtail\mathbf{C}^{qs}B}\,(\mathbf{C}\mathrm{E})$$

is replaced by

$$\dfrac{\dfrac{\dfrac{\Sigma}{\vec{x}:\Gamma\vdash t:\mathcal{b}\wedge\mathcal{d}\rightarrowtail A}\quad \dfrac{\Pi[\mathcal{b}\mapsto\mathcal{b}\wedge\mathcal{d}]}{\vec{x}:\Gamma,y:A\vdash u:\mathcal{b}\wedge\mathcal{d}\rightarrowtail B}}{\vec{x}:\Gamma\vdash u[t/x]:\mathcal{b}\wedge\mathcal{d}\rightarrowtail B}\,(\mathrm{subst})\qquad \mu(\mathcal{d})\geq qs}{\vec{x}:\Gamma\vdash\nu a.u[t/x]:\mathcal{b}\rightarrowtail\mathbf{C}^{qs}B}\,(\mathbf{C}\mathrm{I})$$

Observe that the normalization step above corresponds to two reduction steps $\{\lambda y.u\}\nu a.t \rightarrowtriangle_{\mathsf{p}} \nu a.(\lambda y.u)t \rightarrowtriangle_{\beta} \nu a.u[t/x]$ of the corresponding term.

Other conversion rules for $\mathbf{G}_{\mathsf{iCPL}_0}$ can be considered, permuting the rule (m) with other rules of the calculus, and corresponding to the permutative reductions illustrated in Fig. 4. Indeed, a corresponding conversion rule for $\mathbf{G}_{\mathsf{iCPL}_0}$ can be defined for any of the permutations (i)-$(\oplus\nu)$. One cannot construct conversion rules corresponding to $(\nu\lambda)$ and $(\nu\mathsf{f})$: for the first, this would require that $(A\to\mathbf{C}^{q}B)\vdash\mathbf{C}^{q}(A\to B)$, which is not derivable from $\mathbf{G}_{\mathsf{iCPL}_0}$ (see Remark 3); for the second, because, as is easily seen by inspecting the rules in Fig. 5, no decorating term is of the form $(\nu a.t)u$. A more precise correspondence between permuting rules for terms and conversion rules for proofs can be obtained by replacing $\mathbf{G}_{\mathsf{iCPL}_0}$ with a suitable *type system* extending the standard simply typed λ-calculus with counting quantifiers. However, the study of type systems for $\Lambda^{\{\}}_{\mathsf{PE}}$ goes beyond the scope of this text, and will be developed in a future paper.

6.3. A Probabilistic Normalization Theorem

The remaining step consists in showing that whenever $\vec{x}:\Gamma\vdash t:\mathcal{b}\rightarrowtail A$ is derivable, the counting quantifiers in the formula A actually reveal the *probability of termination* of t.

More precisely, for any formula A, let $\lceil A\rceil\in(0,1]$ be defined by $\lceil \mathcal{p}\rceil=1$, $\lceil A\to B\rceil=\lceil B\rceil$ and $\lceil\mathbf{C}^{q}A\rceil=q\cdot\lceil A\rceil$. We will show that any term of type A has probability $\lceil A\rceil$ of yielding a head normal value, as stated in the following result, and more directly in its immediate corollary.

Theorem 7 (Probabilistic Normalization)**.** *If $\vec{x}:\Gamma\vdash t:\mathcal{b}\rightarrowtail A$, then for all $\omega\in(2^{\mathbb{N}})^{\mathrm{FN}(\mathcal{b})}$, if $\omega\vDash\mathcal{b}$, then* $\mathrm{HNV}(\pi^{\omega}_{X}(t))\geq\lceil A\rceil$.

Corollary 4. *If $\vdash t:\top\rightarrowtail A$, then* $\mathrm{HNV}(t)\geq\lceil A\rceil$.

The proof of Theorem 7 is obtained by adapting the standard technique of *reducibility predicates* to the quantitative framework provided by counting quantifiers.

To define such predicates, let us first associate, with any **PNF** $t\in\mathcal{T}$, a distribution $\mathcal{D}^{1}_{t}:\mathcal{T}\to[0,1]$ defined as δ_t if $t\in\mathrm{HNV}$, and as $\mathcal{D}^{1}_{t}(u)=\mu\{\omega\in 2^{\mathbb{N}}\mid\pi^{\omega}_{\{a\}}(t')=u\}$, if $t=\nu a.t'$.

Lemma 26. *For all $t,u,w\in\mathcal{T}$, $\mathcal{D}^{1}_{t}(u)=\mathcal{D}^{1}_{tw}(uw)$.*

Proof. If t is an HNV, tw is a pseudo-value, so $\mathscr{D}^1_t = \delta_t$ and $\mathscr{D}^1_{tw} = \delta_{tw}$, which implies $\mathscr{D}^1_t(u) = 1 \Leftrightarrow t = u \Leftrightarrow tw = uw \Leftrightarrow \mathscr{D}^1_{tw} = tw$. If $t = \nu a.t'$, since we can suppose $a \notin \mathrm{FN}(w)$, from $\pi^{\omega}_{\{a\}}(t'w) = \pi^{\omega}_{\{a\}}(t')w$ we deduce $\{\omega \mid \pi^{\omega}_{\{a\}}(t'w) = uw\} = \{\omega \mid \pi^{\omega}_{\{a\}}(t') = u\}$, from which the claim follows. $\square$

Definition 19 (Probabilistic Reducibility Predicates)**.** *For any formula A of iCPL_0, finite set of names X, $r \in (0,1]$ and $S \subseteq (2^{\mathbb{N}})^X$, the set $\mathrm{Red}^{X,r}_A(S) \subseteq \Lambda^X_{\oplus,\nu,\{\}}$ is defined inductively as follows:*

$$\mathrm{Red}^{X,r}_{p}(S) = \{t \mid \forall \omega \in S\ \mathrm{HNV}_{\twoheadrightarrow}(\pi^{\omega}_X(t)) \geq r\}$$

$$\mathrm{Red}^{X,r}_{A\to B}(S) = \{t \mid \forall S' \subseteq S,\ \forall u \in \mathrm{HRed}^{X,1}_A(S'),\ tu \in \mathrm{HRed}^{X,r}_B(S')\}$$

$$\mathrm{Red}^{X,r}_{\mathbf{C}^q A}(S) = \left\{ t \;\middle|\; \forall \omega \in S \quad \sum_{u \in \mathrm{Red}^{\emptyset,r}_A} \mathscr{D}^1_{\pi^{\omega}_X(t)}(u) \geq q \right\}$$

Several properties of the sets $\mathrm{Red}^{X,r}_A(S)$ are proved as in standard normalization arguments based on reducibility. In particular, the following two lemmas are easily established by induction

Lemma 27. *(i.) $t \in \mathrm{HRed}^{X,r}_A(S)$ iff for all $\omega \in S$, $\pi^{\omega}_X(t) \in \mathrm{HRed}^{\emptyset,r}_A$;*
(ii.) $t \in \mathrm{NRed}^{X,r}_A(S)$ iff for all $\omega \in S$, $\pi^{\omega}_X(t) \in \mathrm{NRed}^{\emptyset,r}_A$.

Lemma 28. *If $t \in \mathrm{Red}^{X,r}_A(S)$ and $u \twoheadrightarrow^*_{\mathsf{h}} t$, then $u \in \mathrm{Red}^{X,r}_A(S)$.*

The two lemmas below are proved in A.4, and provide the essential ingredients to show that the introduction rule ($\mathbf{C}$I) preserves reducibility.

Lemma 29. *For any formula A, terms $t, u_1, \dots, u_n$ with $\mathrm{FN}(t) \subseteq X \cup \{a\}$, $\mathrm{FN}(u_i) \subseteq X$, and measurable set $S \subseteq (2^{\mathbb{N}})^{X\cup\{a\}}$ and $S' \subseteq (2^{\mathbb{N}})^X$, if*

1. *$tu_1 \dots u_n \in \mathrm{Red}^{X\cup\{a\},r}_A(S)$*
2. *for all $\omega \in S'$, $\mu(\Pi^{\omega}(S)) \geq s$*

then $(\nu a.t)u_1 \dots u_n \in \mathrm{Red}^{X,r}_{\mathbf{C}^s A}(S')$.

Lemma 30. *For all formulas A and B, $\mathrm{Red}^{X,r}_{\mathbf{C}^{q_1}\dots\mathbf{C}^{q_k}(A\to B)}(S) = \mathrm{Red}^{X,r}_{A\to\mathbf{C}^{q_1}\dots\mathbf{C}^{q_k}B}(S)$.*

Let us call a term t *ν-safe* if it never reduces to a term of the form $\nu a.t'$.

Definition 20 (Neutral Terms)**.** *For any set $S \subseteq (2^{\mathbb{N}})^X$, the set $\mathrm{Neut}(S)$ is defined by induction as follows:*

- *for any variable x, $x \in \mathrm{Neut}(S)$*
- *if $t \in \mathrm{Neut}(S)$ and for all $\omega \in S$ $\pi^{\omega}_X(u)$ is ν-safe and $\mathrm{HNV}_{\twoheadrightarrow}(u) = 1$, $\{t\}u \in \mathrm{Neut}(S)$*
- *if $t \in \mathrm{Neut}(S)$, then for all $u \in \Lambda^X_{\oplus,\nu,\{\}}$, $tu \in \mathrm{Neut}(S)$.*

It is easily checked that for all $t \in \mathrm{Neut}(S)$ and $\omega \in S$, $\mathrm{HNV}(\pi_X^\omega(t)) = 1$. Furthermore, the following properties can be easily established by a mutual induction.

Lemma 31. *For any formula A,*

i. if $t \in \mathrm{Red}_A^{X,r}(S)$ and $\omega \in S$, then $\mathrm{HNV}(\pi_X^\omega(t)) \geq \lceil A \rceil \cdot r$

ii. $\mathrm{Neut}(S) \subseteq \mathrm{Red}_A^{X,1}(S)$.

The following property is established by induction as well.

Lemma 32. *For any formula A and term t name-closed and ν-safe, if $t \in \mathrm{Red}_A^{\emptyset,r}$, then $\mathrm{HNV}_{\rightarrow}(t) = 1$.*

Proof.
- If $A = \mathscr{p}$, then $t \in \mathrm{Red}_A^{\emptyset,r}$ implies $\mathrm{HNV}_{\rightarrow}(t) \geq r$, hence at least one term u with $\mathrm{HNV}(u) = 1$ is found by progressively computing the tree of choices of t. However, since t is ν-safe, can actual choice $\oplus$ can occur in a head-reduction of t, so in fact $t \rightarrow_{\mathsf{h}}^* u$, which implies $\mathrm{HNV}_{\rightarrow}(t) = 1$.
- if $A = B \to C$, then from Lemma 31 we deduce $x \in \mathrm{Red}_B^{\emptyset,1}$, whence $tx \in \mathrm{Red}_C^{\emptyset,r}$, which implies by IH that $\mathrm{HNV}_{\rightarrow}(tx) = 1$, which in turn forces $\mathrm{HNV}_{\rightarrow}(t) = 1$.
- if $A = \mathbf{C}_a^q B$, then, since t is ν-safe, $\mathscr{D}_t^1 = \delta_t$, whence from $t \in \mathrm{Red}_A^{\emptyset,r}$ it follows $t \in \mathrm{Red}_B^{\emptyset,r}$, so by the IH we deduce $\mathrm{HNV}_{\rightarrow}(t) = 1$.

□

The lemma below provides the essential ingredient to show that the rule ($\mathbf{C}$E) is reducibility preserving:

Lemma 33. *If $t \in \mathrm{Red}_{A \to B}^{X,q}(S)$ and $u \in \mathrm{Red}_{\mathbf{C}^s A}^{X,1}(S)$, then $\{t\}u \in \mathrm{Red}_{\mathbf{C}^s B}^{X,q}(S)$.*

Proof. Let $\omega \in S$. We distinguish two cases:

i. $\pi_X^\omega(u) \rightarrow_{\mathsf{h}}^* \nu a.u'$ then, letting $U := \mathrm{Red}_A^{X,1}(S)$ and $W := \mathrm{Red}_B^{X,q}(S)$, we have that $\sum_{w \in W} \mathscr{D}^1_{\pi_X^\omega(\nu a.tu')}(w) = \sum_{w \in W} \mu(\{\omega' \mid \pi_{X \cup \{a\}}^{\omega+\omega'}(tu') = w\} = \mu\{\omega' \mid \pi_{X \cup \{a\}}^{\omega+\omega'}(tu') \in W\} = \mu\{\omega' \mid t\pi_{X \cup \{a\}}^{\omega+\omega'}(u') \in W\} \geq q \cdot \mu\{\omega' \mid \pi_{X \cup \{a\}}^{\omega+\omega'}(u') \in U\} = q \cdot \sum_{w \in U} \mathscr{D}^1_{\nu a.u}(w) \geq q \cdot s$, which proves that $\nu a.tu' \in \mathrm{Red}_{\mathbf{C}^s B}^{X,q}(S)$. Since $\{t\}u \rightarrow_{\mathsf{h}}^* \nu a.tu'$, we conclude then $\{t\}u \in \mathrm{Red}_{\mathbf{C}^s B}^{X,q}(S)$ by Lemma 28

ii. $\pi_X^\omega(u)$ is ν-safe; then from $\pi_X^\omega(u) \in \mathrm{Red}_A^{\emptyset,q}$ we deduce, by Lemma 32, that $\mathrm{HNV}_{\rightarrow}(\pi_X^\omega(u)) = 1$. We deduce then that $\{t\}u \in \mathrm{Neut}(S)$, and by Lemma 31 we conclude $\{t\}u \in \mathrm{Red}_{\mathbf{C}^s B}^{X,q}(S)$.

□

Theorem 7 can now be deduced from Proposition 6 below, together with Lemma 31 (ii.).

Proposition 6. *If $\vec{x} : \Gamma \vdash t : \mathscr{b} \rightarrowtail A$ is derivable, where $\Gamma = A_1, \ldots, A_n$, then for all $S \subseteq [\![\mathscr{b}]\!]_X$, for all $u_i \in \mathrm{Red}_{A_i}^{X,1}(S)$, $t[u_1/x_1, \ldots, u_m/x_m] \in \mathrm{Red}_A^{X,1}(S)$.*

Proof. The proof is by induction on a derivation of $\vec{x} : \Gamma \vdash t : \mathscr{b} \rightarrowtail A$. We only consider a few interesting cases:

- If $t = \{t_1\}t_2$ and the last rule is

$$\frac{\vec{x} : \Gamma \vdash t_1 : \mathscr{c} \rightarrowtail A \to B \qquad \vec{x} : \Gamma \vdash t_2 : \mathscr{d} \rightarrowtail \mathbf{C}^s A \qquad \mathscr{b} \vDash \mathscr{c} \wedge \mathscr{d}}{\vec{x} : \Gamma \vdash t : \mathscr{b} \rightarrowtail \mathbf{C}^s B} (\{\})$$

  then by IH for all $u_i \in \mathrm{Red}^{X,1}_{A_i}(S)$, $t_1[u_1/x_1, \ldots, u_n/x_n] \in \mathrm{Red}^{X,1}_{A\to B}(S)$ and $t_2[u_1/x_1, \ldots, u_n/x_n] \in \mathrm{Red}^{X,1}_{\mathbf{C}^s A}(S)$; then by Lemma 30 and Lemma 33 we deduce $(\{t_1\}t_2)[u_1/x_1, \ldots, u_n/x_n] = \{t_1[u_1/x_1, \ldots, u_n/x_n]\}(t_2[u_1/x_1, \ldots, u_n/x_n]) \in \mathrm{Red}^{X,1}_{\mathbf{C}^s B}(S)$.
- If $t = \nu a.u$ and the last rule is

$$\frac{\vec{x} : \Gamma \vdash u : \mathscr{c} \wedge \mathscr{d} \rightarrowtail A \qquad \vDash \mu(\mathscr{d}) \geq r \qquad \mathscr{b} \vDash \mathscr{c}}{\vec{x} : \Gamma \vdash \nu a.u : \mathscr{b} \rightarrowtail \mathbf{C}^r A} (\mu)$$

  where $a \notin \mathrm{FN}(\mathscr{c})$, then let $S \subseteq [\![\mathscr{b}]\!]_X \subseteq [\![\mathscr{c}]\!]_X$ and $u_i \in \mathrm{Red}^{X,1}_{A_i}(S)$. Let $T = \{g + f \mid \omega \in S\} \subseteq (2^{\mathbb{N}})^{X\cup\{a\}}$, which is measurable since counter-image of a measurable set through the projection function; then since $\mathrm{FN}(u_i) \subseteq X$, we deduce using Lemma 27 that $u_i \in \mathrm{Red}^{X\cup\{a\},1}_{A_i}(T \cap [\![\mathscr{d}_j]\!]_{X\cup\{a\}})$. By IH and the hypothesis we deduce then that:

  1. $u[u_1/x_1, \ldots, u_n/x_n] \in \mathrm{Red}^{X\cup\{a\},1}_{A}(T \cap [\![\mathscr{d}_j]\!]_{X\cup\{a\}})$
  2. for all $\omega \in S$, $\mu(\Pi^\omega(T \cap [\![\mathscr{d}_j]\!]_{X\cup\{a\}})) \geq r$.

  Hence, by Lemma 29 we conclude that $(\nu a.u)[u_1/x_1, \ldots, u_n/x_n] \in \mathrm{Red}^{X,1}_{\mathbf{C}^r A}(S)$.

□

7. Related Work

As discussed in the introduction, our results provide, to the best of the authors' knowledge, the first clear correspondences between a class of probability logics, on the one side, and a hierarchy of probabilistic complexity classes as well as probabilistic extension of the λ-calculus, on the other one. Yet, this is not to say that our logic and calculi come from nowhere.

While, as discussed in Section 4, our use of the term "counting quantifier" is inspired from Wagner's work on the counting hierarchy, different kinds of measure-theoretic quantifiers have been investigated in the literature, with the intuitive meaning of "A is true *for almost all* x" (see [51, 68] and more recently [49, 50]), or "A is true for the *majority of* x" [55, 82, 81]. Despite the extensive literature on logical systems enabling (in various ways and for different purposes) some forms of probabilistic reasoning, there is not much about logics tied to computational aspects, as CPL is. Most of the recent logical formalisms have been developed in the realm of modal logic, like e.g. [52, 53, 7, 5, 6, 26, 37, 38]. Remarkably, some of them are developed together with (axiomatic) proof systems, [7, 27, 38]. Another class of probabilistic modal logics have been designed to model Markov chains and [44, 39, 45]. With the sole exception of *Riesz modal logic* [28], we are not aware of sequent calculi for probability logic.

Intuitionistic modal logic has been related, in the Curry-Howard sense, to monadic extensions of the λ-calculus [22, 8, 1, 83, 16]. However, in these correspondences modal operators are related to *qualitative* properties of programs (typically, tracing algebraic effects), as opposed to the *quantitative* properties expressed by counting quantifiers. Nevertheless, as discussed in Section 5, our semantics for iCPL can be related to standard Kripke' structures for intuitionistic modal logic [58, 65].

On the other hand, quantitative semantics arising from linear logic have been largely used in the study of probabilistic λ-calculi, as in [18, 23, 25]. Notably, *probabilistic coherence spaces* [35, 23, 24] have been shown to provide a fully abstract model of probabilistic PCF. While we are not aware of correspondences relating probabilistic programs with proofs in linear logic, it seems that the proof-theory of counting quantifiers could be somehow related to that of *bounded exponentials* [36, 21] and, more generally, to the theory of *graded* monads and co-monads [41, 12, 29, 42].

As already observed, the calculus $\Lambda^{\{\}}_{\mathsf{PE}}$ is strongly inspired by [20]. In the same paper a simple type system is also introduced, ensuring strong normalization (and thus leaving out all cases in which normalization is inherently probabilistic, as discussed in Section 6). Beyond this one, several type systems for probabilistic λ-calculi have been introduced in the recent literature. Among these, systems based on *type distributions* [19], i.e. where a single derivation assigns several types to a term, each with some probability, and systems based on *oracle intersection types* [11], where type derivations capture single evaluations as determined by an oracle. The type discipline provided by iCPL_0 sits in between these two approaches: (i) similarly to the former (and differently from the latter), in it decorated proofs, seen as typing derivations, can capture a finite number of different evaluations, without using distributions of types, (ii) as for the latter, typing derivations reflect the dependency of evaluation on oracles, although the latter are manipulated in a collective way by means of Boolean constraints.

Finally, [79] is worth mentioning, as it presents a dependent type theory enriched with a probabilistic choice operator, yielding a calculus with both term and type distributions. Interestingly, a fragment of this system enjoys a sort of CHC with so-called *Markov Logic Networks* [61], a class of probabilistic graphical models specified by means of first-order logic formulas.

8. Conclusions

The main contribution of this paper is not the introduction of counting propositional logic *per se*. After all, the idea of considering logics with measure-theoretic quantifiers is definitely not a new one.Instead, our principal result consists in showing that counting quantifiers can play nicely with propositional logic in characterizing the counting hierarchy, and in designing type disciplines for a (randomized) λ-calculus in the spirit of the Curry-Howard correspondence. In this way, logic can somehow catch up with some old and recent results both in computational complexity and in programming language theory.

Many problems and questions remain open, or could not be described in this paper, due to space reasons. For example, the proof theory of CPL has just been briefly delineated, but the dynamics of the introduced formal systems (namely, the underlying cut-elimination procedure) certainly deserves to be investigated, despite not being crucial for the results in this paper. Moreover, the Curry-Howard correspondence discussed in Section 6 can be used to design type systems for the probabilistic event λ-calculus based on counting quantifiers and assuring quantitative bounds on probabilistic termination. In particular, ongoing work suggests that a complete characterization of the probabilistic behavior of programs could be obtained by enriching usual simple types with counting quantifiers as well as an *intersection* operator.

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A. Postponed Proofs

A.1. Postponed Proofs from Section 3

Proof of Lemma 2. By induction on $\boldsymbol{b}$:

- if $\boldsymbol{b} = \boldsymbol{x}_i^a$ or $\boldsymbol{b} = \neg \boldsymbol{x}_i^a$, then $k = 1$, $\boldsymbol{d}_i = \boldsymbol{b}$ and $\boldsymbol{e}_i = \top$
- if $\boldsymbol{b} = \boldsymbol{x}_i^b$, where $b \neq a$, then $k = 2$, $\boldsymbol{d}_0 = \top$, $\boldsymbol{d}_1 = \bot$ and $\boldsymbol{e}_0 = \boldsymbol{b}$, $\boldsymbol{e}_1 = \neg \boldsymbol{b}$
- if $\boldsymbol{b} = \boldsymbol{b}_1 \vee \boldsymbol{b}_2$ then by induction hypothesis $\boldsymbol{b}_1 = \bigvee_{i=0}^{k_1-1} \boldsymbol{d}_i^1 \wedge \boldsymbol{e}_i^1$ and $\boldsymbol{b}_2 = \bigvee_{j=0}^{k_2-1} \boldsymbol{d}_j^2 \wedge \boldsymbol{e}_j^2$. Then,

$$
\begin{aligned}
b &\equiv \left(\bigvee_{i=0}^{k_1-1} d_i^1 \wedge e_i^1\right) \vee \left(\bigvee_{j=0}^{k_2-1} d_j^2 \wedge e_j^2\right)\\
&\equiv \left(\bigvee_{i=0}^{k_1-1} d_i^1 \wedge e_i^1 \wedge \top\right) \vee \left(\bigvee_{j=0}^{k_2-1} d_j^2 \wedge e_j^2 \wedge \top\right)\\
&\equiv \left(\bigvee_{i=0}^{k_1-1} d_i^1 \wedge e_i^1 \wedge \bigvee_j e_j^2\right) \vee \left(\bigvee_{j=0}^{k_2-1} d_j^2 \wedge e_j^2 \wedge \bigvee_i e_i^1\right)\\
&\equiv \left(\bigvee_{i=0,j=0}^{k_1-1,k_2-1} (d_i^1 \wedge e_i^1 \wedge e_j^2)\right) \vee \left(\bigvee_{j=0,i=0}^{k_2-1,k_1-1} d_j^2 \wedge e_j^2 \wedge e_i^1\right)\\
&\equiv \bigvee_{i=0,j=0}^{k_1-1,k_2-1} (d_i^1 \vee d_j^2) \wedge (e_i^1 \wedge e_j^2)
\end{aligned}
$$

Let $k = k_1 \cdot k_2$. We can identify any $l \leq k-1$ with a pair (i,j), $i < k_1$ and $j < k_2$. Let $d_{i,j} = d_i^1 \vee d_j^2$ and $e_{i,j} = e_i^1 \vee e_j^2$. Then, we have that:

$$
b = \bigvee_{i=0,j=0}^{k_1-1,k_2-1} d_{i,j} \wedge e_{i,j}.
$$

Observe that for $(i,j) \neq (i',j')$, $e_{i,j} \wedge e_{i',j'} \equiv \bot$. Moreover, $\bigvee_{i,j} e_{i,j} \equiv \bigvee_{i,j} e_i^1 \vee e_j^2 \equiv (\bigvee_i e_i^1) \vee (\bigvee_j e_j^2) \equiv \top \vee \top \equiv \top$.

- if $b = b_1 \wedge b_2$, then by IH $b_1 \equiv \bigvee_{i=0}^{k_1-1} d_i^1 \wedge e_i^1$ and $b_2 \equiv \bigvee_{j=0}^{k_2-1} d_j^2 \wedge e_j^2$. Then

$$
\begin{aligned}
b &\equiv \left(\bigvee_{i=0}^{k_1-1} d_i^1 \wedge e_i^1\right) \wedge \left(\bigvee_{j=0}^{k_2-1} d_j^2 \wedge e_j^2\right)\\
&\equiv \bigvee_{i=0,j=0}^{k_1-1,k_2-1} d_i^1 \wedge e_i^1 \wedge d_j^2 \wedge e_j^2\\
&\equiv \bigvee_{i=0,j=0}^{k_1-1,k_2-1} (d_i^1 \wedge d_j^2) \wedge (e_i^1 \wedge e_j^2)
\end{aligned}
$$

As in the case above let $k_1 \cdot k_2$. We can identify any $l \leq k-1$ with a pair (i,j), $i < k_1$ and $j < k_2$. Let $d_{i,j} = d_i^1 \wedge d_j^2$ and $e_{i,j} = e_i^1 \wedge e_j^2$. We have then that

$$
b \equiv \bigvee_{i=0,j=0}^{k_1-1,k_2-1} d_{i,j} \wedge e_{i,j}
$$

As in the previous case we have that for $(i,j) \neq (i',j')$, $e_{i,j} \wedge e_{i',j'} \equiv \bot$ and $\bigvee_{i,j} e_{i,j} \equiv \bigvee_{i,j} e_i^1 \wedge e_j^2 \equiv (\bigvee_i e_i^1) \wedge (\bigvee_j e_j^2) \equiv \top \wedge \top \equiv \top$.

$\square$

A.2. Postponed Proofs from Section 4

In order to prove Lemma 10, some auxiliary results have to be preliminarily considered.

Lemma 34. *Given* $\mathcal{Z} \subseteq (2^{\mathbb{N}})^{X\cup Y}$ *and* $\omega \in \mathcal{X} \subseteq (2^{\mathbb{N}})^{X}$, *then* $\Pi_\omega(\mathcal{Z}) = \Pi_\omega(\mathcal{Z} \cap (X)^{\Uparrow Y})$.

Proof. $\subseteq$ If $\omega' \in \Pi_\omega(\mathcal{Z})$, then $\omega + \omega' \in \mathcal{Z}$. Since $\omega \in \mathcal{X}$, by $(X)^{\Uparrow Y}$ definition, $\omega + \omega' \in (X)^{\Uparrow Y}$. As a consequence, $\omega+\omega' \in \mathcal{Z} \cap (X)^{\Uparrow Y}$ and $\omega' \in \Pi_\omega(\mathcal{Z} \cap (X)^{\Uparrow Y})$.

$\supseteq$ Trivial, since the projection operator is monotone. □

Lemma 35. *Let* $\omega \in (2^{\mathbb{N}})^{X}$, $\mathcal{Z} \subseteq (2^{\mathbb{N}})^{X\cup Y}$, *and* $\mathcal{X} \subseteq (2^{\mathbb{N}})^{X}$ $(X \cap Y = \emptyset)$. *If* $\mu(\Pi_\omega(\mathcal{Z} \cap (\mathcal{X})^{\Uparrow Y})) > 0$, *then* $\omega \in \mathcal{X}$.

Proof. By contraposition. Assume $\omega \notin \mathcal{X}$. There are two possible cases.

- If $\mathcal{Z} \cap (X)^{\Uparrow Y} = \emptyset$, then $\mu(\Pi_\omega(\mathcal{Z} \cap (X)^{\Uparrow Y})) = \mu(\Pi_\omega(\emptyset)) = \mu(\emptyset) = 0$.
- Otherwise, $\mu(\Pi_\omega(\mathcal{Z} \cap (X)^{\Uparrow Y})) = \mu(\{\omega' \in (2^{\mathbb{N}})^{Y} \mid \omega+\omega' \in (\mathcal{Z} \cap (X)^{\Uparrow Y})\}) =^{\omega \notin \mathcal{X}} \mu(\emptyset) = 0$. Indeed by definition, $(X)^{\Uparrow Y} = \{\xi + \xi' \in (2^{\mathbb{N}})^{X\cup Y} | \xi \in \mathcal{X}\}$. So (since $X \cap Y = \emptyset$), if $\omega \notin \mathcal{X}$, $\{\omega' \in (2^{\mathbb{N}})^{Y} \mid \omega+\omega' \in (X)^{\Uparrow Y}\} = \emptyset$. Trivially, also $\{\omega' \in (2^{\mathbb{N}})^{Y} \mid \omega + \omega' \in \mathcal{Z} \cap (X)^{\Uparrow Y}\} = \emptyset$.

□

Lemma 36. *Let* $a \notin X, \mathrm{FN}(A) \subseteq X$, *and* $\omega \in (2^{\mathbb{N}})^{X}$:

- *if* $\omega \in [\![A]\!]_X$, *then* $\Pi_\omega([\![A]\!]_{X\cup\{a\}}) = (2^{\mathbb{N}})^{\{a\}}$
- *if* $\omega \notin [\![A]\!]_X$, *then* $\Pi_\omega([\![A]\!]_{X\cup\{a\}}) = \emptyset$

(*or equivalently,* $[\![A]\!]_{X\cup\{a\}} = [\![A]\!]_X^{\Uparrow\{a\}}$).

Proof. Since $a \notin \mathrm{FN}(A)$, there are two possible cases:

- if $\omega \in [\![A]\!]_X$, then $\omega + \omega' \in [\![A]\!]_{X\cup\{a\}}$
- if $\omega \notin [\![A]\!]_X$, then $\omega + \omega' \notin [\![A]\!]_{X\cup\{a\}}$.

Thus, if $\omega \in [\![A]\!]_X$, $\Pi_\omega([\![A]\!]_{X\cup\{a\}}) = \{\omega' \in (2^{\mathbb{N}})^{\{a\}} \mid \omega + \omega' \in [\![A]\!]_{X\cup\{a\}}\} = \{\omega' \in (2^{\mathbb{N}})^{\{a\}} \mid \omega \in [\![A]\!]_X\} = (2^{\mathbb{N}})^{\{a\}}$. If $\omega \notin [\![A]\!]_X$, we have $\Pi_\omega([\![A]\!]_{X\cup\{a\}}) = \{\omega' \in (2^{\mathbb{N}})^{\{a\}} \mid \omega + \omega' \in [\![A]\!]_{X\cup\{a\}}\} = \{\omega' \in (2^{\mathbb{N}})^{\{a\}} \mid \omega \in [\![A]\!]_X\} = \emptyset$. □

Lemma 10 can now be deduced from the following result:

Proposition 7. *Let* $a \notin X, \mathrm{FN}(A) \subseteq X, \omega \in (2^{\mathbb{N}})^{X}$, *and* $q > 0$. *Then,*

1. $\mu(\Pi_\omega([\![A]\!]_{X\cup\{a\}})) \geq q$ *iff* $\omega \in [\![A]\!]_X$
2. $\mu(\Pi_\omega([\![A]\!]_{X\cup\{a\}}) \cap \Pi_\omega([\![B]\!]_{X\cup\{a\}})) \geq q$ *iff* $\omega \in [\![A]\!]_X$ *and* $\mu(\Pi_\omega([\![B]\!]_{X\cup\{a\}})) \geq q$
3. $\mu(\Pi_\omega([\![A]\!]_{X\cup\{a\}}) \cup \Pi_\omega([\![B]\!]_{X\cup\{a\}})) \geq q$ *iff* $\omega \in [\![A]\!]_X$ *or* $\mu(\Pi_\omega([\![B]\!]_{X\cup\{a\}})) \geq q$

4. $\mu(\Pi_\omega(\llbracket A\rrbracket_{X\cup\{a\}}) \cup \Pi_\omega(\llbracket B\rrbracket_{X\cup\{a\}}) < q$ *iff* $\omega \notin \llbracket A\rrbracket_X$ *and* $\mu(\Pi_\omega(\llbracket B\rrbracket_{X\cup\{a\}})) < q$.

Proof. We only prove cases (1) and (2), as the other are proved similarly. (1) There are two possible cases:

- $\omega \in \llbracket A\rrbracket_X$. Then $\Pi_\omega(\llbracket A\rrbracket_{X\cup\{a\}}) = \{\omega' \in (2^{\mathbb{N}})^{\{a\}} \mid \omega \in \llbracket A\rrbracket_X\} = (2^{\mathbb{N}})^{\{a\}}$. So, for every q, $\Pi_\omega(\llbracket A\rrbracket_{X\cup\{a\}}) \geq q$.
- $\omega \notin \llbracket A\rrbracket_X$. Then $\Pi_\omega(\llbracket A\rrbracket_{X\cup\{a\}}) = \{\omega' \in (2^{\mathbb{N}})^{\{a\}} \mid \omega \in \llbracket A\rrbracket_X\} = \emptyset$. So, for every q ($q \neq 0$), $\Pi_\omega(\llbracket A\rrbracket_{X\cup\{a\}}) < q$.

(2) There are two cases to be taken into account:

- $\omega \in \llbracket A\rrbracket_X$. From Lemma 36 ($a \notin X$), $\Pi_\omega(\llbracket A\rrbracket_{X\cup\{a\}}) = (2^{\mathbb{N}})^{\{a\}}$, so

$$\begin{aligned}\mu(\Pi_\omega(\llbracket A\rrbracket_{X\cup\{a\}}) \cap \Pi_\omega(\llbracket B\rrbracket_{X\cup\{a\}})) &= \mu((2^{\mathbb{N}})^{\{a\}} \cap \Pi_\omega(\llbracket B\rrbracket_{X\cup\{a\}}))\\ &= \mu(\Pi_\omega(\llbracket B\rrbracket_{X\cup\{a\}})).\end{aligned}$$

- $\omega \notin \llbracket A\rrbracket_X$. From Lemma 36, $\Pi_\omega(\llbracket A\rrbracket_{X\cup\{a\}}) = \emptyset$, so

$$\mu(\Pi_\omega(\llbracket A\rrbracket_{X\cup\{a\}}) \cap \Pi_\omega(\llbracket B\rrbracket_{X\cup\{a\}})) = \mu(\emptyset \cap \Pi_\omega(\llbracket B\rrbracket_{X\cup\{a\}})) = 0.$$

$\Leftarrow$ If $\omega \in \llbracket A\rrbracket_X$ and $\mu(\Pi_\omega(\llbracket B\rrbracket_{X\cup\{a\}})) \geq q$, then for the first clause $\mu(\Pi_\omega(\llbracket A\rrbracket_{X\cup\{a\}}) \cap \Pi_\omega(\llbracket B\rrbracket_{X\cup\{a\}})) = \mu(\Pi_\omega(\llbracket A\rrbracket_{X\cup\{a\}}))$. Therefore, $\mu(\Pi_\omega(\llbracket A\rrbracket_{X\cup\{a\}}) \cap \Pi_\omega(\llbracket B\rrbracket_{X\cup\{a\}})) \geq q$.

$\Rightarrow$ If $\omega \notin \llbracket A\rrbracket_X$, then for the second clause $\mu(\Pi_\omega(\llbracket A\rrbracket_{X\cup\{a\}}) \cap \Pi_\omega(\llbracket B\rrbracket_{X\cup\{a\}})) = 0$. Therefore, $\mu(\Pi_\omega(\llbracket A\rrbracket_{X\cup\{a\}}) \cap \Pi_\omega(\llbracket B\rrbracket_{X\cup\{a\}})) < q$.
If $\omega \in \llbracket A\rrbracket_X$ and $\mu(\Pi_\omega(\llbracket B\rrbracket_{X\cup\{a\}})) < q$, for the first clause $\mu(\Pi_\omega(\llbracket A\rrbracket_{X\cup\{a\}} \cap \Pi_\omega(\llbracket B\rrbracket_{X\cup\{a\}})) = \mu(\Pi_\omega(\llbracket B\rrbracket_{X\cup\{a\}}))$. So, $\mu(\Pi_\omega(\llbracket A\rrbracket_{X\cup\{a\}}) \cap \Pi_\omega(\llbracket B\rrbracket_{X\cup\{a\}})) < q$. ☐

A.3. Postponed Proofs from Section 5

Proof of Lemma 21. Let us fix an enumeration $(a_i)_{i\in\mathbb{N}}$ of $\mathscr{A}$. For any $p, q \in \mathbb{N}$. Let $\mathscr{F}_{p,q}$ be the set of all formulas B such that $\mathrm{FN}(B) \subseteq \{a_0, \dots, a_{q-1}\}$ and $|B| \leq p$. Let us fix, for all $p, q \in \mathbb{N}$ an enumeration $(C_n^{p,q})_{n\in\mathbb{N}}$ of $\mathscr{F}_{p,q}$.

For any natural number N, let $[N] := \{0, \dots, N-1\}$. Given $p \leq p'$ and $q \leq q'$, and finite matrices $s \in (2^{[p+1]})^{[q+1]}$ $s' \in (2^{[p'+1]})^{[q'+1]}$, let $s \sqsubseteq s'$ if for all $i \leq p$ and $j \leq q$ $s(j)(i) = s'(j)(i)$. We will often indicate as p_s and q_s the (unique) natural numbers such that $s \in (2^{[p+1]})^{[q+1]}$. Moreover, for all $\omega \in (2^{\mathbb{N}})^X$, let $s \sqsubseteq \omega$ hold if for all $i \leq p_s$ and $j \leq q_s$, $s(j)(i) = \omega(a_j)(i)$. Observe that if $s \sqsubseteq \omega$, $\mathsf{Th}(\omega) \vdash \mathsf{Th}(s)$. For all p, q and $s \in (2^{[p+1]})^{[q+1]}$, we define a set of formulas $\Gamma^{\langle p,q,s\rangle} \subseteq \mathscr{F}_{p,q}$ such that for all $p \leq p'$ and $q \leq q'$, $s \in (2^{[p+1]})^{[q+1]}$ and $s' \in (2^{[p'+1]})^{[q'+1]}$, $s \sqsubseteq s'$ implies $\Gamma^{\langle p,q,s\rangle} \subseteq \Gamma^{\langle p',q',s'\rangle}$.

We let $\Gamma^{\langle p,q,s\rangle} := \bigcup_n \Gamma_n^{\langle p,q,s\rangle}$, where the sets $\Gamma_n^{\langle p,q,s\rangle}$ are defined by a triple induction on p, q and n as follows:

- if $p = q = 0$ and $b = \omega_0(0)$,

- $\Gamma_0^{\langle 0,0,b\rangle} := \Gamma \cup \{\neg^{(1-b)}\boldsymbol{x}_0^0\}$;
- $\Gamma_{n+1}^{\langle 0,0,b\rangle} := \Gamma_n \cup \{C_n^{0,0}\}$, if this is A-consistent, and $\Gamma_{n+1}^{\langle 0,0,b\rangle} := \Gamma_n^{\langle 0,0,b\rangle}$ otherwise.

If $b \neq \omega_0(0)$, the definition is the same, with $\perp$ in place of A

- if $p > 0$ and $q = 0$, $\Gamma^{\langle p,0,s\rangle} = \bigcup_n \Gamma_n^{\langle p,0,s\rangle}$, where if $s \sqsubseteq \omega_0$,
  - $\Gamma_0^{\langle p,0,s\rangle} := \Gamma^{\langle p-1,0,s|_{p-1,0}\rangle} \cup \{\neg^{(1-s(p)(0))}\boldsymbol{x}_p^0\}$;
  - $\Gamma_{n+1}^{\langle p,0,s\rangle} := \Gamma_n^{\langle p,0,s\rangle} \cup \{C_n^{p,0}\}$ if this is A-consistent, and $\Gamma_{n+1}^{\langle p,0,s\rangle} := \Gamma_n^{\langle p,0,s\rangle}$ otherwise;

  and if $s \not\sqsubseteq \omega_0$, the definition is similar, with $\perp$ in place of A;
- if $p = 0$ and $q > 0$, $\Gamma^{\langle 0,q,s\rangle} = \bigcup_n \Gamma_n^{\langle 0,q,s\rangle}$, where if $s \sqsubseteq \omega_0$,
  - $\Gamma_0^{\langle 0,q,s\rangle} := \Gamma^{\langle 0,q-1,s|_{0,q-1}\rangle} \cup \{\neg^{(1-s(0)(q))}\boldsymbol{x}_0^q\}$;
  - $\Gamma_{n+1}^{\langle 0,q,s\rangle} := \Gamma_n^{\langle 0,q,s\rangle} \cup \{C_n^{0,q}\}$ if this is A-consistent, and $\Gamma_{n+1}^{\langle 0,q,s\rangle} := \Gamma_n^{\langle 0,q,s\rangle}$ otherwise;

  and if $s \not\sqsubseteq \omega_0$, the definition is similar, with $\perp$ in place of A
- if both $p > 0$ and $q > 0$, $\Gamma^{\langle p,q,s\rangle} = \bigcup_n \Gamma_n^{\langle p,q,s\rangle}$, where if $s \sqsubseteq \omega_0$,
  - $\Gamma_0^{\langle p,q,s\rangle} := \Gamma^{\langle p-1,q,s|_{p-1,q}\rangle} \cup \Gamma^{\langle p,q-1,s|_{p,q-1}\rangle} \cup \{\neg^{(1-s(p)(q))}\boldsymbol{x}_p^q\}$;
  - $\Gamma_{n+1}^{\langle p,q,s\rangle} := \Gamma_n^{\langle p,q,s\rangle} \cup \{C_n^{p,q}\}$ if this is A-consistent, and $\Gamma_{n+1}^{\langle p,q,s\rangle} := \Gamma_n^{\langle p,q,s\rangle}$ otherwise;

  and if $s \not\sqsubseteq \omega_0$, the definition is similar, with $\perp$ in place of A.

In the following, whenever this creates no confusion, we will indicate $\Gamma^{\langle p,q,s\rangle}$ simply as Γ^s. For all finite matrices s, s', the following hold:
a. $\mathsf{Th}(s) \subseteq \Gamma^s$
b. if $s \sqsubseteq \omega_0$, Γ^s is A-consistent
c. if $s \not\sqsubseteq \omega_0$, Γ^s is consistent
d. if $s \sqsubseteq s'$, $\Gamma^s \subseteq \Gamma^{s'}$
e. if $\Gamma^s \vdash B$, then for all $t \sqsupseteq s$ such that $B \in \mathscr{F}_{p_t,q_t}$, $B \in \Gamma^t$.
Facts a.-d. are verified by construction, so we only prove e.: suppose there exist $B_1, \dots, B_n \in \Gamma^s$ such that $B_1, \dots, B_n \vdash_{\mathbf{G}_{\mathsf{iCPL}}} B$ and let $t \sqsupseteq s$ be such that $B \in \mathscr{F}_{p_t,q_t}$. Suppose $B \notin \Gamma^t$: then for some $k \in \mathbb{N}$, $\Gamma_k^t \cup \{B\} \vdash \perp$; yet, from d. it follows that $B_1, \dots, B_n \in \Gamma^t$, and thus $\Gamma^t \vdash \perp$, contradicting c.

Given matrices $s_1, \dots, s_n \sqsubseteq \omega$, let $\bigvee^\omega \{s_1, \dots, s_n\}$ indicate the smallest submatrix of ω extending all $s_1, \dots, s_n$ (i.e. the restriction of ω to $p = \max\{p_{s_1}, \dots, p_{s_n}\}$ and $q = \max\{q_{s_1}, \dots, q_{s_n}\}$.

For any matrix s, let $\Gamma^{\dagger s} = \{A^s \mid A \in \Gamma^s\} = \{\mathsf{Th}(s) \to A \mid A \in \Gamma^s\}$. Let $\Gamma^\dagger = \bigcup_s \Gamma^{\dagger s}$ and let Γ^* be the deductive closure of $\Gamma^\dagger$. We will establish the following properties of Γ^*:

α. Γ^* is super-consistent
β. $\mathsf{Th}(\omega_0) \cup \Gamma^*$ is A-consistent
γ. $\Gamma \subseteq \Gamma^*$
δ. if $B^\omega \notin \Gamma^*$ and $B^\omega = B^s$, then $\mathsf{Th}(s) \cup \Gamma^* \cup \{B\} \vdash \bot$ moreover, if $B^{\omega_0} \notin \Gamma^*$ and $B^{\omega_0} = B^s$, then $\mathsf{Th}(s) \cup \Gamma^* \cup \{B\} \vdash A$
ϵ. Γ^* is $\vee$-closed
η. Γ^* is $\mathbf{C}$-closed.

This will conclude the proof of the theorem.

Let us preliminarily observe that any formula B^ω can be written as B^s for a unique $s \sqsubseteq \omega$ such that $p_s = |B|$ and q_s is minimum with the property that $\mathrm{FN}(B) \subseteq \{a_0, \dots, a_{q_s-1}\}$.

α. Let us show that $\Gamma^\dagger$ is super-consistent. This immediately implies that Γ^* is super-consistent too. Suppose $\mathsf{Th}(\omega) \cup \Gamma^\dagger \vdash \bot$; then there exist $s_1, \dots, s_n$ and $B_1 \in \Gamma^{s_1}, \dots, B_n \in \Gamma^{s_n}$, such that $\mathsf{Th}(\omega), B_1^{s_1}, \dots, B_n^{s_n} \vdash \bot$; we can suppose w.l.o.g. that $s_1, \dots, s_n \sqsubseteq \omega$, since if $s_i \not\sqsubseteq \omega$, $\mathsf{Th}(\omega) \vdash \neg\mathsf{Th}(s_i)$, and thus $\mathsf{Th}(\omega) \vdash B_i^{s_i}$.
We have then $B_i^{s_i} = B_i^\omega$; by letting $s = \bigvee^\omega \{s_1, \dots, s_n\}$ we thus have $B_i^{s_i} = B_i^s$ and $\mathsf{Th}(s), B_1^s, \dots, B_n^s \vdash \bot$, which implies $\mathsf{Th}(s), B_1, \dots, B_n \vdash \bot$, and since $\mathsf{Th}(s) \cup \{B_1, \dots, B_n\} \subseteq \Gamma^s$, we deduce $\bot \in \Gamma^s$, contradicting a.

β. The argument is similar to the one for α.

γ. Let $B \in \Gamma$ and s be such that $B^\omega = B^s$, $p_s = |B|$ and $\mathrm{FN}(B) \subseteq \{a_0, \dots, a_{q_s-1}\}$. For any $s' \in (2^{[p_s+1]})^{[q_s+1]}$, $B^{s'} \in \Gamma^{\dagger s'} \subseteq \Gamma^\dagger$. Hence, using the fact that $\vdash \bigvee_{s \in (2^{[p_s+1]})^{[q_s+1]}} \mathsf{Th}(s)$, we deduce that $\{B^s \mid s \in (2^{[p_s+1]})^{[q_s+1]}\} \vdash B$, and thus that $\Gamma^\dagger \vdash B$, which implies $B \in \Gamma^*$.

δ. Let s be such that $B^\omega = B^s$. We consider the case of $\omega = \omega_0$; the case $\omega \neq \omega_0$ is proved similarly with $\bot$ in place of A. We will prove the contrapositive, i.e. that if $\mathsf{Th}(s) \cup \Gamma^\dagger \cup \{B\}$ is A-consistent, then $B^{\omega_0} = B^s \in \Gamma^*$, from which δ. follows. Suppose that $\mathsf{Th}(\omega) \cup \Gamma^\dagger \cup \{B\}$ is A-consistent. Suppose $B \notin \Gamma^s$: this implies $B \notin \Gamma$ and that for some $k \in \mathbb{N}$, $\Gamma_k^s \cup \{B\} \vdash A$. But this forces then $\Gamma^s \cup \{B\} \vdash A$; since for all $F \in \Gamma^s$, $F^s \in \Gamma^\dagger$, and $\mathsf{Th}(s), F^s \vdash F$, we deduce then $\mathsf{Th}(s) \cup \Gamma^\dagger \cup \{B\} \vdash A$, which is absurd. We conclude then that $B \in \Gamma^s$, and thus $B^s = B^\omega \in \Gamma^\dagger \subseteq \Gamma^*$.

ϵ. Again, we consider the case of $\omega = \omega_0$; the case $\omega \neq \omega_0$ is proved similarly with $\bot$ in place of A. Let $s \sqsubseteq \omega_0$ be such that $(B \vee C)^\omega = (B \vee C)^s$ and suppose $(B \vee C)^s \in \Gamma^*$ but neither $B^s \in \Gamma^*$ nor $C^s \in \Gamma^*$; then by δ., $\mathsf{Th}(s) \cup \Gamma^* \cup \{B\} \vdash A$ and $\mathsf{Th}(s) \cup \Gamma^* \cup \{C\} \vdash A$, hence $\mathsf{Th}(s) \cup \Gamma^* \cup \{B \vee C\} \vdash A$; since $(B \vee C)^s \in \Gamma^*$ we have $\mathsf{Th}(s) \cup \Gamma^* \vdash B \vee C$, so we deduce $\mathsf{Th}(s) \cup \Gamma^* \vdash A$, and since $\mathsf{Th}(s) \subseteq \mathsf{Th}(\omega_0)$, we have $\mathsf{Th}(\omega_0) \cup \Gamma^* \vdash A$, contradicting β.

η. Once more, we consider the case of $\omega = \omega_0$; the case $\omega \neq \omega_0$ is proved similarly with $\bot$ in place of A. Let $s \sqsubseteq \omega_0$ be such that $(\mathbf{C}_a^q B)^{\omega_0} = (\mathbf{C}_a^q B)^s$ and suppose $(\mathbf{C}_a^q B)^s \in \Gamma^*$. We can suppose w.l.o.g. that $a = a_q$ with $q > q_s$. Hence we can suppose that the formula B^{ω_0} is of the form $B^{s+s'+s''}$, for some finite matrices s', s'', with $s'' \in 2^{[|B|+1]}$. By Lemma 19 we have that $\Gamma^s \vdash B \leftrightarrow (\bigvee_{v \in 2^{[|B|+1]}} \mathsf{Th}(v) \wedge B_{s+v})$, where B_{s+v} has no free name. Let $S = \{v \in 2^{[|B|+1]} \mid B_{s+v} \in \Gamma^s\}$ and $d = \bigvee_{v \in S} \mathsf{Th}(v)$. Observe that for any $v \in S$, $\Gamma^{s+s'} \vdash \mathsf{Th}(v) \to B$. Let $v \in S$ and ω_v be such that $v \subseteq \omega_v$. Then $\Gamma^{s+s'} \vdash \mathsf{Th}(\omega_v) \to B$. Using e. we have then that for all $t \in 2^{[|B|+1]}$, $\mathsf{Th}(\omega_v) \to B \in \Gamma^{s+s'+t}$, so in particular, $\mathsf{Th}(\omega_v) \to B \in \Gamma^{s+s'+s''}$, so $(\mathsf{Th}(\omega_v) \to B)^{\omega_0} \in \Gamma^{\dagger} \subseteq \Gamma^*$. We thus have that for all $v \in S$, $\Gamma^* \vdash (\mathsf{Th}(v) \to B)^{\omega_0}$, which implies $\Gamma^* \vdash (\boldsymbol{d} \to B)^{\omega_0}$ and thus $(\boldsymbol{d} \to B)^{\omega_0} \in \Gamma^*$. At this point, if $\mu(\boldsymbol{d}) \geq q$ we are done. Otherwise, suppose $\mu(\boldsymbol{d}) < q$; if $v \notin S$, then $B_{s+v} \notin \Gamma^s$, which implies that for some k, $\Gamma^s_k \cup \{B_{s+v}\} \vdash A$; this implies that $\Gamma^s \vdash (\mathsf{Th}(v) \wedge B_{s+v}) \to A$; we deduce then that $\Gamma^s \vdash B \to ((\bigvee_{v \in S} \mathsf{Th}(v) \wedge B_{s+v}) \vee A)$, and in particular that $\Gamma^s \vdash B \to (\boldsymbol{d} \vee A)$; this means that there exist formulas $B_1, \dots, B_n \in \Gamma^s \subseteq \mathscr{F}_{p_s,q_s}$, hence containing no occurrences of the name a, such that $B_1, \dots, B_n \vdash B \to (\boldsymbol{d} \vee A)$; from this we deduce first $\mathsf{Th}(s), B_1^s, \dots, B_n^s \vdash B \to (\boldsymbol{d} \vee A)$, and then, using ($\mathbf{C}E_2$), $\mathsf{Th}(s), \mathbf{C}^q_a B, B_1^s, \dots, B_n^s \vdash \mathbf{C}^q_a(\boldsymbol{d} \vee A)$. Now, since $s \subseteq \omega_0$, $\mathbf{C}^q_a A, B_1^s, \dots, B_n^s \in \Gamma^*$ and $\mathbf{C}^q_a(\boldsymbol{d} \vee A) \vdash (\mathbf{C}^q_a \boldsymbol{d}) \vee A$ (Axiom ($\mathbf{C}\vee$), since we can suppose w.l.o.g. that $a \notin \mathrm{FN}(A)$), we have $\mathsf{Th}(\omega_0) \cup \Gamma^* \vdash (\mathbf{C}^q_a \boldsymbol{d}) \vee A$; moreover, from $\mu(\boldsymbol{d}) < q$ we get (by Axiom ($\mathbf{C}\bot$)) $\vdash \neg \mathbf{C}^q_a \boldsymbol{d}$ and thus $(\mathbf{C}^q_a \boldsymbol{d}) \vee A \vdash A$, so we can conclude $\mathsf{Th}(\omega_0) \cup \Gamma^* \vdash A$, which contradicts β.

□

Proof of Lemma 22. Let us define sets of formulae $\Gamma_{a,i}$ together with $\omega(a)(i)$ as follows:

- let $\omega(a_0)(0) = 1$ if $\Gamma \cup \{\boldsymbol{x}^0_{a_0}\}$ is A-consistent, and $\omega(a_0)(0) = 0$ otherwise; moreover, let $\Gamma_{a_0,0} = \Gamma$
- let $\omega(a_j)(i+1) = 1$ if $\Gamma_{a_j,i} \cup \{\boldsymbol{x}^{i+1}_{a_j}\}$ is A-consistent, and $\omega(a_j)(i+1) = 0$ otherwise; moreover, let $\Gamma_{a_j,i+1} = \Gamma_{a_j,i} \cup \{\neg^{(1-\omega(a_j)(i+1))}\boldsymbol{x}^{i+1}_{a_j}\}$
- let $\omega(a_{j+1})(0) = 1$ if $\bigcup_{k \leq j} \bigcup_i \Gamma_{a_j,i} \cup \{\boldsymbol{x}^0_{a_{j+1}}\}$ is A-consistent, and $\omega(a_{j+1})(0) = 0$ otherwise; moreover, let $\Gamma_{a_{j+1},0} = \bigcup_{k \leq j} \bigcup_i \Gamma_{a_k,i} \cup \{(\neg^{1-\omega(a_{j+1})(0))}\boldsymbol{x}^0_{a_{j+1}}\}$.

By construction we have that for $j \leq j'$ and $i \leq i'$, $\Gamma_{a_j,i} \subseteq \Gamma_{a_{j'},i'}$, and that $\mathsf{Th}(\omega) \cup \Gamma = \bigcup_{a,i} \Gamma_{a,i}$; let us show that $\bigcup_{a,i} \Gamma_{a,i}$ is A-consistent:

- $\Gamma_{0,0} = \Gamma$ is A-consistent by hypothesis
- suppose $\Gamma_{a,i}$ is A-consistent; if $\Gamma_{a,i+1} = \Gamma_{a,i} \cup \{\boldsymbol{x}^{i+1}_a\}$ then by construction $\Gamma_{a,i+1}$ is A-consistent; if $\Gamma_{a,i+1} = \Gamma_{a,i} \cup \{\neg \boldsymbol{x}^{i+1}_a\}$ then by construction $\Gamma_{a,i} \cup \{\boldsymbol{x}^i_a\} \vdash A$; if moreover $\Gamma_{a,i} \cup \{\neg \boldsymbol{x}^i_a\} \vdash A$, the $\Gamma_{a,i} \cup \{\boldsymbol{x}^i_a \vee \neg \boldsymbol{x}^i_a\} \vdash A$ and thus $\Gamma_{a,i} \vdash A$, which is absurd. We conclude then that $\Gamma_{a,i+1}$ is A-consistent
- suppose $\Gamma_{a_k,i}$ is A-consistent for all $k \leq j$ and $i \in \mathbb{N}$; first observe that $\Gamma_{a_{j+1},0} := \bigcup_{k \leq j} \bigcup_i \Gamma_{a_k,i}$ is A-consistent: if $\Gamma_{a_{j+1},0} \vdash A$, then there exist $B_1 \in \Gamma_{a_{k_1},i_1}, \dots, B_n \in \Gamma_{a_{k_n},i_n}$ such that $B_1, \dots, B_n \vdash A$; since for $k \leq k'$ and $i \leq i'$, $\Gamma_{a_k,i} \subseteq \Gamma_{a_{k'},i'}$, we deduce that $\Gamma_{a_{\max\{k_j\}},\max\{i_j\}} \vdash A$, which is absurd. Now, if $\Gamma_{a_{j+1},0}$ contains $\{\boldsymbol{x}^0_{a+1}\}$, then by construction $\Gamma_{a_{j+1},0}$ is A-consistent; if $\Gamma_{a_{j+1},0}$ contains $\neg \boldsymbol{x}^0_{a+1}$, then $\bigcup_{k \leq j} \bigcup_i \Gamma_{a_k,i} \cup \{\boldsymbol{x}^0_{a_{j+1}}\} \vdash A$; hence, if $\Gamma_{a_{j+1},0} \vdash A$, we deduce $\bigcup_{k \leq j} \bigcup_i \Gamma_{a_k,i} \cup \{\boldsymbol{x}^0_{a_{j+1}} \vee \neg \boldsymbol{x}^0_{a_{j+1}}\} \vdash A$, so we conclude $\bigcup_{k \leq j} \bigcup_i \Gamma_{a_k,i}$, against what we proved before.

Now, for each formula $B \in \Gamma$ let us define a formula B^* as follows:

$$B^* := B \vee \left(\bigvee_{a,i \text{ occurring in } B} \neg^{\omega(a)(i)} \boldsymbol{x}_a^i \right).$$

Observe that $\mathsf{Th}(\omega) \cup \{B^*\} \vdash B$ and that for all $\omega' \in (2^{\mathbb{N}})^{\mathscr{A}}$, $\mathsf{Th}(\omega') \cup \{B^*\}$ is consistent. Let then $\Delta = \{B^* \mid B \in \Gamma\}$. It is clear that $\mathsf{Th}(\omega) \cup \Delta \vdash B$ for all $B \in \Gamma$. Suppose $\mathsf{Th}(\omega) \cup \Delta \vdash A$, then there exists Boolean formulas $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k \in \mathsf{Th}(\omega)$ and formulas $B_1, \dots, B_n \in \Gamma$ such that $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k, B_1^*, \dots, B_n^* \vdash A$. Since $B_i \vdash B^*$, this implies then $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k, B_1, \dots, B_n \vdash A$, and thus $\mathsf{Th}(\omega) \cup \Gamma$ is not A-consistent, which is absurd. $\square$

A.4. Postponed Proofs from Section 6

Proof of Lemma 30. For any $j \leq k$, let $U_j := \mathrm{Red}^{X,r}_{\mathbf{C}^{q_{k-j+1}} \dots \mathbf{C}^{q_k}(A \to B)}(S)$ and $V_j := \mathrm{Red}^{X,r}_{A \to \mathbf{C}^{q_{k-j+1}} \dots \mathbf{C}^{q_k} B}(S)$. We show, by induction on $j \leq k$, that $U_j = V_j$. If $j = 0$ then $U_j = A \to B = V_j$. Let then $j > 0$ and $\omega \in S$. Suppose $t \in U_j$ and let $w \in \mathrm{Red}^{X,1}_A(S)$. By IH we have $U_{j-1} \subseteq V_{j-1}$, from which we deduce $q_1 \leq \sum_{u \in U_{j-1}} \mathscr{D}^1_{\pi^\omega_X(t)}(u) \leq \sum_{u \in V_{j-1}} \mathscr{D}^1_{\pi^\omega_X(t)}(u) \overset{[\text{Lemma 26}]}{=} \sum_{u \in V_{j-1}} \mathscr{D}^1_{\pi^\omega_X(t)\pi^\omega_X(w)}(u\pi^\omega_X(w)) = \sum_{u \in V_{j-1}} \mathscr{D}^1_{\pi^\omega_X(tw)}(u\pi^\omega_X(w))$ which implies $t \in V_j$. For the converse direction, suppose $t \in V_j$; then for all $\omega \in (2^{\mathbb{N}})^X$, $q_1 \geq \sum_{v \in V_{j-1}} \mathscr{D}^1_{\pi^\omega_X(t)x}(vx) \overset{[\text{Lemma 26}]}{=} \sum_{v \in V_{j-1}} \mathscr{D}^1_{\pi^\omega_X(t)}(v) \overset{[\text{IH}]}{=} \sum_{v \in U_{j-1}} \mathscr{D}^1_{\pi^\omega_X(t)}(v)$, which proves that $t \in U_j$.

$\square$

In order to prove Lemma 29, we first establish the following:

Lemma 37. *For all terms $t, u_1, \dots, u_n$ with $\mathrm{FN}(t) \subseteq X \cup \{a\}$, $\mathrm{FN}(u_1), \dots, \mathrm{FN}(u_n) \subseteq X$, and measurable set $S \subseteq (2^{\mathbb{N}})^{X \cup \{a\}}$ and $S' \subseteq (2^{\mathbb{N}})^X$, if*

1. *for all $\omega \in S$, $\mathrm{HNV}_{\to}(\pi^\omega_X(tu_1 \dots u_n)) \geq r$;*
2. *for all $\omega \in S'$, $\mu(\Pi^\omega(S)) \geq s$;*

then for all $\omega \in S'$, $\mathrm{HNV}_{\to}(\pi^\omega_X((\nu a.t)u_1 \dots u_n)) \geq rs$.

Proof. Let us first consider the case where $n = 0$. Let $\omega \in S'$. Since any term reduces to a (unique) **PNF**, we can suppose w.l.o.g. that the name-closed term $\pi^\omega_X(\nu a.t) = \nu a.t^*$ is in **PNF**. Then, by Lemma 24 t^* is a tree of a-labeled choices. Observe that for all $\omega' \in (2^{\mathbb{N}})^{\{a\}}$, $\pi^{\omega'}_{\{a\}}(t^*) = \pi^{\omega'}_{\{a\}}(\pi^\omega_X(t)) = \pi^{\omega+\omega'}_{X \cup \{a\}}(t)$.

If N is the cardinality of $\mathrm{Supp}(t^*)$, by 3. and the fact that $s > 0$ we deduce that there exists finitely many terms $w_1, \dots, w_K$ of $\mathrm{Supp}(t^*)$ such that, for some $\omega_i \in (2^{\mathbb{N}})^{\{a\}}$ such that $\omega + \omega_i \in S$, $\pi^{\omega_i}_{\{a\}}(t^*) = w_i$ and $\mathrm{HNV}(w_i) \geq r$. Using 3. we deduce then

$$\sum_{w : \mathrm{HNV}(w) \geq r} \mu\{\omega \mid \pi^\omega_{\{a\}}(t^*) = w\} \geq \sum_{j=1}^{K_i} \mu\{\omega \mid \pi^\omega_{\{a\}}(t^*) = w_i\} \geq s$$

Now, by reducing, for all $i \leq k+1$, each such term w_i inside $\nu a.t^\sharp$, to some **PNF** w'_i such that $\mathrm{HNV}(w'_i) \geq r$, we obtain a new **PNF** $\nu a.t^\sharp$ and we can compute:

$$\begin{aligned}
\sum_{u\in\mathrm{HNV}} \mathcal{D}_{\nu a.t^\sharp}(u) &= \sum_{u\in\mathrm{HNV}} \left(\sum_{t'\in\mathrm{Supp}(t^\sharp)} \mathcal{D}_{t'}(u) \cdot \mu\{\omega' \mid \pi^{\omega'}_{\{a\}}(t^\sharp) = t'\} \right) \\
&= \sum_{t'\in\mathrm{Supp}(t^\sharp)} \left(\sum_{u\in\mathrm{HNV}} \mathcal{D}_{t'}(u) \cdot \mu\{\omega' \mid \pi^{\omega'}_{\{a\}}(t^\sharp) = t'\} \right) \\
&= \sum_{t'\in\mathrm{Supp}(t^\sharp)} \left(\sum_{u\in\mathrm{HNV}} \mathcal{D}_{t'}(u) \right) \cdot \mu\{\omega' \mid \pi^{\omega'}_{\{a\}}(t^\sharp) = t'\} \\
&\geq \sum_{i=1}^{K} \left(\sum_{u\in\mathrm{HNV}} \mathcal{D}_{w'_i}(u) \right) \cdot \mu\{\omega' \mid \pi^{\omega'}_{\{a\}}(t^\sharp) = w'_i\} \\
&\geq \sum_{i=1}^{K} r \cdot \mu\{\omega' \mid \pi^{\omega'}_{\{a\}}(t^\sharp) = w'_i\} \\
&= r \cdot \sum_{i=1}^{K} \mu\{\omega' \mid \pi^{\omega'}_{\{a\}}(t^\sharp) = w'_i\} \\
&= r \cdot \sum_{i=1}^{K} \mu\{\omega' \mid \pi^{\omega'}_{\{a\}}(t^*) = w_i\} \\
&\geq r \cdot s
\end{aligned}$$

Now, from $\nu a.t \twoheadrightarrow^*_{\{\}} \nu a.t^\sharp$ and $\mathrm{HNV}(\nu a.t^\sharp) \geq rs$, we conclude $\mathrm{HNV}(\nu a.t) \geq rs$. For the case in which $n > 0$, we argue as follows: from the hypotheses we deduce by what we have shown that $\mathrm{HNV}(\nu a.(tu_1u_2 \dots u_n)) \geq rs$. We can then conclude by observing that $(\nu a.t)u_1u_2 \dots u_n \twoheadrightarrow_{\mathsf{p}\{\}} \nu a.(tu_1u_2 \dots u_n)$. □

Lemma 29 follows from the result below:

Lemma 38. *For any formula A, terms $t, u_1, \dots, u_n$ with $\mathrm{FN}(t) \subseteq X \cup \{a\}$, $\mathrm{FN}(u_i) \subseteq X$, and measurable sets $S_0 \subseteq (2^{\mathbb{N}})^X$, $S_1 \subseteq (2^{\mathbb{N}})^{X\cup\{a_1\}}$, ..., $S_k \subseteq (2^{\mathbb{N}})^{X\cup\{a_1,\dots,a_k\}}$, if*

1. *$tu_1 \dots u_n \in \mathrm{Red}_A^{X\cup\{a_1,\dots,a_k\},r}(S_k)$;*
2. *for all $i = 0, \dots, k-1$ and $\omega \in S_i$, $\mu(\Pi^\omega(S_{i+1})) \geq s_i$;*

then $(\nu a_1. \dots \nu a_k.t)u_1 \dots u_n \in \mathrm{Red}^{X,r}_{\mathbf{C}^{s_1}\dots\mathbf{C}^{s_k}A}(S_0)$.

Proof. We argue by induction on A:
- if $A = \mathfrak{p}$, the claim follows by applying Lemma 37 a finite number of times.
- if $A = B \to C$, then let $S'' \subseteq S_0$ and $u \in \mathrm{Red}_B^{X,1}(S'')$. For all $i = 0, 1, \dots, k$, let $T_0 = S''$ and $T_{i+1} = \{\omega + \omega' \in S_{i+1} \mid \omega \in T_i\}$. Each set T_i is measurable, since counter-image of a measurable set through a measurable function (the projection function from $(2^{\mathbb{N}})^{X\cup\{a_0,\dots,a_{i+1}\}}$ to $(2^{\mathbb{N}})^{X\cup\{a_0,\dots,a_i\}}$). Using

Lemma 27 one can show by induction on i that $u \in \mathrm{Red}_B^{X,1}(T_i)$. Since $T_k \subseteq S_k$, using the hypothesis 2. we deduce that $tu_1 \dots u_n u \in \mathrm{Red}_C^{X,r}(T_k)$; moreover, for all $\omega \in T_i$, $\mu(\Pi^\omega(T_{i+1})) \geq s_i$. So, by the induction hypothesis $(\nu\vec{a}.t)u_1 \dots u_n u \in \mathrm{Red}^{X,r}_{\mathbf{C}^{s_1}\dots\mathbf{C}^{s_k}C}(T_0) = \mathrm{Red}^{X,r}_{\mathbf{C}^{s_1}\dots\mathbf{C}^{s_k}C}(S'')$. We deduce then that $(\nu\vec{a}.t)u_1 \dots u_n \in \mathrm{Red}^{X,r}_{B\to\mathbf{C}^{s_1}\dots\mathbf{C}^{s_k}C}(S_0)$, so we conclude by Lemma 30.

- if $A = \mathbf{C}^q B$, then, letting $v := \pi_X^\omega(tu_1 \dots u_n)$, $\sum_{u\in\mathrm{Red}_B^{\emptyset,1}} \mathscr{D}_v^1(u) \geq q$. Since we can suppose w.l.o.g. that v is a **PNF**, we must consider two cases. First, if v does not start with ν, then $\mathscr{D}_v^1 = \delta_v$, so we deduce $v \in \mathrm{Red}_B^{\emptyset,1}$, and we conclude by the IH. If $v = \nu b.v'$, there exist terms $v_1, \dots, v_k$ such that $\pi_X^\omega(v_i) \in \mathrm{Red}_B^{\emptyset,1}$ and $\sum_{i=1}^k \mathscr{D}_v^1(\pi_X^\omega(v_i)) = q$. For each v_i, let $\mathscr{b}_i$ be the Boolean formula describing the path leading from v to $\pi_X^\omega(v_i)$, and let $\mathscr{b} = \bigvee_{i=1}^k \mathscr{b}_i$, so that it must be $\mu(\mathscr{b}) \geq q$. Let then t' be the term obtained from $tu_1 \dots u_n$ by deleting all occurrences of the binder νb. Observe that $\nu\vec{a}.\nu b.t' \rightarrowtriangle^*_{\mathsf{p}\{\}} \nu\vec{a}.tu_1 \dots u_n$. Let $S_{k+1} := S_k \cap [\![\mathscr{b}]\!]$ and $s_{k+1} = q$. We now have that $t' \in \mathrm{NRed}_B^{X\cup\{a_1,\dots,a_k,b\}}(S_{k+1})$ and that for all $\omega \in S_i$ $\Pi^\omega(S_{i+1}) \geq s_i$. By IH we deduce then $\nu a_1.\dots.\nu a_k.\nu b.t' \in \mathrm{Red}^{X,1}_{\mathbf{C}^{s_1}\dots\mathbf{C}^{s_k}A}(S_0)$, and we conclude by Lemma 28.

$\square$