Tropical Mathematics and the Lambda Calculus

***Abstract*—In this paper we study an interpretation of the λ-calculus in the framework of tropical mathematics, and we show that it provides a unified framework for both program metrics, as based on the analysis of program sensitivity via Lipschitz-conditions, and for resource analysis, as based on higher-order program differentiation. We discuss applications of this tropical semantics to "best case" resource analysis as well as to other quantitative properties like convergence log-probabilities and differential privacy. Finally, we show that a general foundation for this semantic approach is provided by an abstract correspondence between Lawvere's theory of generalized metric spaces and tropical algebra.**

***Index Terms*—Differential lambda calculus, Tropical semiring, Lawvere quantale, Program metrics, Relational semantics.**

I. Introduction

In recent years, more and more interest in the programming language community has been directed towards the study of *quantitative* properties of programs like computing the number of computation steps or convergence probabilities, as opposed to purely *qualitative* properties like termination or program equivalence. Notably, a significant effort has been made to extend, or adapt, well-established qualitative methods, like type systems, relational logics or denotational semantics, to account for quantitative properties. We can mention, for example, intersection type systems aimed at capturing time or space resources [3], [28] or convergence probabilities [9], [18], relational logics to account for probabilistic properties like e.g. differential privacy [13] or metric preservation [25], [64], as well as the study of denotational models for probabilistic [33], [44] or differential [35] extensions of the λ-calculus. The main reason to look for methods relying on (quantitative extensions of) type-theory or denotational semantics is that these approaches yield *modular* and *compositional* techniques, that is, allow one to deduce properties of complex programs from the properties of their constituent parts.

A. Two kinds of quantitative approaches

Among such quantitative approaches, two different directions have received considerable attention.

On the one hand one there is the approach of *program metrics* [10], [11], [64] and *quantitative equational theories* [55]: when considering probabilistic or approximate computation, rather than asking whether two programs compute *the same* function, it makes more sense to ask whether they compute functions which do not differ *too much*. This has motivated the study of denotational frameworks in which types are endowed with a metric, measuring similarity of behavior; this approach has found applications in e.g. differential privacy [64] and coinductive methods [12], and was recently extended to account for the full λ-calculus [26], [40], [63].

On the other hand, there is the approach based on *differential* [35] or *resource-aware* [17] extensions of the λ-calculus, which is well-connected to the so-called *relational semantics* [30], [48], [53] and has a syntactic counterpart in the study of *non-idempotent* intersection types [28], [56]. This family of approaches have been exploited to account for higher-order program differentiation [35], to establish reasonable *cost-models* for the λ-calculus [2], and have also been shown suitable for the probabilistic setting [9], [18], [48].

In both approaches the notion of *linearity*, in the sense of linear logic [41] (i.e. of using inputs exactly once), plays a crucial role. In metric semantics, linear programs correspond to *non-expansive* maps, that is, to functions that do not increase distances; moreover, the possibility of duplicating inputs leads to interpret *bounded* programs (i.e. programs with a fixed duplication bound) as *Lipschitz-continuous* maps [10]. By contrast, in the standard semantics of the differential λ-calculus, linear programs correspond to linear maps, in the usual algebraic sense, while the possibility of duplicating inputs leads to consider functions defined as *power series*.

A natural question is thus whether these two apparently unrelated ways of interpreting linearity and duplication can be somehow reconciled. At a first glance, there seems to be a "logarithmic" gap between the two approaches: in metric models n times duplication results in a *linear* (hence Lipschitz) function $n \cdot x$, while in differential models this results in a *polynomial* function x^n, hence not Lipschitz. The fundamental motivation of this work is then the observation that this gap is naturally overcome once we interpret these functions in the framework of tropical mathematics, where, as we'll see, the monomial x^n precisely reads as the linear function $n \cdot x$.

B. Tropical mathematics and program semantics

Tropical mathematics was introduced in the seventies by the Brazilian mathematician Imre Simon [70] as an alternative approach to algebra and geometry where the usual ring structure of numbers based on addition and multiplication is replaced by the semiring structure given, respectively, by "min" and "+". For instance, the polynomial $p(x, y) = x^2 + xy^2 + y^3$, when interpreted over the tropical semiring, translates as the piecewise linear function $\varphi(x, y) = \min\{2x, x + 2y, 3y\}$.

In the last decades, tropical geometry evolved into a vast and rich research domain, providing a combinatorial counterpart of usual algebraic geometry, with important connections with optimisation theory [52]. Computationally speaking, working with $\min$ and $+$ is generally easier than working with standard addition and multiplication; for instance, the fundamental (and generally intractable) problem of finding the roots of a polynomial admits a *linear time* algorithm in the tropical case (and, moreover, the tropical roots can be used to approximate the actual roots [60]). The computational nature of tropical notions explains why these are so widely applied in computer science, notably for convex analysis and machine learning (see [54] for a recent survey).

Coming back to our discussion on program semantics, tropical geometry might seem to provide precisely what look for, as it turns the monomials x^n into the Lipschitz map $n \cdot x$. At this point, it is worth mentioning that a tropical variant of relational semantics has already been considered [48], and shown capable of capturing *best-case* quantitative properties, but has not yet been studied in detail. Furthermore, connections between tropical linear algebra and metric spaces have also been observed [37] within the abstract setting of *quantale-enriched* categories [45], [72]. However, a thorough investigation of the interpretation of the λ-calculus within tropical mathematics has not yet been undertaken.

In this paper we demonstrate that the relational interpretation of the λ-calculus based on tropical mathematics does indeed provides the desired bridge between differential and metric semantics. Moreover, we show that the conceptual unification of these two approaches suggests ways in which techniques from resource-analysis could be used in sensitivity analysis and *vice-versa*, paving the way for new applications of tropical geometry to the study of higher-order programs.

C. Contributions

Our contributions in this paper are threefold:

- we study the relational model over the tropical semiring and we show that the functions interpreting simply-typed lambda terms, which correspond to a generalization of *tropical Laurent series* [59], are locally Lipschitz-continuous, thus yielding a full-scale metric semantics for the λ-calculus and its bounded fragments. This is in Sections III and IV.
- Using the relational model as our main source of inspiration, we suggest a few potential applications of tropical methods to the study of quantitative properties of non-deterministic and probabilistic functional programs, like counting best-case computation steps, measuring convergence log-probabilities, and differential privacy. This is in Section V
- We conclude by putting the connection between the tropical, differential and metric viewpoints at the right level of generality. By recalling and suitably extending a well-known correspondence between Lawvere's *generalized metric spaces* [49], [72] and modules over the tropical semi-ring [65], we show that the category of *complete* generalized metric spaces provides a model of the differential λ-calculus which extends the tropical relational model. This is in Section VI.

II. A Bridge Between Two Quantitative Approaches to the λ-calculus

In this section, we discuss in some more detail the two approaches to quantitative semantics we mentioned in the Introduction, at the same time providing an overview of how we aim at bridging them using tropical mathematics.

A. Metric approach: bounded λ-terms as Lipschitz functions

In many situations (e.g. when dealing with computationally difficult problems) one does not look for algorithms to compute a function *exactly*, but rather to approximate it (in an efficient way) within some error bound. In other common situations (e.g. in differential privacy [8], [64]) one needs to verify that an algorithm is not *too sensitive* to errors, that is, a small error in the input will produce a comparably small error in the output.

In all these cases, it is common to consider forms of denotational semantics in which types are endowed with a *behavioral metric*, that is, a metric on programs which account for differences in behavior. A fundamental insight coming from this line of work is that *affine* programs, i.e. programs that may use their input at most once, correspond to *non-expansive* (or 1-Lipschitz) maps, that is, to functions f for which the distances $d(f(x), f(y))$ produced in output are bounded by the distances $d(x, y)$ in input. A more formal way of stating this observation is that the category Met of pseudo-metric spaces and non-expansive maps provides a model of the *linear* simply typed λ-calculus, being a *symmetric monoidal closed* category, and in fact it also models affine terms (since the cartesian and monoidal units coincide).

As observed in [10], [64], the metric approach is not restricted to affine programs, but can be extended to programs with *bounded* duplications. The fundamental intuition is that a program duplicating its input K times will give rise to a K-*Lipschitz* map. For instance, the higher-order program $M = \lambda f.\lambda x.f(f(x))$, which duplicates the functional input f, yields a 2-Lipschitz map between the metric space $\mathbb{R} \multimap \mathbb{R}$ of non-expansive real functions and itself: if f, g are two non-expansive maps differing by at most ϵ (i.e. for which $|f(x) - g(x)| \leqslant \epsilon$ holds for all $x \in \mathbb{R}$), then the application of M to f and g will produce two maps differing by at most 2ϵ. By observing that a r-Lipschitz map between metric spaces X and Y is the same as a non-expansive map between the *re-scaled* space $r \cdot X$ (i.e. with distance $d_r(x, y) = r \cdot d(x, y)$) and Y, the program M above can thus be interpreted as a non-expansive map from $2 \cdot (\mathbb{R} \multimap \mathbb{R})$ to $\mathbb{R} \multimap \mathbb{R}$.

These observations have led to the study of linear λ-calculi with *graded* exponentials, like Fuzz [64], inspired from Girard's Bounded Linear Logic [42], which have been applied to the study of differential privacy [10], [38]. The types of such systems are defined by combining linear constructors with a *graded linear exponential comonad* $!_r(-)$ [47]. In the following sections we will sometimes make reference to a basic graded type system, that we call bSTLC, for bounded higher-order programs, with types defined via $A ::= o \mid !_n A \multimap A$, where $n \in \mathbb{N}$. Intuitively, $!_n A \multimap B$ is the type of functions from A to B that may use their input *at most* n times. More details about bSTLC are provided in the Appendix.

Now, what about the good old, "unbounded", simply typed λ-calculus? Actually, by using unbounded duplications, one might lose the Lipschitz property. For instance, while the functions $M_k = \lambda x.k \cdot x : \mathbb{R} \to \mathbb{R}$ are all Lipschitz-continuous, with Lipschitz constant k, the function $M = \lambda x.x^2$ obtained by "duplicating" x is not Lipschitz anymore: M is, so to say, *too* sensitive to errors. More abstractly, it is well-known that the category Met is *not* cartesian closed, so it is not a model of STLC (yet, several cartesian closed *sub*-categories of Met do exist, see e.g. [21], [26]). Still, one might observe that the program M above is actually Lipschitz-continuous, if not globally, at least *locally* (i.e. over any compact set). Indeed, some cartesian closed categories of locally Lipschitz maps have been produced in the literature [33], [63], and a new example will be exhibited in this paper.

B. Resource approach: differential λ-terms as polynomials

A different family of approaches to linearity and duplication arises from the study of the *differential λ-calculus* [35] (and differential linear logic [30]) and its categorical models. The key ingredient is a *differential operator* D, added to the usual syntax of the λ-calculus. The intuition is that, given M of type $A \to B$ and N of type A, the program $\mathsf{D}[M] \cdot N$, still of type $A \to B$, corresponds to the *linear application* of M to N: this means that N is passed to M so that the latter may use it exactly once (this is why $\mathsf{D}[M] \cdot N$ still has type $A \to B$, since M might need *other* copies of an input of type A).

Interestingly, the categorical study of the operator D has led to the introduction of *cartesian differential categories $C\partial C$* [16], a class of categories providing an abstract axiomatization of differentiation, in the usual mathematical sense. More precisely, a cartesian category $\mathcal{C}$ is a $C\partial C$ when:

- $\mathcal{C}$ is left-additive, i.e. its hom-sets have the structure of commutative monoids, and the cartesian structure is well-behaved w.r.t. this monoid structure;
- $\mathcal{C}$ is equipped with a differential operator $D : \mathcal{C}(X, Y) \to \mathcal{C}(X \times X, Y)$ satisfying some axioms which capture usual properties of differentials (e.g. the linearity of D in one of its two variables, the chain rule, etc.).

Correspondingly, the syntax of the simply typed *differential* λ-calculus (ST∂LC) is defined by enriching STLC with a monoid structure $0, +$ over terms, as well as with D and a notion of *linear substitution* (see [35] or the Appendix for details). The models of ST∂LC are the cartesian *closed* differential categories ($CC\partial C$), which are defined as $C\partial C$ which are also cartesian closed, and in which the monoid structure and the differential operator are both well-behaved with respect to the closed structure [53].

Another intriguing similarity between program derivatives and actual derivatives is provided by the *Taylor expansion* $\mathcal{T}$: in ST∂LC one can expand any application MN as an infinite formal sum of *linear* applications $\mathsf{D}^{(k)}[M] \cdot N^k$, i.e. where N is passed exactly k times to M:

$$\mathcal{T}(MN) := \sum_{k=0}^{\infty} \frac{1}{!k} \cdot (\mathsf{D}^{(k)}[M] \cdot N^k)0 \tag{1}$$

In other words, unbounded duplications correspond to some sort of limit of bounded, but arbitrarily large, ones.

C. Tropical Mathematics: a possible synthesis

At this point, as the Taylor formula decomposes an unbounded application as a limit of bounded ones, one might well ask whether it could be possible to see this formula as interpreting a λ-term as a limit of Lipschitz maps, in some sense, thus bridging the metric and differential approaches. Here, a natural direction to look for is the *relational semantics*, i.e. the somehow canonical "Taylor" semantics for ST∂LC. However, in this semantics, terms with bounded applications correspond to *polynomials*, i.e. to non-Lipschitz functions.

Yet, what if these polynomials were tropical ones, i.e. piecewise linear functions? This way, (1) could really be interpreted as a decomposition of λ-terms via limits (indeed, $\inf$s) of Lipschitz maps. In other words, unbounded term application could be seen as a limit of *more and more sensitive* operations.

This viewpoint, that we develop in the following sections, leads to the somehow unexpected discovery of a bridge between the metric and differential study of higher-order programs. This connection not only suggests the application of optimization methods based on tropical mathematics to the study of the λ-calculus and its quantitative extensions, but it scales to a more abstract level, leading to introduce a differential operator for continuous functors between *generalized* metric spaces (in the sense of [49]).

III. The Tropical Relational Model

A. Tropical linear algebra

At the basis of our approach is the observation that the *tropical semiring* $([0, \infty], \min, +)$, which is at the heart of tropical mathematics, coincides with the *Lawvere quantale* $\mathbb{L} = ([0, \infty], \geqslant, +)$ [45], [72], the structure at the heart of the categorical study of metric spaces initiated by Lawvere himself [49]. Let us recall that a quantale is a complete lattice endowed with a continuous monoid action. In the case of $\mathbb{L}$ the lattice is defined by the reverse order $\geqslant$ on $\mathbb{R}$, and the monoid action is provided by addition. Notice that the lattice join operation of $\mathbb{L}$ coincides with the idempotent semiring operation $\min$. A consequence of these observations is that, as we discussed below, the tropical approach to linear algebra coincides with the study of "$\mathbb{L}$-valued matrices", i.e. of maps of the form $s : X \times Y \to \mathbb{L}$. In particular, a (possibly ∞) metric on a set X is nothing but a "$\mathbb{L}$-valued square matrix" $d : X \times X \to \mathbb{L}$ satisfying axioms like e.g. the triangular law (indeed, such distance matrices correspond to $\mathbb{L}$-*enriched categories*, a viewpoint we explicitly take in Section VI).

The study of matrices with values over the tropical semiring can be seen as a special case of the *quantitative relational semantics* [48], a well-studied semantics of the λ-calculus.

For a fixed *continuous* semi-ring Q [Def. II.5, [48]], the category $Q\mathsf{Rel}$ has sets as objects and set-indexed matrices with coefficients in Q as morphisms, i.e. $Q\mathsf{Rel}(X, Y) = Q^{X \times Y}$ ([48] would call it Q^{Π}). The identity morphism of $Q\mathsf{Rel}$ is the identity matrix $I : X \times X \to Q$, and composition of morphisms $t : X \times Y \to Q$ and $s : Y \times Z \to Q$ is given by $(st)_{a,c} := \sum_{b \in Y} s_{b,c} t_{a,b}$. Notice that the assumption of Q being continuous is used in order to make this potentially infinite sum converge.

As it is expected, Q^X is a Q-semimodule and the bijection $\hat{(\cdot)}$ identifies the set of linear maps from Q^X to Q^Y with $Q\mathsf{Rel}(X,Y)$, any map $f : Q^X \to Q^Y$ being of the form

$$f(x)_b := \sum_{a\in X} \hat{f}_{a,b} x_a \qquad (2)$$

for some matrix $\hat{f} : X \times Y \to Q$.

Remark 1. *Following [29], [45], [48], we chose to see a matrix from X to Y as a map $s : X \times Y \to Q$. Notice that usual linear algebra conventions correspond to working in $Q\mathsf{Rel}^{\mathrm{op}}$: a matrix $X \times Y \to Q$ is usually called a "$Y \times X$-matrix", meaning Y rows and X columns, and the usual matrix-vector product defines a map $Q^Y \to Q^X$.*

Remark 2. *The category $Q\mathsf{Rel}$ is (equivalent to) a subcategory of the category $Q\mathsf{Mod}$ of* complete *Q-semimodules. If $Q\mathsf{Rel}$ corresponds to considering semimodules (the Q^X's) whose vectors are given in coordinates w.r.t. a* fixed base *(the set X), $Q\mathsf{Mod}$ corresponds to considering semimodules in abstract, without fixing a base. We take this viewpoint in Section VI.*

The tropical relational model is thus provided by the category $\mathbb{L}\mathsf{Rel}$ of matrices with values over $\mathbb{L}$ (which, being a quantale, is indeed a continuous semi-ring). It is worth observing that the formula for composition in $\mathbb{L}\mathsf{Rel}$ reads as $(st)_{a,c} := \inf_{b\in Y}\{s_{b,c} + t_{a,b}\}$; similarly, the linear functions $f : \mathbb{L}^X \to \mathbb{L}^Y$ induced by matrices, which we call *tropical linear*, are exactly those of shape $f(x)_b = \inf_{a\in X}\{\hat{f}_{a,b} + x_a\}$, for some matrix $\hat{f}$ from X to Y.

Since $\mathbb{L}$ is a continuous commutative semiring, [Proposition III.3, [48]] immediately applies and gives:

Fact 1. $\mathbb{L}\mathsf{Rel}$ *is a linear $\mathbb{L}$-category.*

Unwrapping [Definition II.9, [48]], this means that: $\mathbb{L}\mathsf{Rel}(X,Y)$ is a continuous $\mathbb{L}$-semimodule, with semimodule operations defined pointwise; $\mathbb{L}\mathsf{Rel}$ is a continuous $\mathbb{L}$-category, i.e. composition of morphisms commutes with $\inf$'s; $\mathbb{L}\mathsf{Rel}$ is linear, i.e. pre- and post-composition with any morphism in any $\mathbb{L}\mathsf{Rel}(X,Y)$ are automorphisms on it.

In the next sections we will see how $\mathbb{L}\mathsf{Rel}$ gives rise to denotational models of several variants of the λ-calculus.

B. Prohibited duplication/erasure: the linear STLC

We now turn to discuss how different variants of the simply typed λ-calculus are interpreted in $\mathbb{L}\mathsf{Rel}$.

The linear simply typed λ-calculus is a restriction of the ordinary λ-calculus in which programs can only use their arguments *exactly once*. This is the notion of linearity taken into account by linear logic, and indeed this calculus can be embedded in its *intuitionistic multiplicative* fragment *IMLL*.

More precisely, the linear λ-calculus is obtained from the ordinary one by adding the constraint that each λ-abstracted variable appears exactly once in the scope of the λ-abstraction.

In order to frame it in a category $\mathcal{C}$, one needs a symmetric tensor product $\otimes$ together with internal hom-set objects $X \multimap Y$ s.t. the evaluation and curry maps yield a *symmetric monoidal adjunction*: $\mathcal{C}(Z \otimes X, Y) \simeq \mathcal{C}(Z, X \multimap Y)$. This gives the notion of Symmetric Monoidal Closed Category (SMCC).

[Section III.A, [48]], immediately gives:

Fact 2. $\mathbb{L}\mathsf{Rel}$ *is a SMCC, thus a model of the linear* STLC.

The monoidal structure of $\mathbb{L}\mathsf{Rel}$ is given by a tensor product $\otimes$ acting on the objects as the Cartesian product of sets, and as the *Kronecker product* of matrices on morphisms. Its closed structures is also given by the Cartesian product on sets, with the usual evaluation and curry maps. Actually, the SMCC $\mathbb{L}\mathsf{Rel}$ is even a model of IMLL.

C. Allowed duplication/erasure: the STLC

In order to interpret the full STLC, we need a Cartesian closed category (CCC). It is well-known [57] that it is always possible to construct a CCC by taking the *co-Kleisli* $\mathcal{C}_!$ of a so-called *Lafont category* $\mathcal{C}$. A SMCC is Lafont when it has finite products and it is equipped with a comonad ! (its *Lafont exponential*) which, at level of objects, sends X to an object $!X$ being the free commutative comonoid on X. Such objects $!X$ represent the *bang* connective of linear logic, granting infinite duplications via the infinite product $X^0 \otimes X \otimes X^2 \otimes X^3 \otimes \cdots$, each factor representing a possible number of duplications. It is well-known that, under mild conditions satisfied by $Q\mathsf{Rel}$, one can explicit this idea via the fact that the map $X \mapsto \mathcal{M}_{\mathrm{fin}}(X)$ (where $\mathcal{M}_{\mathrm{fin}}(X)$ is the set of finite multi-sets on X) lifts to a functor $! : Q\mathsf{Rel} \to Q\mathsf{Rel}$ which is a Lafont-exponential comonad. Specializing [Corollary III.6, [48]] to our case, we have:

Fact 3. $\mathbb{L}\mathsf{Rel}$ *is Lafont.*

Let us recall that the coKleisli category $\mathcal{C}_!$ of a category $\mathcal{C}$ w.r.t. a comonad ! is the category whose elements are the same of $\mathcal{C}$, and $\mathcal{C}_!(X,Y) := \mathcal{C}(!X,Y)$, with composition $\circ_!$ defined by making use of the co-multiplication of !. As already mentioned, one obtains a CCC from a Lafont category by defining the exponential objects as $X \to Y := !X \multimap Y$. In our case, specialising [Theorem III.7, [48]] we have indeed:

Fact 4. *The coKleisli* $\mathbb{L}\mathsf{Rel}_!$ *is CCC, i.e. a model of* STLC.

In our tropical setting, the exponential object $X \multimap Y$ is $!X \times Y$, and the coKleisli composition of $s \in \mathbb{L}^{!Y\times Z}$ and $t \in \mathbb{L}^{!X\times Y}$ is the matrix $s \circ_! t \in \mathbb{L}^{!X\times Z}$ where $(s \circ_! t)_{\mu,c}$ is:

$$\inf_{n\in\mathbb{N}, b_1\ldots,b_n\in Y, \mu=\mu_1+\cdots+\mu_n} \left\{ s_{[b_1,\ldots,b_n],c} + \sum_{i=1}^{n} t_{\mu_i,b_i} \right\}. \qquad (3)$$

As it is well-known, the Cartesian closed structure allows to define a sound interpretation $[\![\Gamma \vdash M : A]\!] \in \mathbb{L}\mathsf{Rel}_!([\![\Gamma]\!], [\![A]\!])$ of terms as morphisms. In our case, we have:

Proposition 1. *The interpretation $[\![M]\!]$ of a λ-term M in* $\mathbb{L}\mathsf{Rel}$ *is a* discrete *matrix, i.e. its coefficients are either 0 or ∞.*

Proof. First easily prove that composition preserves discreteness. Then go by straightforward induction on M, using that the projections and evaluation of $\mathbb{L}\mathsf{Rel}$ are discrete. $\square$

While ordinary λ-terms yield discrete matrices, as discussed in Section V, the other coefficients from $[0, \infty]$ will appear when considering quantitative effects like computation steps or probabilities.

Remark 3. *The exponential ! can be "decomposed" into a family of* graded *exponentials* $!_n : \mathbb{L}\mathsf{Rel} \to \mathbb{L}\mathsf{Rel}$ *($n \in \mathbb{N}$), defined by functors* $X \mapsto \mathcal{M}_{\leqslant n}(X)$*. The sequence* $(!_n)_{n\in\mathbb{N}}$ *can be then turned into a* $\mathbb{N}$*-graded linear exponential comonad on (the SMC)* $\mathbb{L}\mathsf{Rel}$*, satisfying the adjunction:* $\mathbb{L}\mathsf{Rel}(Z \otimes !_n X, Y) \simeq \mathbb{L}\mathsf{Rel}(Z, !_n X \multimap Y)$*. Therefore,* $\mathbb{L}\mathsf{Rel}$*, together with* $(!_n)_{n\in\mathbb{N}}$*, is a model for* bSTLC*. In particular, bounded arrow types are interpreted via* $[\![o]\!] = \{\star\}$ *and* $[\![!_n A \multimap B]\!] := \mathcal{M}_{\leqslant n}([\![A]\!]) \times [\![B]\!]$*. Notice that, for any type* A *of* bSTLC*, its interpretation* $[\![A]\!]$ *is a finite set.*

D. Controlled duplication/erasure: the ST∂LC

Until now we simply specialised well-known results in our tropical case, with the intent of showing how things read in this particular case. Now we go further, by showing that $\mathbb{L}\mathsf{Rel}_!$ actually admits a *differential structure*, turning it into a model of the ST∂LC, i.e. a *CC∂C*. This viewpoint will be further generalised in Section VI.

Let us show the differential structure of $\mathbb{L}\mathsf{Rel}_!$ (remember that the Cartesian product of $\mathbb{L}\mathsf{Rel}_!$ is the disjoint union $+$).

Definition 1. *The* tropical differential operator *is the map* $D : \mathbb{L}\mathsf{Rel}(!X, Y) \to \mathbb{L}\mathsf{Rel}(!(X + X), Y)$ *defined as* $(Dt)_{\mu\oplus\rho,b} = t_{\rho+\mu,b}$ *if* $\#\mu = 1$ *and as* ∞ *otherwise (where a multiset* $\nu \in !(X + X)$ *is identified with a disjoint sum of* $\mu, \rho \in !X$*).*

Remark 4. *For* $t \in \mathbb{L}\mathsf{Rel}(!X, Y)$*, we have:* $D^2 t \in \mathbb{L}\mathsf{Rel}_!((X + X) + (X + X), Y)$*, where* $(D^2 t)_{(\rho\oplus\rho')\oplus(\nu\oplus\nu'),b}$ *equals* $t_{\nu+\nu'+\rho',b}$ *if* $\rho = \varnothing$ *and* $\#\rho' = 1 = \#\nu$*; it equals* $t_{\rho+\nu',b}$ *if* $\rho' = \varnothing = \nu$ *and* $\#\rho = 1$*; it equals* ∞ *otherwise.*

Theorem 2. $\mathbb{L}\mathsf{Rel}_!$ *equipped with* D*, is a CC∂C.*

Proof. Left-additivity (with respect to min) is quite straightforward. The other axioms of CC∂C's are checked in the Appendix. $\square$

This ensures that one can define a sound interpretation of ST∂LC-terms in the standard way (see [Section 4.3, [19]]).

There is more: this model is also well-behaved w.r.t. to the Taylor expansion, as expressed by the following property (see [Definition 4.22, [53]]).

Theorem 3. *In* $\mathbb{L}\mathsf{Rel}_!$ *equipped with* D*, all morphisms can be Taylor expanded, i.e. in* $\mathbb{L}\mathsf{Rel}_!$ *we have that, for all* $t \in \mathbb{L}\mathsf{Rel}_!(Z, X \multimap Y)$ *and* $s \in \mathbb{L}\mathsf{Rel}_!(Z, X)$*:*

$$\mathrm{ev} \circ_! \langle t, s\rangle = \inf_{n\in\mathbb{N}}\{((\dots((\Lambda^- t) \star s) \star \dots) \star s) \circ_! \langle \mathrm{id}, \infty\rangle\}. \tag{4}$$

Here, $u \star s = (Du) \circ_! \langle\langle \infty, s \circ_! \pi_1\rangle, \mathrm{id}\rangle$ corresponds to the application of the derivative of u on s, and Λ^- is the uncurry operator. Hence the right-hand term in (4) corresponds to the inf of the n-th derivative of $\Lambda^- t$ applied to "n copies" of s, i.e. it coincides with the tropical version of the usual Taylor expansion. Now, since $\mathbb{L}\mathsf{Rel}_!$ has countable sums (i.e. all inf's converge), an immediate adaptation of the proof of [Theorem 4.23, [53]] shows:

Corollary 4. *In* $\mathbb{L}\mathsf{Rel}_!$*, the interpretation of the Taylor expansion of a* STLC*-term* M*, given in* (1)*, converges to the one of* M*.*

IV. Tropical Laurent Series

As morphisms in $\mathbb{L}\mathsf{Rel}$ correspond to tropical linear functions, morphisms in the coKleisli category $\mathbb{L}\mathsf{Rel}_!$ correspond to a generalization of tropical Laurent series, i.e. the tropicalization of power series.

In this section we establish some topological and metric properties of tropical Laurent series, this way highlighting the rich and interesting topological and metric structure of the category $\mathbb{L}\mathsf{Rel}_!$. The literature on power series in tropical mathematics is very recent (see e.g. [59]) and several results we prove in this section are, to our knowledge, new.

A. From tropical polynomials to tLs

A tropical polynomial is a piece-wise linear function $f : \mathbb{L} \to \mathbb{L}$ of the form

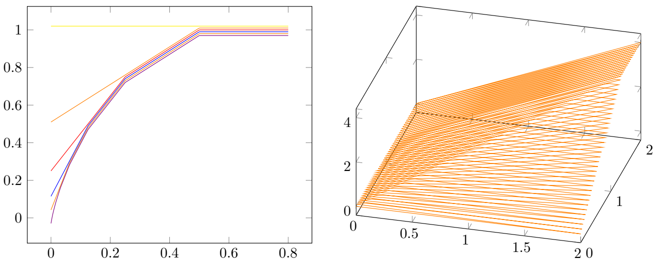
$$f(x) = \min_{i_1,\dots,i_k}\{i_j x + c_{i_j}\} \tag{5}$$

where the i_j are natural numbers and the c_{i_j} are in $\mathbb{L}$. For example, the polynomials $\varphi_n(x) = \min_{i\leqslant n}\{ix + 2^{-i}\}$ are illustrated in Fig. 1a for $n \leqslant 4$. A *tropical root* of a tropical polynomials φ is a point $x \in \mathbb{L}$ where φ is not differentiable. In other words, the roots of φ are the points where the minimum defining φ is attained at least twice (i.e. where the slope of φ changes). For instance, the tropical roots of φ_{n+1} are of the form $2^{-(i+1)}$, for $i \leqslant n$. With this definition, tropical roots mimic the usual factorization property of roots: if x_0 is a root of f, this factorizes as $f(x) = \min\{x, x_0\} + g(x)$. Yet, unlike in standard algebra, tropical roots can be computed in linear time [60].

A *tropical Laurent series* (of one variable $x \in \mathbb{L}$), shortly a *tLs*, is a function that can be expressed as $f(x) = \inf_n\{nx + \hat{f}_n\}$, with $\hat{f}_n$ a sequence in $\mathbb{L}$. In other words, a tLs is a "limit" of tropical polynomials of higher and higher degree. For instance, the function $\varphi(x) := \inf_{n\in\mathbb{N}}\{nx + \frac{1}{2^n}\}$ (illustrated in Fig. 1a), that we will take as our running example, is the "limit" of the polynomials φ_n. Since infs are not in general mins, the behavior of tLS may be less predictable than that of tropical polynomials. For instance, tropical roots for tLs (see [59]) may also include limit points.

As usual, a matrix $t \in \mathbb{L}\mathsf{Rel}_!(X, Y)$ yields a linear map $\mathbb{L}^{!X} \to \mathbb{L}^Y$, but we can also "express it in the base X", i.e. see it as a map $t^! : \mathbb{L}^X \to \mathbb{L}^Y$, by setting $t^!(x) := t \circ_! x$. This is the notion of *non-linear* map generated by the CCC-structure of $\mathbb{L}\mathsf{Rel}_!$. Concretely, we have

$$t^!(x)_b = \inf_{\mu\in !X}\{\mu x + t_{\mu,b}\} \tag{6}$$



(a) (b)

Fig. 1: (a) Plot of the tropical polynomials φ_n, for $n \leqslant 4$ (from top to bottom), and of their limit tLs φ (in violet). The points where the slope changes are the tropical roots of φ, i.e. the points $x = 2^{-(i+1)}$, satisfying $ix + 2^{-i} = (i+1)x + 2^{-(i+1)}$. (b) Plot of $\varphi(x, y) = \min\{2x, 2x + y, 3y\}$.

where $\mu x := \sum_{a \in X} \mu(a) x_a$. These functions correspond then to tLs with possibly infinitely many variables (in fact, as many as the elements of X).

Notice that, by identifying $!\{*\} \simeq \mathbb{N}$ and $\mathbb{L}^{\{*\}} \simeq \mathbb{L}$, the tLs generated by the morphisms in $\mathbb{L}\mathsf{Rel}_!(\{*\}, \{*\})$ are exactly the functions $f : \mathbb{L} \to \mathbb{L}$ of shape $f(x) = \inf_{n \in \mathbb{N}} \{nx + \hat{f}(n)\}$, for some $\hat{f} : \mathbb{N} \to \mathbb{L}$, i.e. usual tLs's of one variable.

Remark 5. *Our running example is indeed of shape $\varphi = t^!$, for $t \in \mathbb{L}^{!\{*\} \times \{*\}}$, $t_{\mu,*} := 2^{-\#\mu}$.*

In a similar way, the tropical polynomials can be identified with the tLs $f : \mathbb{L}^X \to \mathbb{L}^Y$ for which the support $\mathcal{F} = \{\mu \in !X \mid \hat{f}_{\mu,b} \neq \infty\}$ is *finite*, and which have thus shape $f(x)_b := \min_{\mu \in \mathcal{F}} \{\mu x + t_{\mu,b}\}$. This is again the generalisation of usual tropical polynomials to the case of infinitely many variables.

Looking at Fig 1a, we see that the function φ, just like the polynomials φ_1, is non-decreasing and concave. It can be shown that this is indeed always the case:

Proposition 5. *Any tLs $f : \mathbb{L}^X \to \mathbb{L}^Y$ is non-decreasing and concave, w.r.t. the pointwise order.*

Again, by looking at Fig 1a, it appears that φ behaves *locally* like the polynomials φ_n. In particular, for all $\epsilon > 0$, f coincides on $[\epsilon, \infty]$ with some polynomial φ_n. More, precisely, φ coincides with φ_n for $\epsilon \geqslant 2^{-(n+1)}$, the smallest tropical root of φ_n. However, at $x = 0$ we have that $\varphi(x = 0) = \inf_{n \in \mathbb{N}} \frac{1}{2^n} = 0$, and this is the only point where the inf is *not* a min. Also, while the derivative of f is bounded on all $\mathbb{R}_{>0}$, at $x = 0$ it tends to ∞. This phenomenon is reminiscent of [Example 7, [29]], which actually motivated our first investigations. In fact, these properties are shared by all tLs with *finitely* many variables, as shown by the following result.

Remark that $!\{1, \dots, k\}$ can be identified with $\mathbb{N}^k$. So the matrix of a tLs f with finitely many variables $x = x_1, \dots, x_k$ (and one variable $f(x)$ in output) can be given as a $\hat{f} : \mathbb{N}^k \to \mathbb{L}$, and f has shape $f(x) = \inf_{n \in \mathbb{N}^k} \{nx + \hat{f}(n)\}$, where nx is the scalar product.

Theorem 6. *Let $k \in \mathbb{N}$ and $f : \mathbb{L}^k \to \mathbb{L}$ a tLs with matrix $\hat{f} : \mathbb{N}^k \to \mathbb{L}$. For all $0 < \epsilon < \infty$, there is a* finite *$\mathcal{F}_\epsilon \subseteq \mathbb{N}^k$ such that f coincides on all $[\epsilon, \infty]^k$ with the tropical polynomial $P_\epsilon(x) := \min_{n \in \mathcal{F}_\epsilon} \{nx + \hat{f}(n)\}$.*

Proof sketch. Let $\mathcal{F}_\epsilon$ be the set of $n \in \mathbb{N}^k$ such that $\hat{f}(n) < \infty$ and $\hat{f}(m) > \hat{f}(n) + \epsilon$ holds for all $m \prec n$, where $\preceq$ is the pointwise order on $\mathbb{N}^k$. The core of the proof is showing that this set is indeed finite and enough for computing f. $\square$

B. Continuity of tLs

The tLs φ is continuous on $\mathbb{R}_{\geqslant 0}$ (w.r.t. the usual norm of real numbers). By considering the usual norm $\|x\|_\infty := \sup_{a \in X} |x_a|$ on $\mathbb{L}^X$, we can generalise this property by dropping the case of x having some 0 coordinate:

Theorem 7. *All tLs $f : \mathbb{L}^X \to \mathbb{L}$ are continuous on $\mathbb{R}^X_{>0}$, w.r.t. to the norm $\|\cdot\|_\infty$.*

Proof. It follows after adapting [Proposition 4.4, [22]] in order to prove that if a real-valued function on a locally convex topological $\mathbb{R}$-vector space is, locally around x, concave and bounded by a finite constant, then it is continuous at x. $\square$

We conclude this subsection by noticing that $\mathbb{L}^X$ with the usual $+$ and the usual $\cdot$ is a $\mathbb{R}_{\geqslant 0}$-semimodule. Together with the norm $\|\cdot\|_\infty$, it can be proved that it is a Scott-complete *normed cone* (see [68], or the appendix, for such notions). Its cone structure induces an order on it, called its *cone order*: $x \leqslant y$ iff $y = x + z$ for some (unique) $z \in \mathbb{L}^X$. This makes it a Scott-continuous dcpo. Suitable categories of cones have been recently investigated as models of probabilistic computation ([24], [31], [34]). Here we will just mention that:

Theorem 8. *All tLs $\mathbb{L}^X \to \mathbb{L}^Y$ are Scott-continuous on $\mathbb{R}^X_{>0}$, w.r.t. the cone orders.*

C. Lipschitz-continuity of tLs

The norm $\|\cdot\|_\infty$ naturally induces a metric $\|x - y\|_\infty$ over the spaces $\mathbb{L}^X$. We will show that tLs satisfy suitable Lipschitz properties w.r.t. these metrics.

Let us first look at tropical linear functions:

Proposition 9. *All tropical* linear *functions $f : \mathbb{L}^X \to \mathbb{L}^Y$ are non-expansive.*

This result shows that, in analogy with that happens in usual metric semantics, linear programs are interpreted by non-expansive functions.

The following proposition provides a useful characterization of the functional metrics in $\mathbb{L}\mathsf{Rel}$, relying on the bijection between $\mathbb{L}\mathsf{Rel}(X, Y)$ and set of tropical linear functions from $\mathbb{L}^X$ to $\mathbb{L}^Y$.

Proposition 10. *For all tropical linear functions $f, g : \mathbb{L}^X \to \mathbb{L}^Y$, $\|\hat{f} - \hat{g}\|_\infty = \sup_{x \in \mathbb{L}^X} \|f(x) - g(x)\|_\infty$.*

Let us now consider the case of bounded exponentials:

Proposition 11. *If a tLs $f : \mathbb{L}^X \to \mathbb{L}^Y$ arises from a bounded matrix $\hat{f} : !_n X \times Y \to \mathbb{L}$, then f is n-Lipschitz-continuous.*

Proof sketch. This follows from Proposition 49 and the remark that, for all $x \in \mathbb{L}^X$, $\|!_n x - !_n y\|_\infty \leqslant n \cdot \|x - y\|_\infty$, where $!_n x$ is the restriction of $!x$ to $\mathcal{M}_{\leqslant n}(X)$. $\square$

This result is perfectly analogous to what happens in the metric models discussed in Section II, the bounded exponentials $!_n$ playing the role of the re-scaling trick.

Observe that, for any tropical polynomial $\varphi : \mathbb{L}^X \to \mathbb{L}^Y$, the associated matrix has shape $!_{\deg(\varphi)}(X) \times Y \to \mathbb{L}$ (as a monomial $\mu_i x + c_i$ yields a matrix entry on $!_{\#\mu_i} X \times Y$). Hence, using Proposition 11, we have:

Corollary 12. *For any tropical polynomial $\varphi : \mathbb{L}^X \to \mathbb{L}$, φ is $\deg(\varphi)$-Lipschitz continuous.*

Moreover, from Proposition 11, we can also deduce the desired Lipschitz decomposition of the Taylor expansion (using Corollary 4):

Corollary 13. *If $\Gamma \vdash_{\mathrm{STLC}} M : A$, then $[\![M]\!]^! : [\![\Gamma]\!] \to [\![A]\!]$ is an* inf *of* Lipschitz *functions (via its Taylor expansion* (4)*).*

Let us now look at what happens with tLs, i.e. when considering the full exponential !. As consequence of Theorem 6, the tLs with *finitely many* variables are always *locally* Lipschitz on all $\mathbb{R}_{>0}$. Actually, we can prove a more general statement, also covering the infinitary case.

Theorem 14. *All tLs $\mathbb{L}^X \to \mathbb{L}$ are locally Lipschitz on $\mathbb{R}^X_{>0}$.*

Proof sketch. The core of the proof is a convex analysis argument (see the Appendix) showing that an arbitrary function $f : \mathbb{L}^X \to \mathbb{L}$ which is non-decreasing, concave and continuous, must be locally Lipschitz. $\square$

The results just presented translate into the following facts about the interpretation of higher-order programs:

Corollary 15. *For any λ-term M:*

1) *if $\Gamma \vdash_{\mathrm{bSTLC}} M : A$, then $[\![M]\!]^! : [\![\Gamma]\!] \to [\![A]\!]$ is a Lipschitz map.*
2) *if $\Gamma \vdash_{\mathrm{STLC}} M : A$, then $[\![M]\!]^! : [\![\Gamma]\!] \to [\![A]\!]$ is a locally Lipschitz map.*

Proof. As we observed, the interpretation $[\![A]\!]$ of a bounded type is a finite set, hence $[\![M]\!]$ must be a finite matrix and its corresponding linear map $\mathbb{L}^{[\![A]\!]} \to \mathbb{L}^{[\![B]\!]}$ is thus a tropical polynomial, so we apply Corollary 12. Claim 2) follows directly from Theorem 14. $\square$

Finally, let us discuss the differential structure. The differential operator D of $\mathbb{L}\mathsf{Rel}_!$ translates into a differential operator $D_!$ turing a tLs $f : \mathbb{L}^X \to \mathbb{L}^Y$ into a tLs $D_! f : \mathbb{L}^X \times \mathbb{L}^X \to \mathbb{L}^Y$, linear in its first variable, and given by

$$D_! f(x, y)_b = \inf_{a \in X, \mu \in !X} \left\{ \hat{f}_{\mu + a} + x_a + \mu y \right\} \tag{7}$$

One can check that, when f is a tropical polynomial, $D_! f$ coincides with the standard tropical derivative (see e.g. [43]). Moreover, the Taylor formula (4) yields a "tropical" Taylor formula for tLs of the form

$$f(x) = \inf_n \left\{ D_!^{(n)}(f)(!_n x, \infty) \right\} \tag{8}$$

The following result shows that the distance between two tropical maps can be approximated using the terms appearing in their Taylor expansions:

Proposition 16. *For all tLs $f, g : \mathbb{L}^X \to \mathbb{L}^Y$, and for all $n \in \mathbb{N}$, the functions $x \mapsto D_!^{(n)}(f)(!_n x, \infty)$ are n-Lipschitz. Moreover $\|\hat{f} - \hat{g}\|_\infty = \sup_n \|\delta^{(n)} f - \delta^{(n)} g\|_\infty$, where $\delta^{(n)} h$ indicates the matrix of $D_!^{(n)} h$.*

V. Applications

In this section we illustrate a few directions in which the tropical semantics just introduced could be used to analyze quantitative properties of higher-order programs.

Since algebraic and geometric properties in tropical mathematics are usually more tractable from a computational point of view, in several well-known applications (e.g. for optimization problems related to machine learning [54], [62], [75]) one starts from a given model, typically expressed by some polynomial function f, and studies what properties of the model can be deduced from the *tropicalization* of f, noted $\mathtt{t}f$, i.e. the transformation of f into a tropical polynomial.

Here we follow a similar pattern: we consider a program M, which can be expressed in the form of a polynomial or a power series f, and we investigate what quantitative properties of M can be deduced from the properties of $\mathtt{t}f$, that will indeed coincide with the interpretation of M in $\mathbb{L}\mathsf{Rel}_!$.

A. The tropicalization of polynomials and power series

Let us first recall how standard polynomials and power series over $[0, 1]$ can be turned into tLs via the so-called *Maslov dequantization* [50].

Let us fix a positive real $t > 0$. For any function $f : [0, 1] \to [0, 1]$ which can be written as a parameterized power series of the form $f_t(x) = \sum_n t^{c_n} x^n$, we let its *tropicalization* $\mathtt{t}f : \mathbb{L} \to \mathbb{L}$ be the tLs defined as follows:

$$\mathtt{t}f(\alpha) = \inf_n \{ n\alpha + c_n \} \tag{9}$$

Clearly, for any $t > 0$, there is a one-to-one correspondence between the representations of power series in parameterized form and the associated tLs. Moreover, f and $\mathtt{t}f$ can be related by a limit passage as follows: the functions $\phi_t(x) = -\log_t x$ and $\varphi_t(\alpha) = t^{-\alpha}$ define continuous bijections between $[0, 1]$ and $[0, \infty]$ and, by letting $\mathtt{t}_t f : [0, \infty] \to [0, \infty]$ be defined by $\mathtt{t}_t f(\alpha) = \phi_t \circ f \circ \psi_t$, one has that $\mathtt{t}f = \lim_{t \to 0} \mathtt{t}_t f$. Indeed, one can check that the "parameterized" sums and product $\alpha +^t \beta := \phi_t(\psi_t(\alpha) + \psi_t(\beta)) = -\log_t(t^{-\alpha} + t^{-\beta})$ and $\alpha \times^t \beta := \phi_t(\psi_t(\alpha)\psi_t(\beta)) = -\log_t(t^{-\alpha} t^{-\beta})$ converge respectively to $\min\{\alpha, \beta\}$ and $\alpha + \beta$, when $t \to 0$.

B. Best case analysis and metric reasoning

The possibility of using the relational model over the tropical semiring for "best case" resource analysis has already been explored in [48]. Notably, they considered an interpretation of a language for **PCF** with non-deterministic choice in which each λ-abstraction and each occurrence of the fixpoint operator Y is assigned a "weight" 1, and showed that for any program M of type $\mathbf{nat}$, the value of the interpretation $[\![M]\!] \in \mathbb{L}^{\mathbb{N}}$ on a particular natural number k, i.e. $[\![M]\!](k) \in \mathbb{L}$, corresponds to the *minimum* number of β- or `fix`-redexes reduced in a reductions sequence from M to $\underline{n}$. In the next paragraph we will illustrate an analogous "best case" analysis for probabilistic programs.

What does the metric analysis from the previous sections add to that? Firstly, the possibility of *comparing* different programs with respect to their quantitative properties. For example, in the $\mathbf{PCF}$ semantics recalled above, the distance between two programs M and N of type $\mathbf{nat}$, provides a bound on the difference between the "best case" computation time of M and that of N. For instance, by taking, instead of the ∞-norm metric on $\mathbb{L}^{\mathbb{N}}$, the *non symmetric* distance (or quasi-metric, a viewpoint we explicitly take in Section VI) $q(\mathbf{x},\mathbf{y}) = \sup_n\{\mathbf{y}_n \dot{-} \mathbf{x}_n\}$, a "distance" $q([\![M]\!],[\![N]\!]) \leqslant \epsilon$ would indicate that $[\![M]\!]$ *improves* on $[\![N]\!]$ of at most ϵ steps at each computation.

Secondly, the Lipschitz conditions from Section IV allow us to reason on program distances in a *compositional* way: suppose, as before, that $M, N : A$ are two programs such that M improves on N by ϵ, and let $\mathtt{C}[-] : A \to \mathbf{nat}$ indicate a context; knowing that the interpretation of $\mathtt{C}$ is k-Lipschitz-continuous on some open set containing both $[\![M]\!]$ and $[\![N]\!]$, allows us to immediately deduce that $\mathtt{C}[M]$ improves on $\mathtt{C}[N]$ by $k\epsilon$. Observe that this will typically be the case when the Taylor expansions of $\mathtt{C}[M]$ and $\mathtt{C}[N]$ actually yields a *finite* sum of at most k terms, i.e. when

$$\mathtt{C}[M] = \sum_{i=0}^{k} \mathtt{D}^{(k)}\Big[\lambda x.\mathtt{C}[x]\Big] \cdot M^k \tag{10}$$

and similarly for $\mathtt{C}[N]$. It is here worth recalling that, for ST∂LC, a well-known result [35] is that the Taylor expansion of a closed application MN is always *finite*, although its non-zero coefficients may be arbitrarily high. Notably, these observations suggest to study tropical versions of *finiteness spaces* [29], a variant of the relational semantics modeling strongly normalizing programs via *finite* power series.

C. Convergence log-probabilities

Let us consider a probabilistic extension of STLC, call it STLC$_{\oplus}$: we add a ground type Bool, terms $\mathrm{True}, \mathrm{False}$ of type Bool, terms of shape $M \oplus_p N$ and pM, for $p \in [0,1]$, typed via the rules:

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : A}{\Gamma \vdash M \oplus_p N : A} \qquad\qquad \frac{\Gamma \vdash M : A}{\Gamma \vdash pM : A}$$

We add reduction rules:

$$M \oplus_p N \to pM \quad \textit{and} \quad M \oplus_p N \to (1-p)N$$

so that $M \oplus_p N$ plays the role of a probabilistic coin toss of bias p. We use this calculus as a toy example for our purposes, and it can be seen as a fragment of the PCF considered in [48].

Consider now the following term M of type Bool:

$$(\mathrm{True} \oplus_p \mathrm{False}) \oplus_p ((\mathrm{True} \oplus_p \mathrm{False}) \oplus_p (\mathrm{True} \oplus_p \mathrm{False}))$$

Let us give addresses $\omega \in \{00, 01, 100, 101, 110, 111\}$ to the occurrences of $\mathrm{True}, \mathrm{False}$ in M, by following the tree structure of M, reading 0 as "left" and 1 as "right". The same addresses also represent all the different possible reduction paths from M to a normal form. For instance, 100 represents the reduction which keeps the right part of the outermost $\oplus_p$ and erases the left part, then continues by choosing the left part twice, reaching at the end the occurrence True_{100} in M, i.e. the second occurrence of True in M starting from the left. Calling $q := 1 - p$, there are the following six normal terms reachable from M: $P_{00}(p,q)\mathrm{True}$, $P_{100}(p,q)\mathrm{True}$, $P_{111}(p,q)\mathrm{True}$, $P_{01}(p,q)\mathrm{False}$, $P_{110}(p,q)\mathrm{False}$, $P_{101}(p,q)\mathrm{False}$, where the P's are the following monomials in p,q: $P_{00}(p,q) := p^2$, $P_{100}(p,q) := qp^2$, $P_{111}(p,q) := q^3$, $P_{01}(p,q) := pq$, $P_{110}(p,q) = P_{101}(p,q) := q^2p$. They correspond to the respective reduction path from M to the normal term of the same address. P_ω is then the probability (as a function of p,q) of obtaining the respective occurrence True_ω or False_ω after all the coin tossings in M are performed. Thinking of p,q as parameters, P_ω can thus be read as the *likelihood function* of the event ω. The polynomial $Q_1(p,q) := P_{00}(p,q) + P_{100}(p,q) + P_{111}(p,q)$ gives instead the whole probability of obtaining True after all the tossings, and $Q_0(p,q) := P_{01}(p,q) + P_{110}(p,q) + P_{101}(p,q)$ the one for False.

This way, the probabilistic evaluation of M is presented as a *hidden Markov model* [14], a fundamental statistical model, and notably one to which tropical methods are generally applied [62]. Typical questions in this case would be:

1) What is the *maximum likelihood estimator* for the event "M produces True_ω" (similarly for False_ω)? I.e., which is the choice of the parameters p,q that maximizes the probability of getting True_ω after all tossings in M?
2) Knowing that M produced True (similarly for False), which is the choice of the parameters p,q that maximizes the probability that the produced occurrence of True was a fixed True_{ω_0} (i.e. that among the paths leading to True, the one taken was ω_0)?

A similar argument could be done by replacing True and False by, respectively, a converging and a diverging term (e.g. in a $\mathbf{PCF}$-style language), so 1) would be about finding maximum likelihood estimators for the event "M converges".

Answering 1) and 2) amounts at solving a maximization problem related to P_ω, Q_ω, which is more easily solved by passing to the tropical monomials/polynomial $\mathtt{t}P_\omega, \mathtt{t}Q_\omega$. For 1), we are looking for $p,q \in [0,1]$ s.t. $q = 1-p$ and (p,q) belong to:

$$\begin{aligned}\underset{(x,y)}{\mathrm{argmax}} P_\omega(x,y) &= \underset{(x,y)}{\mathrm{argmin}}\{-\log_c P_\omega(x,y)\} \\ &= \underset{(x,y)}{\mathrm{argmax}}\{(\mathtt{t}P_\omega)(-\log_c x, -\log_c y)\}\end{aligned} \tag{11}$$

where the equalities hold for all $0 < c \neq 1$.

For 2), we are looking for $p,q \in [0,1]$ s.t. $q = 1-p$ and $\max\limits_{\omega \in \{00,100,111\}} P_\omega(x,y) = P_{\omega_0}(x,y)$. This last condition is equivalent to ask that:

$$\min_{\omega \in \{00,100,111\}} -\log_c P_\omega(x,y) = -\log_c P_{\omega_0}(x,y) \qquad \textit{i.e.:}$$
$$(\mathsf{t}Q_1)(-\log_c x, -\log_c y) = (\mathsf{t}P_{\omega_0})(-\log_c x, -\log_c y). \tag{12}$$

In both cases, passing through t makes the problem easier, as this amounts to study tropical polynomials (for instance computing tropical roots can be done in linear time [60]).

Remark 6. *By adapting [Section IV, [48]], it can be seen that* $\mathbb{L}\mathsf{Rel}_!$ *is a model of* $\mathrm{STLC}_\oplus$. *In particular, if we set* $[\![\mathsf{Bool}]\!] := \{0,1\}$, *our running example* M *is interpreted as* $[\![M]\!] \in \mathbb{L}^{\{0,1\}}$ *giving the tropicalised probabilities:* $[\![M]\!]_0 = (\mathsf{t}Q_0)(p, 1-p)$, $[\![M]\!]_1 = (\mathsf{t}Q_1)(p, 1-p)$.

Therefore, assuming the interpretation of a term is known, so that one can extract the associated tropical polynomials from it, one could study the optimisation problems 11, 12 *without* having to actually reduce the term.

For our running example M, we have $\mathsf{t}Q_1(x,y) = \min\{2x, y+2x, 3y\}$ and $\mathsf{t}Q_0(x,y) = \min\{x+y, 2y+x\}$. Studying $\mathsf{t}Q_1$, whose plot is in Fig. 1b, we see that $\mathsf{t}Q_1(x,y) = 3y$ iff $y \leqslant \frac{2}{3}x$, and it coincides with $2x$ otherwise. Remembering that $3y = P_{111}(x,y)$, we can now solve the optimisation problem 12 for $\omega_0 = 111$: via the substitution $x := -\log_c p$, $y := -\log_c(1-p)$, Equation 12 is equivalent to $-\log_c(1-p) \leqslant -\frac{2}{3}\log_c p$, i.e. $1-p \geqslant p^{\frac{2}{3}}$. This means that, for $p \in [0,1]$ s.t. $1-p \geqslant p^{\frac{2}{3}}$ (for example, $p = \frac{1}{4}$), the most likely occurrence of True to obtain, knowing that M sampled True in its normal form, is True_{111}. Remembering that $2x = P_{00}(x,y)$, for the other values of p (for example, $p = \frac{1}{2}$), the most likely True to be sampled is the occurrence True_{00}. We have thus answered question 2) above.

Remark 7. *Taking the usual* M *as example, the* $p \in [0,1]$ *s.t.* $(p, 1-p)$ *is a tropical root of* $\mathsf{t}Q_1$ *or of* $\mathsf{t}Q_0$ *provide, by definition, the values of the bias of* $\oplus_p$ *for which there are at least two different equiprobable paths of tossings from* M *to its normal form. Moreover, looking at Equation 12, we see that the* $p \in [0,1]$ *s.t.* $(-\log_c p, -\log_c(1-p))$ *is a tropical root of, say,* $\mathsf{t}Q_1$ *are the values of the bias of* $\oplus_p$ *for which there are at least two different equiprobable occurrences of* True *that are smapled by* M *during its tossings to normal form, knowing that some* True *was sampled.*

D. Resource analysis for differential privacy

The typical situation in differential privacy is where one considers a probabilistic query $f : \mathsf{db} \to [0,1]^X$ over some database, and one requires that f should not be *too sensitive* to small changes in the output, in other to prevent potential leaks of private information about individual items in db (for instance, an element $x \in \mathsf{db}$ could be the list of students of some university and $f(x)$ could indicate the percentage of female students in $x : \mathsf{db}$).

More formally, differential privacy is defined as follows:

Definition 2. *Let* $f : \mathsf{db} \to [0,1]^X$ *and* $\epsilon \in \mathbb{R}_{\geqslant 0}$. *$f$ is said* ϵ-differentially private *if for all* $x, x' \in \mathsf{db}$ *differing by* L *items, for all* $s \in X$, $f(x)(s) = e^{\epsilon L} f(x')(s)$.

A well-studied approach to ensure differential privacy for higher-order programs is via linear type systems with graded exponentials like Fuzz [64]: these systems provide an *a priori* warrant that well-typed programs are Lipschitz.

Tropical semantics suggests that resource analysis could also be used to provide bounds for differential privacy. Suppose our probabilistic query f can be expressed as a power series (this is what happens e.g. in *probabilistic coherent spaces* [33]). Then, if we discover, either by studying differential properties of f, or by using methods from convex analysis as suggested in the previous section (e.g. Theorem 6), that the tropicalization $\mathsf{t}f$ satisfies a Lipschitz condition, we may use this fact to deduce that f is differentially private, as shown by the result below.

Let us equip the space $[0,1]^X$ with the metric $d(x,y) = \sup_{a \in X} |\log x_a - \log y_a|$. Then we have the following:

Proposition 17. *Let* $f : [0,1]^X \to [0,1]^Y$ *be a function expressed by a power series. If* $\mathsf{t}f$ *is* ϵ*-Lipschitz over some open set* U, *then* f *is* ϵ*-differentially private over* $\psi_t(U)$, *for small enough* t.

Proof sketch. We consider $X = Y = \{\star\}$ for simplicity. Using the fact that $\lim_{t \to 0} \mathsf{t}_t f = \mathsf{t}f$, as well as the commutation $f(\phi_t(\alpha)) = \phi_t(\mathsf{t}_t f(\alpha))$ (which follows from the definition of t_t), one can see that the Lipschitz condition $|\mathsf{t}f(\phi_t(x)) - \mathsf{t}f(\phi_t(y))| \leqslant \epsilon|\phi_t(x) - \phi_t(y)|$ translates into the condition $f(x) \leqslant e^{\epsilon d(x,y)} f(y)$ (supposing $y \geqslant x$ and $f(y) \neq 0$). $\square$

VI. Generalized Metric Spaces and Modules over the Lawvere Quantale

As we have seen, the morphisms of $\mathbb{L}\mathsf{Rel}$ can be seen as continuous functions between the $\mathbb{L}$-modules $\mathbb{L}^X$, when the latter are taken with the metric induced by the ∞-norm. This viewpoint gives a metric flavor to $\mathbb{L}\mathsf{Rel}$, and allowed us to relate differential and metric structure. Yet, how far can this correspondence be pushed? In particular, is this correspondence restricted to $\mathbb{L}$-modules of the form $\mathbb{L}^X$ (i.e. with a fixed base), or does it hold in some sense for arbitrary $\mathbb{L}$-modules? Is this correspondence restricted to the ∞-norm metric, or does it hold for other metrics too?

An answer to these questions comes from an elegant categorical correspondence between tropical linear algebra and the theory of *generalized metric spaces*, initiated by Lawvere's pioneering work [49], and at the heart of the emergent field of *monoidal topology* [45], [72]. In this section we first reconstruct this correspondence, by combining some folklore results with more abstract ones from recent literature in enriched category theory [37], [69], [71]. Then we show that this correspondence can be lifted to a model of the full differential λ-calculus, by suitably generalizing the construction of the Lafont exponential of $\mathbb{L}\mathsf{Rel}$.

A. $\mathbb{L}$-modules

A $\mathbb{L}$-module is a triple $(M, \preceq, \star)$ where $(M, \preceq)$ is a sup-lattice, and $\star : \mathbb{L} \times M \to M$ is a continuous (left-)action of $\mathbb{L}$ on it, where continuous means that $\star$ commutes with both joins in $\mathbb{L}$ and in M.A $\mathbb{L}$-module homomorphism is a map $f : M \to N$ commuting with both joins and the $\mathbb{L}$-action. We let $\mathbb{L}\mathsf{Mod}$ indicate the category of $\mathbb{L}$-modules and their homomorphisms.

$\mathbb{L}$ is the most basic example of $\mathbb{L}$-module. Any $\mathbb{L}$-module M has a dual M^{op}, with reversed order and (right-)action $x \multimapinv \epsilon = \bigvee\{y \mid \epsilon \star y \succeq x\}$. Other basic examples of $\mathbb{L}$-modules are the sets $\mathbb{L}^X$, with order and action defined pointwise.

While the $\mathbb{L}^X$ have a fixed base, for an arbitrary $\mathbb{L}$-module one can retrieve a base via the *Yoneda embedding* $\mathbf{Y} : M \to \mathbb{L}\mathsf{Mod}(M^{\mathrm{op}}, \mathbb{L})$, where $\mathbf{Y}(x)(y) = \inf\{\epsilon \mid \epsilon \star y \succeq x\}$.

Proposition 18 (cf. [69], [72])**.** *For any $\mathbb{L}$-module M, the Yoneda embedding has a left-adjoint* $\sup(f) = \bigvee_{x \in M} f(x) \star x$.

Like $\mathbb{L}\mathsf{Rel}$, the category $\mathbb{L}\mathsf{Mod}$ has the relevant structure to interpret the linear λ-calculus:

Proposition 19. $\mathbb{L}\mathsf{Mod}$ *is a SMCC.*

The hom-sets $\mathbb{L}\mathsf{Mod}(M, N)$ have a natural $\mathbb{L}$-module structure, defined pointwise. The tensor product of $\mathbb{L}$-modules $M \otimes N$ can be defined as the quotient of the usual tensor product of sup-lattices under the smallest congruence containing all pairs $(\{(\epsilon \star x, y)\}, \{(x, \epsilon \star y)\})$ (see e.g. [65]). Notably, any element of $M \otimes N$ can be identified with a join of basic tensors $x \otimes y$, corresponding to the equivalence class of the pair $\langle x, y\rangle$. Beyond the required adjointness of the internal hom and the tensor, one can check that $\mathbb{L}\mathsf{Mod}$ is actually **-autonomous*, since it satisfies $(M^{\mathrm{op}})^{\mathrm{op}} \simeq M$ and $\mathrm{Hom}(M, N^{\mathrm{op}}) \simeq (M \otimes N)^{\mathrm{op}}$. Finally, $\mathbb{L}\mathsf{Mod}$ has *biproducts*: products and coproducts are both given by the Cartesian product of the underlying posets, with action defined pointwise.

Remark 8. *The SMCC structure of* $\mathbb{L}\mathsf{Mod}$ *coincides with that of* $\mathbb{L}\mathsf{Rel}$ *for the modules* $\mathbb{L}^X$*: we already know that* $\mathbb{L}\mathsf{Mod}(\mathbb{L}^X, \mathbb{L}^Y) \simeq \mathbb{L}^{X \times Y}$*, and one can prove* $\mathbb{L}^X \otimes \mathbb{L}^Y \simeq \mathbb{L}^{X \times Y}$ *(cf. [65]).*

B. $\mathbb{L}$-categories

Lawvere was the first to observe that a metric space can be described as a $\mathbb{L}$-*enriched* category. Indeed, spelling out the definition, a $\mathbb{L}$-enriched category (in short, a $\mathbb{L}$-category) is given by a set X together with a "hom-set" $X(-,-) : X \times X \to \mathbb{L}$ satisfying

$$0 \geqslant X(x, x) \qquad (\mathbb{L}\text{-cat } 1)$$

$$X(y, z) + X(x, y) \geqslant X(x, z) \qquad (\mathbb{L}\text{-cat } 2)$$

This structure is often referred to as a *generalized metric space* [45], [49], [72]. Notice that a $\mathbb{L}$-enriched *functor* between $\mathbb{L}$-categories is just a non-expansive map $f : X \to Y$, as functoriality reads as $Y(f(x), f(y)) \leqslant X(x, y)$.

A $\mathbb{L}$-category is *skeletal* [72] when $X(x, y) = 0$ implies $x = y$, and *symmetric* when it coincides with its opposite category $X^{\mathrm{op}}(x, y) := X(y, x)$, i.e. when $X(x, y) = X(y, x)$. A metric space, in the usual sense, is thus the same as a skeletal and symmetric $\mathbb{L}$-category. Notice that any $\mathbb{L}$-category X induces a skeletal category X^{sk} by quotienting points under $X(x, y) = 0$, and a symmetric one by letting $X^{\mathrm{sym}}(x, y) = \max\{X(x, y), X^{\mathrm{op}}(x, y)\}$.

$\mathbb{L}$ has a canonical $\mathbb{L}$-enriched structure given by $\mathbb{L}(r, s) = s \dot{-} r$ (where "$\dot{-}$" indicates truncated subtraction), and the Euclidean distance coincides with its symmetrization $\mathbb{L}^{\mathrm{sym}}(x, y)$.

For any $\mathbb{L}$-category X, the presheafs $[X^{\mathrm{op}}, \mathbb{L}]$ on X form another $\mathbb{L}$-category, with metric $[X^{\mathrm{op}}, \mathbb{L}](f, g) = \sup_{x \in X} \mathbb{L}(f(x), g(x))$. Notice that, when X has the discrete metric, the metric space $[X, \mathbb{L}]^{\mathrm{sym}}$ coincides with $\mathbb{L}^X$ with the ∞-norm metric. The *Yoneda embedding* is the faithful functor $\mathbf{Y} : X \to [X^{\mathrm{op}}, \mathbb{L}]$ given by $\mathbf{Y}(x)(y) = X(y, x)$.

Actually, an important example of $\mathbb{L}$-categories are precisely the $\mathbb{L}$-modules:

Proposition 20. *Any $\mathbb{L}$-module $(M, \preceq, \star)$ is a $\mathbb{L}$-category via*

$$M(x, y) = \inf\{\epsilon \mid \epsilon \star x \geqslant y\} \qquad (13)$$

Moreover, a homomorphism of $\mathbb{L}$-modules is a functor of the associated $\mathbb{L}$-categories.

Observe that, since the distance $M(x, y)$ coincides with the Yoneda embedding $\mathbf{Y}$ in $\mathbb{L}\mathsf{Mod}$, the latter also coincides with the Yoneda embedding of the associated $\mathbb{L}$-category (this justifies the use of a unique symbol for both embeddings).

C. Complete $\mathbb{L}$-categories correspond to $\mathbb{L}$-modules

Lawvere also observed that Cauchy-completeness can be expressed in categorical language as the representability in X of certain presheaves in $[X^{\mathrm{op}}, \mathbb{L}]$ [49], [66]. Category theory suggests a yet stronger notion of completeness, corresponding to the existence of all *weighted colimits* (for a comparison of different notions of completeness on $\mathbb{L}$-categories, see [66], [74]).

First, let us recall that functors of shape $\Phi : X \times Y^{\mathrm{op}} \to \mathbb{L}$ are called *distributors* and usually noted $\Phi : Y \nrightarrow X$.

Definition 3. *Let X, Y, Z be $\mathbb{L}$-categories, $\Phi : Z \nrightarrow Y$ be a distributor and $f : Y \to X$ be a functor. A functor $g : Z \to X$ is the* Φ-weighted colimit of f over X*, noted* $\mathrm{colim}(\Phi, f)$*, if for all $z \in Z$ and $x \in X$*

$$X(g(z), x) = \sup_{y \in Y}\{X(f(y), x) \dot{-} \Phi(y, z)\} \qquad (14)$$

A functor $f : X \to Y$ is continuous *if it commutes with all existing weighted colimits in X, i.e.* $f(\mathrm{colim}(\Phi, g)) = \mathrm{colim}(\Phi, f \circ g)$*. A $\mathbb{L}$-enriched category X is said* categorically complete *(or just* complete*) if all weighted colimits over X exist.*

We let $\mathbb{L}\mathsf{CCat}$ indicate the category of complete and skeletal $\mathbb{L}$-categories and continuous functors.

A useful alternative characterization of complete $\mathbb{L}$-categories is the following:

Proposition 21 (cf. [69], [71]). *A $\mathbb{L}$-category is complete iff* $\mathbf{Y}$ *has a left-adjoint.*

Proof. For one side, if X is complete, one can define $\sup : [X^{\mathrm{op}}, \mathbb{L}] \to X$ as a weighted colimit via $X(\sup x, b) = \sup_{a \in X} X(a,b) - x_a$. Conversely, if a left-adjoint sup exists, one can define $\mathrm{colim}(\Phi, f) := \sup \Psi$, where $\Psi_a = \inf_{b \in X} X(a, f(b)) + \Phi_b$. □

Indeed, using this fact, together with Proposition 18, we arrive at the following:

Proposition 22. *For any $\mathbb{L}$-module, the associated $\mathbb{L}$-category is complete.*

Beyond limits of Cauchy sequences, another important example of colimit is the following:

Definition 4. *Let X be a $\mathbb{L}$-category, $x \in X$ and $\epsilon \in \mathbb{L}$. The* tensor of x and ϵ, *if it exists, is the colimit $\epsilon \otimes x := \mathrm{colim}([\epsilon], \Delta x)$, where $[\epsilon] : \{\star\} \nrightarrow \{\star\}$ is the constantly ϵ distributor and $\Delta x : \{\star\} \to X$ is the constant functor.*

In a complete $\mathbb{L}$-category all tensors exist and give rise to a $\mathbb{L}$-module structure:

Proposition 23. *Any complete $\mathbb{L}$-category X is a $\mathbb{L}$-module, with order given by $x \preceq_X y$ iff $X(y,x) = 0$, and action given by tensors $\epsilon \otimes x$. Moreover, a continuous functor between complete $\mathbb{L}$-categories is the same as a homomorphism of the associated $\mathbb{L}$-modules.*

To conclude our correspondence between $\mathbb{L}$-modules and complete $\mathbb{L}$-categories, it remains to observe that the two constructions leading from one structure to the other are one the inverse of the other: for any $\mathbb{L}$-module $(M, \preceq, \star)$, $x \preceq_M y$ iff $M(y,x) = 0$ iff $x \preceq y = 0 \star y$, and, from $M(\epsilon \star x, y) = M(x,y) \dot{-} \epsilon$, we deduce $\epsilon \otimes x = \epsilon \star x$. Conversely, for any complete $\mathbb{L}$-category X and $x, y \in X$, one can check that $X(y,x) = \inf\{\epsilon \mid X(\epsilon \otimes y, x) = 0\}$.

This leads to the following:

Theorem 24 (cf. [71]). *$\mathbb{L}\mathsf{Mod}$ and $\mathbb{L}\mathsf{CCat}$ are isomorphic categories.*

Since $\mathbb{L}\mathsf{Mod}$ (and thus $\mathbb{L}\mathsf{CCat}$ too) is a SMCC, it is worth making its metric structure explicit. Given complete $\mathbb{L}$-categories X, Y, we have that the distance on the hom-set $\mathbb{L}\mathsf{Mod}(X,Y)$ is given by

$$\mathbb{L}\mathsf{Mod}(X,Y)(f,g) = \sup_{x \in X} Y(f(x), g(x)); \tag{15}$$

and the distance on $X \otimes Y$ is given by

$$(X \otimes Y)(\alpha, \beta) = \sup_i \inf_j \left\{ X(x_i, x'_j) + Y(y_j, y'_j) \right\}, \tag{16}$$

where $\alpha = \bigvee_i x_i \otimes y_i$ and $\beta = \bigvee_j x'_j \otimes y'_j$, from which we deduce that, for basic tensors $x \otimes y, x' \otimes y'$, their distance is just $X(x,x') + Y(y,y')$.

D. Exponential and differential structure

We now want to show how the correspondence $\mathbb{L}\mathsf{Mod} \simeq \mathbb{L}\mathsf{CCat}$ lifts to a model of the differential λ-calculus, extending the co-Kleisli category $\mathbb{L}\mathsf{Rel}_!$.

First, we need to define a Lafont exponential ! over $\mathbb{L}\mathsf{Mod}$, and for this we use a well-known recipe from [48], [58], that is: we first define a symmetric algebra $\mathrm{Sym}_n(M)$ as the equalizer of all permutative actions on n-tensors $M \otimes \cdots \otimes M$; then, exploiting suitable properties of $\mathbb{L}\mathsf{Mod}$, we may define ! as the product of the symmetric algebras.

For any $\mathbb{L}$-module M, $n \in \mathbb{N}$ and permutation $\sigma \in \mathfrak{S}_n$, define the homomorphism $\langle \sigma \rangle : M^{\otimes_n} \to M^{\otimes_n}$ by letting $\langle \sigma \rangle (x_1 \otimes \cdots \otimes x_n) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$ on basic tensors, and extending by continuity on the whole tensor module.

Definition 5 (symmetric tensor algebra). *For any $\mathbb{L}$-module M and $n \in \mathbb{N}$, let $\mathrm{Sym}_n(M)$ indicate the $\mathbb{L}$-module obtained by quotienting $M^{\otimes_n}$ via the least congruence generated by the action $\langle \sigma \rangle$ of permutations $\sigma \in \mathfrak{S}_n$.*

To prove that the map $[x] \mapsto x : \mathrm{Sym}_n(M) \to M^{\otimes_n}$ is the equalizer of the diagram formed by all morphisms $\langle \sigma \rangle$, it is useful to provide an alternative characterization of it.

Definition 6. *Let M be a $\mathbb{L}$-module and $n \in \mathbb{N}$. An element $x \in M^{\otimes_n}$ is said* permutation-invariant *(in short,* p-invariant*) if for all $\sigma \in \mathfrak{S}_n$, $\langle \sigma \rangle (x) = x$. A* $\mathbb{L}$-multiset *(with n elements) is an element of $M^{\otimes_n}$ of the form $[x_1, \dots, x_n] := \bigvee_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$, where $x_1, \dots, x_n \in M$.*

Proposition 25. *Any $\mathbb{L}$-multiset is p-invariant. Moreover, the set $!_n M$ of p-invariant elements of $M^{\otimes_n}$ is a $\mathbb{L}$-submodule of $M^{\otimes_n}$, whose elements can be written as joins of $\mathbb{L}$-multisets.*

Since $!_n M$ is included in M, using the properties of p-invariant one can easily deduce that $!_n M$ provides the desired equalizer. It suffices then to show that $!_n M$ is isomorphic to the symmetric algebra.

Proposition 26. *The inclusion morphism $!_n M \to M^{\otimes_n}$ is the equalizer of the diagram formed by all $M^{\otimes_n} \overset{\langle\sigma\rangle}{\to} M^{\otimes_n}$. Moreover, $!_n M \simeq \mathrm{Sym}_n(M)$.*

The module $!_n M$ is a complete $\mathbb{L}$-category with distance function defined on $\mathbb{L}$-multisets as below:

$$(!_n M)(\alpha, \beta) = \sup_{\sigma \in \mathfrak{S}_n} \inf_{\tau \in \mathfrak{S}_n} \sum_{i=1}^n X(x_{\sigma(i)}, y_{\tau(i)}) \tag{17}$$

where $\alpha = [x_1, \dots, x_n]$ and $\beta = [y_1, \dots, y_n]$.

Using the fact that $\prod_i M_i \otimes N \simeq (\prod_i M_i) \otimes N$ holds in $\mathbb{L}\mathsf{Mod}$ (see e.g. [65]), we obtain the following:

Theorem 27. *The functor $!M := \prod_{n \in \mathbb{N}} !_n M$ is a Lafont exponential in $\mathbb{L}\mathsf{Mod}$. Hence, $\mathbb{L}\mathsf{Mod}_!$ (equivalently, $\mathbb{L}\mathsf{CCat}_!$) is cartesian closed.*

Also in the case of exponentials, the constructions for $\mathbb{L}\mathsf{Mod}$ generalize those of $\mathbb{L}\mathsf{Rel}$:

Proposition 28. $!_n(\mathbb{L}^X) \simeq \mathbb{L}^{\mathcal{M}_{\leq n}(X)}$, $!(\mathbb{L}^X) \simeq \mathbb{L}^{\mathcal{M}_{\mathrm{fin}}(X)}$. *In particular, $\mathbb{L}\mathsf{Mod}_!(\mathbb{L}^X, \mathbb{L}^Y) \simeq \mathbb{L}\mathsf{Rel}_!(X,Y)$.*

To conclude, we mush show that $\mathbb{L}\mathsf{Mod}$ is a CC∂C. Firstly, $\mathbb{L}\mathsf{Mod}$ is a left-additive-CCC, since each hom-set is a commutative monoid with respect to ∞ and $\min$, and the cartesian closed structure is well-behaved with respect to that.

Secondly, $\mathbb{L}\mathsf{Mod}$ can be equipped with a differential operator E defined, for $f :!M \to N$, by

$$Ef(\alpha) = \bigvee \left\{ f(\beta \cup [x]) \;\middle|\; \iota_n(\beta) \otimes \iota_1(x) \leqslant S(\alpha) \right\} \quad (18)$$

where $\iota_k : M_k \to \prod_{i \in I} M_i$ is the injection morphism given by $\iota_k(x)(k) = x$ and $\iota_k(x)(i \neq k) = \infty$, and $S :!(M \times N) \to !M \otimes !N$ is the Seely isomorphism [58], and E satisfies all required axioms.

Notice that, when $f \in \mathbb{L}\mathsf{Mod}_!(\mathbb{L}^X, \mathbb{L}^Y) \simeq \mathbb{L}\mathsf{Rel}_!(X, Y)$, its derivative Ef coincides with the derivative $D_! f$ described in Section IV.

All this leads then to:

Theorem 29. $\mathbb{L}\mathsf{Mod}_!$ *(equivalently,* $\mathbb{L}\mathsf{CCat}_!$*), equipped with* E*, is a CC∂C.*

VII. Related Work

The connections between differential λ-calculus (and differential linear logic), relational semantics, and non-idempotent intersection types are very well-studied (see [28], and more recently, [56] for a more abstract perspective, and [39], [61] for a 2-categorical, or proof-relevant, extension). As we said, the relational semantics over the tropical semi-ring was quickly explored in [48], to provide a "best case" resource analysis of a $\mathbf{PCF}$-like language with non-deterministic choice. *Probabilistic coherent spaces* [33], a variant of the relational semantics, provide an interpretation of higher-order probabilistic programs as analytic functions. In [32] it was observed that such functions satisfy a local Lipschitz condition somehow reminiscent of our examples in Section IV.

The study of linear or bounded type systems for sensitivity analysis was initiated in [42] and later developed [27], [64], [67]. Related approaches, although not based on metrics, are provided by *differential logical relations* [25] and *change action* models [7].

More generally, the literature on program metrics in denotational semantics is vast. Since at least [73] metric spaces, also in Lawvere's generalized sense [49], have been exploited as an alternative framework to standard, domain-theoretic, denotational semantics. While standard categories of metric spaces are not models of the full simply typed λ-calculus, several constructions of cartesian closed categories of metric spaces can be found in the literature. For instance, *ultra*-metric spaces form a CCC, and have been shown to model $\mathbf{PCF}$ [36]. Also *partial* metrics, introduced in [20], have been shown to provide models of STLC, under suitable generalizations [40]. More generally, [21] provides a general characterization of exponentiable objects in categories of (generalized) metric spaces, and [26], [63] provide other ways to construct CCCs of (generalized) metrics, including one based on locally Lipschitz maps, using ideas from differential logical relations and quantitative equational theories [55].

Motivated by connections with computer science and fuzzy set-theory, the abstract study of generalized metric spaces in the framework of *quantale-* or even *quantaloid*-enriched categories has led to a vast literature in recent years [45], [72], and its connections with tropical mathematics have been explored e.g. in [37], [74]. Moreover, applications of quantale-modules to both logic and computer science have also been studied [1], [65].

Connections between program metrics and the differential λ-calculus have been already suggested in [63]; moreover, *cartesian difference categories* [6] have been proposed as a way to relate derivatives in differential categories with those found in change action models.

Finally, applications of tropical mathematics in computer science abound, originally for automata theory [46], [70], and more recently in optimization methods for machine learning and other statistical models (see e.g. [54], [62], [75]), optimization [4], [5], and convex analysis [51]. While our discussion in Section V is inspired by a well-known application of tropical polynomials to Hidden Markov Models [62], the vast literature in this domain lets us think that other ways to apply tropical semantics to the analysis of higher-order programs might be studied.

VIII. Conclusion and Future Work

The main goals of this paper are two. Firstly, to demonstrate the existence of a conceptual bridge between two different well-studied quantitative approaches to higher-order programs, and to highlight the possibility of transferring results and techniques from one approach to the other. Secondly, to suggest that tropical mathematics, a field which has been largely and successfully applied in computer science, could be used to study quantitative properties of higher-order programs.

While the first goal was here developed in detail, and at different levels of abstraction, for the second goal we only sketched a few interesting directions (best-case analysis, log-probabilities, differential privacy). We believe that exploring these ideas in more depth could be a fruitful direction; moreover, since both generalized metrics and quantale-modules have been largely studied in computer science, a natural question is if the generalized approach of Section VI could lead to new applications of metric and tropical methods to the λ-calculus.

Finally, since bounds on the Taylor expansion translate into Lipschitz conditions, two interesting directions to explore are provided by (non-idempotent) intersection types and finiteness spaces [29], as both methods are in principle capable of capturing *finitary* bounds on the Taylor expansion. Notably, knowing that the application of a program M to N expands as a finite sum of linear applications may allow one to predict how sensitive M will be "around N".

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$$\frac{}{x :_1 A \vdash x : A}$$

$$\frac{\Gamma \vdash M : A}{\Gamma, x :_0 B \vdash M : A} \qquad \frac{\Gamma, x :_n B, y_m : B \vdash M : A}{\Gamma, x :_{n+m} B \vdash M[x/y] : A}$$

$$\frac{\Gamma, x :_n A \vdash M : B}{\Gamma \vdash \lambda x.M :!_n A \multimap B} \qquad \frac{\Gamma \vdash M : A \multimap B \qquad \Delta \vdash N : A}{\Gamma + \Delta \vdash MN : B}$$

$$\frac{\Gamma \vdash M : A}{n\Gamma \vdash M : !_n A}$$

Fig. 2: Typing rules for bSTLC.

APPENDIX

We give here the proofs of the several results in Sections III, Section IV and Section VI.

A. *Section III-C: the language* bSTLC

In Section 4 we considered a variant bSTLC of STLC with linear simple types and a graded exponential $!_n A$, for all $n \in \mathbb{N}$, corresponding to a somehow simplified version of the language Fuzz [64].

The types are generated by the grammar below:

$$A := X \ \mid !_n A \multimap A$$

Type judgements are of the form $\Gamma \vdash t : A$, where a context Γ is a list of declarations of the form $x :_n A$, with $n \in \mathbb{N}$. We define the following operation $\Gamma + \Delta$ as follows:

$$\begin{aligned} () + () &= () \\ (\Gamma, x :_m A) + (\Delta, x :_n A) &= (\Gamma + \Delta), x :_{m+n} A \\ (\Gamma, x :_n A) + \Delta &= (\Gamma + \Delta), x :_n A \qquad (x \notin \Delta) \\ \Gamma + (\Delta, x :_n A) &= (\Gamma + \Delta), x :_n A \qquad (x \notin \Gamma) \end{aligned}$$

Moreover, we let $m\Gamma$ be the context made all judgements $x :_{mn} A$, where $(x :_n A) : \Gamma$.

The typing rules of bSTLC are illustrated in Fig. 2.

B. *Proofs from Section III-C:* $(\mathbb{L}\mathsf{Rel}, !_n)$ *is a model of* bSTLC

Given SMCs $\mathcal{C}, \mathcal{D}$, let $\mathbf{SMC}_l(\mathcal{C}, \mathcal{D})$ indicate the category of symmetric lax monoidal functors and monoidal natural transformations between them. $\mathbf{SMC}_l(\mathcal{C}, \mathcal{D})$ is itself a SMC, with monoidal structure defined pointwise.

The set $\mathbb{N}$ can be seen as the category with objects the natural numbers and a morphism between r and r' precisely when $r \leqslant r'$. Moreover, $\mathbb{N}$ can be seen as a SMC in two ways:

- we indicate as $\mathbb{N}^+$ the SMC with monoidal product given by addition;
- we indicate as $\mathbb{N}^\times$ the SMC with monoidal product given by multiplication.

Definition 7 (cf. [47])**.** *A* $\mathbb{N}$*-graded linear exponential comonad on a symmetric monoidal category* $\mathcal{C}$ *is a tuple* $(D, w, c, \epsilon, \delta)$ *where:*

- $D : \mathbb{N} \to \mathbf{SMC}_l(\mathcal{C}, \mathcal{C})$ *is a functor. We write* $m_r : \{\star\} \to D(r)(\{\star\})$ *and* $m_{r,A,B} : D(r)(A) \otimes D(r)(B) \to D(r)(A \otimes B)$ *for the symmetric lax monoidal structure of* $D(r)$*;*
- $(D, w, c) : \mathbb{N}^+ \to \mathbf{SMC}_l(\mathcal{C}, \mathcal{C})$ *is a symmetric colax monoidal functor;*
- $(D, \epsilon, \delta) : \mathbb{N}^\times \to (\mathbf{SMC}_l, \mathrm{Id}, \circ)$ *is a colax monoidal functor.*

further satisfying the axioms below:

$$w_A = w_{D(s)(A)} \circ \delta_{0,s,A} \tag{19}$$

$$w_A = D(s)(w_A) \circ \delta_{s,0,A} \tag{20}$$

$$(\delta_{r,s,A} \otimes \delta_{r',s,A}) \circ c_{rs,r's,A} = c_{r,r',D(s)(A)} \circ \delta_{r+r',s,A} \tag{21}$$

$$m_{s,D(r)(A),D(r')(A)} \circ (\delta_{r,s,A} \otimes \delta_{s,r',A}) \circ c_{sr,sr',A} = D(s)(c_{r,r',A}) \circ \delta_{s,r+r',A} \tag{22}$$

Concretely, the definition above requires 6 natural transformations:

$$\begin{aligned} m_r &: \{\star\} \to D(r)(\{\star\}) \\ m_{r,A,B} &: D(r)(A) \otimes D(r)(B) \to D(r)(A \otimes B) \\ w_A &: D(0)(A) \to \{\star\} \\ c_{r,r',A} &: D(r+r')(A) \to D(r)(A) \otimes D(r')(A) \\ \epsilon_A &: D(1)(A) \to A \\ \delta_{r,r',A} &: D(rr')(A) \to D(r)(D(r')(A)) \end{aligned}$$

subject to the following list of equations:

- $D(r)$ is a lax monoidal functor:

$$m_{r,A\otimes B,C} \circ (m_{r,A,B} \otimes D(r)(C)) = m_{r,A,B\otimes C} \circ (D(r)(A) \otimes m_{r,B,C}) \tag{23}$$

$$m_{r,A,\{\star\}} \circ (D(r)(A) \otimes m_r) = D(r)(A) \tag{24}$$

$$m_{r,\{\star\},B} \circ (m_r \otimes D(r)(B)) = D(r)(B) \tag{25}$$

- (D, w, c) is a symmetric colax monoidal functor:

$$(c_{r,s,-} \otimes D(t)(-)) \circ c_{r+s,t} = (D(r)(-) \otimes c_{s,t,-}) \circ c_{r,s+t} \tag{26}$$

$$(D(r)(-) \otimes w_-) \circ c_{r,0,-} = D(r)(-) \tag{27}$$

$$(w_- \otimes D(r)(-)) \circ c_{0,r,-} = D(r)(-) \tag{28}$$

- (D, ϵ, δ) is a colax monoidal functor:

$$\delta_{r,s,D(t)(-)} \circ \delta_{(rs),t,-} = D(r)(\delta_{s,t,-}) \circ \delta_{r,st,-} \tag{29}$$

$$D(r)(\epsilon_-) \circ \delta_{r,1,-} = D(r)(-) \tag{30}$$

$$\epsilon_{D(r)(-)} \circ \delta_{1,r,-} = D(r)(-) \tag{31}$$

The following definition provides an interpretation of bSTLC in any symmetric monoidal closed category with a $\mathbb{N}$-graded linear exponential comonad.

Definition 8 (interpretation of bSTLC)**.** *Let $\mathcal{C}$ be a symmetric monoidal closed category and $(D, w, c, \epsilon, \delta)$ be a $\mathbb{N}$-graded linear exponential comonad. Let $[\![X]\!]$ be fixed objects of $\mathcal{C}$, one for each ground type X of* bSTLC*.*

One lifts the interpretation to types as $[\![!_n A \multimap B]\!] = D(n)([\![A]\!]) \multimap [\![B]\!]$. Moreover, one extends the interpretation to contexts via $[\![x :_n A]\!] := D(n)([\![A]\!])$ and $[\![\{x_1 :_{n_1} A_1, \ldots, x_k :_{n_k} A_k\}]\!] = \bigotimes_{i=1}^{k} [\![x_i :_{n_i} A_i]\!]$.

Then, one inductively defines an interpretation $[\![\Gamma \vdash M : A]\!] \in \mathcal{C}([\![\Gamma]\!], [\![A]\!])$ of $\Gamma \vdash M : A$ by induction as follows:

- $[\![x \in_1 A \vdash x \in A]\!] = \epsilon_A$;
- $[\![\Gamma, x :_0 B \vdash M : A]\!] = [\![\Gamma \vdash M : A]\!] \circ ([\![\Gamma]\!] \otimes w_{[\![B]\!]})$;
- $[\![\Gamma, x :_{m+n} B \vdash M[x/y] : A]\!] = [\![\Gamma, x :_m B, y :_m B \vdash M : A]\!] \circ ([\![\Gamma]\!] \otimes c_{m,n,[\![B]\!]})$;
- $[\![\Gamma \vdash \lambda x.M :!_n A \multimap B]\!] = \Lambda([\![\Gamma, x :_n A \vdash M : B]\!])$, *where Λ is the isomorphism $\mathcal{C}([\![\Gamma]\!] \otimes D(n)([\![A]\!]), [\![B]\!]) \to \mathcal{C}([\![\Gamma]\!], D(n)([\![A]\!]) \multimap [\![B]\!])$;*
- $[\![\Gamma + \Delta \vdash MN : B]\!] = \mathsf{ev} \circ \big([\![\Gamma \vdash M : A \multimap B]\!] \otimes [\![\Delta \vdash N : A]\!]\big)$, *where* $\mathsf{ev} \in \mathcal{C}(([\![A]\!] \multimap [\![B]\!]) \otimes [\![A]\!], [\![B]\!])$ *is the evaluation morphism of $\mathcal{C}$;*
- $[\![n\Gamma \vdash M :!_n A]\!] =!_n([\![\Gamma \vdash M : A]\!]) \circ \big(\delta_{n,m_1,[\![A_1]\!]} \otimes \cdots \otimes \delta_{n,m_k,[\![A_k]\!]}\big)$, *where $\Gamma = \{x_1 :_{m_1} A_1, \ldots, x_k :_{m_k} A_k\}$.*

Let us now show that bounded multisets defined a $\mathbb{N}$-graded linear exponential comonad over $\mathbb{L}\mathsf{Rel}$.

Definition 9. *We define the following structure $(!_-(-), w, c, \epsilon, \delta)$ over the category $\mathbb{L}\mathsf{Rel}$ as follows:*

- *for any set X and $n \in \mathbb{N}$, let $!_n(X) = \mathcal{M}_{\leqslant n}(X)$;*
- *for all $f : X \times Y \to \mathbb{L}$, let $!_n(f) :!_n(X) \times !_n(Y) \to \mathbb{L}$ be defined by*

$$!_n(f)(\alpha, \beta) = \begin{cases} \min_{\sigma \in \mathfrak{S}_k} \sum_{i=1}^{k} f(x_i, y_{\sigma(i)}) & \textit{if } \alpha = [x_1, \ldots, x_k], \beta = [y_1, \ldots, y_k] \\ \infty & \textit{otherwise} \end{cases}$$

- *$m_r(\star, \{\star\}) = 0$ and $m_r(\star, \varnothing) = \infty$;*
- *$m_{r,A,B} : D(r)(A) \times D(r)(B) \times D(r)(A \times B) \to \mathbb{L}$ is defined by*

$$m_{r,A,B}((\alpha, \beta), \gamma) = \begin{cases} 0 & \textit{if } \alpha = [x_1, \ldots, x_k], \beta = [y_1, \ldots, y_k], \gamma = [(x_1, y_1), \ldots, (x_k, y_k)] \\ \infty & \textit{otherwise} \end{cases}$$

- $w_A : D(0)(A) \times \{\star\} \to \mathbb{L}$ *is given by* $w_A(\varnothing, \star) = 0$ *and is* ∞ *otherwise (observe that* $D(0)(A) \simeq \{\star\}$*);*
- $c_{r,s,A} : D(r+s)(A) \times D(r)(A) \times D(s)(A) \to \mathbb{L}$ *is given by* $c_{r,r',A}(\langle \alpha, \beta, \gamma \rangle) = 0$ *if* $\alpha = \beta + \gamma$, *and is* ∞ *otherwise;*
- $\epsilon_A(\varnothing, a) = \infty$, $\epsilon_A([a], a) = 0$, $\epsilon_A([b], a) = \infty$ $(b \neq a)$,
- $\delta_{r,r',A}(\alpha, B) = 0$ *if* $\alpha = \sum B$ *(where* $\sum B$ *indicates the multiset obtained by the sum of all multisets contained in* B*) and is* ∞ *otherwise.*

Proposition 30. $(!_{-}(-), w, c, \epsilon, \delta)$ *is a* $\mathbb{N}$*-graded linear exponential comonad over* $\mathbb{L}\mathsf{Rel}$*.*

Proof.
- $D(r)$ is a lax monoidal functor:

$$m_{r,A\times B,C} \circ (m_{r,A,B} \times D(r)(C))(\langle \alpha, \beta, \gamma, \delta \rangle) : D(r)(A) \times D(r)(B) \times D(r)(C) \times D(r)(A \times B \times C) \to \mathbb{L}$$

  is equal to 0 precisely when $\alpha = [x_1, \dots, x_k]$, $\beta = [y_1, \dots, y_k]$, $\gamma = [z_1, \dots, z_k]$ and $\delta = [(x_1, y_1, z_1), \dots, (x_k, y_k, z_k)]$, and is ∞ in all other cases.
  Observe that $m_{r,A,B\times C} \circ (D(r)(A) \times m_{r,B,C})(\langle \alpha, \beta, \gamma, \delta \rangle)$ is equal to 0 in the same situation, and is ∞ otherwise.
  We conclude that the two matrices coincide.
  Furthermore, we have that $m_{r,A,\{\star\}} \circ (D(r)(A) \times m_r)(\langle \alpha, \beta \rangle) : D(r)(A) \times \{\star\} \times D(r)(A)$ is equal to 0 precisely when $\alpha = \beta$ and is ∞ otherwise, that is, it coincides with $\mathrm{id}_{D(r)(A)}$.
- (D, w, c) is a symmetric colax monoidal functor.
  $((c_{r,s,A} \times D(t)(A)) \circ c_{r+s,t,A})(\langle \alpha, \beta, \gamma, \delta \rangle) : D(r+s+t)(A) \times D(r)(A) \times D(s)(A) \times D(t)(A)$ is equal to 0 when $\alpha = \beta + \gamma + \delta$, and is ∞ otherwise, and the same holds for $((D(r)(A) \times c_{s,t,A}) \circ c_{r,s+t,A})(\langle \alpha, \beta, \gamma, \delta \rangle)$.
  Furthermore, $((D(r)(A) \times w_A) \circ c_{r,0,A})(\alpha, \beta) : D(r)(A) \times D(r)(A) \to \mathbb{L}$ is equal to 0 when $\alpha = \beta$, and is ∞ otherwise, so it coincides with $\mathrm{id}_{D(r)(A)}$.
- (D, ϵ, δ) is a colax monoidal functor:
  $(\delta_{r,s,D(t)(A)} \circ \delta_{rs,t,A})(\alpha, \Gamma) : D(rst)(A) \times D(r)(D(s)(D(t)(A))) \to \mathbb{L}$ is 0 precisely when $\alpha = \sum\sum\Gamma$, and is ∞ otherwise, and similarly for $(D(r)(\delta_{s,t,A}) \circ \delta_{r,st,A})(\alpha, \Gamma)$.
  Furthermore, $(D(r)(\epsilon_A) \circ \delta_{r,1})(\alpha, \beta) : D(r)(A) \times D(r)(A) \to \mathbb{L}$ is equal to 0 when $\alpha = \beta$ and is ∞ otherwise, so it coincides with $\mathrm{id}_{D(r)(A)}$.

Let us check the further equations:

- $(w_{D(s)(A)} \circ \delta_{0,s,A})(\langle \varnothing, \star \rangle : D(0)(A) \times \{\star\} \to \mathbb{L}$ is 0, precisely like w_A.
- A similar argument holds for the second equation.
- $((\delta_{r,s,A} \times \delta_{r',s,a}) \circ c_{rs,r's,A})(\langle \alpha, \Gamma, \Delta \rangle) : D(rs + r's)(A) \times D(r)(s)(A) \times D(r')(s)(A) \to \mathbb{L}$ is equal to 0 when $\alpha = \sum\Gamma + \sum\Delta$, and is ∞ otherwise.
  Now, using the fact that $D(rs + r's)(A) = D((r + r')s)(A)$, we can check that the same holds for $c_{r,r',D(s)(A)} \circ \delta_{r+r',s,A})(\langle \alpha, \Gamma, \Delta \rangle)$: it is 0 when $\alpha = \sum\Gamma + \Delta = \sum\Gamma + \sum\Delta$.
- A similar argument holds for the fourth equation.

∎

C. Section III-D: Proof of Theorem 2

Theorem 2 says that $(\mathbb{L}\mathsf{Rel}_!, D)$ is a $CC\partial C$. This is proved in Proposition 32. First we need the following:

Proposition 31. $\mathbb{L}\mathsf{Rel}_!$ *is a cartesian closed left-*$\mathbb{L}$*-additive category.*

Proof. Since $\infty \circ f = f \circ \infty = \infty$ and $\min\{g, h\} \circ f = \min\{g \circ f, h \circ f\}$, $\mathbb{L}\mathsf{Rel}_!$ is left-$\mathbb{L}$-additive. A morphism $h \in \mathbb{L}\mathsf{Rel}_!(X, Y)$ that further satisfies $h \circ \min\{f, g\} = \min\{h \circ f, h \circ g\}$ for all object X' and $f, g \in \mathbb{L}\mathsf{Rel}_!(X', X)$, is called *additive*. To show that $\mathbb{L}\mathsf{Rel}_!$ is cartesian closed left-$\mathbb{L}$-additive we must also check that (1) products and projections of additive morphisms are additive, and that (2) $\Lambda(\min\{f, g\}) = \min\{\Lambda(f), \Lambda(g)\}$, $\Lambda(\infty) = \infty$, where $\Lambda : \mathbb{L}^{\mathcal{M}_{\mathsf{fin}}(Z+X)\times Y} \to \mathbb{L}^{\mathcal{M}_{\mathsf{fin}}(Z)\times(\mathcal{M}_{\mathsf{fin}}(X)\times Y)}$ is the isomorphism given by $(\Lambda(f))_{\mu,\nu,y} = f_{\mu\oplus\nu,y}$, where $\mu \oplus \nu$ is defined by $(\mu \oplus \nu)(\langle 0, x \rangle) = \mu(x)$ and $(\mu \oplus \nu)(\langle 1, x \rangle) = \nu(x)$.

1) Let $f \in \mathbb{L}^{\mathcal{M}_{\mathsf{fin}}(X)\times Y}$ and $g \in \mathbb{L}^{\mathcal{M}_{\mathsf{fin}}(X)\times Z}$ be additive; then $\langle f, g \rangle \in \mathbb{L}^{\mathcal{M}_{\mathsf{fin}}(X)\times(Y+Z)}$, which is defined by

$$\langle f, g \rangle_{\mu,\langle i,a \rangle} = \begin{cases} f_{\mu,a} & \text{if } i = 0 \\ g_{\mu,a} & \text{if } i = 1 \end{cases}$$

is also additive. Indeed, for all $h \in \mathbb{L}\mathsf{Rel}_!(X', X)$, in any cartesian category it holds that $\langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle$. Now, if $i = 0$, then for all $h_1, h_2 \in \mathbb{L}^{\mathcal{M}_{\mathsf{fin}}(X')\times X}$,

$$\begin{aligned}(\langle f, g\rangle \circ \min\{h_1, h_2\})_{\rho,\langle i,z\rangle} &= (\langle f \circ \min\{h_1, h_2\}, g \circ \min\{h_1, h_2\}\rangle)_{\rho,\langle i,z\rangle}\\ &= (f \circ \min\{h_1, h_2\})_{\rho,z}\\ &= \min\{(f \circ h_1)_{\rho,z}, (f \circ h_2)_{\rho,z}\}\\ &= \min\{\langle f \circ h_1, g \circ h_1\rangle_{\rho,\langle i,z\rangle}, \langle f \circ h_2, g \circ h_2\rangle_{\rho,\langle i,z\rangle}\}\\ &= \min\{(\langle f, g\rangle \circ h_1)_{\rho,\langle i,z\rangle}, (\langle f, g\rangle \circ h_2)_{\rho,\langle i,z\rangle}\}\end{aligned}$$

and similarly if $i = 1$.

Moreover, suppose $f \in \mathbb{L}\mathsf{Rel}_!(X, Y + Z)$ is additive, and let us show that $\pi_1(f) \in \mathbb{L}\mathsf{Rel}_!(X, Y)$, defined by $(\pi_1(f))_{\mu,y} = f_{\mu,\langle 0,y\rangle}$, is also additive: first observe that $\pi_1(f) = \pi_1 \circ f$, where $\pi_1 \in \mathbb{L}\mathsf{Rel}_!(Y + Z, Y)$ is given by $(\pi_1)_{\mu,y} = 0$ if $\mu = [y] \oplus \varnothing$ and is ∞ otherwise; moreover, $\pi_1(\min\{g, h\}) = \min\{\pi_1(g), \pi_1(h)\}$, since $(\pi_1(\min\{g, h\}))_{\mu,y} = (\min\{g, h\})_{\mu,\langle 0,y\rangle} = \min\{g_{\mu,\langle 0,y\rangle}, h_{\mu,\langle 0,y\rangle}\} = (\min\{\pi_1(g), \pi_1(h)\})_{\mu,y}$. Now, given $h_1, h_2 \in \mathbb{L}\mathsf{Rel}_!(X', X)$, we have that $\pi_1(f)\circ\min\{h_1, h_2\} = (\pi_1\circ f)\circ\min\{h_1, h_2\} = \pi_1\circ(f\circ\min\{h_1, h_2\}) = \pi_1\circ\min\{f\circ h_1, f\circ h_2\} = \pi_1(\min\{f\circ h_1, f\circ h_2\}) = \min\{\pi_1(f \circ h_1), \pi_1(f \circ h_2)\} = \min\{\pi_1(f) \circ h_1, \pi_1(f) \circ h_2\}$.

2) It is clear then that $\Lambda(\infty) = \infty$, and moreover

$$\Lambda(\min\{f, g\})_{\mu,\nu,y} = \min\{f, g\}_{\mu\oplus\nu,y} = \min\{f_{\mu\oplus\nu,y}, g_{\mu\oplus\nu,y}\} = \min(\Lambda(f), \Lambda(g))_{\mu,\nu,y}$$

□

For any morphism $f \in \mathbb{L}\mathsf{Rel}_!(X, Y)$, let us define a morphism $D(f) \in \mathbb{L}\mathsf{Rel}_!(X + X, Y)$, i.e. $D(f) \in \mathbb{L}^{\mathcal{M}_{\mathsf{fin}}(X+X),Y}$, by

$$D(f)_{\mu,y} = \begin{cases} f_{\mu'+x,y} & \text{if } \mu = [x] \oplus \mu' \\ \infty & \text{otherwise} \end{cases}$$

Proposition 32. *The category $\mathbb{L}\mathsf{Rel}_!$, endowed with the operator D, is a cartesian closed differential category.*

Proof. We must check axioms (D1)-(D7) of cartesian differential categories plus axiom (D-curry) (cf. [53]).

(D1) $D(\min\{f, g\}) = \min\{D(f), D(g)\}$ and $D(\infty) = \infty$: while the latter is obvious, for the former we have $D(\min\{f, g\})_{[x]\oplus\nu,y} = \min\{f, g\}_{\nu+x,y} = \min\{f_{\nu+x,y}, g_{\nu+x,y}\} = \min\{D(f), D(g)\}_{[x]\oplus\nu,y}$, and if $\mu \neq [x] \oplus \nu$, $D(\min\{f, g\})_{\mu,y} = \infty = \min\{\infty, \infty\} = \min\{D(f), D(g)\}_{\mu,y}$.

(D2) $D(f) \circ \langle\min\{h, k\}, v\rangle = \min\{D(f) \circ \langle h, v\rangle, D(f) \circ \langle k, v\rangle\}$, and $D(f) \circ \langle\infty, v\rangle = \infty$: we can compute

$$\begin{aligned}(D(f) \circ \langle\min\{h, k\}, v\rangle)_{\mu,y} &= \inf\Big\{ \sum_{i=1}^{n} \min\{h, k\}_{\rho_i,w_i} + \sum_{j=1}^{m} v_{\nu_j,z_j} + f_{[z_1,\dots,z_m]+w,y} \\ &\qquad \mid \mu = \sum_{i=1}^{n} \rho_i + \sum_{j=1}^{m} \nu_j, [w] = [w_1, \dots, w_n]\Big\}\\ &= \inf\Big\{ \min\{h, k\}_{\rho,w} + \sum_{j=1}^{m} v_{\nu_j,z_j} + f_{[z_1,\dots,z_m]+w,y} \\ &\qquad \mid \mu = \rho + \sum_{j=1}^{m} \nu_j\Big\}\\ &= \min\Big\{ \inf\big\{h_{\rho,w} + \sum_{j=1}^{m} v_{\nu_j,z_j} + f_{[z_1,\dots,z_m]+w,y} \mid \mu = \rho + \sum_{j=1}^{m} \nu_j\big\},\\ &\qquad \inf\big\{k_{\rho,w} + \sum_{j=1}^{m} v_{\nu_j,z_j} + f_{[z_1,\dots,z_m]+w,y} \mid \mu = \rho + \sum_{j=1}^{m} \nu_j\big\}\Big\}\\ &= \min\Big\{(D(f) \circ \langle h, v\rangle)_{\mu,y}, (D(f) \circ \langle k, v\rangle)_{\mu,y}\Big\}\\ &= \Big(\min\big\{D(f) \circ \langle h, v\rangle, D(f) \circ \langle k, v\rangle\big\}\Big)_{\mu,y}\end{aligned}$$

where, in the first equation, the condition $[w_1, \dots, w_n] = [w]$ (i.e. $n = 1$) is forced by the fact that, otherwise, the application of $D(f)$ would give ∞. Moreover, we have

$$(D(f) \circ \langle \infty, v \rangle)_{\mu,y} = \inf \left\{ \infty + \sum_{j=1}^{m} v_{\nu_j, z_j} + f_{[z_1, \dots, z_m] + w, y} \mid \mu = \rho + \sum_{j=1}^{m} \nu_j \right\} = \infty$$

(D3) $D(\mathrm{id}) = \pi_1$, $D(\pi_i) = \pi_i \circ \pi_1$: recall that $\mathrm{id}_{[x],x} = 0$ and $\mathrm{id}_{\mu,x} = \infty$, if $\mu \neq [x]$. Moreover $(\pi_1)_{\mu,x} = 0$ if $\mu = [x] \oplus \varnothing$, and is ∞ otherwise, and π_2 is defined similarly. Hence $D(\mathrm{id})_{[x]\oplus\nu,y} = \mathrm{id}_{\nu+x,y}$ is 0 precisely when $x = y$ and $\nu = \varnothing$, and in all other cases is ∞. This shows that $D(\mathrm{id}) = \pi_1$.
$D(\pi_1) \in \mathbb{L}^{\mathcal{M}_{\mathsf{fin}}((X+Y)+(X+Y))\times Y}$ is given by $D(\pi_1)_{[x\oplus\varnothing]\oplus(\mu\oplus\nu),y} = (\pi_1)_{(\mu\oplus\nu)+\langle 0,x\rangle,y}$, which is 0 precisely when $(\mu \oplus \nu) + \langle 0, x \rangle = y \oplus \varnothing$, i.e. when $x = y$ and $\mu = \nu = \varnothing$; in all other cases one can check that $D(\pi_1)_{\rho,y} = \infty$, so we conclude $D(\pi_1) = \pi_1 \circ \pi_1$. One can argue similarly for π_2.

(D4) $D(\langle f, g \rangle) = \langle D(f), D(g) \rangle$: we have

$$D(\langle f, g \rangle)_{[x]\oplus\mu,\langle 0,y\rangle} = (\langle f, g \rangle)_{\mu+x,\langle 0,y\rangle} = f_{\mu+x,y} = D(f)_{[x]\oplus\mu,y}$$
$$D(\langle f, g \rangle)_{[x]\oplus\mu,\langle 1,y\rangle} = (\langle f, g \rangle)_{\mu+x,\langle 1,y\rangle} = g_{\mu+x,y} = D(g)_{[x]\oplus\mu,y}$$

from which we deduce $D(\langle f, g \rangle)_{[x]\oplus\mu,\langle i,y\rangle} = \langle D(f), D(g) \rangle_{[x]\oplus\mu,\langle i,y\rangle}$ by the definition of $\langle \_, \_ \rangle$. If $\rho \neq [x] \oplus \mu$, then $D(\langle f, g \rangle)_{\rho,\langle i,y\rangle} = \infty = \langle \infty, \infty \rangle = \langle D(f), D(g) \rangle_{\rho,\langle i,y\rangle}$ (where the equation $\infty = \langle \infty, \infty \rangle$ is to be read as an equality between the functions $X + Y \longrightarrow Q$ defined by $\langle i, y \rangle \mapsto \infty$ and by $\begin{matrix}\langle 0, x \rangle \mapsto \infty \\ \langle 1, y \rangle \mapsto \infty\end{matrix}$, respectively).

(D5) $D(f \circ g) = D(f) \circ \langle D(g), g \circ \pi_2 \rangle$: we can compute

$$\begin{aligned}
\Big(D(f) \circ \langle D(g), g \circ \pi_2 \rangle\Big)_{[x]\oplus\mu,z} &= \inf \Big\{ D(g)_{[x]\oplus\mu',w} + \sum_i g_{\mu_i,w_i} + D(f)_{[w]\oplus[w_1,\dots,w_n],z} \\
&\qquad \mid w, w_i \in Y, \mu = \mu' + \sum_i \mu_i, \Big\} \\
&= \inf \Big\{ g_{\mu'+x,w} + \sum_i g_{\mu_i,w_i} + f_{[w_1,\dots,w_n]+w,z} \\
&\qquad \mid w, w_i \in Y, \mu = \mu' + \sum_i \mu_i \Big\} \\
&= \inf \Big\{ \sum_i g_{\mu_i,w_i} + f_{[w_1,\dots,w_n],z} \mid w_1, \dots, w_n \in Y, \mu + x = \sum_i \mu_i \Big\} \\
&= (f \circ g)_{\mu+x,y} = D(f \circ g)_{[x]\oplus\mu,z}
\end{aligned}$$

if $\rho \neq [x] \oplus \mu$, then $D(f \circ g)_{\rho,z} = \infty$ and from the first equation above it follows that also $(D(f) \circ \langle D(g), g \circ \pi_2 \rangle)_{\rho,z} = \infty$.

(D6) $D(D(f)) \circ \langle \langle g, \infty \rangle, \langle h, k \rangle \rangle = D(f) \circ \langle g, k \rangle$: observe that

$$\Big(D(D(f))\Big)_{[\langle 1,x'\rangle]\oplus([x]\oplus\mu),z} = \big(D(f)\big)_{[x]\oplus(\mu+x'),z} = f_{\mu+x'+x,z}$$
$$\Big(D(D(f))\Big)_{[\langle 0,x\rangle]\oplus(\varnothing\oplus\mu),z} = \big(D(f)\big)_{[x]\oplus\mu,z} = f_{\mu+x,z}$$

and in all other cases $(D(D(f)))_{\mu,z} = \infty$. Using this fact we can compute:

$$\begin{aligned}
\Big(D(D(f)) \circ \langle \langle g, \infty \rangle, \langle h, k \rangle \rangle\Big)_{\mu,z} &= \min \left\{ \begin{matrix} \inf \left\{ \begin{matrix} \infty_{\rho_1,x'} + h_{\rho_2,x} + \sum_i k_{\mu_i,w_i} + f_{[w_1,\dots,w_n]+x'+x,z} \\ \mid x, x', w_i \in Y, \mu = \rho_1 + \rho_2 + \sum_i \mu_i \end{matrix} \right\}, \\ \inf \left\{ \begin{matrix} g_{\rho,x} + \sum_i k_{\mu_i,w_i} + f_{[w_1,\dots,w_n]+x,z} \\ \mid x, w_i \in Y, \mu = \rho + \sum_i \mu_i \end{matrix} \right\} \end{matrix} \right\} \\
&= \inf \left\{ g_{\rho,x} + \sum_i k_{\mu_i,w_i} + f_{[w_1,\dots,w_n]+x,z} \mid x, w_i \in Y, \mu = \rho + \sum_i \mu_i \right\} \\
&= \Big(D(f) \circ \langle g, k \rangle\Big)_{\mu,z}
\end{aligned}$$

(D7) $D(D(f)) \circ \langle \langle \infty, h \rangle, \langle g, k \rangle \rangle = D(D(f)) \circ \langle \langle \infty, g \rangle, \langle h, k \rangle \rangle$: by computations similar to the case above we obtain

$$\begin{aligned}&\Big(D(D(f))\circ\langle\langle\infty,h\rangle,\langle g,k\rangle\rangle\Big)_{\mu,z}\\&\qquad=\inf\left\{h_{\rho',x'}+g_{\rho,x}+\sum_i k_{\mu_i,w_i}+f_{[w_1,\dots,w_n]+x'+x,z}\mid x,x',w_i\in Y,\mu=\rho'+\rho+\sum_i\mu_i\right\}\\&\qquad=\inf\left\{g_{\rho,x}+h_{\rho',x'}+\sum_i k_{\mu_i,w_i}+f_{[w_1,\dots,w_n]+x+x',z}\mid x,x',w_i\in Y,\mu=\rho+\rho'+\sum_i\mu_i\right\}\\&\qquad=\Big(D(D(f))\circ\langle\langle\infty,g\rangle,\langle h,k\rangle\rangle\Big)_{\mu,z}\end{aligned}$$

(D-curry) $D(\Lambda(f))=\Lambda(D(f)\circ\langle\pi_1\times\infty,\pi_2\times\mathrm{id}\rangle)$: by observing that both morphisms are in $\mathbb{L}\mathsf{Rel}_!(X+X,Z^Y)=\mathbb{L}^{\mathcal{M}_{\mathsf{fin}}(X+X)\times\mathcal{M}_{\mathsf{fin}}(Y)\times Z}$, and that $\langle\pi_1\times\infty,\pi_2\times\mathrm{id}\rangle\in\mathbb{L}\mathsf{Rel}_!((X+X)+Y,(X+Y)+(X+Y))$, we can compute:

$$\begin{aligned}&\big(\Lambda(D(f)\circ\langle\pi_1\times\infty,\pi_2\times\mathrm{id}\rangle)\big)_{[x]\oplus\mu,\nu,z}\\&\quad=\big(D(f)\circ\langle\pi_1\times\infty,\pi_2\times\mathrm{id}\rangle\big)_{([x]\oplus\mu)\oplus\nu,z}\\&\quad=\inf\left\{(\pi_1)_{[x]\oplus\varnothing,x}+\sum_i(\pi_2)_{\varnothing\oplus[w_i],w_i}+\sum_j(\mathrm{id})_{[z_j],z_j}+D(f)_{([x]\oplus\varnothing)\oplus(\mu\oplus\nu)}\mid \begin{matrix}\mu=[w_1,\dots,w_n],\\ \nu=[z_1,\dots,z_m]\end{matrix}\right\}\\&\quad=\inf\left\{0+0+0+D(f)_{([x]\oplus\varnothing)\oplus(\mu\oplus\nu)}\mid \begin{matrix}\mu=[w_1,\dots,w_n],\\ \nu=[z_1,\dots,z_m]\end{matrix}\right\}\\&\quad=\big(D(f)\big)_{([x]\oplus\varnothing)\oplus(\mu\oplus\nu),z}\\&\quad=f_{(\mu+x)\oplus\nu,z}\\&\quad=\big(\Lambda(f)\big)_{\mu+x,\nu,z}=\big(D(\Lambda(f))\big)_{[x]\oplus\mu,\nu,z}\end{aligned}$$

If $\rho\neq[x]\oplus\mu$, then $(D(\Lambda(f)))_{\rho,\nu,z}=\infty$ and $(\Lambda(D(f)\circ\langle\pi_1\times\infty,\pi_2\times\mathrm{id}\rangle))_{\rho,\nu,z}=(D(f)\circ\langle\pi_1\times\infty,\pi_2\times\mathrm{id}\rangle)_{\rho\oplus\nu,z}$, and one can check that also this is ∞, using the second equation above and the fact that $(\pi_1)_{\rho\oplus\varnothing,x}=\infty$.

□

D. *Section III-D: proof of Theorem 3*

Theorem 3 states the validity of Taylor expansion in $\mathbb{L}\mathsf{Rel}_!$.

We must check for it the following equation, given $f\in\mathbb{L}\mathsf{Rel}_!(C,B^A)$ and $g\in\mathbb{L}\mathsf{Rel}_!(C,A)$:

$$\mathsf{ev}\circ\langle f,g\rangle=\inf_{n\in\mathbb{N}}\left\{((\cdots(\Lambda^-(f)\underbrace{\star g)\cdots)\star g}_{n\text{ times}})\circ\langle\mathrm{id},\infty\rangle\right\}$$

where:

1) $\mathsf{ev}\in\mathbb{L}\mathsf{Rel}_!(B^A+A,B)$ is the canonical *evaluation* morphism;
2) $\Lambda^-(\_):=\mathsf{ev}\circ(\_\times\mathrm{id})$ is the *uncurry* operator;
3) given $f\in\mathbb{L}\mathsf{Rel}_!(C+A,B)$ and $g\in\mathbb{L}\mathsf{Rel}_!(C,A)$, $f\star g\in\mathbb{L}\mathsf{Rel}_!(C+A,B)$ is the morphism obtained by differentiating f in its second component and applying g in that component, i.e.
$$f\star g=D(f)\circ\langle\langle\infty,g\circ\pi_1\rangle,\mathrm{id}_{C+A}\rangle.$$

We do it in the following 4 steps.

1) Let us compute the morphism ev explicitly: $\mathsf{ev}\in\mathbb{L}^{\mathcal{M}_{\mathsf{fin}}((\mathcal{M}_{\mathsf{fin}}(A)\times B)+A)\times B}$ is given by
$$\mathsf{ev}_{\mu,y}=\begin{cases}0&\text{if }\mu=[\langle\rho,y\rangle]\oplus\rho\\\infty&\text{otherwise}\end{cases}$$
and observe that, given $f\in\mathbb{L}\mathsf{Rel}_!(C,B^A)$ and $g\in\mathbb{L}\mathsf{Rel}_!(C,A)$,
$$\big(\mathsf{ev}\circ\langle f,g\rangle\big)_{\chi,y}=\inf\left\{\sum_{i=1}^m g_{\chi_i,x_i}+f_{\chi',\langle[x_1,\dots,x_m],y\rangle}\mid x_1,\dots,x_m\in A,\chi=\chi'+\sum_{i=1}^m\chi_i,\right\}$$
2) Let us compute the morphism Λ^- explicitly: given $g\in\mathbb{L}\mathsf{Rel}_!(C,B^A)$, $\Lambda^-(g)\in\mathbb{L}\mathsf{Rel}_!(C+A,B)$ is given by
$$\big(\Lambda^-(g)\big)_{\rho\oplus\mu,y}=g_{\rho,\langle\mu,y\rangle}$$

3) Let us compute the morphism $\star$ explicitly: $f \star g$ is given by

$$(f \star g)_{\rho\oplus\mu,y} = \inf\left\{g_{\rho',x} + f_{\rho''\oplus(\mu+x)} \mid x \in A, \rho = \rho' + \rho''\right\}$$

4) We can now conclude: given the definition of $\mathsf{ev} \circ \langle f, g\rangle$, to check the Taylor equation it is enough to check that, for all $N \in \mathbb{N}$,

$$\left(((\cdots(\Lambda^-(f) \underbrace{\star g)\cdots) \star g}_{N \text{ times}}) \circ \langle \mathrm{id}, \infty\rangle\right)_{\chi,y} = \inf\left\{ \sum_{i=1}^{N} g_{\chi_i,x_i} + f_{\chi',\langle[x_1,\dots,x_N],y\rangle} \mid \begin{array}{c} x_1,\dots,x_N \in A, \\ \chi = \chi' + \sum_{i=1}^N \chi_i \end{array}\right\}$$

Let us show, by induction on N, the following equality, from which the desired equality easily descends:

$$\big((\cdots(\Lambda^-(f) \underbrace{\star g)\cdots) \star g}_{N \text{ times}}\big)_{\chi\oplus\mu,y} = \inf\left\{ \sum_{i=1}^{N} g_{\chi_i,x_i} + f_{\chi',\langle\mu+[x_1,\dots,x_N],y\rangle} \mid \begin{array}{c} x_1,\dots,x_N \in A, \\ \chi = \chi' + \sum_{i=1}^N \chi_i \end{array}\right\}$$

- if $N = 0$, the right-hand term reduces to $f_{\chi,\langle\mu,y\rangle} = (\Lambda^-(f))_{\chi\oplus\mu,y}$;
- otherwise, let $F = ((\cdots(\Lambda^-(f) \underbrace{\star g)\cdots) \star g}_{N-1 \text{ times}})$, so that by I.H. we have

$$F_{\chi\oplus\mu,y} = \inf\left\{ \sum_{i=1}^{N-1} g_{\chi_i,x_i} + f_{\chi',\langle\mu+[x_1,\dots,x_{N-1}],y\rangle} \mid \begin{array}{c} x_1,\dots,x_{N-1} \in A, \\ \chi = \chi' + \sum_{i=1}^{N-1} \chi_i \end{array}\right\}$$

Then we have

$$\begin{aligned}
\big(F \star g\big)_{\chi\oplus\mu,y} &= \inf\left\{g_{\chi',x} + F_{\chi''\oplus(\mu+x)} \mid x \in A, \chi = \chi' + \chi''\right\} \\
&= \inf\left\{ g_{\chi',x} + \inf\left\{ \sum_{i=1}^{N-1} g_{\chi_i,x_i} + f_{\chi'',\langle\mu+[x_1,\dots,x_{N-1}]+x,y\rangle} \mid \begin{array}{c} x_1,\dots,x_{N-1} \in A, \\ \chi^* = \chi'' + \sum_{i=1}^{N-1} \chi_i \end{array}\right\} \mid \begin{array}{c} x \in A, \\ \chi = \chi' + \chi^* \end{array}\right\} \\
&= \inf\left\{ g_{\chi',x} + \sum_{i=1}^{N-1} g_{\chi_i,x_i} + f_{\chi'',\langle\mu+[x_1,\dots,x_{N-1}]+x,y\rangle} \mid \begin{array}{c} x, x_1,\dots,x_{N-1} \in A, \\ \chi = \chi' + \chi'' + \sum_{i=1}^{N-1} \chi_i \end{array}\right\} \\
&= \inf\left\{ \sum_{i=1}^{N} g_{\chi_i,x_i} + f_{\chi',\langle\mu+[x_1,\dots,x_N],y\rangle} \mid \begin{array}{c} x_1,\dots,x_N \in A, \\ \chi = \chi' + \sum_{i=1}^{N} \chi_i \end{array}\right\}.
\end{aligned}$$

E. Section III-D: The language and the semantics of the ST∂LC

The language we mainly consider in the paper is the differential λ-calculus ST∂LC, that we briefly define here, following [Section 3, [19]].

The *terms* and the *sums*[1] are mutually generated by the following grammars:

$$\begin{aligned} &\textit{Terms: } M, N := x \mid \lambda x.M \mid M\mathbb{T} \mid \mathsf{D}[M]\cdot N \qquad \textit{(Call } \Lambda^\partial \textit{ the set of such terms)} \\ &\textit{Sums: } \mathbb{T} := 0 \mid M \mid M + \mathbb{T} \end{aligned}$$

quotiented by usual α-equivalence plus other equivalences. Among those, we ask that $+$ is commutative, associative, with neutral element 0 (thus the name "sums"), and the constructors $\lambda x.(\cdot)$, $(\cdot)\mathbb{T}$ and $\mathsf{D}[\cdot]\cdot(\cdot)$ commute with $+$. One says that those constructors are *linear*. Remark that $M(\cdot)$ is *not* set to be linear. So, while $\lambda x.0 = 0\mathbb{T} = \mathsf{D}[0]\cdot N = \mathsf{D}[M]\cdot 0 = 0$, we have $M0 \neq 0$ in general. This is crucial for the definition of the Taylor expansion. One also needs to quotient for an equation, called "permutative equality", which basically says that the order of *consecutive* linear applications $\mathsf{D}[\cdot]\cdot(\cdot)$ does not count. Finally, for simplicity we follow here the tradition of "eliminating the coefficients", that is, we quotient for the idempotency of the sum. Sums are, then, just (*finite*) subsets of Λ^∂.

Simple types are as usual generated by the grammar:

$$A, B := X \mid A \to B$$

where X varies in a fixed set of ground types.

The typing system is given in Figure 3.

We finally recall the sound interpretation of ST∂LC-terms in a $CC\partial C$ category, following [Section 4.3, [19]].

Definition 10. *Let $\mathcal{C}$ be a $CC\partial C$ and $[\![X]\!]$ fixed objects of $\mathcal{C}$, one for each ground type X of the* ST∂LC.

[1] Usually what we call here "terms" are called "simple terms", and what we call here "sums" are called "differential terms".

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x \in A}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \qquad \frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash \mathbb{T} : A}{\Gamma \vdash M\mathbb{T} : B}$$

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : B}{\Gamma \vdash \mathsf{D}[M] \cdot N : A \to B}$$

$$\frac{}{\Gamma \vdash 0 : A} \qquad \frac{\Gamma \vdash M_1 : A \quad \cdots \quad \Gamma \vdash M_n : A \qquad (n \geqslant 2)}{\Gamma \vdash M_1 + \cdots + M_n : A}$$

Fig. 3: Typing rules for ST∂LC.

One lifts as usual the interpretation to types and contexts as $[\![A \to B]\!] := [\![A]\!] \to [\![B]\!]$ *(the exponential object in* $\mathcal{C}$*) and* $[\![x_1 : A_1, \ldots, x_n : A_n]\!] := \Pi_{i=1}^n [\![x_i : A_i]\!]$.

One inductively defines an interpretation $[\![\Gamma \vdash \mathbb{T} : A]\!] \in \mathcal{C}([\![\Gamma]\!], [\![A]\!])$ *of* $\Gamma \vdash \mathbb{T} : A$ *by induction as:*

- $[\![\Gamma, x : A \vdash x : A]\!] := \pi_2$
- $[\![\Gamma, x : A \vdash y : B]\!] := [\![\Gamma \vdash y : B]\!] \circ \pi_1$ *if* $y \neq x$
- $[\![\Gamma \vdash \lambda x.M : A \to B]\!] := \Lambda([\![\Gamma, x : A \vdash M : B]\!])$
- $[\![\Gamma \vdash M\mathbb{T} : B]\!] := \mathsf{ev} \circ \langle [\![\Gamma \vdash M : A \to B]\!], [\![\Gamma \vdash \mathbb{T} : A]\!] \rangle$
- $[\![\Gamma \vdash \mathsf{D}[M] \cdot N : A \to B]\!] := \Lambda\left(\Lambda^-([\![\Gamma \vdash M : A \to B]\!]) \star [\![\Gamma \vdash N : A]\!]\right)$
- $[\![\Gamma \vdash 0 : A]\!] :=: 0$ *(the* 0 *element of* $\mathcal{C}([\![\Gamma]\!], [\![A]\!])$*)*
- $[\![\Gamma \vdash M + \mathbb{T} : A]\!] := [\![\Gamma \vdash M : A]\!] + [\![\Gamma \vdash \mathbb{T} : A]\!]$ *(the* $+$ *operation on* $\mathcal{C}([\![\Gamma]\!], [\![A]\!])$*)*

One then defines a "differential" extension of usual β-reduction (see again [Section 3, [19]], or [35]), and thanks to the fact that $\mathcal{C}$ is a $CC\partial C$, the interpretation above can can be proved to be sound for it ([Theorem 4.12, [19]]).

In the paper we write $[\![M]\!]$, but of course we mean $[\![\Gamma \vdash M : A]\!]$.

F. Section III-D: proof of Corollary 4

In the last lines of Section III-C we state Corollary 4 which is about the Taylor expansion of a ordinary λ-terms M. In the main paper we do not define it, let us briefly recall it here, and the we prove the Corollary. For details, the reader can look at the many papers on the subject pointed in the bibliography.

If sums (as defined in the above lines of Section E) are *finite* subsets of Λ^∂, the Taylor expansion considers in general infinite subsets of Λ^∂. Since we decided to "eliminate the coefficients", what we will define is often called the "qualitative" Taylor expansion.

Definition 11. *The (qualitative)* Taylor expansion *is the following inductively defined map* $\Theta : \Lambda \to \mathcal{P}\left(\Lambda^\partial\right)$.

$$\begin{array}{rcl} \mathcal{T}(x) & := & x \\ \mathcal{T}(\lambda x.M) & := & \lambda x.\mathcal{T}(M) \\ \mathcal{T}(M\mathbb{T}) & := & \sum\limits_{n \in \mathbb{N}} (\mathsf{D}^n[\mathcal{T}(M)] \cdot \mathcal{T}(\mathbb{T})^n)\, 0 \end{array}$$

where $\mathcal{T}(M + \mathbb{T}) := \mathcal{T}(M) + \mathcal{T}(\mathbb{T})$ *and* $\mathsf{D}^{n+1}[\mathcal{T}(M)] \cdot \mathcal{T}(N)^{n+1} := \mathsf{D}[\mathsf{D}^n[\mathcal{T}(M)] \cdot \mathcal{T}(N)^n] \cdot \mathcal{T}(N)$.

We can now prove Corollary 4.

Theorem 33 (Corollary 4)**.** *In* $(\mathbb{L}\mathsf{Rel}_!, D)$*, we have:*

$$[\![M]\!] = [\![\mathcal{T}(M)]\!]$$

for all $M \in \Lambda$.

Proof. By induction on M. We only give the crucial case, that is $M = P\left(\sum_j N_j\right)$. Looking at the definition of $[\![\cdot]\!]$ we have:

$$
\begin{array}{rcll}
[\![P(\sum_j N_j)]\!] & = & \mathsf{ev} \circ \langle\, [\![P]\!], [\![\sum_j N_j]\!]\,\rangle & \\
& = & \inf_{k\in\mathbb{N}} \left\{ ((\ldots(\Lambda^-([\![P]\!]) \star [\![\sum_j N_j]\!]) \cdots) \star [\![\sum_j N_j]\!]) \circ \langle id, \infty\rangle \right\} & \textit{(Theorem 3)} \\
& = & \inf_{k\in\mathbb{N}} \left\{ ((\ldots(\Lambda^-([\![\mathcal{T}(P)]\!]) \star [\![\sum_j \mathcal{T}(N)_j]\!]) \cdots) \star [\![\sum_j \mathcal{T}(N)_j]\!]) \circ \langle id, \infty\rangle \right\} & \textit{(ind. hyp.)} \\
& = & \inf_{k\in\mathbb{N}} \left\{ \mathsf{ev} \circ \langle\, \Lambda \left((\ldots(\Lambda^-([\![\mathcal{T}(P)]\!]) \star [\![\sum_j \mathcal{T}(N)_j]\!]) \cdots) \star [\![\sum_j \mathcal{T}(N)_j]\!] \right), \infty\,\rangle \right\} & \textit{(general property of CCC's)} \\
& = & \inf_{k\in\mathbb{N}} \left\{ \mathsf{ev} \circ \langle\, [\![\mathsf{D}^n[\mathcal{T}(P)] \cdot \mathcal{T}(\sum_j N_j)^n]\!], \infty\,\rangle \right\} & \\
& = & \inf_{k\in\mathbb{N}} \left\{ [\![(\mathsf{D}^n[\mathcal{T}(P)] \cdot \mathcal{T}(\sum_j N_j)^n)0]\!] \right\} & \\
& = & [\![\mathcal{T}(P(\sum_j N_j))]\!] &
\end{array}
$$

□

G. Section IV-A: Proof of Proposition 5

We just show the concavity, the rest being easy.

A function $f : Q^X \to Q^Y$ is *concave* if for all $\alpha \in [0, 1]$, $x, y \in Q^X$ and $b \in Y$

$$f(\alpha \cdot x + (1-\alpha) \cdot y)_b \geqslant \alpha f(x)_b + (1-\alpha) f(y)_b$$

Proposition 34. *All tLs $f : Q^X \to Q^Y$ are concave.*

Proof. Let us first show that all functions of the form $f(x)_b = \mu x + c$ are concave: we have $f(\alpha x + (1-\alpha)y)_b = \mu(\alpha x) + (1-\alpha)y) + c = \mu(\alpha x) + (1-\alpha)y) + \alpha c + (1-\alpha)c = \alpha(\mu x + c) + (1-\alpha)(\mu y + c) = \alpha f(x)_b + (1-\alpha) f(x)_b$.

To conclude, let us show that if $(f_i)_{i\in I}$ is a family of concave functions from Q^X to Q^Y, the function $f = \inf_{i\in I} f_i$ is also concave: we have $f(\alpha x + (1-\alpha)y)_b = \inf_{i\in I} f_i(\alpha x + (1-\alpha)y)_b \geqslant \inf_{i\in I} \alpha f_i(x)_b + (1-\alpha) f_i(y)_b \geqslant \inf_{i\in I} \alpha f_i(x)_b + \inf_{j\in I}(1-\alpha) f_j(y)_b = \alpha \cdot (\inf_{i\in I} f_i(x)_b) + (1-\alpha) \cdot (\inf_{j\in I} f_j(y)_b) = \alpha f(x)_b + (1-\alpha) f(y)_b$, where we used the fact that given families a_i, b_i of reals, $\inf_i a_i + b_i \geqslant \inf_i a_i + \inf_j b_j$. This follows from the fact that for all $i \in I$, $a_i + b_i \geqslant \inf_i a_i + \inf_i b_i$. $\square$

H. Section IV-A: Proof of Theorem 6

We give below the complete statement of Theorem 6 together with its proof.

First, let us set the following:

Definition 12. *Let $\preceq$ be the pointwise order on $\mathbb{N}^k$ (i.e. for all $m, n \in \mathbb{N}^K$, $m \preceq n$ iff $m_i \leqslant n_i$ for all $1 \leqslant i \leqslant K$). Of course $m \prec n$ holds exactly when $m \preceq n$ and $m_i < n_i$ for at least one $1 \leqslant i \leqslant K$. Finally, we set $m \prec_1 n$ iff $m \prec n$ and $\sum_{i=1}^K n_i - m_i = 1$ (i.e. they differ on exactly one coordinate).*

We will exploit the following:

Remark 9. *If $U \subseteq \mathbb{N}^K$ is infinite, then U contains an infinite ascending chain $m_0 \prec m_1 \prec m_2 \prec \ldots$.*

This is a consequence of König Lemma (KL): consider the directed acyclic graph $(U, \prec_1)$, indeed a K-branching tree; if there is no infinite ascending chain $m_0 \prec m_1 \prec m_2 \prec \ldots$, then in particular there is no infinite ascending chain $m_0 \prec_1 m_1 \prec_1 m_2 \prec_1 \ldots$ so the tree U has no infinite ascending chain; then by KL it is finite, contradicting the assumption.

Theorem 35 (Theorem 6)**.** *Let $k \in \mathbb{N}$ and $f : \mathbb{L}^k \to \mathbb{L}$ a tLs with matrix $\hat{f} : \mathbb{N}^k \to \mathbb{L}$. For all $0 < \epsilon < \infty$, there is $\mathcal{F}_\epsilon \subseteq \mathbb{N}^k$ such that:*

1) *$\mathcal{F}_\epsilon$ is finite*
2) *If $\mathcal{F}_\epsilon = \varnothing$ then $f(x) = +\infty$ for all $x \in \mathbb{L}^k$*
3) *If $f(x_0) = +\infty$ for some $x_0 \in [\epsilon, \infty)^K$ then $\mathcal{F}_\epsilon = \varnothing$*
4) *The restriction of f on $[\epsilon, \infty]^k$ coincides with the tropical polynomial*

$$P_\epsilon(x) := \min_{n\in\mathcal{F}_\epsilon} \{nx + \hat{f}(n)\}$$

where $nx := \sum_{i=1}^k n_i x_i$.

Proof. We let $\mathcal{F}_\epsilon$ to be the complementary in $\mathbb{N}$ of the set:

$$\{n \in \mathbb{N}^K \mid \textit{either } \hat{f}(n) = +\infty \textit{ or there is } m \prec n \textit{ s.t. } \hat{f}(m) \leqslant \hat{f}(n) + \epsilon\}.$$

In other words, $n \in \mathcal{F}_\epsilon$ iff $\hat{f}(n) < +\infty$ and for all $m \prec n$, one has $\hat{f}(m) > \hat{f}(n) + \epsilon$.

1). Suppose that $\mathcal{F}_\epsilon$ is infinite; then, using Remark 9, it contains an infinite ascending chain

$$\{m_0 \prec m_1 \prec \cdots\}.$$

By definition of $\mathcal{F}_\epsilon$ we have then:

$$+\infty > \hat{f}(m_0) > \hat{f}(m_1) + \epsilon > \hat{f}(m_2) + 2\epsilon > \cdots$$

so that $+\infty > \hat{f}(m_0) > \hat{f}(m_i) + i\epsilon \geqslant i\epsilon$ for all $i \in \mathbb{N}$. This contradicts the Archimedean property of $\mathbb{R}$.

2). We show that if $\mathcal{F}_\epsilon = \varnothing$, then $\hat{f}(n) = +\infty$ for all $n \in \mathbb{N}^K$. This immediately entails the desired result. We go by induction on the well-founded order $\prec$ over $n \in \mathbb{N}^K$:

- if $n = 0^K \notin \mathcal{F}_\epsilon$, then $\hat{f}(n) = +\infty$, because there is no $m \prec n$.

- if $n \notin \mathcal{F}_\epsilon$, with $n \neq 0^K$ then either $\hat{f}(n) = +\infty$ and we are done, or there is $m \prec n$ s.t. $\hat{f}(m) \leqslant \hat{f}(n) + \epsilon$. By induction $\hat{f}(m) = +\infty$ and, since $\epsilon < +\infty$, this entails $\hat{f}(n) = +\infty$.

3). If $f(x_0) = +\infty$ with $x_0 \in [\epsilon, \infty)^K$, then necessarily $\hat{f}(n) = +\infty$ for all $n \in \mathbb{N}^K$. Therefore, no $n \in \mathbb{N}^K$ belongs to $\mathcal{F}_\epsilon$.

4). We have to show that $f(x) = P_\epsilon(x)$ for all $x \in [\epsilon, +\infty]^K$. By 1), it suffices to show that we can compute $f(x)$ by taking the $\inf$, that is therefore a $\min$, only in $\mathcal{F}_\epsilon$ (instead of all $\mathbb{N}^K$). If $\mathcal{F}_\epsilon = \varnothing$ then by 2) we are done (remember that

$\min \varnothing := +\infty$). If $\mathcal{F}_\epsilon \neq \varnothing$, we show that for all $n \in \mathbb{N}^K$, if $n \notin \mathcal{F}_\epsilon$, then there is $m \in \mathcal{F}_\epsilon$ s.t. $\hat{f}(m) + mx \leqslant \hat{f}(n) + nx$. We do it again by induction on $\prec_1$:

- if $n = 0^K$, then from $\mathbf{n} \notin \mathcal{F}_\epsilon$, by definition of $\mathcal{F}_\epsilon$, we have $\hat{f}(n) = +\infty$ (because there is no $n' \prec n$). So any element of $\mathcal{F}_\epsilon \neq \varnothing$ works.

- if $n \neq 0^K$, then we have two cases: either $\hat{f}(n) = +\infty$, in which case we are done as before by taking any element of $\mathcal{F}_\epsilon \neq \varnothing$. Or $\hat{f}(n) < +\infty$, in which case (again by definition of $\mathcal{F}_\epsilon$) there is $n' \prec n$ s.t.

$$\hat{f}(n') \leqslant \hat{f}(n) + \epsilon. \tag{32}$$

Therefore we have (remark that the following inequalities hold also for the case $x = +\infty$):

$$\begin{aligned}
\hat{f}(n') + n'x &\leqslant \hat{f}(n) + \epsilon + n'x && \textit{by } (32) \\
&\leqslant \hat{f}(n) + (n - n')x + n'x && \textit{because } \epsilon \leqslant \min x \textit{ and } \min x \leqslant (n - n')x \\
&= \hat{f}(n) + nx.
\end{aligned}$$

Now, if $n' \in \mathcal{F}_\epsilon$ we are done. Otherwise $n' \notin \mathcal{F}_\epsilon$ and we can apply the induction hypothesis on it, obtaining an $m \in \mathcal{F}_\epsilon$ s.t. $\hat{f}(m) + mx \leqslant \hat{f}(n') + n'x$. Therefore this m works. ☐

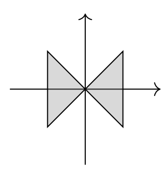
I. Section IV-B: proof of Theorem 7

In order to prove Theorem 7, we place ourselves in the general setting of locally convex topological vector spaces (LCTVS), and prove a general result about convex functions in this setting, adapting, in the main Proposition 36, an argument taken from [Proposition 4.4, [22]].

Definition 13. *A* normed semiring *is the data of a semiring R together with a* norm *on it, that is, a function $|.| : R \to \mathbb{R}$ satisfying the usual axioms of the absolute value function*[2]. *One can immediately check that if R is a ring, then any norm on it induces a metric $d(r, s) :=\mid r - s \mid$ on it (and thus also a topology). A* topological normed ring *is a normed ring which is a topological ring w.r.t. the topology induced by the norm distance.*

Definition 14. *Let R be normed semiring, let A be a topological R-module and let $C \subseteq A$. We say that C is* convex *iff for all $x, y \in C$ and for all $r, s \in R$ s.t. $\mid r \mid + \mid s \mid= 1$, we have $rx + sy \in C$. We say that C is* balanced *iff for all $x \in C$ and for all $r, s \in R$ s.t. $\mid r \mid + \mid s \mid= 1$, we have $rx, sx \in C$. We say that C is a* disk at $x \in A$ *iff it is of shape $C = x + D$ for some D both convex and balanced.*

Example 1. *An example of a balanced but not convex set in (the normed field) $\mathbb{R}^2$.*



Definition 15. *Let R be a normed field and let $\mathbb{V}$ be a topological R-vector space. We say that $\mathbb{V}$ is* locally convex *iff every $x \in \mathbb{V}$ admits a local basis consisting of disks at x. Since the topology of $\mathbb{V}$ is necessarily translation invariant, it is equivalent to only ask that 0 has such a local basis.*

The main result is the following:

Proposition 36. *Let $\mathbb{V}$ be a locally convex topological $\mathbb{R}$-vector space (where $\mathbb{R}$ is endowed with its usual absolute value and the induced topology by it). Fix $x \in \mathbb{V}$, a neighbourhood $V \subseteq \mathbb{V}$ of x and $f : V \to \mathbb{R} \cup \{+\infty, -\infty\}$. If V is convex, f is concave and lower bounded by a finite constant K, then f is continuous at x (w.r.t. the subspace topology on V).*

Proof. Since V is a neighbourhood of x, x admits an open neighbourhood $U \subseteq V$, and by local convexity of $\mathbb{V}$ there is a (non necessary open) disk $D \subseteq U$ at x. Now fix $\epsilon \in (0, 1)$ and $r, s \in \mathbb{R}$ with $0 \leqslant r \leqslant \epsilon$, $s = 1 - r$. Fix also $w \in D$. By the convexity of D we have $sx + rw \in D$, and thus the convexity of f entails that:

$$f(sx + rw) \geqslant sf(x) + rf(w) \geqslant sf(x) + rK = (1 - r)f(x) + rK$$

that is,

$$f(x) - f(sx + rw) \leqslant r(f(x) - K).$$

[2]I.e.: $|x| \geqslant 0$, $|x| = 0$ *iff* $x = 0$, $|xy| = |x|\,|y|$ and $|x + y| \leqslant |x| + |y|$.

Now remark that $x = \dfrac{1}{s+2r}(sx+rw) + \dfrac{r}{s+2r}(2x-w)$, where $\dfrac{1}{s+2r} + \dfrac{r}{s+2r} = 1$, $\dfrac{1}{s+2r} < 1$ and $2x - w \in D$ (because D is a disk at x). Therefore by the convexity of f we have:

$$f(x) \geqslant \frac{1}{s+2r} f(sx+rw) + \frac{r}{s+2r} f(2x-w) \geqslant \frac{1}{s+2r}(f(sx+rw) + rK)$$

that is,

$$f(sx+rw) - f(x) \leqslant r(f(x) - K).$$

We have thus shown that $\mid f(sx+rw) - f(x) \mid \leqslant r(f(x)-K)$ for all $w \in D$. Since this holds also for all $\epsilon \in (0,1)$, and due to the choice of r, s, the points of shape $sx + rw$ span D when w spans D and ϵ spans $(0,1)$. That is, we have shown that $\mid f(w) - f(x) \mid \leqslant r(f(x) - K)$ for all $w \in W$. Since $f(x) - K \geqslant 0$ and $r \leqslant \epsilon$, we have that $\exists \lim_{w \to x} f(w) = f(x)$. $\square$

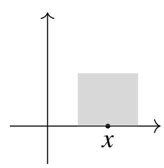
Now we show how to apply this argument to our tropical case $\mathbb{L}^X$, which is *not* a LCTVS (because it is not a vector space). First, we have:

Corollary 37. *Let $f : (\mathbb{R}^X_{>0}, \|.\|_\infty) \to (\overline{\mathbb{R}}_{\geqslant 0}, |.|)$ and $x \in \mathbb{R}^X_{>0}$. If there is a convex neighbourhood $V \subseteq \overline{\mathbb{R}}^X_{>0}$ of x s.t. $f_{\|V}$ is concave, then f is continuous at x.*

Proof. The $\mathbb{R}$-vector space $\mathbb{R}^X$ is topological w.r.t. the topology τ_∞ induced on it by the norm $\|.\|_\infty$, and it is clearly locally convex. Call τ_∞^+ the topology induced by $\|.\|_\infty$ on $\mathbb{R}^X_{>0}$. It is clear that it coincides with τ_∞^+. Since moreover $\mathbb{R}^X_{>0}$ is open in $(\mathbb{R}^X, \tau_\infty)$, the neighbourhood V of x in $(\mathbb{R}^X_{>0}, \|.\|_\infty)$ is also a neighbourhood of x in $(\mathbb{R}^X, \tau_\infty)$. We can therefore apply Proposition 36 to $f_{\|V}$ (which is lower bounded by 0 by definition of f) and obtain that $f_{\|V}$ is continuous at x w.r.t. the subspace topology τ_V induced by τ_∞ on V. But since V is contained in $\mathbb{R}^X_{>0}$, the topology τ_V coincides with the subspace topology induced on V by τ_∞^+. So f is continuous at x w.r.t. τ_∞^+. $\square$

One may wonder if the same proof as above makes it possible to state the previous corollary replacing $\mathbb{R}^X_{>0}$ with $\mathbb{R}^X_{\geqslant 0}$ (so, in particular, taking $x \in \mathbb{R}^X_{\geqslant 0}$). This is not possible because in the proof we crucially use that $\mathbb{R}^X_{>0}$ is open in $(\mathbb{R}^X, \tau_\infty)$, which is not the case of $\mathbb{R}^X_{\geqslant 0}$. In fact, this allowed us to say that V is a neighbourhood of x in $(\mathbb{R}^X, \tau_\infty)$, and therefore to be able to apply Proposition 36. Taking $\mathbb{R}^X_{\geqslant 0}$ instead, this is in general not true: we could have a neighbourhood of x w.r.t. the subspace topology on $\mathbb{R}^X_{\geqslant 0}$ induced by τ_∞, which does not contain any open neighbourhood of x w.r.t. τ_∞ (i.e. it is not a neighbourhood w.r.t. τ_∞).

Example 2. *An example of a neighbourhood of x w.r.t. the subspace topology on $\mathbb{R}^X_{\geqslant 0}$ induced by τ_∞, which is not a neighbourhood w.r.t. τ_∞.*



Finally we have the desired result Theorem 7:

Theorem 38 (Theorem 7)**.** *Tropical Laurent series $f : (\mathbb{R}^X_{\geqslant 0}, \|.\|_\infty) \to (\overline{\mathbb{R}}_{\geqslant 0}, |.|)$ are continuous on $\mathbb{R}^X_{>0}$ w.r.t. the norm $\|\cdot\|_\infty$.*

Proof. We know that tLs's are concave on all their domain. Therefore the result immediate follows by Corollary 37, since the topology induced by $\|.\|_\infty$ on $\mathbb{R}^X_{>0}$ coincides with the subspace topology induced by $(\mathbb{R}^X_{\geqslant 0}, \|.\|_\infty)$ on it. $\square$

Due to the previous discussion about the impossibility of stating Corollary 37 in the case where one of the coordinates of x is 0, the continuity of tLs on the hyperplanes $\mathcal{H}_a := \{x \in \mathbb{R}^X_{\geqslant 0} \mid x_a = 0\}$, for $a \in X$, must be treated separately. Similarly, the continuity at points with infinite coordinates must be treated separately as well. We left it for future investigations.

Remark 10. *We formulated the main argument (Proposition 36) in the general setting of LCTVS's. We did so just to show that it is actually a general argument, but we could have placed ourselves already in a more particular setting of our interest and carry on the proof. For instance, Theorem 14 will be proved by refining this kind of argument (Lemma 50), but not in the setting of LCTVS.*

J. Section IV-B: proof of Theorem 8

In order to prove such result we must quickly recall the theory of normed cones, which we do below.

Here we denote with $\overline{\mathbb{R}}_{\geqslant 0}$ the set $\mathbb{R}_{\geqslant 0} \cup \{\infty\}$ (the same as $\mathbb{L}$, but we prefer to explicit it here).

1) Normed cones:

Definition 16. *An* $\overline{\mathbb{R}}_{\geqslant 0}$-cone *is a commutative* $\overline{\mathbb{R}}_{\geqslant 0}$*-semimodule with cancellative addition*[3].

In [68] cones are required to also have "strict addition", meaning that $x + y = 0 \Rightarrow x = y = 0$. We do not add this requirement since it will automatic hold when considering normed cones.

Remark 11. *The addition of a cone P (which forms a commutative monoid) turns P into a poset by setting:*

$$x \leqslant y \textit{ iff } y = x + z, \textit{ for some } z \in P.$$

By the cancellative property, when such z exists it is unique, and we denote it by $y - x$. Such order is called the cone-order *on P.*

Definition 17. *A* normed $\overline{\mathbb{R}}_{\geqslant 0}$-cone *is the data of a $\overline{\mathbb{R}}_{\geqslant 0}$-cone together with a $\leqslant$-monotone*[4] *norm*[5] *on it.*

In, e.g. [34], a normed $\overline{\mathbb{R}}_{\geqslant 0}$-cone is simply called a cone.

Remark that in a normed $\overline{\mathbb{R}}_{\geqslant 0}$-cone, by monotonicity of the norm, we have: $\|x + y\| = 0 \Rightarrow x = y = 0$. Therefore, as already mentioned, in a normed cone we have: $x + y = 0 \Rightarrow \|x + y\| = 0 \Rightarrow x = y = 0$, that is, addition is strict.

Example 3. $\overline{\mathbb{R}}^X_{\geqslant 0}$ *is a normed cone with the norm* $\|x\|_\infty := \sup\limits_{a \in X} x_a \in \overline{\mathbb{R}}_{\geqslant 0}$.

Remark 12. *The cone-order on $\overline{\mathbb{R}}^X_{\geqslant 0}$ is the pointwise usual order on $\overline{\mathbb{R}}_{\geqslant 0}$.*

Remark also that tLs have no reason to be linear nor sublinear.

Remark 13. *tLs are monotone w.r.t. to the cone order on its domain and codomain. This is clear, using the definition of tLs, because $\mu x \leqslant \mu y$ if $x \leqslant y$.*

Remark 14. *The cone-order on $\overline{\mathbb{R}}^X_{\geqslant 0}$ makes it into a dcpo with least element 0.*

The following is a usual domain theoretical lemma.

Lemma 39. *Let P be a poset and X, I sets. Fix the pointwise order on P^X.*

1) *Let $x^i \in P^X$ for $i \in I$. If $\bigvee\limits_{i \in I} x^i$ exists P^X, then $\bigvee\limits_{i \in I} x^i_a$ exists in P for all $a \in X$, and we have: $\bigvee\limits_{i \in I} x^i_a = \left(\bigvee\limits_{i \in I} x^i\right)_a$.*
2) *Let $x^i_a \in P$ for $i \in I, a \in X$. If $\bigvee\limits_{i \in I} x^i_a$ exists P for all $a \in X$, then $\bigvee\limits_{i \in I} x^i$ exists in P^X, and we have: $\left(\bigvee\limits_{i \in I} x^i\right)_a = \bigvee\limits_{i \in I} x^i_a$.*

A *directed net* in a poset P with indices in a set I is a function $s : I \to P$, denoted by $(s_i)_{i \in I}$, s.t. its image is directed. We say that a directed net in P *admits a sup* iff its image admits a sup in P. We say that a directed net s in a normed cone is *bounded* iff the set $\{\|s_i\| \mid i \in I\}$ is bounded in $\mathbb{R}_{\geqslant 0}$.

Remember the definition of Scott-continuity:

Definition 18. *A function $f : P \to P'$ between posets is* Scott-continuous *iff for all directed net $(s_i)_i$ in P admitting a sup, we have $\exists \bigvee\limits_i f(s_i) = f(\bigvee\limits_i s_i)$ in P'.*

The fundamental result in order to prove Theorem 8 is the following, taken from [68].

Proposition 40. *Let P be a normed $\overline{\mathbb{R}}_{\geqslant 0}$-cone s.t. every bounded directed net in P admits a sup. Let $(v_i)_{i \in I}$ be a directed net in P with an upper bound $v \in P$. Then $\exists \bigvee\limits_{i \in I} v_i \in P$ and, if $\inf\limits_{i \in I} \|v - v_i\| = 0$, one has: $\bigvee\limits_{i \in I} v_i = v$.*

Proof. Remark that $v - v_i$ exists in P by hypothesis and so does $\bigvee\limits_{i \in I} v_i$, thanks to the monotonicity of the norm. Now, since $v \geqslant v_i$ for all i, we have that $v \geqslant \bigvee\limits_{i \in I} v_i$, and so $v - \bigvee\limits_{i \in I} v_i$ exists in P. Fix $i \in I$. Since $v_i \leqslant \bigvee\limits_{i \in I} v_i$, then $v - \bigvee\limits_{i \in I} v_i \leqslant v - v_i$ and, by monotonicity of the norm, $\|v - \bigvee\limits_{i \in I} v_i\| \leqslant \|v - v_i\|$. Since this holds for all $i \in I$, we have: $0 \leqslant \|v - \bigvee\limits_{i \in I} v_i\| \leqslant \inf\limits_{i \in I} \|v - v_i\| = 0$, where the last equality holds by hypothesis. Thus $\|v - \bigvee\limits_{i \in I} v_i\| = 0$, i.e. $v = \bigvee\limits_{i \in I} v_i$. □

[3] I.e.: $x + y = x + y' \Rightarrow y = y'$.

[4] I.e.: $x \leqslant y \Rightarrow \|x\| \leqslant \|y\|$. Remark that requiring this property (for all x, y) is equivalent to requiring that $\|x\| \leqslant \|x + y\|$ for all x, y.

[5] A *norm* on a $\overline{\mathbb{R}}_{\geqslant 0}$-abstract cone P is a map $\|.\| : P \to \overline{\mathbb{R}}$ satisfying the usual axioms of norms: $\|x\| \geqslant 0$, $\|x\| = 0 \Rightarrow x = 0$, $\|rx\| = r\|x\|$ and $\|x + y\| \leqslant \|x\| + \|y\|$.

Definition 19. *A normed $\overline{\mathbb{R}}_{\geqslant 0}$-cone P is* Scott-complete *iff its norm is Scott-continuous (where the codomain $\overline{\mathbb{R}}_{\geqslant 0}$ is endowed with its usual order) and every bounded directed net in P admits a sup. This is equivalent to asking that its closed unit ball is a dcpo.*

Proposition 41. *The normed cone $\overline{\mathbb{R}}^X_{\geqslant 0}$ is Scott-complete.*

Proof. $\overline{\mathbb{R}}^X_{\geqslant 0}$ being a dcpo, all the existences of sup's that we should check do automatically hold, so we only have to show that $\bigvee$ and $\|.\|_\infty$ commute: $\|\bigvee_i x_i\|_\infty = \sup_a \sup_i (x_i)_a = \sup_i \sup_a (x_i)_a = \bigvee_i \|x_i\|_\infty$. □

We finally obtain the desired:

Theorem 42 (Theorem 8)**.** *All tLs $f : \overline{\mathbb{R}}^X_{\geqslant 0} \to \overline{\mathbb{R}}^Y_{\geqslant 0}$ are Scott-continuous on $\mathbb{R}^X_{\geqslant 0}$ w.r.t. the cone-orders on its domain and codomain.*

Proof. Let $(x_i)_i$ a directed net in $\mathbb{R}^X_{\geqslant 0}$ s.t. $\bigvee_i x^i$ exists in $\mathbb{R}^X_{\geqslant 0}$. Then $\inf_i \|\bigvee_i x^i - x^i\|_\infty = 0$, where $\bigvee_i x^i - x^i$ exists because $\bigvee_i x^i \geqslant x^i$ for all i. Since f is $\|.\|_\infty$-continuous on $\mathbb{R}^X_{\geqslant 0}$, then $\inf_i \|f(\bigvee_i x^i) - f(x^i)\|_\infty = 0$, where $f(\bigvee_i x^i) - f(x^i)$ exists because $f(\bigvee_i x^i) \geqslant f(x^i)$ for all i being f monotone (Remark 13). We can therefore apply Proposition 40 to the directed net $(f(x^i))_i$ in $\overline{\mathbb{R}}^Y_{\geqslant 0}$, obtaining that $\bigvee_i f(x^i)$ exists in $\overline{\mathbb{R}}^Y_{\geqslant 0}$ and it coincides with $f(\bigvee_i x^i)$. □

In the main paper we mention that $\mathbb{R}^X_{\geqslant 0}$ is a Scott-complete dcpo. We did not state it as a theorem because it follows from standard and well-known domain theoretical considerations. Let us prove it here anyway.

Recall the definition of Scott-complete dcpo:

Definition 20. *Let P be a dcpo and define a relation by: $x << y$ iff for all directed $D \subseteq P$, if $y \leqslant \bigvee D$, then $x \leqslant d$, for some $d \in D$. Call $\downdownarrows y := \{x \in P \mid x << y\}$. A dcpo P is* Scott-continuous *iff $\downdownarrows x$ is directed and $x = \bigvee \downdownarrows x$ for all $x \in P$.*

Remark 15. *In any dcpo we have: $\downdownarrows x \subseteq \downarrow x$. This immediately follows by considering the directed set $\{x\}$.*

Lemma 43. *In $\overline{\mathbb{R}}^X_{\geqslant 0}$, every set $\downdownarrows x$ is directed.*

Proof. It is immediate that $0 \in \downdownarrows x$, so it is non-empty. Now let $y, y' << x$. Since we are in $\overline{\mathbb{R}}^X_{\geqslant 0}$, there is $y \vee y' \in \overline{\mathbb{R}}^X_{\geqslant 0}$. So we only have to show that $y \vee y' << x$. For that, let D be a directed set in $\overline{\mathbb{R}}^X_{\geqslant 0}$ s.t. $x \leqslant \bigvee D$. Since $y, y' << x$ we find $d, d' \in D$ s.t. $y \leqslant d$, $y' \leqslant d'$. Since D is directed, there is $\hat{d} \in D$ s.t. $y \leqslant d \leqslant \hat{d} \geqslant d' \geqslant y'$. But then, by definition of sup, it must be $\hat{d} \geqslant y \vee y'$ and we are done. □

Remark 16. *The following is a known property (that we will not use): let P be a complete normed cone which is a dcpo w.r.t. its cone-order. Then P is Scott-continuous as a dcpo iff its closed unit ball is Scott-continuous as a dcpo.*

Remark 17. *Consider $\overline{\mathbb{R}}_{\geqslant 0}$ with its usual order (which coincides with its cone-order). It is easily seen (and well known) that $y << x$ iff either $y = 0$ or $y < x$. This immediately implies that $\overline{\mathbb{R}}_{\geqslant 0}$ is Scott-continuous, since $x = \sup_{y<x} y$.*

Lemma 44. *Let $x, y \in \overline{\mathbb{R}}^X_{\geqslant 0}$ (considered as a dcpo with its cone order, which is the pointwise one). Fix $a \in X$. If $y_a << x_a$ and $y_c = 0$ for all $c \neq a$, then $y << x$.*

Proof. Towards a contradiction, assume that there is a directed set D in $\overline{\mathbb{R}}^X_{\geqslant 0}$ with $x \leqslant \bigvee D$ and s.t. $y \nleqslant d$ for all $d \in D$. Call $D_a := \{d_a \mid d \in D\}$ and remark that it is directed, because D is. Also, by Lemma 39, $\bigvee D_a = (\bigvee D)_a$. Therefore from $y_a << x_a \leqslant \bigvee D_a$ we obtain a $d \in D$ s.t. $y_a \leqslant d_a$. By the absurd hypothesis we have $y \nleqslant d$, so there must be $c \in X$ s.t. $y_c \nleqslant d_c$, i.e. (because real numbers are totally ordered) $y_c > d_c$. Therefore it must be $c \neq a$. But then we have $0 = y_c > d_c \geqslant 0$, contradiction. □

Lemma 45. *In $\overline{\mathbb{R}}^X_{\geqslant 0}$ we have: if $y << x$ then $y_a << x_a$ for all $a \in X$.*

Proof. Fix $a \in X$. Let D directed set in $\overline{\mathbb{R}}_{\geqslant 0}$ s.t. $x_a \leqslant \bigvee D$. We look for a $d \in D$ s.t. $y_a \leqslant d$. For $d \in D$, let $x^{a,d} \in \overline{\mathbb{R}}^X_{\geqslant 0}$ defined by $x^{a,d}_c := x_c$ if $c \neq a$ and $x^{a,d}_a := d$. Let $D^x_a := \{x^{a,d} \mid d \in D\}$. By Lemma 39 D^x_a admits sup in $\overline{\mathbb{R}}^X_{\geqslant 0}$ and it is $(\bigvee D^x_a)_c = x_c$ if $c \neq a$ and $(\bigvee D^x_a)_a = \bigvee D$. Hence, $x \leqslant \bigvee D^x_a$. If we prove that D^x_a is directed, we are done: indeed, since $y << x$, there is $d \in D$ s.t. $y \leqslant x^{a,d}$ and thus, in particular, $y_a \leqslant x^{a,d}_a = d$. Let us finally prove that D^x_a is directed: it is clearly non-empty, since D is. Let now $d, d' \in D$. We want to show that there is $\hat{d} \in D$ s.t. $x^{a,d} \leqslant x^{a,\hat{d}} \geqslant x^{a,d'}$. But since D is directed, there is $\hat{d} \in D$ s.t. $d \leqslant \hat{d} \geqslant d'$, and therefore $x^{a,d}_c = x_c = x^{a,\hat{d}}_c = x_c = x^{a,d'}_c$ for all $c \neq a$, and $x^{a,d}_a = d \leqslant \hat{d} = x^{a,\hat{d}}_a = \hat{d} \geqslant d' = x^{a,d'}_a$. □

Lemma 46. *In $\overline{\mathbb{R}}_{\geqslant 0}^X$ we have:* $\bigvee_{y << x_a} y = \bigvee_{y << x} y_a$.

Proof. If we show that $\downarrow x_a = \{y_a \mid y \in \downarrow x\}$, then we are done because in the statement we are taking their sup's. The inclusion ($\supseteq$) immediately follows from Lemma 45. For ($\subseteq$), let $d << x_a$. Then the $y \in \overline{\mathbb{R}}_{\geqslant 0}^X$ defined by $y_c := 0$ if $c \neq a$ and $y_a := d$, is s.t. $y \in \downarrow x$ by Lemma 44. Thus, $d \in \{y_a \mid y \in \downarrow x\}$. ☐

Corollary 47. *The dcpo $\overline{\mathbb{R}}_{\geqslant 0}^X$ is Scott-continuous.*

Proof. The fact that $\downarrow x$ is directed is given by Lemma 43. The fact that $x = \bigvee \downarrow x$ is given by the following equalities: $x_a = \bigvee_{y << x_a} y = \bigvee_{y << x} y_a = \left(\bigvee_{y << x} y \right)_a = (\bigvee \downarrow x)_a$. The first equality follows from Remark 17, the second one from Lemma 46, the third one from Lemma 39. ☐

K. Proofs from Section IV-C

1) Lipschitz continuity: We need a first, preliminary, lemma:

Lemma 48. *Let $u, v : I \to \mathbb{L}$ and suppose $|u(i) - v(i)| \leqslant \delta$, for all $i \in I$. Then $|\inf_{i \in I} u(i) - \inf_{i \in I} v(i)| \leqslant \delta$.*

Proof. Let $A = \inf_{i \in I} u(i)$ and $B = \inf_{i \in I} v(i)$ and suppose $A \geqslant B$. Suppose by way of contradiction $|A - B| > \delta$; then there exists $i \in I$ such that $v(i) < A$ and $|A - v(i)| > \delta$. Indeed, otherwise we would have $|A - B| = \sup\{|A - v(i)| \mid v(i) \leqslant A\} \leqslant \delta$. Now, from $|A - v(i)| > \delta$ and $v(i) < A$ we deduce that $|u(j) - v(i)| > \delta$ for all $j \in I$, and thus in particular that $|u(i) - v(i)| > \delta$, against the assumption. We conclude then $|A - B| \leqslant \delta$. In case $B \geqslant A$, we can argue in a similar way. ☐

Proposition 49. *All tropical linear functions $f : \mathbb{L}^X \to \mathbb{L}^Y$ are non-expansive.*

Proof. Let $f(x)_b = \inf_a\{\widehat{f}_{a,b} + x_a\}$. For all $a \in X$ we have that $|(\widehat{f}_{a,b} + x_a) - (\widehat{f}_{a,b} + y_a)| = |x_a - y_a| \leqslant \|x - y\|_\infty$. Using Lemma 48, we deduce then $|f(x)_b - f(y)_b| = |\inf_a\{\widehat{f}_{a,b} + x_a\} - \inf_a\{\widehat{f}_{a,b} + y_a\}| \leqslant \|x - y\|_\infty$. ☐

Lemma 50. *Let $f : [0, \infty)^X \to Q$ be concave, monotone increasing, and continuous. Let $x \neq y \in [0, \infty)^X$, with $\|x - y\|_\infty < \infty$, and let $S(x, y) = \{\alpha x + (1 - \alpha) y \mid \alpha \in [0, 1]\}$ be the segment generated by x and y. Then f is Lipschitz-continuous over $S(x, y)$.*

Proof. Let us prove the lemma under the assumption that for all $a \in X$, $y_a - x_a \geqslant 1$. From the fact that the claim holds under the assumption, we can deduce the claim of the lemma: indeed for $\alpha \in (0, 1)$ large enough we have that $y' := \frac{y - \alpha x}{1 - \alpha}$ is such that $y \in S(x, y')$ (and thus $S(x, y) \subseteq S(x, y')$) and $y'_a - x_a \geqslant 1$. Hence from our proof we deduce that f is Lipschitz-continuous over $S(x, y')$, and thus a fortiori over $S(x, y)$ too.

Since f is continuous over $[0, \infty)^X$ and $S(x, y)$ is compact, f admits a maximum MAX over $S(x, y)$. For all $z < z' \in S(x, y)$, let $M(z, z') \in Q^X$ be defined by

$$M(z, z')_a = \frac{f(z') - f(z)}{z'_a - z_a}$$

Observe that

$$M(x, y)_a = \frac{f(y) - f(x)}{y_a - x_a} \leqslant f(y) - f(x) \leqslant \mathrm{MAX}$$

using the fact that $y_a - x_a \geqslant 1$.

We now claim that $M(z, z')$ is contravariant in both z and z'. Indeed suppose $z \leqslant z'' < z'$, so that $z = \lambda z' + (1 - \lambda) z''$ for some $\lambda \in (0, 1)$. Then, using the fact that f is concave, we have

$$\begin{aligned} M(z, z')_a &= \frac{f(z') - f(\lambda x + (1 - \lambda) z'')}{z'_a - \lambda x_a - (1 - \lambda) z''_a} \\ &\geqslant \frac{f(z') - \lambda f(z') - (1 - \lambda) f(z'')}{z'_a - \lambda z'_a - (1 - \lambda) z''_a} \\ &= \frac{(1 - \lambda)(f(z') - f(z''))}{(1 - \lambda) z'_a - z''_a} = M(z'', z') \end{aligned}$$

In a similar way it is shown that for $z < z'' \leqslant z'$, $M(z, z') \leqslant M(z, z'')$.

Therefore, for all $z < z' \in S(x, y)$, we have that $M(z, z')_a \leqslant M(x, z')_a \leqslant M(x, y)_a \leqslant \mathrm{MAX}$. From this, using the fact that f is monotone increasing, we deduce that $|f(z') - f(z)| = f(z') - f(z) \leqslant \mathrm{MAX} \cdot |z'_a - z_a|$ and thus that

$$|f(z') - f(z)| \leqslant \mathrm{MAX} \cdot \|z' - z\|_\infty$$

that is, that f is MAX-Lipschitz over $S(x,y)$. $\square$

Proposition 51. *Let $f:[0,\infty)^X \to Q$ be concave, monotone increasing, and continuous. For all $\epsilon \in (0,\infty)$ and $x \in [0,\infty)^X$, f is Lipschitz-continuous over the open ball $B(x,\epsilon)$.*

Proof. Let MAX indicate the maximum of f over $B(x,\epsilon)$. Let $y,z \in B(x,\epsilon)$; then $\|y-z\|_\infty \leqslant 2\epsilon < \infty$, so by the lemma above f is K-Lipschitz over the segment $S(y,z)$ for some $K \leqslant \mathrm{MAX}$, so we deduce $|f(y)-f(z)| \leqslant \mathrm{MAX}\cdot\|y-z\|_\infty$. $\square$

Theorem 52 (local Lipschitz-continuity)**.** *Let $f:[0,\infty)^X \to \mathbb{L}$ be concave, monotone increasing, and continuous. Then f is locally Lipschitz-continuous.*

Proof. For all $x \in [0,\infty)^X$, f is Lipschitz-continuous over the open set $B(x,1)$. $\square$

2) Characterizations of the functional metric :

Proposition 53. *For all maps $f,g:\mathbb{L}^X \to \mathbb{L}^Y$, $\|\widehat{f}-\widehat{g}\|_\infty = \sup\{\|f(x)-g(x)\|_\infty \mid x \in Q\langle X\rangle\}$.*

Proof of Proposition 53. For one side, suppose for all $\mu \in \mathcal{M}_{\mathrm{fin}}(X)$, $b \in Y$, $|\widehat{f}_{\mu,b}-\widehat{g}_{\mu,b}| \leqslant \delta$. Then for all $x \in Q\langle X\rangle$ and $b \in Y$, $|f(x)(b)-g(x)(b)| \leqslant \delta$. Indeed, $|f(x)(b)-g(x)(b)| = |(\inf_\mu \mu x + \widehat{f}_{\mu,b}) - (\inf_\mu \mu x + \widehat{g}_{\mu,b})|$. Since $|\mu x + \widehat{f}_{\mu,b} - \mu x - \widehat{g}_{\mu,b}| = |\widehat{f}_{\mu,b}-\widehat{g}_{\mu,b}| \leqslant \delta$, by Lemma 48 we conclude $|f(x)(b)-g(x)(b)| \leqslant \delta$.

For the other side, suppose for some $\mu \in \mathcal{M}_{\mathrm{f}}(X)$ and $b \in Y$, $|\widehat{f}_{\mu,b}-\widehat{g}_{\mu,b}| > \epsilon$; then, by letting $e_\mu \in Q\langle !X\rangle$ be the (tropical) characteristic function of μ, we have $|f(e_\mu)(b)-g(e_\mu)(b)| = |(\inf_{\mu'} \widehat{f}_{\mu',b}+\mu'(e_\mu)) - (\inf_{\mu'} \widehat{g}_{\mu',b}+\mu'(e_\mu))| = |\widehat{f}_{\mu,b}-\widehat{g}_{\mu,b}| > \epsilon$, so we deduce that $\sup\{\|f(x)-g(x)\|_\infty \mid x \in Q\langle X\rangle\} > \epsilon$. $\square$

The second characterization relates distances with the Taylor expansion. Let us briefly discuss the latter, first. For all $f:\mathbb{L}^X \to \mathbb{L}^Y$, let $\delta^{(n)}f:(\mathbb{L}^X)^n \to \mathbb{L}^Y$ indicate the n-linear function given by

$$\delta^{(n)}f(x^n) = D^{(n)}f(x^n,\infty)$$

Notice that $\widehat{\delta^{(n)}f} \in Q^{X^n\times Y}$ satisfies $\widehat{\delta^{(n)}f}_{a_1,\dots,a_n,b} = \widehat{f}_{[a_1,\dots,a_n],b}$. In other words, $\delta^{(n)}f$ precisely captures the behavior of f when applied to multisets of length n. Moreover, the full behavior of f can be recovered from the functions $\delta^{(n)}f$ using the Taylor expansion which, in its tropical form, reads as:

$$f(x) = \inf_n \left\{ D^{(n)}f(x^n,\infty) \right\} = \inf_n \left\{ \delta^{(n)}f(x^n) \right\}$$

Proposition 54. *For all $f,g:\mathbb{L}^X \to \mathbb{L}^Y$,*

$$\|\widehat{f}-\widehat{g}\|_\infty = \sup_n \|\widehat{\delta^{(n)}f} - \widehat{\delta^{(n)}g}\|_\infty$$

Proof. Let us first show that $\|\widehat{f}-\widehat{g}\|_\infty \leqslant \epsilon$ implies $\|\delta^{(n)}f-\delta^{(n)}g\|_\infty \leqslant \epsilon$, for all $n \in \mathbb{N}$. Notice that $\widehat{\delta^{(n)}f} \in \mathbb{L}^{X^n\times Y}$ satisfies $\widehat{\delta^{(n)}f}_{a_1,\dots,a_n,b} = \widehat{f}_{[a_1,\dots,a_n],b}$. Hence from $\|\widehat{f}-\widehat{g}\| \leqslant \epsilon$ it follows that for all $\mu = [a_1,\dots,a_n]$ and $b \in Y$, $|\widehat{f}_{\mu,b}-\widehat{g}_{\mu,b}| \leqslant \epsilon$, so we deduce that $\|\widehat{\delta^{(n)}f}-\widehat{\delta^{(n)}g}\| \leqslant \epsilon$.

For the converse direction suppose that, for all $n \in \mathbb{N}$, $\|\widehat{\delta^{(n)}f}-\widehat{\delta^{(n)}g}\| \leqslant \epsilon_n$. Then, since the family of coefficients $F_{n,\mu,b} = (\widehat{\delta^{(n)}f})_{a_1,\dots,a_n,b}$ (where $\mu = [a_1,\dots,a_n]$) is in bijection with the coefficients $\widehat{f}_{\mu,b}$, we deduce that $|\widehat{f}_{\mu,b}-\widehat{g}_{\mu,b}| \leqslant \epsilon_{\sharp\mu}$, and thus that $\|\widehat{f}-\widehat{g}\|_\infty \leqslant \sup_n \epsilon_n$. $\square$

L. Proofs from Section VII

Remark 18. *Given $f : X \to \mathbb{L}$, f is a functor precisely when for all $x, y \in X$ $f(x) \leqslant \inf_{x' \in X} f(x') + X(x', x)$. Indeed, if f is a functor then $f(x) \leqslant f(x') + X(x', x)$, since $f(x) - f(x') = \mathbb{L}(f(x'), f(x)) \leqslant X(x', x)$. Conversely, if $f(x) \leqslant f(x') + X(x', x)$ holds for all x', then $\mathbb{L}(f(x), f(x')) = f(x') - f(x) \leqslant X(x', x)$.*

Remark 19 (Yoneda embedding)**.** *The* Yoneda embedding *is the faithful functor $\mathbf{Y} : X \to [X, \mathbb{L}]$ given by $\mathbf{Y}(x)(y) = X(y, x)$. The functoriality and faithfulness of $\mathbf{Y}$ follow from*

$$[X, \mathbb{L}](\mathbf{Y}(x), \mathbf{Y}(x')) = X(x, x') \tag{Yoneda}$$

which is proved as follows: for all $y \in X$ we have

$$\begin{aligned}\mathbb{L}(\mathbf{Y}(x)(y), \mathbf{Y}(x')(y)) &= \mathbf{Y}(x')(y) - \mathbf{Y}(x)(y) \\ &= X(y, x') - X(y, x) \leqslant X(x, x')\end{aligned}$$

where the last step follows from $X(y, x) + X(x, x') \geqslant X(y, x')$. From this we deduce that $[X, \mathbb{L}](\mathbf{Y}(x), \mathbf{Y}(x')) = \sup_{y \in X} \mathbb{L}(\mathbf{Y}(x)(y), \mathbf{Y}(x')(y)) \leqslant X(x, x')$. For the converse direction, we have

$$\begin{aligned}X(x, x') &= X(x, x') - 0 \\ &= X(x, x') - X(x, x) \\ &= \mathbf{Y}(x')(x) - \mathbf{Y}(x)(x) = \mathbb{L}(\mathbf{Y}(x)(x), \mathbf{Y}(x')(x)) \\ &\leqslant [X, \mathbb{L}](\mathbf{Y}(x), \mathbf{Y}(x'))\end{aligned}$$

Remark 20. *The* opposite Yoneda embedding *is the faithful functor $\mathbf{Y}^{\mathrm{op}} : X \to [X, \mathbb{L}]^{\mathrm{op}}$ given by $\mathbf{Y}^{\mathrm{op}}(x)(y) = X(x, y)$. The functoriality and faithfulness of $\mathbf{Y}^{\mathrm{op}}$ follow from*

$$[X, \mathbb{L}](\mathbf{Y}^{\mathrm{op}}(x), \mathbf{Y}^{\mathrm{op}}(x')) = X(x', x) \tag{Yoneda$^{\mathrm{op}}$}$$

which is proved similarly to the case of $\mathbf{Y}$.

1) completeness: Composition with $r : X \nrightarrow Y$ yields a functor

$$r \cdot \_ : \mathbb{L}\mathsf{Rel}(A, X) \longrightarrow \mathbb{L}\mathsf{Rel}(A, Y)$$

which has a right-adjoint

$$\_ \multimapinv r : \mathbb{L}\mathsf{Rel}(A, Y) \longrightarrow \mathbb{L}\mathsf{Rel}(A, X)$$

given, for $s : A \nrightarrow Y$ by

$$(s \multimapinv r)(a, x) = \sup_{y \in Y} s(a, y) \dot{-} r(x, y)$$

For a $\mathbb{L}$-category X, we write $x \simeq y$ when $X(x, y) = X(y, x) = 0$. $\simeq$ coincides with $=$ precisely when x is skeletal.

Proposition 55. *Let X be a skeletal $\mathbb{L}$-category. Then X is complete iff the Yoneda embedding has a left-adjoint.*

Proof. For all $x \in [X, \mathbb{L}]$, let $\sup x$ be defined as a weighted colimit via

$$X(\sup x, b) = \sup_{a \in X} X(a, b) - x_a$$

that is, $\sup x = \operatorname{colim}(x, \mathrm{id}_X)$, where x is seen as a distributor $x : \{\star\} \nrightarrow X$.

Let us check that $\sup : [X, \mathbb{L}] \to X$ is a functor. First, let us check the inequality

$$(X \multimapinv y) \cdot (y \multimapinv x) \geqslant (X \multimapinv x) \tag{33}$$

as follows:

$$\begin{aligned}((X \multimapinv y) \cdot (y \multimapinv x))(a) &= \left(\sup_b X(b, a) - y_b\right) + \left(\sup_b y_b - x_b\right) \\ &\geqslant \sup_b (X(b, a) - y_b) + (y_b - x_b) \\ &= \sup_b (X(b, a) - y_b + y_b) - x_b \\ &= \sup_b X(b, a) - x_a \\ &= (X \multimapinv x)(a)\end{aligned}$$

From (33) we deduce immediately the inequality below:

$$y \multimap\!\!\!\!\!\circ\; x \geqslant (X \circ\!\!- x) \circ\!\!- (X \circ\!\!- y) \tag{34}$$

and we can now compute:

$$\begin{aligned}
[X, \mathbb{L}](x, y) &= \sup_{a \in X} y_a - x_a \\
&\overset{(34)}{\geqslant} \sup_{a \in X} \left(\sup_{b \in X} X(b, a) - x_a \right) - \left(\sup_{b \in X} X(b, a) - y_a \right) \\
&= \sup_{a \in X} X(\sup x, a) - X(\sup y, a) \\
&= [X, \mathbb{L}](\mathbf{Y}^{\mathrm{op}}(\sup y), \mathbf{Y}^{\mathrm{op}}(\sup x)) && (\text{Yoneda}^{\mathrm{op}}) \\
&= X(\sup x, \sup y)
\end{aligned}$$

Then for all $x \in X$, $\sup \mathbf{Y}(x) \simeq x$. Indeed we have

$$\begin{aligned}
X(\sup \mathbf{Y}(x), y) &= \sup_{z \in X} X(z, y) - \mathbf{Y}(x)(z) \\
&= \sup_{z \in X} X(z, y) - X(z, x) \\
&= X(x, y)
\end{aligned}$$

Moreover, for all $x \in [X, \mathbb{L}]$, we have $\mathbf{Y}(\sup x) \geqslant x$:

$$\begin{aligned}
\mathbf{Y}(\sup x)(a) &= X(a, \sup x) \\
&= X(\sup(\mathbf{Y}(a)), \sup x) \\
&\geqslant [X, \mathbb{L}](\mathbf{Y}(a), x) \\
&\geqslant x_a - \mathbf{Y}(a)(a) \\
&= x_a - X(a, a) = x_a
\end{aligned}$$

Conversely, if sup is well-defined and adjoint to $\mathbf{Y}$, then given $\Phi : Y \nrightarrow \{\star\}$ and $f : Y \to X$, we can define $\mathrm{colim}(\Phi, f) := \sup \Psi$, where $\Psi = f^{\circ} \cdot \Phi : X \nrightarrow \{\star\}$, since

$$\begin{aligned}
X(\sup \Psi, y) &= \sup_{z \in X} X(z, y) - \Psi(z) \\
&= \sup_{z \in X} X(z, y) - \inf_{y \in Y} X(z, f(y)) + \Phi(y) \\
&= \sup_{z \in X} \sup_{y \in Y} X(z, y) - X(z, f(y)) - \Phi(y) \\
&= \sup_{z \in X} X(f(z), y) - \Phi(z)
\end{aligned}$$

□

Definition 21 (MacNeill Completion). *Let X be a $\mathbb{L}$-category. For all $f : \{\star\} \nrightarrow X$ and $g : X \nrightarrow \{\star\}$, let $f \supset g$ iff $f = X \circ\!\!- g$ and $g = f \multimap X$. The* MacNeill completion *of X is the $\mathbb{L}$-category $\mathbf{M}(X)$ made of those $f : \{\star\} \nrightarrow X$ such that $f \supset g$ for some $g : X \nrightarrow \{\star\}$, with $\mathbf{M}(X)(f, f') = [X, \mathbb{L}](f, f')$.*

Observe that if $f \supset g$, then $f = X \circ\!\!- (f \multimap X)$, i.e.:

$$f(x) = \sup_{y \in X} \inf_{z \in X} X(x, y) - X(y, z) + f(z) \tag{COH}$$

Proposition 56. *Let X be a skeletal $\mathbb{L}$-category. If X is complete, then $\mathbf{Y}$ is an isomorphism between X and $\mathbf{M}(X)$.*

Proof. For all $x \in X$, one can check that $\mathbf{Y}(x) \in \mathbf{M}(X)$. Indeed, we can check that $\mathbf{Y}(a) \supset \mathbf{Y}^{\mathrm{op}}(a)$:

$$\begin{aligned}
\mathbf{Y}(a)(b) &= X(b, a) \\
&= \sup_{c \in X} X(b, c) - X(a, c) \\
&= \sup_{c \in X} X(b, c) - \mathbf{Y}^{\mathrm{op}}(a)(c) \\
&= (X \circ\!\!- \mathbf{Y}^{\mathrm{op}}(a))(b)
\end{aligned}$$

Since $\sup \mathbf{Y}(a) \simeq a$ holds, it suffices to show that if $x \supset y$, then $\mathbf{Y}(\sup x) = x$:

$$\begin{aligned}\mathbf{Y}(\sup x)(a) &= X(a, \sup x) \\ &= \sup_{b \in X} X(a,b) - X(\sup x, b) \\ &= \sup_{b \in X} \inf_{c \in X} X(a,b) - X(b,c) + x_c \\ &\overset{\text{(COH)}}{=} x_a\end{aligned}$$

□

From now on, all $\mathbb{L}$-categories will be tacitly assumed to be skeletal. As observed in Section VI, any $\mathbb{L}$-category X can be made skeletal by a suitable quotient.

M. Tensors and $\mathbb{L}$-Modules

Definition 22. *A $\mathbb{L}$-category X is said* tensored *if for all $x \in X$ and $\epsilon \in Q$, it admits the tensor $\epsilon \otimes x$.*

Proposition 57. *A tensored $\mathbb{L}$-category X is a $\mathbb{L}$-module $(X, \preceq_X, \otimes)$. A continuous functor of complete $\mathbb{L}$-categories is a $\mathbb{L}$-module morphism between the associated $\mathbb{L}$-modules.*

Proof. We must show that tensors induce a continuous action. Observe that tensors are characterized by the equation

$$X(x \otimes \epsilon, x') = X(x, x') - \epsilon \tag{35}$$

If $\epsilon = 0$, then (35) forces $x \otimes \epsilon \simeq x$. If $\epsilon = \delta + \eta$, then using the fact that $\alpha - (\epsilon + \delta) = (\alpha - \epsilon) - \delta$ we deduce $X((x \otimes \epsilon) \otimes \delta, x') = X(x \otimes \epsilon, x') - \delta = (X(x, x') - \epsilon) - \delta = X(x, x') - (\epsilon + \delta) = X(x \otimes (\epsilon + \delta), x')$, which forces $x \otimes (\epsilon + \delta) \simeq (x \otimes \epsilon) \otimes \delta$.

A continuous functor $f : X \to Y$ commutes with sups and with $\otimes$, and is thus a $\mathbb{L}$-module morphism. □

Lemma 58. i. $\sup_{i \in I} a_i - \epsilon = (\sup_{i \in I} a_i) - \epsilon$.
ii. $\sup_{i \in I}(a_i - \epsilon) - b_i = (\sup_{i \in I} a_i - b_i) - \epsilon$.

Proof. Let $A = \sup_{i \in I} a_i - \epsilon$ and $B = (\sup_{i \in I} a_i) - \epsilon$. Let $J \subseteq I$ be the set of indexes j such that $a_j > \epsilon$. If $J = \varnothing$ then $A = B = 0$. Otherwise, $A = \sup_{j \in J} a_j - \epsilon$ (where "$-$" can be interpreted as subtraction on $\mathbb{R}$, and $B = (\sup_{j \in J} a_j) - \epsilon$ (again with "$-$" being subtraction on $\mathbb{R}$), so $A = B$ follows from the continuity of "$-$" on $\mathbb{R}$.

Let now $A = \sup_{i \in I}(a_i - \epsilon) - b_i$ and $B = (\sup_{i \in I} a_i - b_i) - \epsilon$. Let $J \subseteq I$ be the set of indexes j such that $a_j > b_j + \epsilon$. If $J = \varnothing$, then $A = 0$; suppose $B > 0$, then $\sup_{i \in I} a_i - b_i > \epsilon$, but this implies that we can find $i \in I$ with $a_i > b_i + \epsilon$, against the assumption, so also $B = 0$ holds. If J is non-empty, then $A = \sup_{j \in J}(a_j - \epsilon) - b_j$, where "$-$" is not subtraction on $\mathbb{R}$ and $B = (\sup_{j \in J} a_j - b_j) - \epsilon$, again with "$-$" usual subtraction, so $A = B$ follows from the continuity of "$-$" on $\mathbb{R}$. □

Lemma 59. *In any complete $\mathbb{L}$-category, $x \otimes \epsilon \simeq \sup(\mathbf{Y}(x) + \epsilon)$. In the complete $\mathbb{L}$-category $[X, \mathbb{L}]$, $x \otimes \epsilon = x + \epsilon$.*

Proof. We have

$$\begin{aligned}X(\sup(\mathbf{Y}(x) + \epsilon), x') &= \sup_{y \in X} X(z, x') - (\mathbf{Y}(x)(z) + \epsilon) \\ &= \sup_{y \in X} X(z, x') - (X(z, x) + \epsilon) \\ &= (\sup_{y \in X} X(z, x') - X(z, x)) - \epsilon \\ &= X(x, x') - \epsilon\end{aligned}$$

which shows $x \otimes \epsilon = \sup(\mathbf{Y}(x) + \epsilon)$. In $[X, \mathbb{L}]$ we have $[X, \mathbb{L}](x + \epsilon, x') = \sup_{a \in X}(x_a + \epsilon) - x'_a = (\sup_{a \in X} x_a - x'_a) - \epsilon = [X, \mathbb{L}](x, x') - \epsilon$, which shows $x \otimes \epsilon \simeq x + \epsilon$, and since $[X, \mathbb{L}]$ is skeletal, $x \otimes \epsilon = x + \epsilon$. □

The dual notion of tensors is the *cotensor* $x \multimapinv \epsilon$. Formally, it is defined as a *weighted limit* (whose definition is dual to that of weighted colimit but we do not give details here), and characterized by the equation

$$X(x', x \multimapinv \epsilon) = X(x', x) - \epsilon$$

In other words, in a tensored and cotensored $\mathbb{L}$-category we have $X(x \otimes \epsilon, y) = X(x, y \multimapinv \epsilon)$.

Example 4. *The $\mathbb{L}$-category $[X, \mathbb{L}]$ is cotensored, with $x \multimapinv \epsilon := x - \epsilon$. Indeed we have $[X, \mathbb{L}](x, y \multimapinv \epsilon) = \sup_{a \in X}(y_a - \epsilon) - x_a = (\sup_{a \in X} y_a - x_a) - \epsilon = [X, \mathbb{L}](x, y) - \epsilon$.*

Definition 23. *A $\mathbb{L}$-category X is* order-complete *if it is a sup-lattice with respect to the order $\preceq_X$ (i.e. all joins exist).*

Lemma 60. *Let X be a $\mathbb{L}$-category. If X is order-complete, then*

- *if X is co-tensored, $X(\bigvee_i x_i, y) = \sup_i X(x_i, y)$;*
- *if X is tensored, $X(x, \bigvee_i y_i) = \inf_i X(x, y_i)$.*

Proof. We only prove the second claim, the first being proved similarly.

Let us show that $z \preceq_X z'$ iff for all $w \in X$, $X(w, z') \leqslant X(w, z)$: on one direction we have $X(w, z') \leqslant X(w, z) + X(z, z') = X(w, z)$; on the other direction, we have $X(z, z') \leqslant X(z, z) = 0$.

Using this, since $y_i \preceq_X y := \bigvee_i y_i$ we deduce $X(x, y_i) \leqslant X(x, y)$, and thus $X(x, y) \geqslant \inf_i X(x, y_i)$.

For the converse direction, we argue as follows: let $X(x, y_i) \leqslant \epsilon$ hold for all $i \in I$; then $0 = X(x, y_i) - \epsilon = X(x \otimes \epsilon, y_i)$. Thus $x \otimes \epsilon \preceq_X y_i$, and thus $x \otimes \epsilon \preceq_X y$, that is $X(x \otimes \epsilon, y) = X(x, y) - \epsilon = 0$, and consequently $X(x, y) \leqslant \epsilon$. By letting $\epsilon := X(x, y_i)$ we conclude then $X(x, y) \leqslant X(x, y_i)$, and thus $X(x, y) \leqslant \inf_i X(x, y_i)$.

□

Proposition 61. *If a $\mathbb{L}$-category X is tensored, cotensored and order-complete, then it is complete.*

Proof. For all $x \in [X, \mathbb{L}]$, let $\sup x := \bigvee_{a \in X} a \otimes x_a$. Let us check that $X(\sup x, b) = \sup_{a \in X} X(a, b) - x_a$, using Lemma 60:

$$\begin{aligned} X(\sup x, b) &= \sup_{a \in X} X(a \otimes x_a, b) \\ &= \sup_{a \in X} X(a, b) - x_a \end{aligned}$$

We can thus conclude using Proposition 55. □

Proposition 62. *Let X, Y be two tensored $\mathbb{L}$-categories, and $f : X \to Y$ be a function.*

i. *f is a functor iff f is order-preserving and for all $x \in X$ and $\epsilon \in Q$, $f(x) \otimes \epsilon \preceq_Y f(x \otimes \epsilon)$.*
ii. *f is a continuous functor iff f commutes with joins and for all $x \in X$ and $\epsilon \in Q$, $f(x) \otimes \epsilon = f(x \otimes \epsilon)$.*

Proof. i. If f is a functor then

$$\begin{aligned} Y(f(x) \otimes \epsilon, f(x \otimes \epsilon)) &= Y(f(x), f(x \otimes \epsilon)) - \epsilon \\ &\leqslant X(x, x \otimes \epsilon) - \epsilon \\ &= X(x \otimes \epsilon, x \otimes \epsilon) = 0 \end{aligned}$$

so $Y(f(x) \otimes \epsilon, f(x \otimes \epsilon)) = 0$, which implies $f(x) \otimes \epsilon \preceq_X f(x \otimes \epsilon)$. Moreover, if $x \preceq_X x'$, then $0 \geqslant X(x, x') \geqslant Y(f(x), f(x'))$, whence $f(x) \preceq_Y f(x')$, so f is order-preserving.
Conversely, for all $x, x' \in X$,

$$X(x \otimes X(x, x'), x') = X(x, x') - X(x, x') = 0$$

thus $x \otimes X(x, x') \preceq_X x'$. Since f is order-preserving, it follows that

$$f(x) \otimes X(x, x') \preceq_Y f(x \otimes X(x, x')) \preceq_Y f(x')$$

which implies that

$$Y(f(x), f(x')) - X(x, x') = Y(f(x) \otimes X(x, x'), f(x')) = 0$$

that is $Y(f(x), f(x')) \leqslant X(x, x')$, so f is a functor.

ii. Suppose f is a continuous functors, and let $g : Y \to X$, be its right-adjoint, i.e. satisfying $Y(f(x), y) = X(x, g(y))$. Then

$$\begin{aligned} Y(f(x \otimes \epsilon), y) &= X(x \otimes \epsilon, g(y)) \\ &= X(x, g(y)) - \epsilon \\ &= Y(f(x), y) - \epsilon \end{aligned}$$

which implies that $f(x \otimes \epsilon)$ coincides with the tensor $f(x) \otimes \epsilon$. Moreover, clearly also $f(x) \preceq_Y y$ iff $x \preceq_X g(y)$ holds, which means that f is left-adjoint to g also with respect to the order.
Conversely, suppose the function $f : X \to Y$ preserves joins and tensors. Since f is order-preserving, by i. it is a functor, so we must only prove that it is continuous. Since f preserves joins there exists a function $g : Y \to X$ which is right-adjoint to f with respect to orders, i.e. $f(x) \preceq_Y y$ iff $x \preceq_X g(y)$. We need to prove then that f is left-adjoint to g, i.e. that $Y(f(x), y) = X(x, g(y))$.

On the one hand we have

$$0 = X(x, g(y)) - X(x, g(y)) = X(x \otimes X(x, g(y)), g(y))$$

from which it follows

$$0 = Y(f(x \otimes X(x, g(y)), y) = Y(f(x) \otimes X(x, g(y)), y) = Y(f(x), y) - X(x, g(y))$$

where the first inequality follows from the fact that f and g are adjoint with respect to the order (so $Y(f(x), y) = 0$ iff $X(x, g(y)) = 0$). This implies then $Y(f(x), y) \leqslant X(x, g(y))$.

For the converse inequality,

$$0 = Y(f(x), y) - Y(f(x) - y) = Y(f(x) \otimes Y(f(x), y), y) = Y(f(x \otimes Y(f(x), y)), y)$$

and by a similar reasoning we deduce

$$0 = X(x \otimes Y(f(x), y), g(y)) = X(x, g(y)) - Y(f(x), y)$$

whence $X(x, g(y)) \leqslant Y(f(x), y)$.

□

Theorem 63. *The category* $\mathbb{L}\mathsf{Mod}$ *of* $\mathbb{L}$*-modules and* $\mathbb{L}$*-module morphism coincides with the category* $\mathbb{L}\mathsf{CCat}$ *of complete skeletal* $\mathbb{L}$*-categories and continuous functors.*

Proof. We have already seen that any complete skeletal $\mathbb{L}$-category is a $\mathbb{L}$-module via tensors, and that continuous functors are $\mathbb{L}$-module morphisms. Let us now show that any $\mathbb{L}$-module is a complete skeletal $\mathbb{L}$-category, and that a $\mathbb{L}$-module morphism is a continuous functor.

Let then $M = (M, \preceq, \star)$ be a $\mathbb{L}$-module. Define $M(x, y) = \inf\{\delta \mid x \star \delta \succeq y\}$. It is clear that $M(x, x) = 0$. Let us prove $M(x, y) + M(y, z) \succeq M(x, z)$: from $x \star M(x, y) \succeq y$ and $y \star M(y, z) \succeq z$ we deduce $x \star (M(x, y) + M(y, z)) = (x \star M(x, y)) \star M(y, z) \succeq y \star M(y, z) \succeq z$, and thus $M(x, z) \preceq M(x, y) + M(y, z)$. Observe that $M(x, y) = 0$ iff $x = x \star 0 \geqslant y$, so the order $\preceq_M$ coincides with the order of M.

Let us check that the $\mathbb{L}$-category M is tensored via $x \otimes \epsilon := x \star \epsilon$. Let $A_{x,y} = \{\delta \mid (x \star \epsilon) \star \delta \geqslant y$ and $B_{x,y} = \{\delta - \epsilon \mid x \star \delta \geqslant y\}$. Let us show that $A_{x,y} = B_{x,y}$: if $\delta \in A_{x,y}$, then $\delta = (\epsilon + \delta) - \epsilon$ satisfies $x \star (\epsilon + \delta) = (x \star \epsilon) \star \delta \geqslant y$, whence $\delta \in B_{x,y}$. Conversely, if $\eta = \delta - \epsilon \in B_{x,y}$, then $(x \star \epsilon) \star \eta \geqslant x \star \delta \geqslant y$, whence $\eta \in A_{x,y}$. We conclude then that $M(x \star \epsilon, y) = \inf A_{x,y} = \inf B_{x,y} = \inf\{\delta \mid x \star \delta \geqslant y\} - \epsilon = M(x, y) - \epsilon$.

Let us define the opposite action $x \multimapinv \epsilon = \bigwedge\{y \mid y \star \epsilon \geqslant x\}$. We must show that M is cotensored via $\multimapinv$, for which it suffices to show $M(x \star \epsilon, y) = M(x, y \multimapinv \epsilon)$. Let $C_{x,y} = \{\delta \mid x \star \delta \geqslant y \multimapinv \epsilon\}$. We have that $\delta \in A_{x,y}$ iff $(x \star \delta) \star \epsilon = x \star (\delta + \epsilon) = x \star (\epsilon + \delta) = (x \star \epsilon) \star \delta \geqslant y$ which is equivalent to $x \star \delta \geqslant y \multimapinv \epsilon$. We conclude that $A_{x,y} = C_{x,y}$, from which $M(x \star \epsilon, y) = \inf A_{x,y} = \inf C_{x,y} = M(x, y \multimapinv \epsilon)$.

Since M, as a $\mathbb{L}$-category, is order-complete, tensored and cotensored, it is complete by Proposition 61.

To conclude, notice that if $f : X \to Y$ is a continuous functor, then it commutes with tensors and, by Proposition 62 it commutes with joins, so it is a morphism of the respective $\mathbb{L}$-modules. Conversely, if $f : M \to N$ is a $\mathbb{L}$-module morphism, then, since M and N are both tensored $\mathbb{L}$-categories, the tensor coincides with the actions of M and N, f preserves the joins and the tensor, by Proposition 62, it is a continuous functor of the respective $\mathbb{L}$-categories. □

N. $\mathbb{L}\mathsf{Mod}$ *is a* $*$*-Autonomous Category*

1) The Tensor Product of $\mathbb{L}$*-Modules:* The tensor of $\mathbb{L}$-modules can be introduced as a suitable quotient lattice.

Definition 24 (tensor of $\mathbb{L}$-modules)**.** *Let* M *and* N *be* $\mathbb{L}$*-modules. The* tensor product $M \otimes N$ *of* M *and* N *is the* $\mathbb{L}$*-module defined as* $\mathcal{P}(M \times N)/R^*$*, where* R^* *is the smallest congruence containing the relation* R *defined by:*

$$R' = \left\{ \begin{array}{l|c} ((\bigvee A, y), \bigcup_{a \in A}\{(a, y)\}) & A \subseteq M, y \in N \\ ((x, \bigvee B), \bigcup_{b \in B}\{(x, b)\}) & B \subseteq N, x \in M \\ (\{(x \star \epsilon, y)\}, \{(x, y \star \epsilon)\}) & \epsilon \in Q \end{array} \right\}$$

and the action is defined via $[A] \star \epsilon = \bigvee\{[\{(x \star \epsilon, y)\}] \mid (x, y) \in A\}$.

Let a $\mathbb{L}$-*bimorphism* be a map $f : M \times N \to L$ such that f preserves joins in each variable separately, and moreover $f(x, y \star \epsilon) = f(x \star \epsilon, y)$. A $\mathbb{L}$-bimorphism $f : M \times N \to L$ is *universal* if for all L' and bimorphism $g : M \times N \to L'$ there is a unique sup-lattice homomorphism $h : L \to L'$ such that $g = h \circ f$.

Proposition 64 (universal property of the tensor product, cf. [65])**.** *The tensor product* $M \otimes N$ *is the codomain of the universal* $\mathbb{L}$*-bimorphism* $M \times N \to M \otimes N$.

Remark 21. *For any $x \in M$ and $y \in N$, we indicate as $x \otimes y$ the image of the pair (x, y) under the universal $\mathbb{L}$-bimorphism $\tau : M \times N \to M \otimes N$, or equivalently, as the R^*-equivalence class of (x, y). Since joins in $M \otimes N$ are given by $\bigvee_i [A_i] = [\bigcup_i A_i]$, we have then that any element $[A] \in M \otimes N$ can be written as $[A] = \bigvee\{x \otimes y \mid (x, y) \in A\}$.*

Lemma 65 (cf. [65])**.**
- *$M \otimes N \simeq N \otimes M$;*
- *$(M \otimes N) \otimes R \simeq M \otimes (N \otimes R)$.*

Proposition 66. $\mathbb{L}\mathsf{Mod}(M \otimes N, R) \simeq \mathbb{L}\mathsf{Mod}(M, \mathbb{L}\mathsf{Mod}(N, R))$ *(as an isomorphism of sup-lattices).*

Proof. Given $h : M \otimes N \to R$, for all $x \in M$ define $h_x : N \to R$ by $h_x(y) = h(x \otimes y)$. We then have $h_x(\bigvee_i y_i) = h(x \otimes \bigvee_i y_i) = h(\bigvee_i x \otimes y_i) = \bigvee_i h(x \otimes y_i) = \bigvee_i h_x(y_i)$ and $h_x(y \star \epsilon) = h(x \otimes (y \star \epsilon)) = h((x \otimes y) \star \epsilon) = h(x \otimes y) \star \epsilon = h_x(y) \star \epsilon$, so $h_x \in \mathbb{L}\mathsf{Mod}(N, R)$. Moreover, by a similar argument we have $h_{\bigvee_i x_i}(y) = \bigvee_i h_{x_i}(y)$ and $h_{x \star \epsilon}(y) = h_x(y) \star \epsilon$, so the map $x \mapsto h_x$ is a $\mathbb{L}$-module morphism.

Finally, for any family $h_i : M \otimes N \to R$, we have $(\bigvee_i h_i)_x(y) = \bigvee_i h_i(x \otimes y) = \bigvee_i (h_i)_x(y) = (\bigvee_i h_i)(x \otimes y)$, and thus we have a sup-lattice homomorphism ζ from $\mathbb{L}\mathsf{Mod}(M \otimes N, R)$ to $\mathbb{L}\mathsf{Mod}(M, \mathbb{L}\mathsf{Mod}(N, R))$ given by $\zeta(h) = h_\_$.

Let us show that ζ has an inverse: for all $f \in \mathbb{L}\mathsf{Mod}(M, \mathbb{L}\mathsf{Mod}(N, R))$, define $f' : M \times N \to R$ by $f'(x, y) := f(x)(y)$. This is clearly a bimorphism, so there is a unique homomorphism $h_{f'} : M \otimes N \to M \times N$ such that $f' = h_{f'} \circ \tau$, i.e. such that $h_{f'}(x \otimes y) = f'(x, y) = f(x)(y)$, and thus $\zeta(h_{f'}) = f$. On the other hand, if $f = \zeta(h)$, then the uniqueness of $h_{f'}$ ensures that $h_{f'} = h$. ☐

Proposition 67.
i. $\mathbb{L}\mathsf{Mod}(\mathbb{L}, M) \simeq M$.
ii. $\mathbb{L}\mathsf{Mod}(M, N) \simeq \mathbb{L}\mathsf{Mod}(N^{\mathrm{op}}, M^{\mathrm{op}})$.
iii. $\mathbb{L}\mathsf{Mod}(M, \mathbb{L}^{\mathrm{op}}) \simeq M^{\mathrm{op}}$.
(all isomorphisms of sup-lattices).

Proof. Define $\alpha : M \to \mathbb{L}\mathsf{Mod}(\mathbb{L}, M)$ by $\alpha(x)(\epsilon) = x \star \epsilon$ and $\beta : \mathbb{L}\mathsf{Mod}(\mathbb{L}, M) \to M$ by $\beta(f) = f(0)$. Then we have that $\alpha(\beta(f))(\epsilon) = \alpha(f(0))(\epsilon) = f(0) \star \epsilon = f(\epsilon)$, and $\beta(\alpha(x)) = \beta(\lambda\epsilon.x \star \epsilon) = x \star 0 = x$.

If $f \in \mathbb{L}\mathsf{Mod}(M, N)$, since it preserves joins, it has a right-adjoint $f^* : N^{\mathrm{op}} \to M^{\mathrm{op}}$, such that $f(x) \leqslant y$ iff $x \leqslant f^*(y)$.

By i. $\mathbb{L}\mathsf{Mod}(\mathbb{L}, M^{\mathrm{op}}) \simeq M^{\mathrm{op}}$ and we conclude by ii. ☐

Proposition 68.
i. $\mathbb{L}\mathsf{Mod}(M, N) \simeq (M \otimes N^{\mathrm{op}})^{\mathrm{op}}$.
ii. $M \otimes N \simeq \mathbb{L}\mathsf{Mod}(M, N^{\mathrm{op}})^{\mathrm{op}}$.
iii. $\mathbb{L} \otimes M \simeq M \otimes \mathbb{L} \simeq \mathbb{L}$.

Proof. $\mathbb{L}\mathsf{Mod}(M, N) \simeq \mathbb{L}\mathsf{Mod}(M, \mathbb{L}\mathsf{Mod}(N^{\mathrm{op}}, \mathbb{L}^{\mathrm{op}})) \simeq \mathbb{L}\mathsf{Mod}(M \otimes N^{\mathrm{op}}, \mathbb{L}^{\mathrm{op}}) \simeq (M \otimes N^{\mathrm{op}})^{\mathrm{op}}$. Claim ii. is proved similarly.

For claim iii. $\mathbb{L} \otimes M \simeq M \otimes M \simeq \mathbb{L}\mathsf{Mod}(M, \mathbb{L}^{\mathrm{op}})^{\mathrm{op}} \simeq (M^{\mathrm{op}})^{\mathrm{op}} = M$. ☐

By putting all previous results together we obtain:

Theorem 69. $\mathbb{L}\mathsf{Mod}$ *is a *-autonomous category.*

Further useful properties are the following:

Proposition 70. *[cf. [65]]*
i. *$\mathbb{L}^X \otimes M \simeq M^X$;*
ii. *$\mathbb{L}^X \otimes \mathbb{L}^Y \simeq \mathbb{L}^{X \times Y}$.*

Proof. M^X coincides with the product $\Pi_{x \in X} M$. We have then $\mathbb{L}^X \otimes M \simeq (\Pi_x \mathbb{L}) \otimes M \simeq \Pi_x(\mathbb{L} \otimes M) \simeq \Pi_x M \simeq M^X$.

For ii., using i. we have $\mathbb{L}^X \otimes \mathbb{L}^Y \simeq (\mathbb{L}^Y)^X \simeq \mathbb{L}^{X \times Y}$. ☐

$\mathbb{L}\mathsf{Mod}$ admits biproducts, since products and coproducts exist in $\mathbb{L}\mathsf{Mod}$ and both coincide with the order product:

$$\prod_{i \in I} X_i \simeq \coprod_{i \in I} X_i$$

In particular, the projection and injection morphisms $\pi_i : \prod_i X_i \to X_i$ and $\iota_i : X_i \to \prod_{i \in I} X_i$ are defined by

$$\pi_i\big((x_j)_{j \in I}\big) = x_i \qquad\qquad \iota_i(x)(j) = \begin{cases} x & \text{if } i = j \\ \bot & \text{otherwise} \end{cases}$$

Moreover, biproducts commute with tensors as follows:

Proposition 71. *[cf. [65]]* $\prod_{i \in I} X \otimes Y_i \simeq X \otimes \prod_{i \in I} Y_i$.

Proof sketch. We just recall the isomorphism $h : \prod_{i\in I} X \otimes Y_i \simeq X \otimes \prod_{i\in I} Y_i$, defined as follows:

$$h\Big(i \mapsto \bigvee_{k\in J_i} x_{i,k} \otimes y_{i,k}\Big) = \bigvee_{i\in I, k\in J_i} x_{i,k} \otimes \iota_i\big((y_{j,k})_{j\in I}\big)$$

and its inverse

$$k\Big(\bigvee_{k\in J} x_k \otimes \big(x_{k,i}\big)_{i\in I_k}\Big)(i) = \bigvee_{k\in J, i\in I_k} x_k \otimes \pi_i\big((x_{k,j})_{j\in I_k}\big).$$

□

2) The Tensor Product of $\mathbb{L}$*-Categories:* Thanks to Theorem 63, the *-autonomous structure of $\mathbb{L}\mathsf{Mod}$ translates into a *-autonomous structure for $\mathbb{L}\mathsf{CCat}$.

In Met the "tensor product" of two metric spaces X and Y is just their cartesian product, with the "plus" metric. What can we say about the tensor product in $\mathbb{L}\mathsf{CCat}$?

The goal of this subsection is to describe the $\mathbb{L}$-categorical structure of the tensor product explicitly. The main intuition is that the elements of $X \otimes Y$ can be seen as joins of pairs $x \otimes y$ of elements $x \in X$, $y \in Y$. We will then show that the metric coincides with the "plus" metric over pairs $x \otimes y$, and extends continuously to joins.

Lemma 72. *For all $m, m' \in M$ and $n, n' \in N$ and $\epsilon \in Q$, $(m \otimes n) \star \epsilon \geq (m' \otimes n')$ iff there exists δ_1, δ_2 such that $\delta_1 + \delta_2 = \epsilon$, $m + \delta_1 \geq m'$ and $n + \delta_2 \geq n'$.*

Proof Sketch. Notice that $(m \otimes n) \star \epsilon = [\{(m \star \epsilon, n)\}] = [\{(m \star \delta_1, n \star \delta_2\}]$ for all $\delta_1 + \delta_2 = \epsilon$. Hence, $m' \otimes n' \preceq (m \otimes n) \star \epsilon$ implies that for some $\delta_1 + \delta_2 = \epsilon$, $(m', n') \vee (m \star \delta_1, n \star \delta_2) = (m' \vee (m \star \delta_1), n' \vee (n \star \delta_2)) = (m \star \delta_1, n' \star \delta_2)$, that is, that $m' \preceq_M m \star \delta_1$ and $n' \preceq_n n \star \delta_2$. □

Proposition 73. *For all $m, m' \in N$ and $n, n' \in N$,*

$$(M \otimes N)(m \otimes n, m' \otimes n') = M(m, m') + N(n, n')$$

More generally,

$$(M \otimes N)([A], [B]) = \sup_{(x,y)\in A} \inf_{(x',y')\in B} M(x, x') + N(y, y')$$

Proof. By definition, $(M \otimes N)(m \otimes n, m' \otimes n')$ is given by $\inf A$, where

$$A = \{\epsilon \mid (m \otimes n) \star \epsilon \geqslant m' \otimes n'\}$$

Let us show that A coincides with

$$B = \{\delta_1 + \delta_2 \mid m \star \delta_1 \geqslant m', n \star \delta_2 \geqslant n'\}$$

On the one hand, if $\delta_1 + \delta_2 \in B$, it is clear that $\delta_1 + \delta_2 \in A$. Conversely, if $\epsilon \in A$, then by Lemma 72 $\epsilon = \delta_1 + \delta_2$ with $m \star \delta_1 \geqslant m'$ and $n \star \delta_2 \geqslant n'$, whence $\epsilon \in B$.

We can now conclude as follows:

$$\begin{aligned}(M \otimes N)(m \otimes n, m' \otimes n') &= \inf A \\ &= \inf B \\ &= \inf\{\delta_1 \mid m \star \delta_1 \geqslant m'\} + \inf\{\delta_2 \mid n \star \delta_2 \geqslant n'\} \\ &= M(m, m') + N(n, n').\end{aligned}$$

For the second claim, using the fact that $M \otimes N$, as a $\mathbb{L}$-category, is both tensored and cotensored, using the fact that $[A] = \bigvee_{(x,y)\in A} x \otimes y$ and $[B] = \bigvee_{(x',y')\in B} z \otimes w$, and Lemma 60:

$$\begin{aligned}(M \otimes N)([A], [B]) &= \sup_{(x,y)\in A} (M \otimes N)(x \otimes y, [B]) \\ &= \sup_{(x,y)\in A} \inf_{(x',y')\in B} (M \otimes N)(x \otimes y, x' \otimes y') \\ &= \sup_{(x,y)\in A} \inf_{(x',y')\in B} M(x, x') + N(y, y').\end{aligned}$$

□

O. Exponential and Differential Structure of $\mathbb{L}$Mod

1) Symmetric Algebras in $\mathbb{L}$Mod *:* Given $\mathbb{L}$-multisets A and B, we define the multiset $A \cup B$ as follows:

- if $A = 0$, then $A \cup B = B$;
- if $B = 0$, then $A \cup B = A$;
- if $A = [x_1, \dots, x_n]$ and $B = [y_1, \dots, y_m]$, then $A \cup B = [x_1, \dots, x_n, y_1, \dots, y_m]$.

Proposition 74. *Let X be a $\mathbb{L}$-module and $n \in \mathbb{N}$. Any $\mathbb{L}$-multiset $[x_1, \dots, x_n] \in X^{\otimes n}$ is p-invariant. Moreover, any p-invariant element $x \in X^{\otimes n}$ can be written as a join of $\mathbb{L}$-multisets.*

Proof. For the first claim we have, for all $\sigma \in \mathfrak{S}_n$,

$$\begin{aligned}
\langle\sigma\rangle([x_1, \dots, x_n]) &= \bigvee_{\tau \in \mathfrak{S}_n} \langle\sigma\rangle([x_{\tau(1)}, \dots, x_{\tau(n)}]) \\
&= \bigvee_{\tau \in \mathfrak{S}_n} [x_{\sigma\tau(1)}, \dots, x_{\sigma\tau(n)}]) \\
&= \bigvee_{\tau \in \mathfrak{S}_n} [x_{\tau(1)}, \dots, x_{\tau(n)}]) \\
&= [x_1, \dots, x_n].
\end{aligned}$$

For the second claim, observe that x can always be written as a join of tensors $x = \bigvee_i x_1^i \otimes \cdots \otimes x_n^i$. Moreover, if $x_1^i \otimes \cdots \otimes x_n^i \leqslant x$, since x is p-invariant, for all $\sigma \in \mathfrak{S}_n$, also $x_{\sigma(1)}^i \otimes \cdots \otimes x_n^i \leqslant \langle\sigma\rangle(x) = x$, so we can conclude that $x = \bigvee_i [x_1^i, \dots, x_n^i]$. ☐

Proposition 75. *For any $\mathbb{L}$-module X, the set $!_n X \subseteq X$ of p-invariant elements of $X^{\otimes n}$ is a sub-$\mathbb{L}$-module of X.*

Proof. If $x_i \in X^{\otimes n}$ is a family of p-invariant elements, then $x = \bigvee_i x_i$ is also p-invariant, since $\langle\sigma\rangle(x) = \bigvee_i \langle\sigma\rangle(x_i) = \bigvee_i x_i = x$. Hence $!_n X$ is a sup-lattice. Moreover, for all $x \in !_n X$ and $\epsilon \in Q$, $x \otimes \epsilon$ is also p-invariant, since $\langle\sigma\rangle(x \otimes \epsilon) = \langle\sigma\rangle(x) \otimes \epsilon = x \otimes \epsilon$. We conclude that $!_n X$ is a sup-lattice with a continuous action of $\mathbb{L}$, where both the order and the action are inherited from X, so it is a sub-$\mathbb{L}$-module of X. ☐

The fundamental property of $!_n X$ is the following:

Proposition 76. *For any $\mathbb{L}$-module X and $n \in \mathbb{N}$, the inclusion morphism $\iota : !_n X \longrightarrow X^{\otimes n}$ is the equalizer of the diagram*

$$!_n X \xrightarrow{\iota} X^{\otimes n} \xrightarrow{\langle\sigma\rangle} X^{\otimes n}$$

generated by all actions $\langle\sigma\rangle$, for $\sigma \in \mathfrak{S}_n$.

Proof. It is clear that $\langle\sigma\rangle \circ \iota = \langle\tau\rangle \circ \iota$ holds for all $\sigma, \tau \in \mathfrak{S}_n$. Suppose now $h : C \to X^{\otimes n}$ is another morphism satisfying $\langle\sigma\rangle \circ h = \langle\tau\rangle \circ h$ for all $\sigma, \tau \in \mathfrak{S}_n$. Since $\langle\sigma\rangle \circ h = \langle\mathrm{id}\rangle \circ h = h$, we deduce that $h(c)$ is p-invariant, for all $c \in C$. Hence h splits in a unique way as $C \xrightarrow{h} !_n X \xrightarrow{\iota} X^{\otimes n}$. ☐

Remark 22 (metric structure of $!_n X$)**.** *As $!_n X$ is a sub-$\mathbb{L}$-module of $X^{\otimes n}$, the distance function can be computed explicitly using Proposition 73:*

$$!_n X([x_1, \dots, x_n], [y_1, \dots, y_n]) = \sup_{\sigma \in \mathfrak{S}_n} \inf_{\tau \in \mathfrak{S}_n} \sum_{i=1}^{n} X(x_{\sigma(i)}, y_{\tau(i)})$$

We now show that the $\mathbb{L}$-module $!_n X$ is isomorphic to the symmetric algebra, which is used in the construction of the exponential modality in the relational model.

Definition 25 (symmetric algebra)**.** *For any $\mathbb{L}$-module X and $n \in \mathbb{N}$, we let $\mathrm{Sym}_n(X)$ indicate the $\mathbb{L}$-module defined as $\mathrm{Sym}_n(X) := \frac{X^{\otimes n}}{\sim_n}$, where $\sim_n$ is the least congruence generated by the action $\langle\sigma\rangle$ of permutations $\sigma \in \mathfrak{S}_n$.*

Proposition 77. $!_n X \simeq \mathrm{Sym}_n(X)$.

Proof. First, observe that for any equivalence class $\alpha \in \mathrm{Sym}_n(X)$, the point $\bigvee \alpha$ is p-invariant: since $x \in \alpha$ holds iff $\langle\sigma\rangle(x) \in \alpha$, for all $\sigma \in \mathfrak{S}_n$, it follows that $\langle\sigma\rangle(\bigvee \alpha) = \bigvee \{\langle\sigma\rangle(x) \mid x \in \alpha\} = \bigvee \{x \mid x \in \alpha\} = \bigvee \alpha$, and thus $\bigvee \alpha$ is p-invariant.

Now, let us show that for all $x \in X^{\otimes n}$, $x \sim_n \bigvee [x]$: for all $y \in [x]$, by definition $x \sim_n y$ holds; hence, since $\sim_n$ is a congruence, we have that $x = \bigvee_{y \in [x]} x \sim_n \bigvee_{y \in [x]} y = \bigvee [x]$. Observe that this implies that $[\bigvee [x]] = [x]$.

Let us now show that for all p-invariant point x_0, and for all $y, z \in X^{\otimes n}$, if $y \leqslant x_0$ and $z \sim y$ holds, then $z \leqslant x_0$.

We will exploit the fact that $\sim$ is the least congruence containing the relation $\sim_0$ induced by the action of permutations. More precisely, $\sim$ can be defined explicitly as

$$x \sim y \;\Leftrightarrow\; \exists\alpha.\mathrm{OR}(\alpha) \wedge x \sim^{(\alpha)} y$$

where $\mathrm{OR}(\alpha)$ is the property "α is an ordinal", and the relations $\sim^{(\alpha)}$ are defined by induction as follows:

- $x \sim^{(0)} y$ iff either $x \sim_0 y$, $x = y$ or $y \sim_0 x$ holds;
- $x \sim^{(\alpha+1)} y$ iff one of the following holds:
  - for some z, $x \sim^{(\alpha)} z$ and $z \sim^{(\alpha)} y$ holds;
  - for some set I, and families x_i, y_i, $x = \bigvee x_i, y = \bigvee_i y_i$ and $x_i \sim^{(\alpha)} y_i$ holds for all $i \in I$.
- $x \sim^{(\gamma)} y$ iff $x \sim^{(\delta)} y$ holds for some $\delta < \gamma$, for γ limit.

We will now prove, by induction on an ordinal α, that for all p-invariant point x_0, and for all $y, z \in X^{\otimes_n}$, if $y \leqslant x_0$ and $z \sim^{(\alpha)} y$ holds, then $z \leqslant x_0$. From this the claim will follow.

- if $z \sim^0 y$, then either $z = y$, in which case the claim follows from the hypothesis, or $z = z_1 \otimes \cdots \otimes z_n$ and $y = y_{\sigma(1)} \otimes \cdots \otimes y_{\sigma(n)}$; then from $y \leqslant x_0$ we deduce $z = \langle \sigma^{-1} \rangle(y) \leqslant \langle \sigma^{-1} \rangle(x_0) = x_0$.
- if $z \sim^{\alpha+1} y$ two possibilities arise:
  1) if $z \sim^{\alpha} z' \sim^{\alpha} y$, then by IH we have $z' \leqslant x_0$, and again by IH applied to z' we deduce $z \leqslant x_0$;
  2) $z = \bigvee_i z_i$ and $y = \bigvee_i y_i$, with $z_i \sim^{\alpha} y_i$, then from $y_i \leqslant y \leqslant x_0$, we deduce, by IH, $z_i \leqslant x_0$, and thus $z \leqslant x_0$.
- if $z \sim^{\gamma} y$ for γ limit, then $z \sim^{\beta} y$ for some $\beta < \gamma$, so by IH we deduce $z \leqslant x_0$.

From the argument above we now deduce that for all p-invariant point x_0, and for all $y, z \in X^{\otimes_n}$, if $y \leqslant x_0$ and $z \sim_n y$ holds, then $z \leqslant x_0$. From this we can deduce in turn that for all $x \in X^{\otimes_n}$, $\bigvee[x]$ is the smallest p-invariant over x: suppose x_0 is a p-invariant point and $x \leqslant x_0$; then for all $y \in [x]$, we deduce $y \leqslant x_0$ by the argument above, and we can thus conclude that $\bigvee[x] \leqslant x_0$.

Let now x be p-invariant; as x is the smallest p-invariant over x, we deduce that $x = \bigvee[x]$.

Using the previous facts we can now define an isomorphism $h : !_n X \to \mathrm{Sym}_n(X)$ by letting $h(x) = [x]$, with inverse $k([x]) = \bigvee[x]$. Indeed, we have that $k(h(x)) = \bigvee[x] = x$, and $h(k([x])) = [\bigvee[x]] = [x]$. ☐

The following lemma shows the compatibility of the construction of $!_n X$ with the usual construction of the exponential modality in weighted relational models.

Lemma 78. *For any set S, there exists an isomorphism of $\mathbb{L}$-modules*

$$!_n \mathbb{L}^S \simeq \mathbb{L}^{\mathcal{M}_n(S)}$$

where $\mathcal{M}_n(X)$ indicates the set of multisets of X of cardinality n.

Proof. Let us show that the morphism $h : \mathbb{L}^{\mathcal{M}_n(S)} \to \mathbb{L}^{S \times \cdots \times S}$ defined by

$$h(f)(\langle s_1, \ldots, s_n \rangle) = h([s_1, \ldots, s_n])$$

is the equalizer of the diagram

$$\mathbb{L}^{\mathcal{M}_n(S)} \xrightarrow{h} \mathbb{L}^{S \times \cdots \times S} \xrightarrow{[\sigma]} \mathbb{L}^{S \times \cdots \times S}$$

where $[\sigma](x)(\langle s_1, \ldots, s_n \rangle) = x(\langle x_{\sigma(1)}, \ldots, x_{\sigma(n)} \rangle)$, with σ varying over $\mathfrak{S}_n$.

It is immediate that $h \circ [\sigma] = h \circ [\tau]$, for all $\sigma, \tau \in \mathfrak{S}_n$. Let now $k : C \to \mathbb{L}^{S \times \cdots \times S}$ satisfy $k \circ [\sigma] = k \circ [\tau]$: then for all $c \in C$, $k(c)(\langle s_1, \ldots, s_n \rangle) = k(c)(\langle s_{\sigma(1)}, \ldots, s_{\sigma(n)} \rangle)$, so $k(c)$ actually defines a unique element of $\mathbb{L}^{\mathcal{M}_n(S)}$, and thus k splits in a unique way as $C \xrightarrow{k'} \mathbb{L}^{\mathcal{M}_n(S)} \xrightarrow{h} \mathbb{L}^{S \times \cdots \times S}$.

Now, to conclude it suffices to observe that, by Proposition 70, $\mathbb{L}^{S \times \cdots \times S} \simeq (\mathbb{L}^S)^{\otimes_n}$, and then, since equalizers are unique up to a unique isomorphism, we obtain an isomorphism $\mathbb{L}^{\mathcal{M}_n(S)} \simeq !_n \mathbb{L}^S$. ☐

2) Linear Differential Categories: There exist many equivalent way to describe linear differential structure over symmetric monoidal categories with biproducts. Here we chose the approach via bialgebra modalities (see [15], [23]).

Definition 26 (bialgebra modality)**.** *Let $\mathbb{C}$ be an additive symmetric monoidal category. A* bialgebra modality over $\mathbb{C}$ *is a septuple $(!, \delta, \epsilon, \Delta, e, \nabla, u)$ consisting of:*

1) *a comonad $(!, \delta, \epsilon)$, that is, a functor $!$ together with natural transformations $\delta : !X \to !!X$ and $\epsilon : !X \to X$ satisfying*

$$\epsilon \circ \delta = !\epsilon \circ \delta = 1 \tag{36}$$

$$!\delta \circ \delta = \delta \circ !\delta \tag{37}$$

2) *two natural transformations* $\Delta :!X \to !X \otimes !X$ *and* $e :!X \to \{\star\}$ *such that* $(!X, \Delta, e)$ *is a cocommutative comonoid, that is the following equations hold:*

$$(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta \tag{38}$$

$$(1 \otimes e) \circ \Delta = (e \otimes 1) \circ \Delta = 1 \tag{39}$$

$$\sigma \circ \Delta = \Delta \tag{40}$$

and δ *preserves the comultiplication, that is*

$$(\delta \otimes \delta) \circ \Delta = \Delta \circ \delta \tag{41}$$

3) *two natural transformations* $\nabla :!X \otimes !X \to !X$ *and* $u : \{\star\} \to !X$ *such that* $(!X, \nabla, u)$ *is a commutative monoid, that is, the following equations hold:*

$$\nabla \circ (\nabla \otimes 1) = \nabla \circ (1 \otimes \nabla) \tag{42}$$

$$\nabla \circ (1 \otimes u) = \nabla \circ (u \otimes 1) = 1 \tag{43}$$

$$\nabla \circ \sigma = \nabla \tag{44}$$

4) $(!X, \nabla, u, \Delta, e)$ *is a bialgebra, that is the following equations hold:*

$$e \circ \nabla = e \otimes e \tag{45}$$

$$\Delta \circ u = u \otimes u \tag{46}$$

$$u \circ e = 1 \tag{47}$$

$$\Delta \circ \nabla = (\nabla \otimes \nabla) \circ (1 \otimes \sigma \otimes 1) \circ (\Delta \otimes \Delta) \tag{48}$$

5) ϵ *is compatible with* ∇, *that is*

$$\epsilon \circ \nabla = (\epsilon \otimes e) + (e \otimes \epsilon) \tag{49}$$

Definition 27 (cf. [23])**.** *A* coderelicti*on for a bialgebra modality* $(!, \delta, \epsilon, \Delta, e, \nabla, u)$ *is a natural transformation* $\eta : X \to !X$ *satisfying the following equations:*

$$e \circ \eta = 0 \tag{50}$$

$$\Delta \circ \eta = (\eta \otimes u) + (u \otimes \eta) \tag{51}$$

$$\epsilon \circ \eta = 1 \tag{52}$$

$$\delta \circ \nabla \circ (1 \otimes \eta) = \nabla \circ (\delta \otimes \eta) \circ (1 \otimes \nabla) \circ (\Delta \otimes \eta) \tag{53}$$

Definition 28. *Let* $\mathbb{C}$ *be an additive (i.e. monoid-enriched) symmetric monoidal category with biproducts.* $\mathbb{C}$ *is a* monoidal storage category *if it has a coalgebra modality satisfying the* Seely isomorphisms, *that is, the maps* $e :!\top \to \{\star\}$ *and* $\chi :!(X \times Y) \to !X \otimes !Y$, *with* $\chi = \Delta \circ !(\pi_0) \otimes !(\pi_1)$, *are isomorphisms (whence* $!\top \simeq \{\star\}$ *and* $!(X \times Y) \simeq !X \otimes !Y$*).*

Definition 29. *Let* $\mathbb{C}$ *be an additive symmetric monoidal category. A bialgebra modality* $(!, \delta, \epsilon, \Delta, e, \nabla, u)$ *on* $\mathbb{C}$ *is* additive *if the following equations hold:*

$$\nabla \circ (!f \otimes !g) \otimes \Delta = !(f + g) \tag{54}$$

$$u \circ e = !0 \tag{55}$$

We use the following result:

Theorem 79 (cf. [23])**.** *Every additive symmetric monoidal category with an additive bialgebra modality and finite products satisfies the Seely isomorphisms.*

Definition 30. *A* linear differential category *is an additive symmetric monoidal category* $\mathbb{C}$ *with biproducts and a bialgebra modality with a codereliction and the Seely isomorphisms.*

Theorem 80. *For any linear differential category* $\mathbb{C}$, *the co-Kleisli category* $\mathbb{C}_!$ *is a cartesian closed differential category, with deriving transformation* Df *defined as follows:*

$$!(X \times X) \ \simeq !X \otimes !X \xrightarrow{1 \otimes \epsilon} !X \otimes X \xrightarrow{1 \otimes \eta} !X \otimes !X \xrightarrow{\nabla} !X \xrightarrow{f} Y$$

3) The Free Exponential Modality of $\mathbb{L}\mathsf{Mod}$*:* Using the recipe from [58], together with Proposition 71, the free exponential modality of $\mathbb{L}\mathsf{Mod}$ can be defined as

$$!X := \prod_{n \in \mathbb{N}} !_n X$$

The functorial action of ! is defined, for a morphism $f : X \to Y$, as follows:

$$!f(g)(0) = g(0)$$
$$!f(g)(n+1) = \bigvee \left\{ [f(x_1), \ldots, f(x_{n+1})] \mid [x_1, \ldots, x_{n+1}] \leqslant g(n+1) \right\}$$

The bialgebra modality $(!, \delta, \epsilon, \Delta, e, \nabla, u)$ is defined as follows:

- the comonad $(!, \delta, \epsilon)$ is given by:

$$\epsilon(f) = f(1)$$
$$\delta(f)(n) = \bigvee \left\{ \iota_n([\iota_{i_1}(a_1), \ldots, \iota_{i_n}(a_n)]) \;\middle|\; a_j \in !_{i_j} X, a_1 \cup \cdots \cup a_n \leqslant f(i_1 + \cdots + i_n) \right\}$$

We have $\epsilon(\delta(\alpha)) = \delta(\alpha)(1) = \bigvee\{\iota_n(\alpha(n)) \mid n \in \mathbb{N}\} = \alpha$, and

$$\begin{aligned}
!\epsilon(\delta(\alpha))(n) &= \bigvee\{[\epsilon(\alpha_1), \ldots, \epsilon(\alpha_n)] \mid [\alpha_1, \ldots, \alpha_n] \leqslant \delta(\alpha)(n)\} \\
&= \bigvee \left\{ [\epsilon(\alpha_1), \ldots, \epsilon(\alpha_n)] \;\middle|\; \alpha_i = \iota_{j_i}(a_i), \bigcup_i a_i \leqslant \alpha(n) \right\} \\
&= \bigvee \left\{ [\epsilon(\alpha_1), \ldots, \epsilon(\alpha_n)] \;\middle|\; \alpha_i = \iota_1(x_i), [x_1, \ldots, x_n] \leqslant \alpha(n) \right\} \\
&= \bigvee \left\{ [x_1, \ldots, x_n] \;\middle|\; [x_1, \ldots, x_n] \leqslant \alpha(n) \right\} = \alpha(n)
\end{aligned}$$

Let us now compute $!\delta(\delta(\alpha))$:

$$\begin{aligned}
!\delta(\delta(\alpha))(n) &= \bigvee_i \left\{ [\delta(A_1), \ldots, \delta(A_n)] \;\middle|\; [A_1, \ldots, A_n] \leqslant \delta(\alpha)(n) \right\} \\
&= \bigvee \left\{ [\delta(\iota_{m_1}(a_1)), \ldots, \delta(\iota_{m_n}(a_n))] \;\middle|\; a_j \in !_{m_j} X, \bigcup_{j=1}^{n} a_j \leqslant \alpha(\sum_{l=1}^{n} m_l) \right\} \\
&= \bigvee \left\{ [B_1, \ldots, B_n] \;\middle|\; B_i = \iota_{\sum_j r_j^i}\left([\iota_{r_1^i}(b_1^i), \ldots, \iota_{r_{s_i}^i}(b_{m_i}^i)]\right), \sum_j r_j^i = m_i, \left[\bigcup_j b_j^1, \ldots, \bigcup_j b_j^n\right] \leqslant \alpha(\sum_i m_i) \right\}
\end{aligned}$$

Let us compute $\delta(\delta(\alpha))$:

$$\begin{aligned}
\delta(\delta(\alpha))(n) &= \bigvee \left\{ [\iota_{i_1}(A_1), \ldots, \iota_{i_n}(A_n)] \;\middle|\; A_j \in !_{i_j} X, \bigcup_j A_j \leqslant \delta(\alpha)(\sum_j i_j) \right\} \\
&= \bigvee \left\{ [\iota_{i_1}(A_1), \ldots, \iota_{i_n}(A_n)] \;\middle|\; A_j = [\iota_{r_1^j}(b_1^j), \ldots, \iota_{r_{s_j}^j}(b_{i_j}^j)], \sum_l r_l^j = i_j, \left[\bigcup_l b_l^1, \ldots, \bigcup_l b_l^n\right] \leqslant \alpha(\sum_j i_j) \right\}
\end{aligned}$$

From the two computations it is clear that $!\delta(\delta(\alpha) = \delta(\delta(\alpha))$.

- the cocommutative comonoid structure $(!X, \Delta, e)$ is defined as follows:

$$\Delta(f) = \bigvee \left\{ \iota_n(a) \otimes \iota_m(b) \;\middle|\; n, m \in \mathbb{N}, a \cup b \leqslant f(n+m) \right\}$$
$$e(f) = f(0)$$

Let us check the relevant equations:

$$\begin{aligned}
(\Delta \otimes 1)\Big(\Delta(\alpha)\Big) &= (\Delta \otimes 1)\left(\bigvee \left\{ \iota_n(a) \otimes \iota_m(b) \;\middle|\; a \cup b \leqslant \alpha(n+m) \right\} \right) \\
&= \bigvee \left\{ \iota_{n_1}(a_1) \otimes \iota_{n_2}(a_2) \otimes \iota_m(b) \;\middle|\; a_1 \cup a_2 \cup b \leqslant \alpha(n_1 + n_2 + m) \right\} \\
&= \bigvee \left\{ \iota_n(a) \otimes \iota_{m_1}(b_1) \otimes \iota_{m_2}(b_2) \;\middle|\; a \cup b_1 \cup b_2 \leqslant \alpha(n + m_1 + m_2) \right\} \\
&= (1 \otimes \Delta)\Big(\Delta(\alpha)\Big)
\end{aligned}$$

$$
\begin{aligned}
(e\otimes 1)\Big(\Delta(\alpha)\Big) &= (e\otimes 1)\left(\bigvee\left\{\iota_n(a)\otimes\iota_m(b)\;\Big|\; a\cup b\leqslant \alpha(n+m)\right\}\right)\\
&= \bigvee\left\{\iota_m(b)\;\Big|\; b\leqslant\alpha(m)\right\} = \alpha
\end{aligned}
$$

and one can argue similarly for $(1\otimes e)(\Delta(\alpha))$.

$$
\begin{aligned}
\sigma(\Delta(\alpha)) &= \bigvee\left\{\iota_m(b)\otimes\iota_n(a)\;\Big|\; a\cup b\leqslant\alpha(n+m)\right\}\\
&= \bigvee\left\{\iota_n(a)\otimes\iota_m(b)\;\Big|\; a\cup b\leqslant\alpha(n+m)\right\}\\
&= \Delta(\alpha)
\end{aligned}
$$

Finally, the commutation of Δ and δ:

$$
\begin{aligned}
(\delta\otimes\delta)\Big(\Delta(\alpha)\Big) &= (\delta\otimes\delta)\left(\bigvee\{\iota_n(a)\otimes\iota_m(b)\mid a\cup b\leqslant\alpha(n+m)\}\right)\\
&= \bigvee\left\{\iota_{k_1}([\iota_{i_1}(a_1),\ldots,\iota_{i_{k_1}}(a_{k_1})])\otimes\iota_{k_2}([\iota_{j_1}(b_1),\ldots,\iota_{j_{k_2}}(b_{k_2})])\;\Bigg|\;\bigcup_l a_l\cup\bigcup_l b_l\leqslant\alpha\left(\sum_l i_l+\sum_l j_l\right)\right\}\\
&= \Delta\left(\bigvee\left\{\iota_n([\iota_{i_1}(a_1),\ldots,\iota_{i_n}(a_n)])\;\Bigg|\;\bigcup_j a_j\leqslant\alpha(\sum_j i_j)\right\}\right)\\
&= \Delta(\delta(\alpha))
\end{aligned}
$$

- the commutative monoid structure $(!X,\nabla,u)$ is given by

$$
\nabla(f\otimes g)(n) = \bigvee_{k+h=n} f(k)\cup g(h)
$$

$$
u=\iota_0
$$

Observe that $\nabla(f\otimes g)=\bigvee_{n,k+h=n}\iota_n(f(k)\cup g(h))$.
Let us check the relevant equations:

$$
\begin{aligned}
\nabla\left((\nabla\otimes 1)\left(\bigvee_k\alpha_k\otimes\beta_k\otimes\gamma_k\right)\right) &= \nabla\left(\bigvee_{k,n}\iota_n\left(\bigvee_{u+v=n}\alpha_k(u)\cup\beta_k(v)\right)\otimes\gamma_k\right)\\
&= \bigvee_{k,n}\iota_n\left(\bigvee_{u+v+w=n}\alpha_k(u)\cup\beta_k(v)\cup\gamma_k(w)\right)\\
&= \nabla\left(\bigvee_{n,k}\alpha_k\otimes\iota_n\left(\bigvee_{v+w=n}\beta_k(v)\cup\gamma_k(w)\right)\right)\\
&= \nabla\left((1\otimes\nabla)\left(\bigvee_k\alpha_k\otimes\beta_k\otimes\gamma_k\right)\right)
\end{aligned}
$$

$$
\nabla\left((1\otimes u)\left(\bigvee_k\alpha_k\otimes 0\right)\right) = \nabla\left(\bigvee_k\alpha_k\otimes\iota_0(\epsilon)\right) = \bigvee_{n,k}\iota_n(\alpha_k(n)) = \bigvee_k\alpha_k\otimes 0
$$

and one can argue similarly for $\nabla(u\otimes 1)(\alpha)=\alpha$.

$$
\begin{aligned}
\nabla\left(\sigma\left(\bigvee_k\alpha_k\otimes\beta_k\right)\right) &= \nabla\left(\bigvee_k\beta_k\otimes\alpha_k\right)\\
&= \bigvee_{k,n,p+q=n}\iota_n(\beta_k(p)\cup\alpha_k(q))\\
&= \bigvee_{k,n,p+q=n}\iota_n(\alpha_k(p)\cup\beta_k(q))\\
&= \nabla\left(\bigvee_k\alpha_k\otimes\beta_k\right)
\end{aligned}
$$

Let us check the bialgebra equations:

$$\begin{aligned}
e\left(\nabla\left(\bigvee_k \alpha_k\otimes\beta_k\right)\right) &= e\left(\bigvee_{k,n,p+q=n} \iota_n(\alpha_k(p)\cup\beta_k(q))\right)\\
&= \bigvee_k \alpha_k(0)\cup\beta_k(0)\\
&= \bigvee_k 0\cup 0\\
&= h\left(\bigvee_k \alpha_k(0)\otimes\beta_k(0))\right) = h\left((e\otimes e)\left(\bigvee_k \alpha_k\otimes\beta_k\right)\right)
\end{aligned}$$

where $h:\mathbb{L}\otimes\mathbb{L}\to\mathbb{L}$ indicates the isomorphism $h(\epsilon\otimes\delta)=\epsilon+\delta$.

$$\begin{aligned}
\Delta(u(\epsilon))) = \Delta(\iota_0(\epsilon)) &= \left(\bigvee\left\{\iota_m(a)\otimes\iota_n(b)\;\middle|\; a\cup b\leqslant\iota_0(\epsilon)(n+m)\right\}\right)\\
&= \bigvee\left\{\iota_0(\epsilon_1)\otimes\iota_0(\epsilon_2)\;\middle|\;\epsilon_1+\epsilon_2\leqslant\epsilon\right\}\\
&= (u\otimes u)(\epsilon)
\end{aligned}$$

where we are identifying $\epsilon\in q$ with the equivalence class $\epsilon\otimes 0=\{\langle\epsilon_1,\epsilon_2\rangle \mid \epsilon_1+\epsilon_2=\epsilon\}\in\mathbb{L}\otimes\mathbb{L}$ via the isomorphism h described above.

$$e(u(\epsilon)) = \iota_0(\epsilon)(0) = \epsilon$$

$$\begin{aligned}
&(\nabla\otimes\nabla)(1\otimes\sigma\otimes 1)(\Delta\otimes\Delta)\left(\bigvee_k \alpha_k\otimes\beta_k\right)\\
&= (\nabla\otimes\nabla)(1\otimes\sigma\otimes 1)\left(\bigvee\{\iota_n(a_1)\otimes\iota_m(a_2)\mid a_1\cup a_2\leqslant\alpha_k(n+m)\}\otimes\{\iota_n(b_1)\otimes\iota_m(b_2)\mid b_1\cup b_2\leqslant\beta_k(n+m)\}\right)\\
&= (\nabla\otimes\nabla)(1\otimes\sigma\otimes 1)\left(\bigvee\{\iota_n(a_1)\otimes\iota_m(a_2)\otimes\iota_{n'}(b_1)\otimes\iota_{m'}(b_2)\mid a_1\cup a_2\leqslant\alpha_k(n+m), b_1\cup b_2\leqslant\beta_k(n'+m')\}\right)\\
&= (\nabla\otimes\nabla)\left(\bigvee\{\iota_n(a_1)\otimes\iota_{n'}(b_1)\otimes\iota_m(a_2)\otimes\iota_{m'}(b_2)\mid a_1\cup a_2\leqslant\alpha_k(n+m), b_1\cup b_2\leqslant\beta_k(n'+m')\}\right)\\
&= \bigvee_{k,z,z'}\left\{\left(\iota_z\left(\bigvee_{p+q=z} a_1\cup b_1\right)\right)\otimes\left(\iota_{z'}\left(\bigvee_{p'+q'=z'} a_2\cup b_2\right)\right)\;\middle|\; a_1\cup a_2\leqslant\alpha_k(p+p'), b_1\cup b_2\leqslant\beta_k(q+q')\right\}\\
&= \Delta\left(\bigvee_{k,z}\left\{\iota_z\left(\bigvee_{p+q=z}\alpha_k(p)\cup\beta_k(q)\right)\right\}\right)\\
&= \Delta\left(\nabla\left(\bigvee_k \alpha_k\otimes\beta_k\right)\right)
\end{aligned}$$

Finally, let us check the compatibility of ϵ and ∇, which in $\mathbb{L}\mathsf{Mod}$ reads as $\epsilon\circ\nabla=(\epsilon\otimes e)\vee(e\otimes\epsilon)$:

$$\begin{aligned}
\epsilon\left(\nabla\left(\bigvee_k \alpha_k\otimes\beta_k\right)\right) &= \epsilon\left(\bigvee_{k,n,p+q=n}\iota_n(\alpha_k(p)\cup\beta_k(q))\right)\\
&= \left(\bigvee_k \alpha_k(1)\right)\vee\left(\bigvee_k \beta_k(1)\right)\\
&= \left((\epsilon\otimes e)\left(\bigvee_k \alpha_k\otimes\beta_k\right)\right)\vee\left((e\otimes\epsilon)\left(\bigvee_k \alpha_k\otimes\beta_k\right)\right)\\
&= \left((\epsilon\otimes e)\vee(e\otimes\epsilon)\right)\left(\bigvee_k \alpha_k\otimes\beta_k\right)
\end{aligned}$$

- the codereliction $\eta: X\to !X$ is defined by $\eta=\iota_1$.

Let us check the coddereliction equations:

$$e(\eta(x)) = \iota_1(x)(0) = 0$$

$$\begin{aligned}
\Delta(\eta(x)) &= \Delta(\iota_1(x)) \\
&= \bigvee \{\iota_1(x) \otimes \iota_0(r) \mid r \in Q\} \vee \bigvee \{\iota_0(r) \otimes \iota_1(x) \mid r \in Q\} \\
&= (\iota_1(x) \otimes 0) \vee (0 \otimes \iota_1(x)) \\
&= (\eta \otimes u)(x) \vee (u \otimes \eta)(x) \\
&= \Big(\eta \otimes u) \vee (u \otimes \eta)\Big)(x)
\end{aligned}$$

$$\epsilon(\eta(x)) = \iota_1(x)(1) = x$$

For the last equation, we only check it on basic tensors:

$$\begin{aligned}
&\nabla((\delta \otimes \eta)((1 \otimes \nabla))(\Delta \otimes \eta)(\alpha \otimes x)) \\
&= \nabla((\delta \otimes \eta)((1 \otimes \nabla)) \left(\bigvee \{\iota_n(a) \otimes \iota_m(b) \otimes \iota_1(x) \mid a \cup b \leqslant \alpha(n+m)\}\right) \\
&= \nabla\left((\delta \otimes \eta)\left(\bigvee \{\iota_n(a) \otimes \iota_{m+1}(b \cup \{x\}) \mid a \cup b \leqslant \alpha(n+m)\}\right)\right) \\
&= \nabla\left(\bigvee \left\{[\iota_{i_1}(c_1), \ldots, \iota_{i_r}(c_r)] \otimes \iota_1(\iota_{m+1}(b \cup \{x\})) \;\middle|\; \bigcup_j c_j \cup b \leqslant \alpha\left(\sum_j i_j + m\right)\right\}\right) \\
&= \bigvee \left\{[\iota_{i_1}(c_1), \ldots, \iota_{i_r}(c_r), \iota_{m+1}(b \cup \{x\})] \;\middle|\; \bigcup_j c_j \cup b \leqslant \alpha\left(\sum_j i_j + m\right)\right\} \\
&= \delta\left(\bigvee_n \iota_{n+1}(\alpha(n) \cup \{x\})\right) \\
&= \delta(\nabla(\alpha \otimes \iota_1(x)) \\
&= \delta(\nabla((1 \otimes \eta)(\alpha \otimes x)))
\end{aligned}$$

It remains to check the Seely isomorphisms. Using Theorem 79 it suffices to check that the bialgebra modality defined above is additive (with respect to the "tropical" additive structure given by $\perp$ and $\vee$).

$$\begin{aligned}
\nabla((!f \otimes !g)(\Delta(\alpha))) &= \nabla\left((!f \otimes !g)\left(\bigvee \{\iota_n([x_1, \ldots, x_n]) \otimes \iota_m([y_1, \ldots, y_m]) \mid [x_1, \ldots, x_n, y_1, \ldots y_m] \leqslant \alpha(n+m)\}\right)\right) \\
&= \nabla\left(\bigvee \{\iota_n([f(x_1), \ldots, f(x_n)]) \otimes \iota_m([g(y_1), \ldots, g(y_m)]) \mid [x_1, \ldots, x_n, y_1, \ldots y_m] \leqslant \alpha(n+m)\}\right) \\
&= \bigvee \{\iota_{n+m}([f(x_1), \ldots, f(x_n), g(y_1), \ldots, g(y_m)]) \mid [x_1, \ldots, x_n, y_1, \ldots y_m] \leqslant \alpha(n+m)\} \\
&= \bigvee_n \{\iota_n([f(x_1) \vee g(x_1), \ldots, f(x_n) \vee g(x_1)] \mid [x_1, \ldots, x_n] \leqslant \alpha(n)\} \\
&=!(f \vee g)(\alpha)
\end{aligned}$$

$$u(e(\alpha)) = u(\alpha(0)) = \iota_0(\alpha(0)) = \iota_0(\alpha(0)) \vee \bigvee \{\iota_{n+1}([\underbrace{\perp, \ldots, \perp}_{n+1}]) \mid n \in \mathbb{N}\} = (!\perp)(\alpha)$$

In particular, any $\alpha \in !(X \times Y)$ can be represented as an object α^S of $!X \otimes !Y$ defined as follows:

$$\alpha^S = \bigvee \{\iota_n([x_1, \ldots, x_n]) \otimes \iota_m([y_1, \ldots, y_m]) \mid \iota_{n+m}(\langle x_1, \perp\rangle, \ldots, \langle x_n, \perp\rangle, \langle \perp, y_1\rangle, \ldots, \langle \perp, y_m\rangle) \leqslant \alpha(n+m)\}$$

Theorem 81. $\mathbb{L}\mathsf{Mod}$ *(equivalently,* $\mathbb{L}\mathsf{CCat}$*) is a linear differential category. Hence* $\mathbb{L}\mathsf{Mod}_!$ *(equivalently,* $\mathbb{L}\mathsf{CCat}_!$*) is a cartesian closed differential category.*