REPORT ON PAOLO PISTONE'S PHD MANUSCRIPT

JEAN-BAPTISTE JOINET
PROFESSEUR Á L'UNIVERSITÉ JEAN MOULIN - LYON 3

The thesis work of Paulo Pistone is a 215 pages manuscript (bibliography and annexes included) written in english, entitled "On Proofs and Types in Second Order Logic". Ideas and results are exposed with a very high degree of clarity, in a fluent language, in spite of a few minors typos noticed. Citations and references are numerous, coming from various areas of knowledge (philosophy, mathematical logic, computer science) and are appropriately convoked. The bibliography is rich, accurate and presented in a shape faithful to academic standards.

The manuscript is centrally devoted to *impredicativity*, the form of logical circularity pointed out by Poincaré as being the source of the antinomies of naïve set-theory. Reinvesting the debates around impredicativity which, from the beginning of the XXth century to our times, never ceased to be controversial, Paolo Pistone aims to show that a switching from the "referential" semantics of proofs (coming from the Logic tradition) to the interactional "untyped semantics" (developed from the theoretical computer science view) permits, through the concept of parametricity, to understand and in a sense to tame, the circularity due to impredicativity of second order logical systems.

Relevantly, the manuscript begins with a short "Prelude", devoted to Frege's *Grundgesetze*, analyzing the circularity due to impredicativity in Frege's (wrong) argumentation aiming at justifying his formalism for second order logic.

Chapter 1 then introduces historically the debates over the legitimacy of a second order logic, focusing on what Pistone call the *why* proof-theoretical investigations (attempts for an internal justification of logical rules : Prawitz, Dummett), rather than on set-theoretical or model theoretical interpretations (Quine, Shapiro), and contrasting it with the proof-theoretical investigations inspired by theoretical computer science, where circularity and auto-applications occurring are common instances of standard recursive technics.

Chapter 2, entitled "Arithmetics, logic and type theory" is so to speak a crash (but precise and brillant) presentation of second order arithmetic (Heyting and

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Peano arithmetics HA^2 or PA^2), of second order logic (LJ^2 and LK^2) and of System F.
Pistone starts by recalling some conceptual, historical and philosophical aspects about theses systems. The rest of the chapter, is devoted to present functors (from the first and to the second, and from the second to the third) preserving proofs normalization, hence showing thus that they essentially represent the same proof-theoretical contents. After having presented and studied the "Dedekind functor" (from arithmetics to second order logic), Pistone smartly juggles with all the corollaries of its existence, notably for comparing the arithmetical and the logical hierarchies, rephrasing Gdel incompleteness, linking failure of subformula property and failure of completeness. He then presents and studies the "Forgetful functor" (from second order logic to System F). And, finally, presents the direct interpretation of arithmetics in system F, via the composition of the two functors. In a last subsection, Pistone quickly presents extension of System F which are investigated only in chapter 6.

Chapter 3 confronts in one hand the proof theoretic tradition about inferential content of proofs (proof-theoretic meaning and validity, from Gentzen to Prawitz, Dummett, Martin-Löf), in the other hand the realizability and interactional (Kleene, Gödel...) tradition, restricting himself to the first order case (the second order case, being investigated in chapter 4).
Pistone roots this confrontation in the history of semantical ideas, starting with the debates on the semantical status of axioms from Frege-Hilbert views to Carnap and Wittgenstein's ones on axioms as definitions (Pistone omits more ancient similar views in Poincaré or, even prior, Gergonne). Pistone then focuses on the history of debates about internal normativity for rules (notably with respect to consistency) and in particular on the thesis that introduction rules are self-justifying and provides a definition of validity (based on the fact that a valid derivation should reduce to a derivation ending with an introduction rule for the principal connective of its conclusion : the "last rule condition") and on Prawitz's inversion principle (however not powerful enough to characterize validity).
His historical panorama on Proof-theoretic semantics, goes from Gentzen's germinal observations to Prawitz's notion of validity (here recalled for the implicative fragment of intuitionistic logic, also as an anticipation of the investigations on computability or reducibility properties of λ-terms presented in the next sub-section) and Dummett's semantical conclusions. He then dedicates a section to realizability (mainly) of intuitionistic proofs and reducibility. In a first time, are successively presented Kleene original work, then Kreisel's one. The subsection ends with a synthetic, cristal clear comparaison of the space of realizability interpretations (whose main features are listed as: the polymorphism of realizers, the proof-irrelevance of atomic sentences and the incompleteness with respect to realizable sentences) which underlines in which respects they overcome "Proof-theoretic semantics". In

passing, Pistone also clearly explain why realizability for classical logic needs new ingredients.
He then approaches the Tait-Girard reducibility technics (very clearly presented) and ends with a relevant comparaison of validity and realizability/reducibility approaches, making explicit in what respect the second one(s) corresponds to a true switch of point of view : from epistemic, intuitionistic considerations (the special status imputed to canonical proofs), to purely interactional ones.
Finally, Pistone propose conceptual and philosophical insights on the idea of untyped semantics. He notably and relevantly convokes the kantian distinction between constitutive and regulative norms (p.82) as recently retrieved by Searle. The subsection ends with considerations about the incompleteness of pure interpretations (some realizers are not proofs), distinguishing two sources for it. The first one is just linked with the "proof-irrelevance" of the interpretation in realizability semantics (for atomic atomic propositions and more generally for Harrop's formulas); the second one, more deep, being linked with the overcoming of limits in the arithmetical hierarchy (as showed by Theorem 3.2.2).

Chapter 4 is devoted to evaluate whether second order cut-elimination is able to play a justificative role (as it is the case first-order cut-elimination within a proof-theoretic theory of validity) in spite of the observed circularity. The methodology of the chapter is clear.
Pistone first recalls and carefully analyses how parametrization (in Girard's strong normalization proof for System F) permits to avoid circular quantification over sets. He then recall the existence of an elementary procedure to replace cuts by means of instances of the comprehension rule, which shows that the distinction between cut-free and non cut-free derivation (so as the one between canonical and non canonical derivations), in the second order case, can no more be thought as capturing structural properties of derivations (a "motto" of proof-theoretic semantics).
He then investigates and compares all the debates against and pro impredicativity (Poincaré, Russell, Gödel, Carnap, Martin-Löf, Sundholm), and already points out that some arguments given are based on an incorrect way to define reducibility and validity for second order formulae, i.e. more generally to assign meaning (proof-theoretically) to the impredicative universal quantifier. Pistone shows the substantial harmlessness of the listed standard objections when the viewpoint of the untyped interpretation of proofs is adopted (his critics prefigures the explanation of impredicative quantification, based on the notion of parametric polymorphism, given in the next chapter).
After that, Pistone try to characterizes in appropriate terms the "subtle" form of circularity (different from Poincars notion of vicious circularity) which nevertheless exists : an argument which justifies the fact that a program is a realizer of an impredicative type must employ an impredicative comprehension principle

(p.103). After having examined various classifications about "kinds" of circularity (Dummett's distinction between Pragmatic and Epistemic circularities; Alston's epistemical circularity), he argues that the"subtle" form of circularity identified, is the (somehow harmless) epistemical one.
Pistone dedicates the last section to exhibit the specific epistemic circularity at work in the proof of the second order Hauptsatz, and then shows how the (set-theoretical) frame needed reflects the properties of the type system. In particular, Pistone ends the chapter with showing that shaping the reducibility argument for Martin-Löf's - inconsistent - impredicative type theory leads to consider a "seemingly" inconsistent set-theory ("reflection" of Girard's paradoxes in the set-theory needed).

Chapter 5, entitled "Impredicativity and parametric polymorphism", is the acme of the thesis. Pistone begins by recalling Reynolds result which discards the set-theoretical intuition of universal quantification as set-theoretical intersection (showing the relevance of Carnap's intuitions on universal second order quantification). He then gives a presentation of the dinatural interpretation of polymorphism and provides a syntactic proof of the Π^1-completeness theorem (the closed normal λ-terms in the reducibility of the universal closure of a simple type, are simply typable), by means of a syntactical formulation of the dinatural interpretation (via two given typed λ-terms representing the equations forced by the dinaturality condition) used to show that parametricity implies dinaturality. He then focuses on corollaries of the Π^1-completeness, which permit to describe the structure of closed normal reducible terms (analogous of the "last rule conditions" for reducibility).
This result constitutes the *belvedere* of Pistone's enterprise : reducibility implies parametricity, which implies dinaturality, and then (by Π^1-completeness) the last rule condition. So that a "bridge" between the inferentialist and the interactionist point of view can now be built. As he comments : "an impredicative notion (reducibility for universal types) is needed in order to recover the hierarchical, 'predicative', inner structure of normal derivations".

Chapter 6 develops a constructive point of view on impredicativity, by analyzing the typability problem, but now from an unification view point. Vicious circles in proofs correspond to recursive specification for types and the geometrical structure of these cycles is investigated. A typable term could be not normalizing (the typing analysis does not distinguish between correct and incorrect proofs). The main result of the chapter is theorem 6.2.1, which shows that all strongly normalizable λ-terms are compatible (a notion proposed by Pistone as a generalization of Sophie Malecki's notion of "compatibility" between the constraints forced by recursive type equations, proves) through a combinatorial characterization of typability in System F. Some results are also shown concerning System U^- presented in a prior chapter (some of the results are steps toward a conjecture about characterization of typability in U^-, others are orignal results about System U^-).

Finally, **Chapter 7** sketches future lines of research.

Pistone's thesis could be equally considered as a panoramic work in *history of logic* (the thesis is built over a large, erudite survey on proof theory and type theory, gathering, organizing and explaining a very large amount of works from the XXth and XXIth centuries logic corpus), as a genuine, creative work in *philosophy of logic* (his original conceptual analysis are in a permanent and dense dialogue with the texts of the philosophical and epistemological tradition around logic) and, last but not least, as a valuable contribution to *mathematical logic* (the thesis including original, deep mathematical res lts).

Evidently, Paolo Pistone deeply masters the fields he investigates. His manuscript shows a particularily clear understanding of the results that he gathers, explains or proves and an amazing pedagogical capacity to explain to the reader the conceptual meaning to those results. The numerous historical, philosophical and logical sources that he convokes are relevantly introduced w.r.t. the purpose and demonstrate a wide culture (in mathematical logic, as well as in philosophy of logic). His work presents a synthetic vision over all of the field, a capacity not so frequent at the level of a PhD work. Finally, this philosophical and conceptual dimension of his thesis, results in deep mathematical results (concentrated into chapters 5 and 6).

In short, I consider that Paolo Pistone's dissertation highly deserves to be defended.

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Jean-Baptiste Joinet

ENS

Ecole Normale Supérieure

45, Rue D'Ulm, 75005 Paris

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Report on Paolo Pistone's PhD Thesis : « On Proofs and Types in Second Order Logic »

I has been a pleasure to go back to my old interests in Type Theory, by reading Pistone's thesis. First, it is extremely well written, second it develops concepts and results by “walking on the border” of paradoxes, yet by “staying consistent”: an excellent way to work on maximally expressive and challenging theories.

Paradoxical theories are the main source of inspiration of this work, beginning with Frege's blatant mistake (who ever wrote mathematics with no types ? a function usually goes from a space A to a space B, which are both different from A   B). Too much fuss has then been made on some triviality on barbers, a joke to be told in a barber's shop on saturday. Pistone instead highlights the intrinsic challenge of actual impredicativity, which will provide a guideline from there till very modern systems. As a matter of fact, a similar mistake to Frege's was iterated by Church, at his first formal writing of type-free -calculus, in 1932. Curry's paradox followed, a minor, yet smarter, variant of the barber's joke: once the contradiction was fixed, the paradox became a very expressive operator in the type-free, yet consistent, system – and one could talk of functions in spaces isomorphic to their function spaces.

An increasing depth is then found in Martin-Löf's error, a key issue in the thesis: Martin-Löf dared to propose a very strong and refined form of impredicativity, facing the abyss. As a consequence, the derived paradox, proved by Girard, is particularly deep and allows much richer developments, also explored in the thesis.

This lively beginning of the thesis leads to an elegant philosophical reflection on “explanation”: the distinction between “how” and “why”, made much clearer than usual by the rigorous context of Type Theory. The impact of Computer Science on Logic in this switch from the “why” to the “how”, in Logic, is very clearly presented: second order, impredicative, systems do work as very expressive parametric languages.

The thesis surveys, in a very scholarly fashion, the analytical reflections on universal quantification. The prevailing interpretation of “for all” by … *for all* (the italics is important) is a typical example of the tarskian approach to “truth”, like in the understanding of “the snow is white”, which happens to be true when *the snow is white* (the italics provides the mathematical contribution). The dominating role of arithmetic induction in foundational analyses is at the core of this trivial transfer from “syntax to semantics”: induction is meant to scan all individuals in a well ordered field. In absence of well ordering, “for all”, in mathematics, is not proved by scanning all instances of the intended object, but by proving the *genericity* of the considered object in a proof. This goes together with the delicate issue of proving the right level of generality of a mathematical theorem, based on a structure which is considered specific and it is instead generic (the history of Yoneda lemma is paradigmatic for this: it does not hold only over Set, a specific category, but over *any* Topos satisfying some reasonable conditions: Set is generic, in a suitable context).

A minor disagreement on a reference in the thesis. Poincaré understands that Russell's paradox is just a (very shallow) contradiction, in an otherwise sterile approach (the empty formalization by logic) and that impredicativity is a different and more delicate issue. He and Hilbert, two immense mathematicians, were dealing with the delicate issue of "giving good definitions", a true need after a century of fantastic mathematical inventions and scattered lack of rigor (Cauchy, Poinsot … had made mistakes by not "well defining" uniform continuity, derivability …). Poincaré himself had fallen in a similar mistake in an early proof of the Three-Body problem. Then, when so much noise popped out of barbers who cannot shave themselves, he couldn't stop laughing and declare clumsy formalisms and Russell's joke "not sterile" and proposed to restrict definitions in a predicative way. Yet, his mathematics, beginning with "analysis situ" (topology, Lebesgue measure …), is full of impredicative definitions, as a key expressive tool.

The thesis then extensively focuses on the Curry-Howard correspondence. This is particularly pertinent for the "how" vs. "why" discussion and for the very sound hints given to Krivine's program. Dummett and Prawitz are extensively quoted and their approaches are presented by a close analysis of proofs as (un-)typed programs and of the behavioral content of proofs, i.e. the way in which they interact through the cut-elimination algorithm. An interactionist point of view is then traced back to Kleene's realizability and to the Tait/Girard reducibility technique.

As pointed out, Carnap defended impredicative definitions as a lonely wolf. As Pistone observes, his argument has been taken up by the author of this letter and collaborators, many years later, as a tool to understand the genericity of objects (and structures) in mathematics, a way to (later) analyse genericity also in physics, a form of theoretical and empirical invariance of physical objects. Mathematical and physical, very reasonable, co-constitution of knowledge is largely based on this shared use of genericity of objects and structures. This is also related to the proof theoretic investigation that lead to the Genericity Theorem, generously quoted in the Thesis: second order terms use each input type as a generic ones, or, these terms, as functions from types to terms, never intersect. This is a form of regularity which does not reduce the expressivity of system F at all, it only says how elegantly it allows to deal with the delicate issue of genericity in mathematics. (On this, I would just recommend Pistone to refer to the exact statement of our Genericity Theorem, as the technical notion of "context",   , is very important for the theorem; in TeX: Theorem 9.3 (Genericity.) Let $\Gamma\vdash M, N:\forall X.\sigma$. If $M\tau = _{F_c} N\tau$ for some type τ, $M=_{F_c}N$.

Categories step in the thesis by the notion of dinaturality, as "parametricity implies dinaturality". Then, from a syntactic formulation of a dinaturality criterion, Pistone derives an equational characterization of reducible terms, a very elegant result. In discussions with Sergei Soloviev, Sergei pointed out that the notion of generalized natural transformation (due to Eilenberg and Kelly, used also by Kelly and Mac Lane in "Coherence in Closed Categories", 1971) is more general than the ordinary natural transformations; moreover, it is closed under composition, in contrast to the dinatural transformations to which Pistone refers. This may have some interesting consequences on the author's completeness theorem (5.2.2, 5.2.3).

Further on, Pistone shows that an impredicative notion (reducibility for universal types) is needed in order to recover the hierarchical, "predicative", inner structure of normal derivations. The interest of this result is that it allows to combine, in a uniform frame, the two interpretations of proofs, as programs and as rule-based constructions.

In a concluding part, Pistone proves some original results on "compatibility" between constraints in recursive type equations. Strong normalizing terms are proved to be compatible. Finally, the thesis closely analyses constraints in extensions of system F, by an original analysis of type inference in these extensions.

In summary, the thesis develops first a relevant and rigorous philosophical analysis of impredicativity, based a broad historical and conceptual reconstruction. The novel insight brought by this comparative investigation and the conceptual contributions are as philosofically interesting, as they are belated: for too long, too many trivialities have been said on this matter. Thus, it was not evident, in my opinion, to explore new paths. The concluding chapters contain some technical contributions which are also relevant and open new questions for further work: an interesting conjecture is formulated.

The thesis is an excellent philosophical and mathematical work, thus, it deserves, in my opinion, to be considered for the attribution of the intended doctoral title.

Best regards, sincerely

Giuseppe Longo
http://www.di.ens.fr/users/longo
Centre Cavaillès, République des Savoirs,
CNRS, Collège de France et Ecole Normale Supérieure, Paris,
and Department of Integrative Physiology and Pathobiology,
Tufts University School of Medicine, Boston