15 Local Lipschitz Property

In this section we show that a function $f : Q^X \to Q$ continuous over $(0,\infty)^X$ and concave (hence in particular and tropical analytic function) is locally Lipschitz-continuous over $(0,\infty)^X$.

Actually, in the argument below the continuity of f can be replaced by the property that f admits a maximum ($< \infty$) over any compact set. This property holds for any tropical analytic function f such that $\infty \notin \mathrm{Im}(f)$, independently from its continuity, since for all $\mu \in \mathcal{M}_{\mathrm{f}}(X)$, $f(\mathbf{x}) \leqslant \phi_\mu(\mathbf{x})$, where $\phi_\mu(\mathbf{x}) = \widehat{f}_\mu + \mu\mathbf{x}$, $\widehat{f}_\mu < \infty$, and the latter function is linear, so it admits a maximum over any compact set).

Hence the argument below yields a new proof that tropical analytic functions whose image does not contain ∞ are continuous over $[0,\infty)^X$.

Lemma 40. *Let $f : [0,\infty)^X \to Q$ be concave, monotone increasing, and continuous. Let $\mathbf{x} \neq \mathbf{y} \in [0,\infty)^X$, with $\|\mathbf{x}-\mathbf{y}\|_\infty < \infty$, and let $S(\mathbf{x},\mathbf{y}) = \{\alpha\mathbf{x}+(1-\alpha)\mathbf{y} \mid \alpha \in [0,1]\}$ be the segment generated by $\mathbf{x}$ and $\mathbf{y}$. Then f is Lipschitz-continuous over $S(\mathbf{x},\mathbf{y})$.*

Proof. Let us prove the lemma under the assumption that for all $a \in X$, $\mathbf{y}_a - \mathbf{x}_a \geqslant 1$. From the fact that the claim holds under the assumption, we can deduce the claim of the lemma: indeed for $\alpha \in (0,1)$ large enough we have that $\mathbf{y}' := \frac{\mathbf{y}-\alpha\mathbf{x}}{1-\alpha}$ is such that $\mathbf{y} \in S(\mathbf{x},\mathbf{y}')$ (and thus $S(\mathbf{x},\mathbf{y}) \subseteq S(\mathbf{x},\mathbf{y}')$) and $\mathbf{y}'_a - \mathbf{x}_a \geqslant 1$. Hence from our proof we deduce that f is Lipschitz-continuous over $S(\mathbf{x},\mathbf{y}')$, and thus a fortiori over $S(\mathbf{x},\mathbf{y})$ too.

Since f is continuous over $[0,\infty)^X$ and $S(\mathbf{x},\mathbf{y})$ is compact, f admits a maximum MAX over $S(\mathbf{x},\mathbf{y})$. For all $\mathbf{z} < \mathbf{z}' \in S(\mathbf{x},\mathbf{y})$, let $M(\mathbf{z},\mathbf{z}') \in Q^X$ be defined by

$$M(\mathbf{z},\mathbf{z}')_a = \frac{f(\mathbf{z}') - f(\mathbf{z})}{\mathbf{z}'_a - \mathbf{z}_a}$$

Observe that

$$M(\mathbf{x},\mathbf{y})_a = \frac{f(\mathbf{y}) - f(\mathbf{x})}{\mathbf{y}_a - \mathbf{x}_a} \leqslant f(\mathbf{y}) - f(\mathbf{x}) \leqslant \mathrm{MAX}$$

using the fact that $\mathbf{y}_a - \mathbf{x}_a \geqslant 1$.

We now claim that $M(\mathbf{z},\mathbf{z}')$ is contravariant in both $\mathbf{z}$ and $\mathbf{z}'$. Indeed suppose $\mathbf{z} \leqslant \mathbf{z}'' < \mathbf{z}'$, so that $\mathbf{z} = \lambda\mathbf{z}' + (1-\lambda)\mathbf{z}''$ for some $\lambda \in (0,1)$. Then, using the fact that f is concave, we have

$$\begin{aligned} M(\mathbf{z},\mathbf{z}')_a &= \frac{f(\mathbf{z}') - f(\lambda\mathbf{x} + (1-\lambda)\mathbf{z}'')}{\mathbf{z}'_a - \lambda\mathbf{x}_a - (1-\lambda)\mathbf{z}''_a} \\ &\geqslant \frac{f(\mathbf{z}') - \lambda f(\mathbf{z}') - (1-\lambda)f(\mathbf{z}'')}{\mathbf{z}'_a - \lambda\mathbf{z}'_a - (1-\lambda)\mathbf{z}''_a} \\ &= \frac{(1-\lambda)(f(\mathbf{z}') - f(\mathbf{z}''))}{(1-\lambda)\mathbf{z}'_a - \mathbf{z}''_a} = M(\mathbf{z}'',\mathbf{z}') \end{aligned}$$

In a similar way it is shown that for $\mathbf{z} < \mathbf{z}'' \leqslant \mathbf{z}'$, $M(\mathbf{z},\mathbf{z}') \leqslant M(\mathbf{z},\mathbf{z}'')$.

Therefore, for all $\mathbf{z} < \mathbf{z}' \in S(\mathbf{x},\mathbf{y})$, we have that $M(\mathbf{z},\mathbf{z}')_a \leqslant M(\mathbf{x},\mathbf{z}')_a \leqslant M(\mathbf{x},\mathbf{y})_a \leqslant \mathrm{MAX}$. From this, using the fact that f is monotone increasing, we deduce that $|f(\mathbf{z}') - f(\mathbf{z})| = f(\mathbf{z}') - f(\mathbf{z}) \leqslant \mathrm{MAX} \cdot |\mathbf{z}'_a - \mathbf{z}_a|$ and thus that

$$|f(\mathbf{z}') - f(\mathbf{z})| \leqslant \mathrm{MAX} \cdot \|\mathbf{z}' - \mathbf{z}\|_\infty$$

that is, that f is MAX-Lipschitz over $S(\mathbf{x}, \mathbf{y})$. $\square$

Proposition 41. *Let $f : [0, \infty)^X \to Q$ be concave, monotone increasing, and continuous. For all $\epsilon \in (0, \infty)$ and $\mathbf{x} \in [0, \infty)^X$, f is Lipschitz-continuous over the open ball $B(\mathbf{x}, \epsilon)$.*

Proof. Let MAX indicate the maximum of f over $B(\mathbf{x}, \epsilon)$. Let $\mathbf{y}, \mathbf{z} \in B(\mathbf{x}, \epsilon)$; then $\|\mathbf{y} - \mathbf{z}\|_\infty \leqslant 2\epsilon < \infty$, so by the lemma above f is K-Lipschitz over the segment $S(\mathbf{y}, \mathbf{z})$ for some $K \leqslant$ MAX, so we deduce $|f(\mathbf{y}) - f(\mathbf{z})| \leqslant \text{MAX} \cdot \|\mathbf{y} - \mathbf{z}\|_\infty$. $\square$

Theorem 42 (local Lipschitz-continuity)**.** *Let $f : [0, \infty)^X \to Q$ be concave, monotone increasing, and continuous. Then f is locally Lipschitz-continuous.*

Proof. For all $\mathbf{x} \in [0, \infty)^X$, f is Lipschitz-continuous over the open set $B(\mathbf{x}, 1)$. $\square$

WHAT HAPPENS FOR $\mathbf{x} \in Q^X$?

Remark 3. *If by a* linear *function we indicate a function of the form $f(\mathbf{x}) = \mu\mathbf{x} + c$, then the tropical analytic functions are not in general piecewise linear. For instance, let $f : Q^{\mathbb{N}} \to Q$ be given by $\widehat{f}_{[n]} = 0$ and $\widehat{f}_\mu = \infty$ for μ not a singleton. In other words, $f(\mathbf{x}) = \inf_n \mathbf{x}_n$.*

Let $\mathbf{x}$ be defined by $\mathbf{x}_n = 2^{-n}$. Then for all $\mathbf{y}$ in the segment $S(\mathbf{x}, \mathbf{x} + 1)$, $f(\mathbf{y})$ is an inf which is never a min. More precisely, we have that $\mathbf{y} = \alpha\mathbf{x} + (1 - \alpha)(\mathbf{x} + 1) = \alpha\mathbf{x} + \mathbf{x} - \alpha\mathbf{x} + 1 - \alpha = \mathbf{x} + (1 - \alpha)$, and $f(\mathbf{y}) = \inf_n 2^{-n} + (1 - \alpha) = 1 - \alpha$. We can show that for any linear function $g(\mathbf{x}) = \mu\mathbf{x} + c$, g does not coincide with f over $S(\mathbf{x}, \mathbf{x} + 1)$. Indeed, either g is a constant function, or $g(\mathbf{x} + \alpha) = \mu\mathbf{x} + \sharp\mu \cdot \alpha + c = \left(\sum_{j=i_1,\dots,i_k} \mu(j) \cdot (2^{-j} + \alpha)\right) + c > \alpha$.

Let now $\mathbf{x}^k$ be given by $\mathbf{x}^k_n = 2^{-\min\{n,k\}}$. Notice that $f(\mathbf{x}^k) = 2^{-k}$. Then for all $\mathbf{y}$ in the segment $S(\mathbf{x}, \mathbf{x}^k)$, $f(\mathbf{y}) = f(\alpha\mathbf{x} + (1 - \alpha)\mathbf{x}^k) = \inf_{n \geqslant k} \alpha 2^{-n} + (1 - \alpha)2^{-k} = (1 - \alpha)2^{-k}$. This means that f is linear in all the segment except for its extremal point $\mathbf{x}$. Still, f is continuous on $S(\mathbf{x}, \mathbf{x}^k)$, since $\lim_{\mathbf{y} \to \mathbf{x}} f(\mathbf{y}) = \lim_{\alpha \to 1}(1 - \alpha)2^{-k} = 0 = f(\mathbf{x})$.

15.1 Minimum Spaces

Definition 6. *Let $I \subseteq Q$ and $x \in I$. x is said to be a* strong minimum *of I if the following conditions hold:*

- *x is a minimum of I, i.e. for all $y \in I$, $x \leqslant y$;*
- *there exists $\epsilon > 0$ such that for all $y \neq x \in I$, $x + \epsilon \leqslant y$.*

Definition 7. *Given $\mathbf{x}, \mathbf{y} \in Q^X$, let us define*

$$\mathbf{x} \perp_{\mathsf{m}} \mathbf{y} \textit{ iff } \{\mathbf{x}_a + \mathbf{y}_a \mid a \in X\} \textit{ has a strong minimum}$$

A minimum space *is a pair $X = (|X|, \mathsf{m}(X))$, where $|X|$ is a set and $\mathsf{m}(X) \subseteq Q^X$ is a Q-module satisfying $\mathsf{m}(X) = \mathsf{m}(X)^{\perp_{\mathsf{m}}\perp_{\mathsf{m}}}$.*

Given minimum spaces X and Y, the minimum spaces $X^\perp$ and $X \otimes Y$ are defined by $|X^\perp| = |X|$ and $\mathsf{m}(X^\perp) = \mathsf{m}(X)^{\perp_{\mathsf{m}}}$, and by $|X \otimes Y| = |X| \times |Y|$ and $\mathsf{m}(X \otimes Y) = \{\mathbf{x} \cdot \mathbf{y} \mid \mathbf{x} \in \mathsf{m}(X), \mathbf{y} \in \mathsf{m}(Y)\}^{\perp_{\mathsf{m}}\perp_{\mathsf{m}}}$, where $(\mathbf{x} \cdot \mathbf{y})_{\langle a,b \rangle} = \mathbf{x}_a \cdot \mathbf{y}_b$.

A minimum linear map *from X to Y is a linear map $f : Q^X \to Q^Y$ such that $\widehat{f} \in \mathsf{m}(X \otimes Y^\perp)^\perp$.*

Observe that all $\mathbf{x} \in Q^X$ with finite and non-empty support are in $\mathsf{m}(X)$: for all $\mathbf{y} \in Q^X$, $\inf_{a \in X} \mathbf{x}_a + \mathbf{y}_a = \min_{a=a_1,\ldots,a_n} \mathbf{x}_a + \mathbf{y}_a$.

Definition 8 (piecewise linear function)**.** *A function $f : Q^X \to Q$ is said* piecewise linear *if for all $\mathbf{x} \in Q^X$ there exists a segment S containing $\mathbf{x}$ and a linear function $g_S : Q^X \to Q$ such that $f(\mathbf{y}) = g_S(\mathbf{y})$ for all $\mathbf{y} \in S$.*

Proposition 43. *Minimum linear maps $f : \mathsf{m}X \to \mathsf{m}Y$ are piecewise linear functions.*

Proof. Let $\mathbf{x} \in \mathsf{m}(X)$ and $b \in Y$, $\widehat{f} \perp_{\mathsf{m}} (\mathbf{x} \cdot e_b)$, that is $f(\mathbf{x}) = \inf_{a \in X} \widehat{f}_{a,b} + \mathbf{x}_a$ coincides with the strong minimum of $\{\widehat{f}_{a,b} + \mathbf{x}_a \mid a \in X\}$, say $\widehat{f}_{a_0,b} + \mathbf{x}_{a_0}$. By definition then, there exists $\epsilon > 0$ such that for all $a \in X$, either $\widehat{f}_{a,b} + \mathbf{x}_a = \widehat{f}_{a_0,b} + \mathbf{x}_{a_0}$, or $\widehat{f}_{a,b} + \mathbf{x}_a \geqslant \widehat{f}_{a_0,b} + \mathbf{x}_{a_0} + \epsilon$. Let us partition X as $X_1 \oplus X_2$ where $a \in X_1$ iff $\widehat{f}_{a,b} + \mathbf{x}_a = \widehat{f}_{a_0,b} + \mathbf{x}_{a_0}$, and $a \in X_2$ otherwise. Hence $\mathbf{x}$ can be written as $\min\{\mathbf{x}_1, \mathbf{x}_2\}$, where $(\mathbf{x}_1)_a = \mathbf{x}_a$ if $a \in X_1$ and is ∞ otherwise, and $(\mathbf{x}_2)_a = \mathbf{x}_a$ if $a \in X_2$ and is ∞ otherwise.

Then for all $\mathbf{y} \in S(\min\{\mathbf{x}_1 + \epsilon, \mathbf{x}_2\}, \min\{\mathbf{x}_1, \mathbf{x}_2 - \epsilon\})$, $f(\mathbf{y}) = \widehat{f}_{a_0,b} + \mathbf{y}_{a_0}$, where "$-$" indicates truncated subtraction (i.e. with $x - y = 0$ if $y \geqslant x$): indeed, there exists $\alpha \in (0,1)$ such that $\mathbf{y}_a = \mathbf{x}_a + \alpha\epsilon$ if $a \in X_1$, and $\mathbf{y}_a = \mathbf{x}_a + (1-\alpha)\epsilon$ if $a \in X_2$. Then, if $a \in X_1$, $\widehat{f}_{a,b} + \mathbf{y}_a = \widehat{f}_{a,b} + \mathbf{x}_a + \alpha\epsilon = \widehat{f}_{a_0,b} + \mathbf{x}_{a_0} + \alpha\epsilon = \widehat{f}_{a_0,b} + \mathbf{y}_{a_0}$; if $a \in X_2$, $\widehat{f}_{a,b} + \mathbf{y}_a = \widehat{f}_{a,b} + (\mathbf{x}_a - (1-\alpha)\epsilon) \geqslant (\widehat{f}_{a,b} + \mathbf{x}_a) - (1-\alpha)\epsilon \geqslant \widehat{f}_{a_0,b} + \mathbf{x}_{a_0} + \alpha\epsilon = \widehat{f}_{a_0,b} + \mathbf{y}_{a_0}$, using the fact that $x + (y - z) \geqslant (x + y) - z$ and that $\widehat{f}_{a,b} + \mathbf{x}_a \geqslant \widehat{f}_{a_0,b} + \mathbf{x}_{a_0} + \epsilon$.

We conclude then that f is linear on a segment containing $\mathbf{x}$. $\square$

Example 9. *On $Q^{\mathbb{N}}$ define a minimum space by letting $\mathsf{m}Q^{\mathbb{N}}$ be made of vectors of the form $\mathbf{x}_n = nx$, for some $x \in Q$. Notice that $\mathbf{y} \in (\mathsf{m}Q^{\mathbb{N}})^{\perp}$ iff for all $x \in Q$ $\inf_n \mathbf{y}_n + nx$ is a strong minimum. The case $x = 0$ implies then that $\inf_n \mathbf{y}_n$ is a strong minimum, say y_{n_0}. Hence in general $\inf_n \mathbf{y}_n + nx = \min\{\mathbf{y}_k + kx \mid k \leqslant n_0\}$. As a consequence, a linear map $f : \mathsf{m}Q^{\mathbb{N}} \to Q$ must be a tropical polynomial.*

16 Q-modules and non-expansive functions

On a Q-module $\mathcal{M}$ one has two canonical operations:

- the *action* $\star : Q \times \mathcal{M} \to \mathcal{M}$;
- the *semi-distance* $\ominus : \mathcal{M} \times \mathcal{M} \to Q$, given by

$$\mathbf{x} \ominus \mathbf{y} = \inf\{\epsilon \mid \mathbf{y} \star \epsilon \geqslant \mathbf{x}\}$$

  Notice that $\mathbf{x} \ominus \mathbf{y} \leqslant \delta$ iff $\mathbf{y} \star \delta \geqslant \mathbf{x}$.

Notice that the semi-distance defines a *quasi-metric* $q_{\mathcal{M}}(\mathbf{x}, \mathbf{y}) = \mathbf{x} \ominus \mathbf{y}$:

QM1 $q_{\mathcal{M}}(\mathbf{x}, \mathbf{x}) = 0$;

QM2 $q_{\mathcal{M}}(\mathbf{x}, \mathbf{z}) + q_{\mathcal{M}}(\mathbf{z}, \mathbf{y}) \geqslant q_{\mathcal{M}}(\mathbf{x}, \mathbf{y})$.

This induces a *Q-enriched category* on $\mathcal{M}$, with objects the elements of $\mathcal{M}$ and hom-sets $\mathcal{M}(\mathbf{x}, \mathbf{y}) = q_{\mathcal{M}}(\mathbf{x}, \mathbf{y})$.

One obtains a canonical metric on $\mathcal{M}$ by symmetrizing $\ominus$:

$$d_{\mathcal{M}}(\mathbf{x}, \mathbf{y}) = \max\{q_{\mathcal{M}}(\mathbf{x}, \mathbf{y}), q_{\mathcal{M}}(\mathbf{y}, \mathbf{x})\}$$

Proposition 44. **M1** *$d_{\mathcal{M}}(\mathbf{x}, \mathbf{y}) = 0$ iff $\mathbf{x} = \mathbf{y}$;*

M2 *$d_{\mathcal{M}}(\mathbf{x}, \mathbf{y}) = d_{\mathcal{M}}(\mathbf{y}, \mathbf{x})$;*

M3 *$d_{\mathcal{M}}(\mathbf{x}, \mathbf{z}) + d_{\mathcal{M}}(\mathbf{z}, \mathbf{y}) \geqslant d_{\mathcal{M}}(\mathbf{x}, \mathbf{y})$.*

Proof. The only thing which does not follow trivially from QM1, QM2, is that if $d_{\mathcal{M}}(\mathbf{x}, \mathbf{y}) = 0$, then $\mathbf{x} \geqslant \mathbf{y}$ and $\mathbf{y} \geqslant \mathbf{x}$, and since $\geqslant$ is an order, $\mathbf{x} = \mathbf{y}$. □

This induces a *involutive Q-enriched category* on $\mathcal{M}$, with $\mathcal{M}(\mathbf{x}, \mathbf{y}) = d_{\mathcal{M}}(\mathbf{x}, \mathbf{y})$.

If $\mathcal{M}$ is *complete* (as a poset), then a third canonical operation can be defined

- the *op-action* $\multimapdotinv : \mathcal{M} \times Q \to \mathcal{M}$, given by

$$\mathbf{x} \multimapdotinv \epsilon = \inf\{\mathbf{y} \mid \mathbf{y} \star \epsilon \geqslant \mathbf{x}\}$$

In particular, we have a dual Q-module $\mathcal{M}^{\mathrm{op}}$ with inverted order and action given by $\multimapdotinv$. Moreover, any map $f : \mathcal{M} \to \mathcal{N}$ admits an *adjoint* $f^* : \mathcal{N}^{\mathrm{op}} \to \mathcal{M}^{\mathrm{op}}$ given by

$$f^*(\mathbf{y}) = \inf\{\mathbf{x} \mid f(\mathbf{x}) \geqslant \mathbf{y}\}$$

Given Q-modules $\mathcal{M}, \mathcal{N}$, we indicate as:

- $Q\mathrm{Mod}(\mathcal{M}, \mathcal{N})$ the set of Q-module morphisms from $\mathcal{M}$ to $\mathcal{N}$, which are the monotone functions $f : \mathcal{M} \to \mathcal{N}$ that commute with $\star$ and all existing infs (also called *linear continuous maps*).
- $Q\mathrm{Met}(\mathcal{M}, \mathcal{N})$ the set of continuous non-expansive maps, i.e. maps that commute with all existing infs, and such that $d_{\mathcal{N}}(f(\mathbf{x}), f(\mathbf{y})) \leqslant d_{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ (from the viewpoint of Q-categories, these are the *continuous functors*).

- if $\mathcal{M}, \mathcal{N}$ are complete, $Q\mathrm{Met}^*(\mathcal{M}, \mathcal{N})$ is the set of continuous non-expansive maps with a continuous non-expansive adjoint (from the viewpoint of Q-categories, these are the *left-adjoint functors*).

Proposition 45. *For all complete Q-modules $\mathcal{M}$ and $\mathcal{N}$, $Q\mathrm{Mod}(\mathcal{M}, \mathcal{N}) \simeq Q\mathrm{Mod}(\mathcal{N}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}})$.*

Proposition 46. *i. All Q-module morphisms are non-expansive.*

ii. if $f : \mathcal{M} \to \mathcal{N}$ is non-expansive, then it is sublinear.

Proof. We have that $f(x) \ominus f(y) \leqslant x \ominus y$ iff $f(y) \star (x \ominus y) \geqslant f(x)$, but since f is a Q-module morphism $f(y) \star (x \ominus y) = f(y \star (x \ominus y)) \geqslant f(x)$, since f is monotone and $y \star (x \ominus y) \geqslant x$.

We have that $f(x \star \epsilon) \ominus f(x) \leqslant (x \star \epsilon) \ominus x = \epsilon$, which implies $f(x \star \epsilon) \leqslant f(x) \star \epsilon$. $\square$

Remark 4. *Neither finiteness spaces nor minimum spaces yield complete Q-modules. For instance, if $\mathbf{x}_i$ is a family of finitary sets, then $\inf_i \mathbf{x}_i$ needs not be finitary, since its support is $\bigcup_i |\mathbf{x}_i|$, and finitary sets are not closed by arbitrary (not even filtered) unions.*

Exponential Structure Let us focus on the category $Q\mathrm{Mod}$.

For all Q-module $\mathcal{M}$, we have the *Yoneda embedding*:

$$\mathcal{Y} : \mathcal{M} \to Q^{\mathcal{M}}$$

given by $\mathcal{Y}(\mathbf{x})(\mathbf{y}) = q_{\mathcal{M}}(\mathbf{x}, \mathbf{y})$.

Lemma 47. *$\mathcal{Y}$ is an injective Q-module morphism.*

Proof. Let us first show that it is injective: if $\mathcal{Y}(\mathbf{x}) = \mathcal{Y}(\mathbf{x}')$, then for all $\mathbf{y}$, $\mathbf{x} \ominus \mathbf{y} = \mathbf{x}' \ominus \mathbf{y}$, so in particular $\mathbf{x} \ominus \mathbf{x}' = \mathbf{x}' \ominus \mathbf{x}' = 0$ and $\mathbf{x}' \ominus \mathbf{x} = \mathbf{x} \ominus \mathbf{x} = 0$, whence $\mathbf{x} \leqslant \mathbf{x}'$ and $\mathbf{x}' \leqslant \mathbf{x}$, that is, $\mathbf{x} = \mathbf{x}'$. That $\mathcal{Y}$ is a Q-module morphism follows from the fact that $\mathbf{x} \ominus (-)$ is, for all $\mathbf{x} \in \mathcal{M}$. $\square$

For all matrix $A \in Q^{X \times Y}$, let $A^\dagger : Q^X \to Q^Y$ be the function defined by $A^\dagger(\mathbf{x})_b = \inf_c A_{c,b} + \mathbf{x}_c$.

Lemma 48. *For all $f \in Q\mathrm{Mod}(\mathcal{M}, \mathcal{N})$ there exists $\widehat{f} \in Q^{\mathcal{M} \times \mathcal{N}}$ such that*

$$\mathcal{Y}(f(a)) = (\widehat{f})^\dagger(\mathcal{Y}(a))$$

Proof. We let $\widehat{f}_{a,b} = q_{\mathcal{N}}(f(a), b) = \mathcal{Y}(f(a))(b)$. Then we have $\inf_c \widehat{f}_{c,b} + \mathcal{Y}(a)(c) = \inf_c \mathcal{Y}(f(c))(b) + \mathcal{Y}(a)(c) = \mathcal{Y}(f(a))(b)$.

$\square$

Definition 9 (Isbell hull)**.** *Let $\mathcal{M}$ be a Q-module. The* Isbell hull *of $\mathcal{M}$ is the complete Q-module* $\mathrm{Isb}(\mathcal{M}) \subseteq Q^{\mathcal{M}}$ *made of those $\mathbf{x} \in Q^{\mathcal{M}}$ such that for some $\mathbf{y} \in Q^{\mathcal{M}}$, and for all $a \in \mathcal{M}$*

$$\mathbf{x}_a = \inf_{b \in \mathcal{M}} q_{\mathcal{M}}(a, b) + \mathbf{y}_b$$

Lemma 49. *For all $a \in \mathcal{M}$, $\mathcal{Y}(a) \in \mathrm{Isb}(\mathcal{M})$.*

Proof. We have that

$$\begin{aligned}\mathcal{Y}(a)(c) &= q_{\mathcal{M}}(a, c)\\ &= \inf_c q_{\mathcal{M}}(c, b) + q_{\mathcal{M}}(a, b)\\ &= \inf_c q_{\mathcal{M}}(c, b) + \mathcal{Y}(a)(b)\end{aligned}$$

□

Definition 10. *Let $\mathcal{M}$ be a Q-module. We define a Q-module $!\mathcal{M}$ as follows:*

- *for all μ, ν finite multisets on $\mathcal{M}$, let*

$$q^*_{\mathcal{M}}(\mu, \nu) = \begin{cases} \min_{\sigma \in \mathfrak{S}(n)} \sum_{i=1}^n q_{\mathcal{M}}(\mu_i, \nu_{\sigma(i)}) & \text{if } \ell(\mu) = \ell(\nu) = n \\ \infty & \text{otherwise} \end{cases}$$

- $!\mathcal{M} := \{\mathbf{x} \in Q^{\mathcal{M}_{\mathrm{f}}(\mathcal{M})} \mid \exists y \in Q^{\mathcal{M}_{\mathrm{f}}(\mathcal{M})}, \mathbf{x}_\mu = \inf_\nu q^*_{\mathcal{M}}(\mu, \nu) + \mathbf{y}_\nu\}$.

Lemma 50. *i. $!\mathcal{M}$ is a Q-module.*

ii. For all $a \in \mathcal{M}$, $!\mathcal{Y}(a) \in !\mathcal{M}$, where $!\mathcal{Y}(a)_{[b_1,\dots,b_n]} = \sum_{i=1}^n \mathcal{Y}(a)(b_i)$.

Proof. TODO. □

In a similar way we can define Q-modules $!_n\mathcal{M}$, by restricting to those multisets μ with $\sharp\mu \leqslant n$, and show $!_n\mathcal{Y}(a) \in !_n\mathcal{M}$ for all $a \in \mathcal{M}$.

Given $f \in Q\mathrm{Mod}(!\mathcal{M}, \mathcal{N})$ we define $\mathsf{D}(f) \in Q\mathrm{Mod}(\mathcal{M} \times !\mathcal{M}, \mathcal{N})$ as follows: let $\theta : \mathcal{M} \times !\mathcal{M} \to !\mathcal{M}$ be defined by

$$\theta(a, \mathbf{x})_\mu = \min_{b \in \mu} q_{\mathcal{M}}(a, b) + \mathbf{x}_{\mu - b}$$

We let then $\mathsf{D}(f)(a, \mathbf{x}) = f(\theta(a, \mathbf{x}))$.