# 16 Complete $Q$-categories and $Q$-modules.

In the following we let $Q$ indicate the Lawvere quantale $([0,\infty], \geqslant, +, 0)$. $Q$ can be seen as:

- a category $Q_0$ with a unique object and morphisms being the elements of $Q$, with identity $0$ and composition $+$;
- a degenerate category $Q_1$ with objects the elements of $Q$, and a morphism between $x$ and $y$ whenever $x \geqslant y$;
- a 2-category $Q_2$ with 0-cells and 1-cells as in $Q_0$, and 2-cells the morphisms of $Q_1$.

These categorical formulations of $Q$ will help us here and there.

We let $x - y$ indicate truncated subtraction. Notice that $x \leqslant (x-y)+y$ and $(x+y)-y \leqslant x$. This corresponds to the fact that "$\_ - y$", as a functor from $Q_1$ to itself, is *right-adjoint* to the functor "$\_ + y$". Equivalently, $x + y \geqslant z$ iff $x \geqslant z - y$.

$Q\mathsf{Rel}$ is the category of sets and $Q$-relations, where a $Q$-relation $R : X \to\!\!\!\!\!\!\mapstochar\;\; Y$ is a function $R : X \times Y \to Q$, with identity $1_X : X \to\!\!\!\!\!\!\mapstochar\;\; X$ given by

$$1_X(x,y) = \begin{cases} 0 & \text{if } x = y \\ \infty & \text{otherwise} \end{cases}$$

and composition of $R : X \to\!\!\!\!\!\!\mapstochar\;\; Y$ and $S : Y \to\!\!\!\!\!\!\mapstochar\;\; Z$ given by

$$(S \cdot R)(x,z) = \inf_{y \in Y} R(x,y) + S(y,z)$$

**Remark 4.** *$Q\mathsf{Rel}$ is also a (degenerate) 2-category, where for $R, S : X \to\!\!\!\!\!\!\mapstochar\;\; Y$, $R \Rightarrow S$ iff for all $x \in X$, $y \in Y$, $R(x,y) \geqslant S(x,y)$.*

Composition with $R : X \to\!\!\!\!\!\!\mapstochar\;\; Y$ in $\mathsf{QRel}$ yields a functor

$$R \cdot \_ : Q\mathsf{Rel}(A, X) \longrightarrow Q\mathsf{Rel}(A, Y)$$

which has a right-adjoint

$$\_ \multimapinv R : Q\mathsf{Rel}(A, Y) \longrightarrow Q\mathsf{Rel}(A, X)$$

given, for $S : A \to\!\!\!\!\!\!\mapstochar\;\; Y$ by

$$(S \multimapinv R)(a,x) = \sup_{y \in Y} S(a,y) - R(x,y)$$

## 16.1 $Q$-categories.

A *$Q$-category*, or *generalized (quasi-)metric space* is a small $Q$-enriched category. Concretely, it is given by a pair $(X, q)$ made of a set $X$ and a $Q$-relation $q : X \to\!\!\!\!\!\!\mapstochar\;\; X$ satisfying:

$$q(x,x) = 0 \tag{Q1}$$
$$q(x,z) + q(z,y) \geqslant q(x,y) \tag{Q2}$$

A $Q$-category is *skeletal* if it further satisfies

$$q(x,y) = 0 \ \Rightarrow \ x = y \tag{Q3}$$

and *symmetric* if it satisfies

$$q(x,y) = q(y,x) \tag{Q4}$$

For simplicity, in the following we indicate the distance function of a $Q$-category $X$ simply as $X(x,y)$.

**Remark 5.** *In more abstract terms, a $Q$-category can be identified with a* monad *in $Q\mathsf{Rel}$, seen as a 2-category: it is given by a 1-cell $q : X \nrightarrow X$ together with 2-cells $1_X \Rightarrow q$ and $q \cdot q \Rightarrow q$.*

We let $x \simeq y$ indicate the equivalence relation defined by $X(x,y) = X(x',y)$ holds for all $y \in X$. Notice that $x \simeq y$ coincides with $x = y$ precisely when $X$ is skeletal.

Every $Q$-category $X$ is also a preorder $(X, \preceq_X)$, with $x \preceq_X y$ iff $X(x,y) = 0$. Moreover, if $X$ is skeletal, then $\preceq_X$ is an order.

For every $Q$-category $X$, the category $X^{\mathrm{op}}$ has same objects as $X$, and $X^{\mathrm{op}}(x,y) = X(y,x)$.

**Example 10.** *$Q$ is a $Q$-category with the distance $Q(x,y) = y - x$. The category $Q^{\mathrm{op}}$ has distance $Q^{\mathrm{op}}(x,y) = Q(y,x)$.*

**Example 11.** *For every $Q$-category $X$, the distance $\mathcal{S}(X)(x,y) = \max\{X(x,y), X^{\mathrm{op}}(x,y)\}$ defines a symmetric $Q$-category $\mathcal{S}(X)$. In the case of $Q$, $\mathcal{S}(Q)$ is just the Euclidean distance.*

A *functor* between $Q$-categories $X$ and $Y$ is just a non-expansive function $f : X \to Y$, that is, satisfying $Y(f(x), f(y)) \leqslant X(x,y)$.

**Remark 6.** *Given $f : X \to Q$, $f$ is a functor precisely when for all $x, y \in X$ $f(x) \leqslant \inf_{x' \in X} f(x') + X(x',x)$. Indeed, if $f$ is a functor then $f(x) \leqslant f(x') + X(x',x)$, since $f(x) - f(x') = Q(f(x'), f(x)) \leqslant X(x',x)$. Conversely, if $f(x) \leqslant f(x') + X(x',x)$ holds for all $x'$, then $Q(f(x), f(x')) = f(x') - f(x) \leqslant X(x',x)$.*

**Example 12.** *For every $Q$-category $X$, the $Q$-category $[X,Q]$ has objects all functors from $X$ to $Q$ (where the latter is seen as a $Q$-category), and $[X,Q](f,g) = \sup_{x \in X} Q(f(x), g(x))$.*

*When $X$ has the discrete metric, $[X,Q] = Q^X$ and $\mathcal{S}([X,Q])(f,g) = \|f - g\|_\infty$.*

**Example 13.** *For every $Q$-categories $X, Y$ we can define:*

- *a $Q$-category $X \otimes Y$ with $(X \otimes Y)(\langle x,y \rangle, \langle x',y' \rangle) = X(x,x') + Y(y,y')$;*
- *a $Q$-category $X \times Y$ with $(X \times Y)(\langle x,y \rangle, \langle x',y' \rangle) = \max\{X(x,x'), Y(y,y')\}$.*

## 16.2 $Q$-distributors.

A *$Q$-distributor* $\Phi : X \nrightarrow Y$ is a $Q$-relation $\Phi : X \times Y \to Q$ satisfying

$$\Phi \cdot X \leqslant \Phi \qquad\qquad Y \cdot \Phi \leqslant \Phi$$

**Remark 7.** *Using Remark 6, it follows that $\Phi$ is a $Q$-distributor precisely when it is a functor $\Phi : X^{\mathrm{op}} \otimes Y \to Q$: the condition $\Phi(x, y) \leqslant X(x, x') + \Phi(x', y)$ implies that $\Phi$ is a functor over $X^{\mathrm{op}}$, while the condition $\Phi(x, y) \leqslant \Phi(x, x') + Y(x', y)$ implies that $\Phi$ is a functor over $Y$. But this implies then $\Phi(x, y) \leqslant X(x, x') + \Phi(x', y') + Y(y', y)$ and thus $Q(\Phi(x', y'), \Phi(x, y)) \leqslant X(x, x') + Y(y', y)$.*

The identity distributor $1_X : X \nrightarrow X$ is given by $(1_X)_{x,y} = X(x, y)$. Distributors $\Phi : X \nrightarrow Y$ and $\Psi : Y \nrightarrow Z$ compose via $\Psi \cdot \Phi : X \nrightarrow Z$. We let $Q\mathsf{Dist}$ indicate the category of $Q$-categories and $Q$-distributors.

**Remark 8.** *Usually distributors $\Phi : X \nrightarrow Y$ are presented as $Q$-relations $\Phi : Y \times X \to Q$ (notice the inversion of $X$ and $Y$), that is, as functors $\Phi : X \times Y^{\mathrm{op}} \to Q$. We chose to invert the presentation of distributors for uniformity with the usual notations in models of the differential $\lambda$-calculus.*

Any functor $f : X \to Y$ induces two adjoint distributors $f^\circ : X \nrightarrow Y$ and $f_\circ : Y \nrightarrow X$ given by $f^\circ(x, y) = Y(f(x), y)$ and $f_\circ(y, x) = Y(f(y), x)$. Here adjoint means that $1_X \leqslant f_\circ \cdot f^\circ$ and $f^\circ \cdot f_\circ \geqslant 1_Y$.

**Remark 9** (Yoneda embedding)**.** *The* Yoneda embedding *is the faithful functor $\mathcal{Y} : X \to [X, Q]$ given by $\mathcal{Y}(x)(y) = X(y, x)$. The functoriality and faithfulness of $\mathcal{Y}$ follow from*

$$[X, Q](\mathcal{Y}(x), \mathcal{Y}(x')) = X(x, x') \tag{Yoneda}$$

*which is proved as follows: for all $y \in X$ we have*

$$\begin{aligned} Q(\mathcal{Y}(x)(y), \mathcal{Y}(x')(y)) &= \mathcal{Y}(x')(y) - \mathcal{Y}(x)(y) \\ &= X(y, x') - X(y, x) \leqslant X(x, x') \end{aligned}$$

*where the last step follows from $X(y, x) + X(x, x') \geqslant X(y, x')$. From this we deduce that $[X, Q](\mathcal{Y}(x), \mathcal{Y}(x')) = \sup_{y \in X} Q(\mathcal{Y}(x)(y), \mathcal{Y}(x')(y)) \leqslant X(x, x')$. For the converse direction, we have*

$$\begin{aligned} X(x, x') &= X(x, x') - 0 \\ &= X(x, x') - X(x, x) \\ &= \mathcal{Y}(x')(x) - \mathcal{Y}(x)(x) = Q(\mathcal{Y}(x)(x), \mathcal{Y}(x')(x)) \\ &\leqslant [X, Q](\mathcal{Y}(x), \mathcal{Y}(x')) \end{aligned}$$

**Remark 10.** *The* opposite Yoneda embedding *is the faithful functor $\mathcal{Y}^{\mathrm{op}} : X \to [X, Q]^{\mathrm{op}}$ given by $\mathcal{Y}^{\mathrm{op}}(x)(y) = X(x, y)$. The functoriality and faithfulness of $\mathcal{Y}^{\mathrm{op}}$ follow from*

$$[X, Q](\mathcal{Y}^{\mathrm{op}}(x), \mathcal{Y}^{\mathrm{op}}(x')) = X(x', x) \tag{Yoneda$^{\mathrm{op}}$}$$

*which is proved similarly to the case of $\mathcal{Y}$.*

## 16.3 cocompleteness

Let $\Phi : Y \nrightarrow Z$ be a distributor and $f : Y \to X$ be a functor. A functor $g : Z \to X$ is called the $\Phi$-*weighted colimit of $f$* if it satisfies, for all $z \in Z$ and $x \in X$,

$$X(g(z), x) \ = \ \sup_{y \in Y} X(f(y), x) - \Phi(y, z)$$

If this colimit exists, we write it as $\mathrm{colim}(\Phi, f)$.

A $Q$-category $X$ is said *cocomplete* if it admits all weighted colimits, and a functor $f : X \to Y$ of $Q$-categories is said *cocontinuous* if it commutes with all weighted colimits, meaning that $f \circ \mathrm{colim}(\Phi, g) = \mathrm{colim}(\Phi, f \circ g)$.

We let $Q\mathsf{CCat}$ indicate the category of skeletal and cocomplete $Q$-categories and cocontinuous functors.

**Proposition 44.** *Let $X$ be a $Q$-category. Then $X$ is cocomplete iff the Yoneda embedding has a left-adjoint.*

*Proof.* For all $\mathbf{x} \in [X, Q]$, let $\sup \mathbf{x}$ be defined as a weighted colimit via

$$X(\sup \mathbf{x}, y) = \sup_{z \in X} X(z, y) - \mathbf{x}_z$$

that is, $\sup \mathbf{x} = \mathrm{colim}(\mathbf{x}, \mathrm{id}_X)$, where $\mathbf{x}$ is seen as a distributor $\mathbf{x} : \mathbf{1} \nrightarrow X$.

Let us check that $\sup : [X, Q] \to X$ is a functor. First, let us check the inequality

$$(X \multimapinv \mathbf{y}) \cdot (\mathbf{y} \multimapinv \mathbf{x}) \geqslant (X \multimapinv \mathbf{x}) \tag{1}$$

as follows:

$$\begin{aligned}
((X \multimapinv \mathbf{y}) \cdot (\mathbf{y} \multimapinv \mathbf{x}))(a) &= \left(\sup_b X(b, a) - \mathbf{y}_b\right) + \left(\sup_b \mathbf{y}_b - \mathbf{x}_b\right) \\
&\geqslant \sup_b (X(b, a) - \mathbf{y}_b) + (\mathbf{y}_b - \mathbf{x}_b) \\
&= \sup_b (X(b, a) - \mathbf{y}_b + \mathbf{y}_b) - \mathbf{x}_b \\
&= \sup_b X(b, a) - \mathbf{x}_a \\
&= (X \multimapinv \mathbf{x})(a)
\end{aligned}$$

From (1) we deduce immediately the inequality below:

$$\mathbf{y} \multimapinv \mathbf{x} \geqslant (X \multimapinv \mathbf{x}) \multimapinv (X \multimapinv \mathbf{y}) \tag{2}$$

and we can now compute:

$$\begin{aligned}
[X, Q](\mathbf{x}, \mathbf{y}) &= \sup_{a \in X} \mathbf{y}_a - \mathbf{x}_a \\
&\overset{(2)}{\geqslant} \sup_{a \in X} \left(\sup_{b \in X} X(b, a) - \mathbf{x}_a\right) - \left(\sup_{b \in X} X(b, a) - \mathbf{y}_a\right) \\
&= \sup_{a \in X} X(\sup \mathbf{x}, a) - X(\sup \mathbf{y}, a) \\
&= [X, Q](\mathcal{Y}^{\mathrm{op}}(\sup \mathbf{y}), \mathcal{Y}^{\mathrm{op}}(\sup \mathbf{x})) && (\text{Yoneda}^{\mathrm{op}}) \\
&= X(\sup \mathbf{x}, \sup \mathbf{y})
\end{aligned}$$

Then for all $x \in X$, $\sup \mathcal{Y}(x) \simeq x$. Indeed we have

$$\begin{aligned}
X(\sup \mathcal{Y}(x), y) &= \sup_{z \in X} X(z, y) - \mathcal{Y}(x)(z) \\
&= \sup_{z \in X} X(z, y) - X(z, x) \\
&= X(x, y)
\end{aligned}$$

Moreover, for all $\mathbf{x} \in [X, Q]$, we have $\mathcal{Y}(\sup \mathbf{x}) \geqslant \mathbf{x}$:

$$\begin{aligned}
\mathcal{Y}(\sup \mathbf{x})(a) &= X(a, \sup \mathbf{x}) \\
&= X(\sup(\mathcal{Y}(a)), \sup \mathbf{x}) \\
&\geqslant [X, Q](\mathcal{Y}(a), \mathbf{x}) \\
&\geqslant \mathbf{x}_a - \mathcal{Y}(a)(a) \\
&= \mathbf{x}_a - X(a, a) = \mathbf{x}_a
\end{aligned}$$

Conversely, if sup is well-defined and adjoint to $\mathcal{Y}$, then given $\Phi : Y \nrightarrow \mathbf{1}$ and $f : Y \to X$, we can define $\operatorname{colim}(\Phi, f) := \sup \Psi$, where $\Psi = f^{\circ} \cdot \Phi : X \nrightarrow \mathbf{1}$, since

$$\begin{aligned}
X(\sup \Psi, y) &= \sup_{z \in X} X(z, y) - \Psi(z) \\
&= \sup_{z \in X} X(z, y) - \inf_{y \in Y} X(z, f(y)) + \Phi(y) \\
&= \sup_{z \in X} \sup_{y \in Y} X(z, y) - X(z, f(y)) - \Phi(y) \\
&= \sup_{z \in X} X(f(z), y) - \Phi(z)
\end{aligned}$$

□

**Definition 9** (MacNeill Completion)**.** *Let $X$ be a $Q$-category. For all $f : \mathbf{1} \nrightarrow X$ and $g : X \nrightarrow \mathbf{1}$, let $f \subset g$ iff $f = X \multimapdotinv g$ and $g = f \multimap X$.*

*The* MacNeill completion *of $X$ is the $Q$-category $\mathbf{M}(X)$ made of those $f : \mathbf{1} \nrightarrow X$ such that $f \subset g$ for some $g : X \nrightarrow \mathbf{1}$, with $\mathbf{M}(X)(f, f') = [X, Q](f, f')$.*

Observe that if $f \subset g$, then $f = X \multimapdotinv (f \multimap X)$, i.e.:

$$f(x) = \sup_{y \in X} \inf_{z \in X} X(x, y) - X(y, z) + f(z) \tag{COH}$$

**Proposition 45.** *Let $X$ be a $Q$-category. If $X$ is cocomplete, then $\mathcal{Y}$ is an isomorphism between $X$ and $\mathbf{M}(X)$.*

*Proof.* For all $x \in X$, one can check that $\mathcal{Y}(x) \in \mathbf{M}(X)$. Indeed, we can check that $\mathcal{Y}(x) \subset \mathcal{Y}^{\mathrm{op}}(x)$:

$$\begin{aligned}
\mathcal{Y}(x)(y) &= X(y, x) \\
&= \sup_{z \in X} X(y, z) - X(x, z) \\
&= \sup_{z \in X} X(y, z) - \mathcal{Y}^{\mathrm{op}}(x)(z) \\
&= (X \multimapdotinv \mathcal{Y}^{\mathrm{op}}(x))(y)
\end{aligned}$$

Since $\sup \mathcal{Y}(x) \simeq x$ holds, it suffices to show that if $\mathbf{x} \subset \mathbf{y}$, then $\mathcal{Y}(\sup \mathbf{x}) = \mathbf{x}$:

$$\begin{aligned}
\mathcal{Y}(\sup \mathbf{x})(a) &= X(a, \sup \mathbf{x}) \\
&= \sup_{b \in X} X(a, b) - X(\sup \mathbf{x}, b) \\
&= \sup_{b \in X} \inf_{c \in X} X(a, b) - X(b, c) + \mathbf{x}_c \\
&\overset{\text{(COH)}}{=} \mathbf{x}_a
\end{aligned}$$

□

## 16.4 Tensors and $Q$-Modules

Among weighted colimits, one is of big importance for us. Any $\epsilon \in Q$ generates a constant distributor $(\epsilon) : \mathbf{1} \nrightarrow \mathbf{1}$, and any point $x \in X$ generates a constant functor $\Delta x : \mathbf{1} \to X$. Given a $Q$-category $X$, a point $x \in X$ and $\epsilon \in Q$, the *tensor of* $x$ *and* $\epsilon$, if it exists, is defined as

$$\epsilon \otimes x := \mathrm{colim}((\epsilon), \Delta x)$$

A $Q$-category $X$ is *tensored* if for all $x \in X$ and $\epsilon \in Q$, it admits the tensor $\epsilon \otimes x$.

**Proposition 46.** *A tensored $Q$-category $X$ is a $Q$-module $(X, \preceq_X, \otimes)$. A cocontinuous functor of cocomplete $Q$-categories is a $Q$-module morphism between the associated $Q$-modules.*

*Proof.* We must show that tensors induce a continuous action. Observe that tensors are characterized by the equation

$$X(x \otimes \epsilon, x') = X(x, x') - \epsilon \tag{3}$$

If $\epsilon = 0$, then (3) forces $x \otimes \epsilon \simeq x$. If $\epsilon = \delta + \eta$, then using the fact that $\alpha - (\epsilon + \delta) = (\alpha - \epsilon) - \delta$ we deduce $X((x \otimes \epsilon) \otimes \delta, x') = X(x \otimes \epsilon, x') - \delta = (X(x, x') - \epsilon) - \delta = X(x, x') - (\epsilon + \delta) = X(x \otimes (\epsilon + \delta), x')$, which forces $x \otimes (\epsilon + \delta) \simeq (x \otimes \epsilon) \otimes \delta$.

A cocontinuous functor $f : X \to Y$ commutes with sups and with $\otimes$, and is thus a $Q$-module morphism. $\square$

**Lemma 47.** *i.* $\sup_{i \in I} a_i - \epsilon = (\sup_{i \in I} a_i) - \epsilon$.

*ii.* $\sup_{i \in I}(a_i - \epsilon) - b_i = (\sup_{i \in I} a_i - b_i) - \epsilon$.

*Proof.* Let $A = \sup_{i \in I} a_i - \epsilon$ and $B = (\sup_{i \in I} a_i) - \epsilon$. Let $J \subseteq I$ be the set of indexes $j$ such that $a_j > \epsilon$. If $J = \varnothing$ then $A = B = 0$. Otherwise, $A = \sup_{j \in J} a_j - \epsilon$ (where "$-$" can be interpreted as subtraction on $\mathbb{R}$, and $B = (\sup_{j \in J} a_j) - \epsilon$ (again with "$-$" being subtraction on $\mathbb{R}$), so $A = B$ follows from the continuity of "$-$" on $\mathbb{R}$.

Let now $A = \sup_{i \in I}(a_i - \epsilon) - b_i$ and $B = (\sup_{i \in I} a_i - b_i) - \epsilon$. Let $J \subseteq I$ be the set of indexes $j$ such that $a_j > b_j + \epsilon$. If $J = \varnothing$, then $A = 0$; suppose $B > 0$, then $\sup_{i \in I} a_i - b_i > \epsilon$, but this implies that we can find $i \in I$ with $a_i > b_i + \epsilon$, against the assumption, so also $B = 0$ holds. If $J$ is non-empty, then $A = \sup_{j \in J}(a_j - \epsilon) - b_j$, where "$-$" is not subtraction on $\mathbb{R}$ and $B = (\sup_{j \in J} a_j - b_j) - \epsilon$, again with "$-$" usual subtraction, so $A = B$ follows from the continuity of "$-$" on $\mathbb{R}$. $\square$

**Lemma 48.** *In any cocomplete $Q$-category, $x \otimes \epsilon \simeq \sup(\mathcal{Y}(x) + \epsilon)$. In the cocomplete $Q$-category $[X, Q]$, $\mathbf{x} \otimes \epsilon = \mathbf{x} + \epsilon$.*

*Proof.* We have

$$\begin{aligned} X(\sup(\mathcal{Y}(x) + \epsilon), x') &= \sup_{y \in X} X(z, x') - (\mathcal{Y}(x)(z) + \epsilon) \\ &= \sup_{y \in X} X(z, x') - (X(z, x) + \epsilon) \\ &= (\sup_{y \in X} X(z, x') - X(z, x)) - \epsilon \\ &= X(x, x') - \epsilon \end{aligned}$$

which shows $x \otimes \epsilon = \sup(\mathcal{Y}(x) + \epsilon)$. In $[X, Q]$ we have $[X, Q](\mathbf{x} + \epsilon, \mathbf{x}') = \sup_{a \in X}(\mathbf{x}_a + \epsilon) - \mathbf{x}'_a = (\sup_{a \in X} \mathbf{x}_a - \mathbf{x}'_a) - \epsilon = [X, Q](\mathbf{x}, \mathbf{x}') - \epsilon$, which shows $\mathbf{x} \otimes \epsilon \simeq \mathbf{x} + \epsilon$, and since $[X, Q]$ is skeletal, $\mathbf{x} \otimes \epsilon = \mathbf{x} + \epsilon$. ☐

The dual notion of tensors is the *cotensor* $x \multimapinv \epsilon$. Formally, it is defined as a *weighted limit* (whose definition is dual to that of weighted colimit but we do not give details here), and characterized by the equation

$$X(x', x \multimapinv \epsilon) = X(x', x) - \epsilon$$

In other words, in a tensored and cotensored $Q$-category we have $X(x \otimes \epsilon, y) = X(x, y \multimapinv \epsilon)$.

**Example 14.** *The $Q$-category $[X, Q]$ is cotensored, with $\mathbf{x} \multimapinv \epsilon := \mathbf{x} - \epsilon$. Indeed we have $[X, Q](\mathbf{x}, \mathbf{y} \multimapinv \epsilon) = \sup_{a \in X}(\mathbf{y}_a - \epsilon) - \mathbf{x}_a = (\sup_{a \in X} \mathbf{y}_a - \mathbf{x}_a) - \epsilon = [X, Q](\mathbf{x}, \mathbf{y}) - \epsilon$.*

**Definition 10.** *A $Q$-category $X$ is* order-complete *if it is a sup-lattice with respect to the order $\preceq_X$ (i.e. all joins exist).*

**Lemma 49.** *Let $X$ be a $Q$-category. If $X$ is order-complete, then*

- *if $X$ is co-tensored, $X(\bigvee_i x_i, y) = \sup_i X(x_i, y)$;*
- *if $X$ is tensored, $X(x, \bigvee_i y_i) = \inf_i X(x, y_i)$.*

*Proof.* We only prove the second claim, the first being proved similarly.

Let us show that $z \preceq_X z'$ iff for all $w \in X$, $X(w, z') \leqslant X(w, z)$: on one direction we have $X(w, z') \leqslant X(w, z) + X(z, z') = X(w, z)$; on the other direction, we have $X(z, z') \leqslant X(z, z) = 0$.

Using this, since $y_i \preceq_X y := \bigvee_i y_i$ we deduce $X(x, y_i) \leqslant X(x, y)$, and thus $X(x, y) \geqslant \inf_i X(x, y_i)$.

For the converse direction, we argue as follows: let $X(x, y_i) \leqslant \epsilon$ hold for all $i \in I$; then $0 = X(x, y_i) - \epsilon = X(x \otimes \epsilon, y_i)$. Thus $x \otimes \epsilon \preceq_X y_i$, and thus $x \otimes \epsilon \preceq_X y$, that is $X(x \otimes \epsilon, y) = X(x, y) - \epsilon = 0$, and consequently $X(x, y) \leqslant \epsilon$. By letting $\epsilon := X(x, y_i)$ we conclude then $X(x, y) \leqslant X(x, y_i)$, and thus $X(x, y) \leqslant \inf_i X(x, y_i)$.

☐

**Proposition 50.** *If a $Q$-category $X$ is tensored, cotensored and order-complete, then it is cocomplete.*

*Proof.* For all $\mathbf{x} \in [X, Q]$, let $\sup \mathbf{x} := \bigvee_{a \in X} a \otimes \mathbf{x}_a$. Let us check that $X(\sup \mathbf{x}, b) = \sup_{a \in X} X(a, b) - \mathbf{x}_a$, using Lemma 49:

$$\begin{aligned} X(\sup \mathbf{x}, b) &= \sup_{a \in X} X(a \otimes \mathbf{x}_a, b) \\ &= \sup_{a \in X} X(a, b) - \mathbf{x}_a \end{aligned}$$

We can thus conclude using Proposition 44. ☐

**Proposition 51.** *Let $X, Y$ be two tensored $Q$-categories, and $f : X \to Y$ be a function.*

*i. $f$ is a functor iff $f$ is order-preserving and for all $x \in X$ and $\epsilon \in Q$, $f(x) \otimes \epsilon \preceq_Y f(x \otimes \epsilon)$.*

*ii. $f$ is a cocontinuous functor iff $f$ commutes with joins and for all $x \in X$ and $\epsilon \in Q$, $f(x) \otimes \epsilon = f(x \otimes \epsilon)$.*

*Proof.* i. If $f$ is a functor then

$$\begin{aligned} Y(f(x) \otimes \epsilon, f(x \otimes \epsilon)) &= Y(f(x), f(x \otimes \epsilon)) - \epsilon \\ &\leqslant X(x, x \otimes \epsilon) - \epsilon \\ &= X(x \otimes \epsilon, x \otimes \epsilon) = 0 \end{aligned}$$

so $Y(f(x)\otimes\epsilon, f(x\otimes\epsilon)) = 0$, which implies $f(x)\otimes\epsilon \preceq_X f(x\otimes\epsilon)$. Moreover, if $x \preceq_X x'$, then $0 \geqslant X(x, x') \geqslant Y(f(x), f(x'))$, whence $f(x) \preceq_Y f(x')$, so $f$ is order-preserving.

Conversely, for all $x, x' \in X$,

$$X(x \otimes X(x, x'), x') = X(x, x') - X(x, x') = 0$$

thus $x \otimes X(x, x') \preceq_X x'$. Since $f$ is order-preserving, it follows that

$$f(x) \otimes X(x, x') \preceq_Y f(x \otimes X(x, x')) \preceq_Y f(x')$$

which implies that

$$Y(f(x), f(x')) - X(x, x') = Y(f(x) \otimes X(x, x'), f(x')) = 0$$

that is $Y(f(x), f(x')) \leqslant X(x, x')$, so $f$ is a functor.

ii. Suppose $f$ is a cocontinuous functors, and let $g : Y \to X$, be its right-adjoint, i.e. satisfying $Y(f(x), y) = X(x, g(y))$. Then

$$\begin{aligned} Y(f(x \otimes \epsilon), y) &= X(x \otimes \epsilon, g(y)) \\ &= X(x, g(y)) - \epsilon \\ &= Y(f(x), y) - \epsilon \end{aligned}$$

which implies that $f(x \otimes \epsilon)$ coincides with the tensor $f(x) \otimes \epsilon$. Moreover, clearly also $f(x) \preceq_Y y$ iff $x \preceq_X g(y)$ holds, which means that $f$ is left-adjoint to $g$ also with respect to the order.

Conversely, suppose the function $f : X \to Y$ preserves joins and tensors. Since $f$ is order-preserving, by i. it is a functor, so we must only prove that it is cocontinuous. Since $f$ preserves joins there exists a function $g : Y \to X$ which is right-adjoint to $f$ with respect to orders, i.e. $f(x) \preceq_Y y$ iff $x \preceq_X g(y)$. We need to prove then that $f$ is left-adjoint to $g$, i.e. that $Y(f(x), y) = X(x, g(y))$.

On the one hand we have

$$0 = X(x, g(y)) - X(x, g(y)) = X(x \otimes X(x, g(y)), g(y))$$

from which it follows

$$0 = Y(f(x \otimes X(x, g(y))), y) = Y(f(x) \otimes X(x, g(y)), y) = Y(f(x), y) - X(x, g(y))$$

where the first inequality follows from the fact that $f$ and $g$ are adjoint with respect to the order (so $Y(f(x), y) = 0$ iff $X(x, g(y)) = 0$). This implies then $Y(f(x), y) \leqslant X(x, g(y))$.

For the converse inequality,

$$0 = Y(f(x), y) - Y(f(x) - y) = Y(f(x) \otimes Y(f(x), y), y) = Y(f(x \otimes Y(f(x), y)), y)$$

and by a similar reasoning we deduce

$$0 = X(x \otimes Y(f(x), y), g(y)) = X(x, g(y)) - Y(f(x), y)$$

whence $X(x, g(y)) \leqslant Y(f(x), y)$.

□

**Theorem 52.** *The category $Q\mathsf{Mod}$ of $Q$-modules and $Q$-module morphism coincides with the category $Q\mathsf{CCat}$ of cocomplete skeletal $Q$-categories and cocontinuous functors.*

*Proof.* We have already seen that any cocomplete skeletal $Q$-category is a $Q$-module via tensors, and that cocontinuous functors are $Q$-module morphisms. Let us now show that any $Q$-module is a cocomplete skeletal $Q$-category, and that a $Q$-module morphism is a cocontinuous functor.

Let then $M = (M, \preceq, \star)$ be a $Q$-module. Define $M(x, y) = \inf\{\delta \mid x \star \delta \succeq y\}$. It is clear that $M(x, x) = 0$. Let us prove $M(x, y) + M(y, z) \succeq M(x, z)$: from $x \star M(x, y) \succeq y$ and $y \star M(y, z) \succeq z$ we deduce $x \star (M(x, y) + M(y, z)) = (x \star M(x, y)) \star M(y, z) \succeq y \star M(y, z) \succeq z$, and thus $M(x, z) \preceq M(x, y) + M(y, z)$. Observe that $M(x, y) = 0$ iff $x = x \star 0 \geqslant y$, so the order $\preceq_M$ coincides with the order of $M$.

Let us check that the $Q$-category $M$ is tensored via $x \otimes \epsilon := x \star \epsilon$. Let $A_{x,y} = \{\delta \mid (x \star \epsilon) \star \delta \geqslant y$ and $B_{x,y} = \{\delta - \epsilon \mid x \star \delta \geqslant y\}$. Let us show that $A_{x,y} = B_{x,y}$: if $\delta \in A_{x,y}$, then $\delta = (\epsilon + \delta) - \epsilon$ satisfies $x \star (\epsilon + \delta) = (x \star \epsilon) \star \delta \geqslant y$, whence $\delta \in B_{x,y}$. Conversely, if $\eta = \delta - \epsilon \in B_{x,y}$, then $(x \star \epsilon) \star \eta \geqslant x \star \delta \geqslant y$, whence $\eta \in A_{x,y}$. We conclude then that $M(x \star \epsilon, y) = \inf A_{x,y} = \inf B_{x,y} = \inf\{\delta \mid x \star \delta \geqslant y\} - \epsilon = M(x, y) - \epsilon$.

Let us define the opposite action $x \multimapinv \epsilon = \bigwedge\{y \mid y \star \epsilon \geqslant x\}$. We must show that $M$ is cotensored via $\multimapinv$, for which it suffices to show $M(x \star \epsilon, y) = M(x, y \multimapinv \epsilon)$. Let $C_{x,y} = \{\delta \mid x \star \delta \geqslant y \multimapinv \epsilon\}$. We have that $\delta \in A_{x,y}$ iff $(x \star \delta) \star \epsilon = x \star (\delta + \epsilon) = x \star (\epsilon + \delta) = (x \star \epsilon) \star \delta \geqslant y$ which is equivalent to $x \star \delta \geqslant y \multimapinv \epsilon$. We conclude that $A_{x,y} = C_{x,y}$, from which $M(x \star \epsilon, y) = \inf A_{x,y} = \inf C_{x,y} = M(x, y \multimapinv \epsilon)$.

Since $M$, as a $Q$-category, is order-complete, tensored and cotensored, it is cocomplete by Proposition 50.

To conclude, notice that if $f : X \to Y$ is a cocontinuous functor, then it commutes with tensors and, by Proposition 51 it commutes with joins, so it is a morphism of the respective $Q$-modules. Conversely, if $f : M \to N$ is a $Q$-module morphism, then, since $M$ and $N$ are both tensored $Q$-categories, the tensor coincides with the actions of $M$ and $N$, $f$ preserves the joins and the tensor, by Proposition 51, it is a cocontinuous functor of the respective $Q$-categories. □

## 17 $Q$Mod is a $*$-Autonomous Category

Let us first observe that:

- the hom-set $Hom(M, N)$ of two $Q$-modules is a $Q$-module with order and action defined pointwise;
- for any $Q$-module $M = (M, \preceq, \star)$, there is a $Q$-module $M^{\mathrm{op}} = (M, \succeq, \multimapinv)$, with $\multimapinv$ defined as in the proof of Theorem 52.

Let $M, N$ be two $Q$-modules. For all $A \in Q^{M\times N}$, we define the function

$$H_A : Q^M \longrightarrow Q^N$$

via

$$H_A(\mathbf{x})(b) = \inf_{a\in M} \mathbf{x}_a + A(a, b)$$

**Lemma 53.** *$H_A = H_{A'}$ iff $A = A'$.*

*Proof.* We only need to prove one direction, so suppose $A \neq A'$ and let $a, b$ be such that $A(a, b) \neq A'(a, b)$. Let $\mathbf{x}$ be defined by $\mathbf{x}_a = 1$ and $\mathbf{x}_{a'} = \infty$ for all $a' \neq a$. Then $H_A(\mathbf{x})(b) = A(a, b) \neq A'(a, b) = H_{A'}(\mathbf{x})(b)$. □

**Proposition 54.** *$Q^{M\times N}$ and $Hom(Q^M, Q^N)$ are isomorphic $Q$-modules.*

*Proof.* The map $A \mapsto H_A$ is injective, as shown above. We need to check that it commutes with joins:

$$\begin{aligned}
H_{\bigvee_i A_i}(\mathbf{x})(b) &= \inf_a \mathbf{x}_a + \bigvee_i A_i(a, b)\\
&= \inf_a \bigvee_i \mathbf{x}_a + A_i(a, b)\\
&= \bigvee_i \inf_a \mathbf{x}_a + A_i(a, b)\\
&= \bigvee_i H_{A_i}(\mathbf{x})(b)
\end{aligned}$$

(recall that infs are actually joins in $Q$!)

We must prove that $H$ is surjective: for all $f \in Hom(Q^M, Q^N)$, let $k_f \in Q^{M\times N}$ be given by $k_f(a, b) = f(e_a)(b)$.

Then we have

$$\begin{aligned}
H_{k_f}(\mathbf{x})(b) &= \inf_a \mathbf{x}_a + k_f(a, b)\\
&= \inf_a \mathbf{x}_a + f(e_a)(b)\\
&= \left(\inf_a \mathbf{x}_a + f(e_a)\right)(b)\\
&= \left(\inf_a f(\mathbf{x}_a + e_a)\right)(b)\\
&= f(\inf_a \mathbf{x}_a + e_a)(b)\\
&= f(\mathbf{x})(b)
\end{aligned}$$

and conversely

$$k_{H_A}(a,b) = H_A(e_a)(b) = \inf_{a'}(e_a)_{a'} + A(a',b) = A(a,b)$$

$\square$

## 17.1 The Tensor Product of $Q$-Modules

The description of the tensor product of $Q$-modules requires some work. Let us first recall some important definitions:

**Definition 11** (congruence on a sup-lattice)**.** *Let $(L, \leqslant)$ be a sup-lattice. An equivalence relation $R \subseteq L \times L$ is said a* congruence *if it satisfies the following property:*

$$(\forall i \in I\ x_i R y_i) \ \Rightarrow \left(\bigvee_i x_i\right) R \left(\bigvee_i y_i\right) \qquad \text{(congruence)}$$

**Lemma 55.** *For all suplattices $(L, \leqslant)$, if $R$ is a congruence, then $(L/R, \leqslant_R)$ is a sup-lattice, where $[x] \leqslant_R [y]$ iff $(x \vee y) R y$ (i.e. $[x \vee y] = [y]$), and $\bigvee_i [x_i] = [\bigvee_i x_i]$.*

*Proof.* Let us check that $\leqslant_R$ is an order. It is clear that $[x] \leqslant_R [x]$ holds. If $[x] \leqslant_R [y]$ and $[y] \leqslant_R [z]$ both hold, then $(x \vee y)Ry$ and $(y \vee z)Rz$ hold; then, since $R$ is a congruence $((x \vee y) \vee (y \vee z))R(y \vee z)Rz$, and moreover $x \vee (y \vee z)R(x \vee z)$, whence $(x \vee z)R(x \vee y \vee z)Rz$, so $[x] \leqslant_R [z]$. If $[x] \leqslant_R [y]$ and $[y] \leqslant_R [x]$, then $xR(x \vee y)Ry$, thus $[x] = [y]$.

Let us now check the definition of joins. From $[x_i] \vee [\bigvee_i x_i] = [x_i \vee \bigvee_i x_i] = [\bigvee_i x_i]$ we deduce $[x_i] \leqslant_R [\bigvee x_i]$. Suppose now $[x_i] \leqslant [y]$ holds for all $i \in I$, that is, $(x_i \vee y)Ry$; then, since $R$ is a congruence, $(\bigvee_i (x_i \vee y))Ry$, that is, $((\bigvee_i x_i) \vee y)Ry$, which implies $[\bigvee_i x_i] \leqslant_R [y]$. We conclude then that $\bigvee_i [x_i] = [\bigvee_i x_i]$. $\square$

**Lemma 56.** *Let $(L, \leqslant)$ be a sup-lattice. For any relation $R \subseteq L \times L$, the relation $R^*$ defined as follows:*

$$xR^*y \text{ iff } \exists I\ \exists x_i y_i \ \text{ s.t. } \left\{ \begin{matrix} x = \bigvee_{i \in I} x_i \\ y = \bigvee_{i \in I} y_i \\ x_i R y_i\ (\forall i \in I) \end{matrix} \right\}$$

*is a congruence, and is the smallest congruence containing $R$.*

*Proof.* Suppose $x_i R^* y_i$ holds for all $i \in I$. Then for each $i \in I$ there exists a set $J_i$ and sequences $x_{ij}, y_{ij}$ such that $\bigvee_{j \in J_i} x_{ij} = x_i$, $\bigvee_{j \in J_i} y_{ij} = y_i$ and $x_{ij} R y_{ij}$.

Let then $K = \prod_{i \in I} \{i\} \times J_i$; then for all $(i,j) \in K$, $x_{ij} R y_{ij}$, so we deduce that $\bigvee_{i \in I} x_i = \bigvee_{(i,j) \in K} x_{ij} R^* \bigvee_{(i,j) \in K} y_{ij} = \bigvee_{i \in I} y_i$, which proves that $R^*$ is a congruence.

Suppose now $S$ is a congruence containing $R$ and let $xR^*y$. Then there exists a set $I$ so that $x = \bigvee_{i \in I} x_i$, $y = \bigvee_{i \in I} y_i$ and $x_i R y_i$; now, since $S$ contains $R$, we deduce $x_i S y_i$ for all $i \in I$, and since $S$ is a congruence, $xSy$ holds, which proves that $R^* \subseteq S$. $\square$

We can now introduce the tensor of $Q$-modules, that will be defined as a suitable quotient lattice.

**Definition 12** (tensor of $Q$-modules)**.** *Let $M$ and $N$ be $Q$-modules. The* tensor product *$M \otimes_Q N$ of $M$ and $N$ is the $Q$-module defined as $\mathcal{P}(M \times N)/R^*$, where $R^*$ is the smallest congruence containing the relation $R$ defined by:*

$$R' = \left\{ \begin{array}{l|c} ((\bigvee A, y), \bigcup_{a \in A}\{(a, y)\}) & A \subseteq M, y \in N \\ ((x, \bigvee B), \bigcup_{b \in B}\{(x, b)\}) & B \subseteq N, x \in M \\ (\{(x \star \epsilon, y)\}, \{(x, y \star \epsilon)\}) & \epsilon \in Q \end{array} \right\}$$

*and the action is defined via $[A] \star \epsilon = \bigvee\{[\{(x \star \epsilon, y)\}] \mid (x, y) \in A\}$.*

Let a *$Q$-bimorphism* be a map $f : M \times N \to L$ such that $f$ preserves joins in each variable separately, and moreover $f(x, y \star \epsilon) = f(x \star \epsilon, y)$. A $Q$-bimorphism $f : M \times N \to L$ is *universal* if for all $L'$ and bimorphism $g : M \times N \to L'$ there is a unique sup-lattice homomorphism $h : L \to L'$ such that $g = h \circ f$.

**Proposition 57** (universal property of the tensor product)**.** *The tensor product $M \otimes_Q N$ is the codomain of the universal $Q$-bimorphism $M \times N \to M \otimes_Q N$.*

**Remark 11.** *For any $x \in M$ and $y \in N$, we indicate as $x \otimes_Q y$ the image of the pair $(x, y)$ under the universal $Q$-bimorphism $\tau : M \times N \to M \otimes_Q N$, or equivalently, as the $R^*$-equivalence class of $(x, y)$. Since joins in $M \otimes_Q N$ are given by $\bigvee_i [A_i] = [\bigcup_i A_i]$, we have then that any element $[A] \in M \otimes_Q N$ can be written as $[A] = \bigvee\{x \otimes y \mid (x, y) \in A\}$.*

**Lemma 58.**
- *$M \otimes_Q N \simeq N \otimes_Q M$;*
- *$(M \otimes_Q N) \otimes_Q R \simeq M \otimes_Q (N \otimes_Q R)$.*

**Proposition 59.** *$Hom(M \otimes_Q N, R) \simeq Hom(M, Hom(N, R))$ (as an isomorphism of sup-lattices).*

*Proof.* Given $h : M \otimes_Q N \to R$, for all $x \in M$ define $h_x : N \to R$ by $h_x(y) = h(x \otimes_Q y)$. We then have $h_x(\bigvee_i y_i) = h(x \otimes_Q \bigvee_i y_i) = h(\bigvee_i x \otimes_Q y_i) = \bigvee_i h(x \otimes_Q y_i) = \bigvee_i h_x(y_i)$ and $h_x(y \star \epsilon) = h(x \otimes_Q (y \star \epsilon)) = h((x \otimes_Q y) \star \epsilon) = h(x \otimes_Q y) \star \epsilon = h_x(y) \star \epsilon$, so $h_x \in Hom(N, R)$. Moreover, by a similar argument we have $h_{\bigvee_i x_i}(y) = \bigvee_i h_{x_i}(y)$ and $h_{x \star \epsilon}(y) = h_x(y) \star \epsilon$, so the map $x \mapsto h_x$ is a $Q$-module morphism.

Finally, for any family $h_i : M \otimes_Q N \to R$, we have $(\bigvee_i h_i)_x(y) = \bigvee_i h_i(x \otimes_Q y) = \bigvee_i (h_i)_x(y) = (\bigvee_i h_i)(x \otimes y)$, and thus we have a sup-lattice homomorphism $\zeta$ from $Hom(M \otimes_Q N, R)$ to $Hom(M, Hom(N, R))$ given by $\zeta(h) = h_\_$.

Let us show that $\zeta$ has an inverse: for all $f \in Hom(M, Hom(N, R))$, define $f' : M \times N \to R$ by $f'(x, y) := f(x)(y)$. This is clearly a bimorphism, so there is a unique homomorphism $h_{f'} : M \otimes_Q N \to M \times N$ such that $f' = h_{f'} \circ \tau$, i.e. such that $h_{f'}(x \otimes y) = f'(x, y) = f(x)(y)$, and thus $\zeta(h_{f'}) = f$. On the other hand, if $f = \zeta(h)$, then the uniqueness of $h_{f'}$ ensures that $h_{f'} = h$. $\square$

**Proposition 60.**
*i. $Hom(Q, M) \simeq M$.*

*ii. $Hom(M, N) \simeq Hom(N^{\mathrm{op}}, M^{\mathrm{op}})$.*

*iii. $Hom(M, Q^{\mathrm{op}}) \simeq M^{\mathrm{op}}$.*
*(all isomorphisms of sup-lattices).*

*Proof.* Define $\alpha : M \to Hom(Q, M)$ by $\alpha(x)(\epsilon) = x \star \epsilon$ and $\beta : Hom(Q, M) \to M$ by $\beta(f) = f(0)$. Then we have that $\alpha(\beta(f))(\epsilon) = \alpha(f(0))(\epsilon) = f(0) \star \epsilon = f(\epsilon)$, and $\beta(\alpha(x)) = \beta(\lambda\epsilon.x \star \epsilon) = x \star 0 = x$.

If $f \in Hom(M, N)$, since it preserves joins, it has a right-adjoint $f^* : N^{\mathrm{op}} \to M^{\mathrm{op}}$, such that $f(x) \leqslant y$ iff $x \leqslant f^*(y)$.

By i. $Hom(Q, M^{\mathrm{op}}) \simeq M^{\mathrm{op}}$ and we conclude by ii. □

**Proposition 61.** *i.* $Hom(M, N) \simeq (M \otimes_Q N^{\mathrm{op}})^{\mathrm{op}}$.

*ii.* $M \otimes_Q N \simeq Hom(M, N^{\mathrm{op}})^{\mathrm{op}}$.

*iii.* $Q \otimes_Q M \simeq M \otimes_Q Q \simeq Q$.

*Proof.* $Hom(M, N) \simeq Hom(M, Hom(N^{\mathrm{op}}, Q^{\mathrm{op}})) \simeq Hom(M \otimes N^{\mathrm{op}}, Q^{\mathrm{op}}) \simeq (M \otimes N^{\mathrm{op}})^{\mathrm{op}}$. Claim ii. is proved similarly.

For claim iii. $Q \otimes_Q M \simeq M \otimes_Q M \simeq Hom(M, Q^{\mathrm{op}})^{\mathrm{op}} \simeq (M^{\mathrm{op}})^{\mathrm{op}} = M$. □

By putting all previous results together we obtain:

**Theorem 62.** $Q\mathsf{Mod}$ *is a* *-*autonomous category.*

**Proposition 63.** *i.* $Q^X \otimes_Q M \simeq M^X$*;*

*ii.* $Q^X \otimes_Q Q^Y \simeq Q^{X \times Y}$.

*Proof.* $M^X$ coincides with the product $\Pi_{x \in X} M$. We have then $Q^X \otimes_Q M \simeq (\Pi_x Q) \otimes_Q M \simeq \Pi_x (Q \otimes_Q M) \simeq \Pi_x M \simeq M^X$.

For ii., using i. we have $Q^X \otimes_Q Q^Y \simeq (Q^Y)^X \simeq Q^{X \times Y}$. □

## 17.2 The Tensor Product of $Q$-Categories

Thanks to Theorem 52, the *-autonomous structure of $Q\mathsf{Mod}$ translates into a *-autonomous structure for $Q\mathsf{CCat}$.

In $\mathsf{Met}$ the "tensor product" of two metric spaces $X$ and $Y$ is just their cartesian product, with the "plus" metric. What can we say about the tensor product in $Q\mathsf{CCat}$?

The goal of this subsection is to describe the $Q$-categorical structure of the tensor product explicitly. The main intuition is that the elements of $X \otimes_Q Y$ can be seen as joins of pairs $x \otimes y$ of elements $x \in X$, $y \in Y$. We will then show that the metric coincides with the "plus" metric over pairs $x \otimes y$, and extends continuously to joins.

**Lemma 64.** *For all* $m, m' \in M$ *and* $n, n' \in N$ *and* $\epsilon \in Q$, $(m \otimes n) \star \epsilon \succeq (m' \otimes n')$ *iff there exists* $\delta_1, \delta_2$ *such that* $\delta_1 + \delta_2 = \epsilon$, $m + \delta_1 \succeq m'$ *and* $n + \delta_2 \succeq n'$.

*Proof Sketch.* Notice that $(m \otimes n) \star \epsilon = [\{(m \star \epsilon, n)\}] = [\{(m \star \delta_1, n \star \delta_2\}]$ for all $\delta_1 + \delta_2 = \epsilon$. Hence, $m' \otimes n' \preceq (m \otimes n) \star \epsilon$ implies that for some $\delta_1 + \delta_2 = \epsilon$, $(m', n') \vee (m \star \delta_1, n \star \delta_2) = (m' \vee (m \star \delta_1), n' \vee (n \star \delta_2)) = (m \star \delta_1, n' \star \delta_2)$, that is, that $m' \preceq_M m \star \delta_1$ and $n' \preceq_n n \star \delta_2$. □

**Proposition 65.** *For all* $m, m' \in N$ *and* $n, n' \in N$,

$$(M \otimes_Q N)(m \otimes_Q n, m' \otimes_Q n') = M(m, m') + N(n, n')$$

*More generally,*

$$(M \otimes_Q N)([A], [B]) = \sup_{(x,y) \in A} \inf_{(x',y') \in B} M(x, x') + N(y, y')$$

*Proof.* By definition, $(M \otimes_Q N)(m \otimes_Q n, m' \otimes_Q n')$ is given by $\inf A$, where

$$A = \{\epsilon \mid (m \otimes_Q n) \star \epsilon \geqslant m' \otimes_Q n'\}$$

Let us show that $A$ coincides with

$$B = \{\delta_1 + \delta_2 \mid m \star \delta_1 \geqslant m', n \star \delta_2 \geqslant n'\}$$

On the one hand, if $\delta_1 + \delta_2 \in B$, it is clear that $\delta_1 + \delta_2 \in A$. Conversely, if $\epsilon \in A$, then by Lemma 64 $\epsilon = \delta_1 + \delta_2$ with $m \star \delta_1 \geqslant m'$ and $n \star \delta_2 \geqslant n'$, whence $\epsilon \in B$.

We can now conclude as follows:

$$\begin{aligned}
(M \otimes_Q N)(m \otimes_Q n, m' \otimes_Q n') &= \inf A \\
&= \inf B \\
&= \inf\{\delta_1 \mid m \star \delta_1 \geqslant m'\} + \inf\{\delta_2 \mid n \star \delta_2 \geqslant n'\} \\
&= M(m, m') + N(n, n').
\end{aligned}$$

For the second claim, using the fact that $M \otimes_Q N$, as a $Q$-category, is both tensored and cotensored, using the fact that $[A] = \bigvee_{(x,y) \in A} x \otimes y$ and $[B] = \bigvee_{(x',y') \in B} z \otimes w$, and Lemma 49:

$$\begin{aligned}
(M \otimes_Q N)([A], [B]) &= \sup_{(x,y) \in A} (M \otimes_Q N)(x \otimes y, [B]) \\
&= \sup_{(x,y) \in A} \inf_{(x',y') \in B} (M \otimes_Q N)(x \otimes y, x' \otimes y') \\
&= \sup_{(x,y) \in A} \inf_{(x',y') \in B} M(x, x') + N(y, y').
\end{aligned}$$

$\square$