11 Tropical Finiteness Spaces

Let us recall that a finiteness space is a pair $X = (|X|, \mathsf{F}(X))$ where $|X|$ is a set and $\mathsf{F}(X) \subseteq \mathcal{P}(|X|)$ satisfies $\mathsf{F}(X) = \mathsf{F}(X)^{\perp\perp}$, where for all $u, v \in \mathcal{P}(|X|)$, $u \perp v$ iff $u \cap v$ is finite.

The lemma below illustrates two simple properties of finitary sets, that we will silently exploit in several places in the following.

Lemma 20. *Let X be a finiteness space.*

i. *$\mathsf{F}(X)$ contains all finite subsets of X;*

ii. *if $u \in \mathsf{F}(X)$ and $v \subseteq u$, then $v \in \mathsf{F}(X)$.*

Proof. Let u be a finite subset of $|X|$. Since for all $v \in \mathsf{F}(X)^{\perp}$, $u \cap v$ is finite, we have $u \in \mathsf{F}(X)^{\perp\perp} = \mathsf{F}(X)$. Suppose now $u \in \mathsf{F}(X)$ and $v \subseteq u$. Then for all $w \in \mathsf{F}(X)^{\perp}$, $v \cap w \subseteq u \cap w$ and since the latter is finite, so is the former, whence $v \in \mathsf{F}(X)$. $\square$

Given finiteness spaces X, Y, the finiteness spaces $X \otimes Y$, $X \multimap Y$ and $!X$ are defined by

$$\begin{aligned} |X \otimes Y| = |X \multimap Y| &= |X| \times |Y|, \quad |!X| = \mathcal{M}_{\mathrm{f}}(X) \\ \mathsf{F}(X \otimes Y) &= \{u \times v \mid u \in \mathsf{F}(X), v \in \mathsf{F}(Y)\} \\ \mathsf{F}(X \multimap Y) &= \mathsf{F}(X \otimes Y^{\perp})^{\perp} \\ \mathsf{F}(!X) &= \bigcup_{u \in \mathsf{F}(X)} \mathcal{P}(\mathcal{M}_{\mathrm{f}}(u)) \end{aligned}$$

A morphism of finiteness spaces X, Y is a set $t \in \mathsf{F}(!X \multimap Y) = \mathsf{F}(!X \otimes Y)^{\perp}$. In other words $t \subseteq \mathcal{P}(\mathcal{M}_{\mathrm{f}}(X) \times Y)$ is such that for all $u \in \mathsf{F}(!X)$ and $v \in \mathsf{F}(Y)^{\perp}$, $t \cap u \times v$ is finite.

Given $t : X \to Y$, we define maps $F_t : \mathsf{F}(X) \to \mathsf{F}(Y)$ and $F_t^{\perp} : \mathsf{F}(Y)^{\perp} \to \mathsf{F}(X)^{\perp}$ by:

$$\begin{aligned} F_t(u) &= \{b \in Y \mid \exists \mu \in u\ (\mu, b) \in F\} \\ F_t^{\perp}(v) &= \{\mu \in \mathcal{M}_{\mathrm{f}}(X) \mid \exists b \in Y\ (\mu, b) \in F\} \end{aligned}$$

Given a finiteness space X, $Q\langle X\rangle$ is the Q-module of vectors $\mathbf{x} \in Q^X$ such that $\mathrm{supp}(\mathbf{x}) \in \mathsf{F}(X)$, i.e. $\mathrm{supp}(\mathbf{x}) = \{a \in |X| \mid \mathbf{x}_a \neq \infty\}$. Moreover, for all $u \in \mathsf{F}(X)$, we indicate with $Q\langle u\rangle$ the set of $\mathbf{x} \in Q\langle X\rangle$ such that $\mathrm{supp}(\mathbf{x}) = u$.

The category Fin_Q has objects being finiteness spaces and arrows between X and Y being tropical analytic functions $f : Q\langle X\rangle \to Q\langle Y\rangle$ such that $\widehat{f} \in Q\langle !X \multimap Y\rangle$, that is, such that $\mathrm{supp}(\widehat{f}) \in \mathsf{F}(!X \multimap Y)$.

The following fundamental property holds:

Proposition 21 (Fundamental Property of Finitary Tropical Functions)**.** *Let $f : Q\langle X\rangle \longrightarrow Q\langle Y\rangle$. For all $u \in \mathsf{F}(X)$ and $v \in \mathsf{F}(Y)^{\perp}$, there exists a finite set $S_{u,v} \subseteq \mathcal{M}_{\mathrm{f}}(X)$ such that for all $\mathbf{x} \in Q\langle u\rangle$ and $b \in v$,*

$$f(\mathbf{x})_b = \min\{\widehat{f}_{\mu,b} + \mu\mathbf{x} \mid \mu \in S_{u,v}\}$$

Proof. Since $\text{supp}(\widehat{f}) \in \mathsf{F}(!X \multimap Y)$ it follows that $\text{supp}(\widehat{f}) \cap (\mathcal{M}_{\text{f}}(\text{supp}(X)) \times v)$ is finite. To conclude, notice that $\text{supp}(\widehat{f}) \cap (\mathcal{M}_{\text{f}}(\text{supp}(X)) \times v) = \bigcup_{b \in v} (\mathcal{M}_{\text{f}}(\text{supp}(\mathbf{x})) \cap \widehat{f}^{\perp}(\{b\})) \times \{b\}$, so we let $S_{u,v}$ be the equipotent set $\bigcup_{b \in v} (\mathcal{M}_{\text{f}}(\text{supp}(\mathbf{x})) \cap \widehat{f}^{\perp}(\{b\}))$. □

In the following, we will indicate as $P^f_{u,v}(\mathbf{x})$ the tropical polynomial defined by f over the finitary sets $u \in \mathsf{F}(X)$ and $v \in \mathsf{F}(Y)^{\perp}$.

Metric Structure Let us now consider metric properties. First observe that, for all $\mathbf{x}, \mathbf{y} \in Q\langle X \rangle$, the following conditions are equivalent:

- $\text{supp}(\mathbf{x}) = \text{supp}(\mathbf{y})$,
- $\|\mathbf{x} - \mathbf{y}\|_{\infty} < \infty$.

Hence open balls with center $\mathbf{x}$ all live in $Q\langle \text{supp}(\mathbf{x}) \rangle$. This leads to the following:

Proposition 22. *For all $f \in Q\langle X \rangle \to Q\langle Y \rangle$, $u \in \mathsf{F}(X)$ and $v \in \mathsf{F}(X)^{\perp}$, there exists $L_{u,v}$ such that, for all $b \in v$, the restriction of $f(\_)_b$ to $Q\langle u \rangle$ is $L_{u,v}$-Lipschitz continuous.*

Proof. This follows from the fundamental property and the remark that the tropical polynomial $P^f_{u,v}(\mathbf{x})$ is a $L_{u,v}$-Lipschitz function, where $L_{u,v} = \deg(P^f_{u,v})$. □

Corollary 23. *For all $f : Q\langle X \rangle \to Q\langle Y \rangle$ and $b \in Y$, the function $f(\_)_b$ is locally Lipschitz-continuous.*

Proof. For all $\mathbf{x} \in Q\langle X \rangle$, f is Lipschitz continuous, with Lipschitz constant $\deg(P^f_{\text{supp}(x),\{b\}})$, on any open ball of center $\mathbf{x}$ and finite radius. □

Does this imply continuity of f EVERYWHERE??

We now show different alternative characterizations of the metrics on functional spaces, i.e. spaces of the form $Q\langle !X \multimap Y \rangle$. Importantly, none of these characterization depends on the finitary structure, so it holds also in the "full" category $Q^{\Pi}_{!}$.

Proposition 24. *For all maps $f, g : Q\langle X \rangle \to Q\langle Y \rangle$,*

$$\|\widehat{f} - \widehat{g}\|_{\infty} = \sup\{\|f(\mathbf{x}) - g(\mathbf{x})\|_{\infty} \mid \mathbf{x} \in Q\langle X \rangle\}$$

Before proving the proposition we need one preliminary lemma:

Lemma 25. *Let $u, v : I \to Q$ and suppose $|u(i) - v(i)| \leqslant \delta$, for all $i \in I$. Then $|\inf_{i \in I} u(i) - \inf_{i \in I} v(i)| \leqslant \delta$.*

Proof. Let $A = \inf_{i \in I} u(i)$ and $B = \inf_{i \in I} v(i)$ and suppose $A \geqslant B$. Suppose by way of contradiction $|A - B| > \delta$; then there exists $i \in I$ such that $v(i) < A$ and $|A - v(i)| > \delta$. Indeed, otherwise we would have $|A - B| = \sup\{|A - v(i)| \mid v(i) \leqslant A\} \leqslant \delta$. Now, from $|A - v(i)| > \delta$ and $v(i) < A$ we deduce that $|u(j) - v(i)| > \delta$ for all $j \in I$, and thus in particular that $|u(i) - v(i)| > \delta$, against the assumption. We conclude then $|A - B| \leqslant \delta$. In case $B \geqslant A$, we can argue in a similar way. □

Proof of Proposition 24. For one side, suppose for all $\mu \in \mathcal{M}_{\text{fin}}(X), b \in Y$, $|\widehat{f}_{\mu,b} - \widehat{g}_{\mu,b}| \leqslant \delta$. Then for all $\mathbf{x} \in Q\langle X\rangle$ and $b \in Y$, $|f(\mathbf{x})(b) - g(\mathbf{x})(b)| \leqslant \delta$. Indeed, $|f(\mathbf{x})(b) - g(\mathbf{x})(b)| = |(\inf_\mu \mu\mathbf{x} + \widehat{f}_{\mu,b}) - (\inf_\mu \mu\mathbf{x} + \widehat{g}_{\mu,b})|$. Since $|\mu\mathbf{x} + \widehat{f}_{\mu,b} - \mu\mathbf{x} - \widehat{g}_{\mu,b}| = |\widehat{f}_{\mu,b} - \widehat{g}_{\mu,b}| \leqslant \delta$, by the Lemma above we conclude $|f(\mathbf{x})(b) - g(\mathbf{x})(b)| \leqslant \delta$.

For the other side, suppose for some $\mu \in \mathcal{M}_{\mathrm{f}}(X)$ and $b \in Y$, $|\widehat{f}_{\mu,b} - \widehat{g}_{\mu,b}| > \epsilon$; then, by letting $e_\mu \in Q\langle !X\rangle$ be the (tropical) characteristic function of μ, we have $|f(e_\mu)(b) - g(e_\mu)(b)| = |(\inf_{\mu'} \widehat{f}_{\mu',b} + \mu'(e_\mu)) - (\inf_{\mu'} \widehat{g}_{\mu',b} + \mu'(e_\mu))| = |\widehat{f}_{\mu,b} - \widehat{g}_{\mu,b}| > \epsilon$, so we deduce that $\sup\{\|f(\mathbf{x}) - g(\mathbf{x})\|_\infty \mid \mathbf{x} \in Q\langle X\rangle\} > \epsilon$. □

Remark 1. *Let (X, a) and (Y, b) be two metric spaces, and let $\mathcal{O}_a(X)$ and $\mathcal{O}_b(X)$ indicate the topologies generated by such metrics. Then, letting $(\mathcal{C}(X, Y), d)$ be the metric space given by $d(f, g) = \sup\{b(f(x), g(x)) \mid x \in X\}$, it is well-known that $\mathcal{O}_d(\mathcal{C}(X, Y))$ coincides with the* compact-open *topology, which has basic open sets of the form*

$$\langle K, V\rangle = \{f \mid f^{-1}(V) \subseteq K\}$$

with K a compact set in X and V open in Y. So, a natural question is if this is also the case in our model, where morphisms between $Q\langle X\rangle$ and $Q\langle Y\rangle$ do not coincide with all *the continuous functions.*

The second characterization relates distances with the Taylor expansion. Let us briefly discuss the latter, first. For all $f : Q\langle X\rangle \to Q\langle Y\rangle$, let $\delta^{(n)} f : Q\langle X^n\rangle \to Q\langle Y\rangle$ indicate the n-linear function given by

$$\delta^{(n)} f(\mathbf{x}^n) = \mathsf{D}^{(n)} f(\mathbf{x}^n, \infty)$$

Notice that $\widehat{\delta^{(n)} f} \in Q^{X^n \times Y}$ satisfies $\widehat{\delta^{(n)} f}_{a_1,\dots,a_n,b} = \widehat{f}_{[a_1,\dots,a_n],b}$. In other words, $\delta^{(n)} f$ precisely captures the behavior of f when applied to multisets of length n. Moreover, the full behavior of f can be recovered from the functions $\delta^{(n)} f$ using the Taylor expansion which, in its tropical form, reads as:

$$f(\mathbf{x}) = \inf_n \left\{\mathsf{D}^{(n)} f(\mathbf{x}^n, \infty)\right\} = \inf_n \left\{\delta^{(n)} f(\mathbf{x}^n)\right\} \qquad \text{(Tropical Taylor)}$$

With respect to the classical version of Taylor, the tropical Taylor does not include the coefficients $\frac{1}{!n}$. This is due to the fact that tropical sum is idempotent. In a sense, this seems more similar to the Taylor expansion coming from the resource λ-calculus.

More generally, a salient feature of tropical semantics, compared to the classical one, is that it validates idempotency rule

$$M + M = M$$

with the consequence that, contrarily to the standard differential λ-calculus, term reduction cannot *create* new coefficients: for instance, while computing the second derivative of $\lambda x.xx$ in the classical calculus, a coefficient 2 arises:

$$\begin{aligned} \mathsf{D}^{(2)}(\lambda x.xx) \cdot y^2 &= \lambda x.0 + (\mathsf{D}y \cdot y)x + (\mathsf{D}y \cdot y)x + (\mathsf{D}^{(2)}x \cdot y^2)x \\ &= \lambda x.2 \cdot (\mathsf{D}y \cdot y)x + (\mathsf{D}^{(2)}x \cdot y^2)x \end{aligned}$$

Instead, in the tropical version, we would just get $\lambda x.(\mathsf{D}y \cdot y)x + (\mathsf{D}^{(2)}x \cdot y^2)x$.

We will now show that the distance between two functions in the tropical model is entirely determined by the distance between their "Taylor coefficients", that is, the coefficients of the matrices of $\delta^{(n)}f, \delta^{(n)}g$.

Proposition 26. *For all $f, g : Q\langle X\rangle \to Q\langle Y\rangle$,*

$$\|\widehat{f} - \widehat{g}\|_\infty = \sup_n \|\widehat{\delta^{(n)}f} - \widehat{\delta^{(n)}g}\|_\infty$$

Proof. Let us first show that $\|\widehat{f} - \widehat{g}\|_\infty \leqslant \epsilon$ implies $\|\delta^{(n)}f - \delta^{(n)}g\|_\infty \leqslant \epsilon$, for all $n \in \mathbb{N}$. Notice that $\widehat{\delta^{(n)}f} \in Q^{X^n \times Y}$ satisfies $\widehat{\delta^{(n)}f}_{a_1,\dots,a_n,b} = \widehat{f}_{[a_1,\dots,a_n],b}$. Hence from $\|\widehat{f} - \widehat{g}\| \leqslant \epsilon$ it follows that for all $\mu = [a_1, \dots, a_n]$ and $b \in Y$, $|\widehat{f}_{\mu,b} - \widehat{g}_{\mu,b}| \leqslant \epsilon$, so we deduce that $\|\widehat{\delta^{(n)}f} - \widehat{\delta^{(n)}g}\| \leqslant \epsilon$.

For the converse direction suppose that, for all $n \in \mathbb{N}$, $\|\widehat{\delta^{(n)}f} - \widehat{\delta^{(n)}g}\| \leqslant \epsilon_n$. Then, since the family of coefficients $F_{n,\mu,b} = (\widehat{\delta^{(n)}f})_{a_1,\dots,a_n,b}$ (where $\mu = [a_1, \dots, a_n]$) is in bijection with the coefficients $\widehat{f}_{\mu,b}$, we deduce that $|\widehat{f}_{\mu,b} - \widehat{g}_{\mu,b}| \leqslant \epsilon_{\sharp\mu}$, and thus that $\|\widehat{f} - \widehat{g}\|_\infty \leqslant \sup_n \epsilon_n$. $\square$

12 Tropical Coefficients as Log-Probabilities

For classical program, a non-deterministic sum $\alpha \cdot M + \beta \cdot N$ is naturally interpreted as a *probabilistic* one, as soon as $\alpha + \beta \leqslant 1$. However, when we move to the tropical setting, we get something like $\min\{M - \log\alpha, N - \log\beta\}$: what do these new coefficients $-\log\alpha, -\log\beta \in Q$ indicate? We provide a possible answer, which relates the tropicalization of probabilistic programs with well-known approaches in statistical learning, based on *log-probability maximization*.

On Log-Probabilities and Likelihood The typical situation is when one has a parametric joint probabilistic distribution $P(\vec{x} \mid \vec{\alpha})$, where $\vec{\alpha} \in [0,1]$ are a list of probabilistic parameters, which is expressed as a polynomial of the parameters $\vec{\alpha}$ (this kind of situations are typical in so-called *hidden Markov models*). For instance, suppose x_1, x_2, x_3 indicate three flip coins biased according to $\alpha_1, \alpha_1, \alpha_2$, respectively. Then the value

$$P(\mathtt{HTH} \mid \alpha_1, \alpha_2) = \alpha_1 \cdot (1 - \alpha_1) \cdot \alpha_2 = \alpha_1\alpha_2 - \alpha_1^2\alpha_2$$

indicates the probability of getting head, tail, head by flipping twice the α_1-coin and once the α_2-coin. Similarly, we can compute parametric probabilities for more complex events, e.g. getting tail precisely once out of three flip coins, of which two according to α_1 and one according to α_2:

$$P(\{\mathtt{THH}, \mathtt{HTH}, \mathtt{HHT}\} \mid \vec{\alpha}) = P(\mathtt{THH} \mid \vec{\alpha}) + P(\mathtt{HTH} \mid \vec{\alpha}) + P(\mathtt{HHT} \mid \vec{\alpha})$$

At this point, two problems are naturally posed:

1. can we find parameters $\vec{\alpha}$ which *maximize* the probability of getting head, tail, head?
2. if we find out that tail showed up precisely once, can we guess which, among the tree possible configurations THH, HTH, HHT, is the *most likely* to have occurred?

To answer both these questions it is often preferred to compute log-probabilities instead of actual probabilities, since this leads to the same result, but with much less heavy computations. For example, Problem (1) corresponds to finding parameters $\vec{\alpha}$ which maximize the polynomial function $P(\mathtt{HTH} \mid \alpha_1, \alpha_2)$. Since such functions can be very complicated, it is convenient to *minimize*, instead, the logarithm

$$-\log P(\mathtt{HTH} \mid \alpha_1, \alpha_2) = -\log \alpha_1 - \log(1-\alpha_1) - \log \alpha_2$$

which is a tropical monomial. Indeed, since the logarithm is a monotone increasing function, $\log(P)$ has the same maximum and minimum points as P, but the former involves addition where the latter involves possibly heavy multiplications and powers.

By reasoning in a similar way for Problem (2), it turns out that finding the most likely configuration corresponds to finding the configuration at which the minimum

$$\min\{-\log P(\mathtt{THH}), -\log P(\mathtt{HTH}), -\log P(\mathtt{HHT})\} = \mathrm{trop}(P(\{\mathtt{THH}, \mathtt{HTH}, \mathtt{HHT}\} \mid \vec{\alpha}))$$

is attained, and thus again to compute some tropical polynomial over the probabilistic parameters $\vec{\alpha}$.

So, in general, to address the two problems considered it is convenient to study the tropical polynomial

$$\mathrm{trop}(P)(\vec{x} \mid \vec{\alpha}) = \min_{\vec{j} \in S} \Big\{ \sum_l j_l \cdot \alpha_l + c_l \Big\}$$

which has the following intuitive reading: this polynomial describes the log-probability of some event E, where:

- each index $\vec{j} \in S$ corresponds to one of the possible outcomes $e_j \in E$, the tropical monomial $\sum_l j_l \cdot \alpha_l + c_l$ corresponding to its log-probability; in hidden Markov models these possible outcomes are called the *hidden* variables of the model;
- the study of the tropical polynomial $\mathrm{trop}(P)$ describes:
  1. how to properly choose $\vec{\alpha}$ to maximize one particular outcome in $e_j \in E$;
  2. how, depending on $\vec{\alpha}$, the likelihood of any particular outcome varies: if the event E has obtained, and the $\vec{\alpha}$ are fixed, the most likely events are the e_j such that the minimum is attained at index j. Notably, two (or more) events are equally most likely precisely when $\vec{\alpha}$ is a *tropical root* of the polynomial.

Probabilistic Programs and Tropical Polynomials Let us now see how this discussion applies to the case of the λ-calculus. Let us enrich the differential λ-calculus with probabilistic sum operators $M \oplus_\alpha N$ where $\alpha \in [0,1]$, together with a reduction rule:

$$M \oplus_\alpha N \;\to\; \alpha \cdot M + (1-\alpha) \cdot N \qquad (\oplus)$$

(How to do this extension precisely should be checked in Thomas Leventis' work). Moreover, enrich simply typed rules by a new rule

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : A}{\Gamma \vdash M \oplus_\alpha N : A}$$

Suppose M is a probabilistic simply typed term containing probabilistic operators $\oplus_\alpha$, for $\alpha \in \{\alpha_1, \dots, \alpha_k\}$, where the operator $\oplus_{\alpha_i}$ occurs m_i times. Let $m := \sum_i m_i$. Then by applying the rule $(\oplus)$ to fire all operators $\oplus_\alpha$, we obtain a term of the form

$$\sum_{\sigma = b_1 \dots b_m} P_\sigma(\alpha_1, \dots, \alpha_k) \cdot N_\sigma$$

where:

- $\sigma = b_1^{i_1}, \dots, b_m^{i_m}$ is a string of 0,1 corresponding to probabilistic choices (with $b_j^{i_j}$ corresponding to a probabilistic choice for α_{i_j}) that led from M to N_i (i.e. the σs are the hidden variables of the model);
- the N_σ are normal λ-terms; the $P_\sigma(\vec{\alpha})$ are polynomials obtaining by progressively multiplying by either α_i or $1-\alpha_i$, depending on the probabilistic choice in σ.

Moreover, by applying the commutativity rule as well as the rule $\alpha \cdot M + \beta \cdot M = (\alpha + \beta) \cdot M$, we can reorganize this term, so that it is now of the form

$$\sum_{j=1}^{m} Q_j(\vec{\alpha}) \cdot N_j$$

where the N_j are now pairwise distinct normal forms.

If we tropicalize all this, we obtain log-probabilities as follows:

$$\begin{aligned} \mathrm{trop}(M) &\to^* \min_{\sigma = b_1 \dots b_m} \{\mathrm{trop}(P_\sigma)(-\log \vec{\alpha}) \cdot \mathrm{trop}(N_\sigma)\} \\ &\to^* \min_{j=1,\dots,n} \{\mathrm{trop}(Q_j)(-\log \vec{\alpha}) \cdot \mathrm{trop}(N_j)\} \end{aligned}$$

that we can interpret as follows:

- $\mathrm{trop}(P_\sigma)$ is the log-probability associated with the choice sequence σ. Hence, minimizing this polynomial in $\vec{\alpha}$ amounts at *maximing* the chances of getting N_σ via σ;
- for each $j = 1, \dots, m$, $\mathrm{trop}(Q_j) = \min\{\mathrm{trop}(P_{\sigma_l)} \mid M \to_{\sigma_l} N_j\}$ describes the log-probabilities associated with the reductions leading to N_j, and, for each choice of parameters $\vec{\alpha}$, the word σ where the minimum is attained corresponds to the reduction which, once we know N_j was obtained, is *most likely* to have produced this result.

Again, studying the tropical polynomial $Q_j(\vec{\alpha})$ corresponds to studying the *likelihood* of all choice strategies that might lead to N_j, by varying the parameters $\vec{\alpha}$, and for given $\vec{\alpha}$, two or more strategies can be equally likely precisely when $\vec{\alpha}$ happens to be a tropical root of Q_j.

Example 7. *Consider the following term*

$$M = P_1(\alpha) \cdot I + P_2(\alpha) \cdot \Omega$$

where

$$\begin{aligned} P_1(\alpha) &= \alpha^2 + (1-\alpha)\alpha^2 + (1-\alpha)^3 \\ P_2(\alpha) &= \alpha(1-\alpha) + 2(1-\alpha)^2\alpha \end{aligned}$$

This term can be obtained by reducing the following probabilistic term

$$\Big(I \oplus_\alpha \Omega\Big) \oplus_\alpha \Big((I \oplus_\alpha \Omega) \oplus_\alpha (\Omega \oplus_\alpha I)\Big)$$

In particular, each monomial in P_1 corresponds to one possible choice sequence for an α-biased flip coin that leads to I in the term above, and each monomial in P_2 corresponds to one possible choice sequence for an α-biased filp coin that leads to Ω.

By tropicalizing the term, we obtain

$$\mathrm{trop}(M) = \min\{\mathrm{trop}(P_1)(x,y), \mathrm{trop}(P_2)(x,y)\}$$

where $x = -\log\alpha$, $y = -\log(1-\alpha)$. For instance, we can study the tropical polynomial

$$\mathrm{trop}(P_1) = \min\{2x, 2x+y, 3y\}$$

whose three monomials M_1, M_2, M_3 correspond to the tree positions of I in M. This corresponds to a piecewise linear function where:

- *the tropical roots are all points of the form $(x, 0)$, $(x, \frac{2}{3}x)$, (x, x). From this we deduce that, if we fix the parameter α, since $2x \leqslant 2x + y$ and $3y \leqslant 2x$ in as far $y \leqslant \frac{2}{3}x$, we have that:*

  - as far as $1 - \alpha \geqslant \alpha^{\frac{2}{3}}$ (from $-\log(1-\alpha) \leqslant -\frac{2}{3}\log\alpha$), the rightmost I is the most likely;

  - otherwise, the leftmost I is the most likely.

So for example, if $\alpha = \frac{1}{2}$, the leftmost I is most likely, but if $\alpha = \frac{1}{4}$ (so that $1 - \alpha = \frac{3}{4}$ is greater than $\alpha^{\frac{2}{3}} \approx 0.39$), then the most likely is the rightmost I.