Taylor Expansion To check the validity of Taylor expansion we must further check the following equation, given $f \in Q_!^\Pi(C, B^A)$ and $g \in Q_!^\Pi(C, A)$:

$$\mathsf{ev} \circ \langle f, g \rangle = \inf_{n \in \mathbb{N}} ((\cdots (\Lambda^-(f) \underbrace{\star g) \cdots) \star g}_{n \text{ times}}) \circ \langle \mathrm{id}, \infty \rangle$$

where:

1. $\mathsf{ev} \in Q_!^\Pi(B^A + A, B)$ is the canonical *evaluation* morphism;
2. $\Lambda^-(\_) := \mathsf{ev} \circ (\_ \times \mathrm{id})$ is the *uncurry* operator;
3. given $f \in Q_!^\Pi(C + A, B)$ and $g \in Q_!^\Pi(C, A)$, $f \star g \in Q_!^\Pi(C + A, B)$ is the morphism obtained by differentiating f in its second component and applying g in that component, i.e.

$$f \star g = \mathsf{D}(f) \circ \langle \langle \infty, g \circ \pi_1 \rangle, \mathrm{id}_{C+A} \rangle.$$

Let us first compute the three morphisms ev, Λ^- and $\star$ explicitly:

1. $\mathsf{ev} \in Q^{\mathcal{M}_f((\mathcal{M}_f(A) \times B) + A) \times B}$ is given by

$$\mathsf{ev}_{\mu, y} = \begin{cases} 0 & \text{if } \mu = [\langle \rho, y \rangle] \oplus \rho \\ \infty & \text{otherwise} \end{cases}$$

   and observe that, given $f \in Q_!^\Pi(C, B^A)$ and $g \in Q_!^\Pi(C, A)$,

$$\big(\mathsf{ev} \circ \langle f, g \rangle\big)_{\chi, y} = \inf \Big\{ \sum_{i=1}^{m} g_{\chi_i, x_i} + f_{\chi', \langle [x_1, \dots, x_m], y \rangle} \mid x_1, \dots, x_m \in A, \chi = \chi' + \sum_{i=1}^{m} \chi_i, \Big\}$$

2. given $g \in Q_!^\Pi(C, B^A)$, $\Lambda^-(g) \in Q_!^\Pi(C + A, B)$ is given by

$$\big(\Lambda^-(g)\big)_{\rho \oplus \mu, y} = g_{\rho, \langle \mu, y \rangle}$$

3. $f \star g$ is given by

$$(f \star g)_{\rho \oplus \mu, y} = \inf \Big\{ g_{\rho', x} + f_{\rho'' \oplus (\mu + x)} \mid x \in A, \rho = \rho' + \rho'' \Big\}$$

Given the definition of $\mathsf{ev} \circ \langle f, g \rangle$, to check the Taylor equation it is enough to check that, for all $N \in \mathbb{N}$,

$$\left(((\cdots (\Lambda^-(f) \underbrace{\star g) \cdots) \star g}_{N \text{ times}}) \circ \langle \mathrm{id}, \infty \rangle \right)_{\chi, y} = \inf \Big\{ \sum_{i=1}^{N} g_{\chi_i, x_i} + f_{\chi', \langle [x_1, \dots, x_N], y \rangle} \mid \begin{array}{c} x_1, \dots, x_N \in A, \\ \chi = \chi' + \sum_{i=1}^{N} \chi_i \end{array} \Big\}$$

Let us show, by induction on N, the following equality, from which the desired equality easily descends:

$$\big((\cdots (\Lambda^-(f) \underbrace{\star g) \cdots) \star g}_{N \text{ times}}\big)_{\chi \oplus \mu, y} = \inf \Big\{ \sum_{i=1}^{N} g_{\chi_i, x_i} + f_{\chi', \langle \mu + [x_1, \dots, x_N], y \rangle} \mid \begin{array}{c} x_1, \dots, x_N \in A, \\ \chi = \chi' + \sum_{i=1}^{N} \chi_i \end{array} \Big\}$$

- if $N = 0$, the right-hand term reduces to $f_{\chi,\langle\mu,y\rangle} = (\Lambda^-(f))_{\chi\oplus\mu,y}$;
- otherwise, let $F = ((\cdots(\Lambda^-(f) \underbrace{\star g)\cdots) \star g}_{N-1 \text{ times}})$, so that by I.H. we have

$$F_{\chi\oplus\mu,y} = \inf\left\{ \sum_{i=1}^{N-1} g_{\chi_i,x_i} + f_{\chi',\langle\mu+[x_1,\ldots,x_{N-1}],y\rangle} \mid \begin{array}{c} x_1,\ldots,x_{N-1} \in A, \\ \chi = \chi' + \sum_{i=1}^{N-1} \chi_i \end{array} \right\}$$

Then we have

$$\begin{aligned}
\big(F \star g\big)_{\chi\oplus\mu,y} &= \inf\left\{g_{\chi',x} + F_{\chi''\oplus(\mu+x)} \mid x \in A, \chi = \chi' + \chi''\right\} \\
&= \inf\left\{ g_{\chi',x} + \inf\left\{ \sum_{i=1}^{N-1} g_{\chi_i,x_i} + f_{\chi'',\langle\mu+[x_1,\ldots,x_{N-1}]+x,y\rangle} \mid \begin{array}{c} x_1,\ldots,x_{N-1} \in A, \\ \chi^* = \chi'' + \sum_{i=1}^{N-1} \chi_i \end{array} \right\} \;\middle|\; \begin{array}{c} x \in A, \\ \chi = \chi' + \chi^* \end{array} \right\} \\
&= \inf\left\{ g_{\chi',x} + \sum_{i=1}^{N-1} g_{\chi_i,x_i} + f_{\chi'',\langle\mu+[x_1,\ldots,x_{N-1}]+x,y\rangle} \mid \begin{array}{c} x, x_1,\ldots,x_{N-1} \in A, \\ \chi = \chi' + \chi'' + \sum_{i=1}^{N-1} \chi_i \end{array} \right\} \\
&= \inf\left\{ \sum_{i=1}^{N} g_{\chi_i,x_i} + f_{\chi',\langle\mu+[x_1,\ldots,x_N],y\rangle} \mid \begin{array}{c} x_1,\ldots,x_N \in A, \\ \chi = \chi' + \sum_{i=1}^{N} \chi_i \end{array} \right\}.
\end{aligned}$$