Notes on the Maslov dequantisation

Davide Barbarossa
Dipartimento di Informatica: Scienza ed Ingegneria
Università di Bologna
Bologna, Italia

March 19, 2023

Fix $t > 0$. Define $\phi_t : [0,1] \to [0,+\infty]$, $\phi_t(0 < x) := -t \ln x$, $\phi_t(0) := +\infty$. It is bijective with inverse $\phi_t^{-1} : [0,+\infty] \to [0,1]$, $\phi_t^{-1}(0 < \alpha < +\infty) := e^{-\alpha/t}$, $\phi_t^{-1}(0) := 1$, $\phi_t^{-1}(+\infty) := 0$.

Call $\widetilde{\mathbb{L}} := ([0,1], \tilde{+}, \cdot)$, where $x \tilde{+} y := (x+y)/2$. This is a commutative non-associative idempotent semiring[1] with unit 1 and pseudo-zero[2] 0.

Call $\mathbb{L}_t := ([0,+\infty], +^t, \cdot^t)$, where $\alpha +^t \beta := \phi_t(\phi_t^{-1}(\alpha) \tilde{+} \phi_t^{-1}(\beta)) = -t \ln(e^{-\alpha/t} + e^{-\beta/t}) + t \ln 2$, and $\alpha \cdot^t \beta := \phi_t(\phi_t^{-1}(\alpha)\phi_t^{-1}(\beta)) = \alpha + \beta$. This is a commutative non-associative idempotent semiring with unit 0 and pseudo-zero $+\infty$. $\mathbb{L}_1$ is usually called the *unit-log-semiring*.

Remark that the bijection $\phi_t : \widetilde{\mathbb{L}} \to \mathbb{L}_t$ is an isomorphism of semirings with unit and pseudo-zeros, as $\phi_t(x \tilde{+} y) = \phi_t(x) +^t \phi_t(y)$ and $\phi_t(x \cdot y) = \phi_t(x) \cdot^t \phi_t(y)$. Therefore, all the $\mathbb{L}_t$, for $t > 0$, are isomorphic.

Call $\mathbb{L} := ([0,+\infty], \inf, +)$ the *Lawvere tropical semiring*. This is a commutative idempotent semiring with unit 0 and zero $+\infty$.

[1] Here by *semiring* I mean a set endowed with an associative multiplication, an associative and commutative addition, with distributivity of the multiplication over addition. By *non-associative semiring* I mean as before but without requiring associative addition. By *idempotent* I mean with $x + x = x$.

[2] For a non-associative semiring R, a *pseudo-zero* 0 is an element s.t. $x + 0 = f(x) = 0 + x$ and $x \cdot 0 = 0 = 0 \cdot x$, where f is a fixed function $R \to R$ not depending on x.

Since $\mathbb{L}$ is associative, $\mathbb{L}_t$ and $\mathbb{L}$ cannot be isomorphic, for no $t > 0$ (this can also be proven, without considering the associativity, by showing that there cannot exist a bijective homomorphism of semiring with units and pseudo-zeros between them).

It can be proven that $\alpha +^t \beta \underset{t \to 0}{\longrightarrow} \min\{\alpha, \beta\}$ (and trivially $\alpha \cdot^t \beta = \alpha + \beta \underset{t \to 0}{\longrightarrow} \alpha + \beta$). In this sense, one says that $\mathbb{L}_t \underset{t \to 0}{\longrightarrow} \mathbb{L}$. Looking at $\mathbb{L}$ as this limit is usually referred to as seeing $\mathbb{L}$ as the *Maslov dequantisation* of $\widetilde{\mathbb{L}}$ (i.e. of the $\mathbb{L}_t$).

Remark that this limit deforms a *isomorphic family* of commutative *non-associative* idempotent semirings with units and *pseudo-zeros*, to a *non-isomorphic* commutative idempotent semiring with unit and *zero*.