13 Taylor Coefficients as Quantitative Effects

Discrete Vectors. Let us call a vector $\mathbf{x} \in Q^X$ *discrete* if $\mathrm{Im}(\mathbf{x}) \in \{0, \infty\}$.

Lemma 27. *For all function $f : Q^X \to Q^Y$, if $\widehat{f}$ is discrete, then for all discrete $\mathbf{x} \in Q^X$, $f(\mathbf{x})$ is discrete.*

Proof. If $\widehat{f}$ and $\mathbf{x}$ are both discrete, then for all $y \in Y$ and $\mu \in \mathcal{M}_{\mathrm{f}}(X)$, $\widehat{f}_{\mu,y} + \mu \cdot \mathbf{x} \in \{0, \infty\}$: indeed, either $\widehat{f}_{\mu,y} = \infty$, either $\mu \cdot \mathbf{x} = \infty$ (which means that for some a in the support of μ, $x_a = \infty$, or both are 0. Hence $f(\mathbf{x}) = \inf_\mu \widehat{f}_{\mu,y} + \mu \cdot \mathbf{x}$ is an inf computed in the complete lattice $\{0, \infty\}$, so it is in $\{0, \infty\}$. ☐

Observe that the converse of the lemma above needs not hold in general: let $f : Q\langle \mathbf{1} \rangle \to Q\langle \mathbf{1} \rangle$ be the polynomial $f(x) = \min\{0, x + 1\} = 0$; then f is obviously finitary-preserving but $\widehat{f}_1 = 1$, so $\widehat{f}$ is not discrete. THIS EXAMPLE IS PROBLEMATIC, as it seems to imply that currying is not injective. Maybe we must be more careful on the definition of currying!

Lemma 28. *For all $f : Q^X \to Q^Y$ and $g : Q^Y \to Q^Z$, if $\widehat{f}$ and $\widehat{g}$ are discrete, then $\widehat{g \circ f}$ is discrete.*

Proof. $\widehat{g \circ f}_{\mu,y} = \inf\{\sum_i \widehat{f}_{\mu_i, z_i} + g_{[z_1, \ldots, z_k], y} \mid \mu = \sum_i \mu_i, z_1, \ldots, z_k \in Y\}$ which, by hypothesis, is an inf over a set of discrete values, and so it is discrete. ☐

Importantly, the interpretation of an ordinary λ-term is always a discrete matrix.

Proposition 29. *For all ordinary simply typed λ-term $M : \sigma \to \tau$, the matrix of the interpretation $[\![M]\!]$ of M in the tropical model is discrete.*

Proof. By induction on M:

- if $M = x$, then $[\![M]\!]$ is a projection, which has a discrete matrix;
- if $M = \lambda x.M'$, we simply apply the induction hypothesis to M';
- if $M = NP$, then $[\![M]\!] = \mathsf{ev}([\![N]\!], [\![P]\!])$ so we use the induction hypothesis, the fact that $\widehat{\mathrm{ev}}$ is discrete as well as Lemma 28.

☐

EXTEND THIS LEMMA TO: resource λ-terms, PCF? System T?

Measuring Duplications of Discrete Functions. For all finitary functions $f : Q\langle X \rangle \to Q\langle Y \rangle$, define a new function f^* by letting $\widehat{f^*}_{\mu,y} = \widehat{f}_{\mu,y} + \sharp\mu$. Intuitively, f^* keeps track, for each possible "choice" of a multiset μ, the weight of this multiset. This intuition is justified by the lemma below:

Lemma 30. *f^* is a finitary function, i.e. $f^* : Q\langle X \rangle \to Q\langle Y \rangle$. Moreover, for all discrete $\mathbf{x} \in Q\langle X \rangle$ and $y \in Y$, if $f(\mathbf{x})_y < \infty$, then $f^*(\mathbf{x})_y = \sharp\mu$, where $\mu = \mathrm{argmin}\{\sharp\nu \mid \widehat{f}_{\nu,y} < \infty\} = \mathrm{argmin}\{\sharp\nu \mid f(\mathbf{x})_y = \widehat{f}_{\nu,y} + \nu \cdot \mathbf{x}\}$.*

Proof. Since $|\widehat{f}| = |\widehat{f^*}|$, it follows that f^* is finitary. By finiteness, $f(\mathbf{x})_y = \inf_\mu \mu\cdot\mathbf{x}+\widehat{f}_{\mu,y} = \min_{i=1,\dots,k} \mu_i\cdot\mathbf{x}+\widehat{f}_{\mu_i,y}$, and since $\widehat{f}$ is discrete, either $\widehat{f}_{\mu_i,y} = \infty$ for all i (whence $f(\mathbf{x})_y = \infty$), or $f(\mathbf{x})_y = \min\{\mu_j \cdot \mathbf{x} + \widehat{f}_{\mu_j,y} \mid \widehat{f}_{\mu_j,y} < \infty\}$.

Now, $f^*(\mathbf{x})_y = \min\{\mu_j \cdot \mathbf{x} + \widehat{f}_{\mu_i,y} + \sharp\mu_i \mid \widehat{f}_{\mu_j,y} < \infty\}$, whence if $f(\mathbf{x})_y = \infty$, then $f^*(\mathbf{x})_y = \infty$, and if $f(\mathbf{x})_y = 0$, then $f^*(\mathbf{x})_y = \min\{\sharp\mu_i \mid \widehat{f}_{\mu_j,y} < \infty\}$. $\square$

Observe that the transformation $f \mapsto f^*$ is not stable under composition. In general $(g \circ f)^*(\mathbf{x})_b \leqslant g^*(f^*(\mathbf{x}))_b$, for example, we have

$$(\widehat{\mathrm{id}^*})_{\mu,y} = \begin{cases} 1 & \text{if } \mu = [y] \\ \infty & \text{otherwise} \end{cases}$$

from which it follows that $\mathrm{id}^*(\mathrm{id}^*(\mathbf{x})) = \mathbf{x} + 2 > \mathbf{x} + 1 = \mathrm{id}^*(\mathbf{x}) = (\mathrm{id} \circ \mathrm{id})^*(\mathbf{x})$.

A crucial property of the map $f \mapsto f^*$ is given by the following "commutation" with the differential operator.

Lemma 31. *For all $i \in \mathbb{N}$, $\mathsf{D}^{(i)}(f^*)(\mathbf{x}^i, \infty)_b = \mathsf{D}^{(i)}(f)(\mathbf{x}^i, \infty)_b + i$.*

Proof. By observing that for any function $f : Q^X \to Q^Y$ and $i \in \mathbb{N}$,

$$\left(\widehat{\mathsf{D}^{(i)}(f^*)}\right)_{[x_1,\dots,x_i]\oplus\mu,y} = \left(\widehat{f^*}\right)_{\mu+[x_1,\dots,x_i],y} = \left(\widehat{f}\right)_{\mu+[x_1,\dots,x_i],y} + \sharp\mu + i$$

we deduce that for all $\mathbf{x}$,

$$\begin{aligned}
\mathsf{D}^{(i)}(f^*)(\mathbf{x}^i, \infty)_b &= \inf\left\{\widehat{f^*}_{[a_1,\dots,a_i],y} + [a_1,\dots,a_i]\cdot\mathbf{x} \mid a_1,\dots,a_i \in X\right\} \\
&= \inf\left\{\widehat{f}_{[a_1,\dots,a_i],y} + [a_1,\dots,a_i]\cdot\mathbf{x} + i \mid a_1,\dots,a_i \in X\right\} \\
&= \inf\left\{\widehat{f}_{[a_1,\dots,a_i],y} + [a_1,\dots,a_i]\cdot\mathbf{x} \mid a_1,\dots,a_i \in X\right\} + i \\
&= \mathsf{D}^{(i)} f(\mathbf{x}^i, \infty)_b + i.
\end{aligned}$$

$\square$

This leads to the following result, which illustrates how the map $f \mapsto f^*$ allows one to extract duplication bounds from ordinary λ-terms.

Theorem 32. *For all ordinary λ-terms $M : \sigma \to \mathbf{Nat}$ and $N : \sigma$, $([\![M]\!]^*([\![N]\!]))_k = n$, where $MN \simeq_\beta (\mathsf{D}^{(n)}M \cdot N^n)0 \simeq_\beta \underline{k}$ and for all $m \neq n$, $(\mathsf{D}^{(m)}M \cdot N^m)0 \simeq_\beta 0$.*

Proof. By SN arguments we know that MN reduces to a unique normal form $\underline{k}$. Moreover, by Ehrhard's and Regnier's argument, we know that there exists a unique n such that $(\mathsf{D}^{(n)}M \cdot N^n)0 \not\simeq_\beta 0$ and such that $(\mathsf{D}^{(n)}M \cdot N^n)0 \simeq_\beta \underline{k}$. For all $m \neq n$, since $(\mathsf{D}^{(m)}M \cdot N^m)0 \simeq_\beta 0$, by soundness we deduce that $[\![(\mathsf{D}^{(m)}M \cdot N^m)0]\!] = \infty$ and, using Prop. 29, $[\![(\mathsf{D}^{(n)}M \cdot N^n)0]\!]_k = 0$.

Moreover, from the soundness of the Taylor expansion we deduce that

$$\begin{aligned}
([\![M]\!]([\![N]\!]))_k &= \left(\inf_i \left\{\mathsf{D}^{(i)}[\![M]\!]([\![N]\!]^i, \infty)\right\}\right)_k \\
&= \left(\inf_i \left\{[\![(\mathsf{D}^{(i)}M \cdot N^i)0]\!]\right\}\right)_k \\
&= [\![(\mathsf{D}^{(n)}M \cdot N^n)0]\!]_k = 0
\end{aligned}$$

and thus we deduce, using Lemma 31,

$$\begin{aligned}
(\llbracket M\rrbracket^*(\llbracket N\rrbracket))_k &= \left(\inf_i \left\{\mathsf{D}^{(i)}\llbracket M\rrbracket^*(\llbracket N\rrbracket^i, \infty)\right\}\right)_k \\
&= \left(\inf_i \left\{\mathsf{D}^{(i)}\llbracket M\rrbracket(\llbracket N\rrbracket^i, \infty) + i\right\}\right)_k \\
&= \left(\inf_i \left\{\llbracket(\mathsf{D}^{(i)} M \cdot N^i)0\rrbracket + i\right\}\right)_k \\
&= \llbracket(\mathsf{D}^{(n)} M \cdot N^n)0\rrbracket_k + n = n.
\end{aligned}$$

$\square$

Corollary 33. *For all ordinary λ-terms $M : \sigma \to \mathbf{Nat}$ and $N : \sigma$, with $MN \simeq_\beta \underline{k}$, for all $\epsilon < \infty$, the function $\llbracket M\rrbracket : Q\langle\llbracket\sigma\rrbracket\rangle \to Q\langle\mathbb{N}\rangle$ is $(\llbracket M^*\rrbracket(\llbracket N\rrbracket))_k$-Lipschitz over the open ball of center $\llbracket N\rrbracket$ and radius ϵ.*

Concretely, this corollary says that if we pick up a point $\mathbf{y} \in Q\langle\llbracket\sigma\rrbracket\rangle$ such that $\|\llbracket N\rrbracket - \mathbf{y}\| \leqslant \epsilon$ (here $\mathbf{y}$ could be the interpretation of a non necessarily ordinary λ-term, e.g. the term $\epsilon \cdot M$ or a probabilistic term), then $|f(\mathbf{y})| = |\llbracket MN\rrbracket| = \{k\}$ and the value $f(\mathbf{y})_k$ is bounded by $(\llbracket M^*\rrbracket(\llbracket N\rrbracket))_k \cdot \epsilon$.

This should be related with the fact that for $M : \sigma \to \tau$, $\lambda x.(\mathsf{D}^{(i)} M \cdot x^i)0 : !_i\sigma \multimap \tau$: we are capturing the Lipschitz-constant of M over x.