# 8 $\mathsf{Trop}_!$ is a cartesian differential category

We here focus on the category $Q^{\Pi}_!$, that is, the matricial category associated with $\mathsf{Trop}_!$. Let us recall that a morphism of $Q^{\Pi}_!(X,Y)$ is a matrix $f \in Q^{\mathcal{M}_{\mathrm{f}}(X)\times Y}$, and that composition of $f \in Q^{\Pi}_!(X,Y)$ and $g \in Q^{\Pi}_!(Y,Z)$ is given by

$$(g\circ f)_{\mu,z} = \inf\left\{ \sum_{i=1}^{n} f_{\mu_i,y_i} + g_{[y_1,\dots,y_n],z} \mid n \in \mathbb{N}, y_1,\dots,y_n \in Y, \mu = \mu_1 + \dots + \mu_n \right\}$$

Notice that $\infty\circ f = f\circ\infty = \infty$. However, while $\min\{g,h\}\circ f = \min\{g\circ f, h\circ f\}$ holds, as it can be computed using the continuity of $\mu$, inf and $+$, it is not true in general that $h\circ\min\{f,g\} = \min\{h\circ f, h\circ g\}$ (as a consequence of the fact that $\sum_{i=1}^{n}\min\{f_{\mu_i,y_i}, g_{\mu_i,y_i}\} \neq \min\{\sum_{i=1}^{n} f_{\mu_i,y_i}, \sum_{i=1}^{n} g_{\mu_i,y_i}\}$). In other words, morphisms in $Q^{\Pi}_!$ are only *left-$Q$-additive.* A morphism $h \in Q^{\Pi}_!(X,Y)$ that satisfies $h\circ\min\{f,g\} = \min\{h\circ f, h\circ g\}$ for all object $X'$ and $f,g \in Q^{\Pi}_!(X',X)$, is called *additive.*

**Proposition 18.** *$Q^{\Pi}_!$ is a cartesian closed left-$Q$-additive category.*

*Proof.* While left-$Q$-additivity was already proved, to show that $Q^{\Pi}_!$ is cartesian closed left-$Q$-additive we must also check that (1) products and projections of additive morphisms are additive, and that (2) $\Lambda(\min\{f,g\}) = \min\{\Lambda(f),\Lambda(g)\}$, $\Lambda(\infty) = \infty$, where $\Lambda : Q^{\mathcal{M}_{\mathrm{f}}(Z+X)\times Y} \to Q^{\mathcal{M}_{\mathrm{f}}(Z)\times(\mathcal{M}_{\mathrm{f}}(X)\times Y)}$ is the isomorphism given by $(\Lambda(f))_{\mu,\nu,y} = f_{\mu\oplus\nu,y}$, where $\mu\oplus\nu$ is defined by $(\mu\oplus\nu)(\langle 0,x\rangle) = \mu(x)$ and $(\mu\oplus\nu)(\langle 1,x\rangle) = \nu(x)$.

1. Let $f \in Q^{\mathcal{M}_{\mathrm{f}}(X)\times Y}$ and $g \in Q^{\mathcal{M}_{\mathrm{g}}(X)\times Z}$ be additive; then $\langle f,g\rangle \in Q^{\mathcal{M}_{\mathrm{f}}(X)\times(Y+Z)}$, which is defined by

$$\langle f,g\rangle_{\mu,\langle i,a\rangle} = \begin{cases} f_{\mu,a} & \text{if } i = 0 \\ g_{\mu,a} & \text{if } i = 1 \end{cases}$$

is also additive. Indeed, for all $h \in Q^{\Pi}_!(X',X)$, in any cartesian category it holds that $\langle f,g\rangle\circ h = \langle f\circ h, g\circ h\rangle$. Now, if $i=0$, then for all $h_1,h_2 \in Q^{\mathcal{M}_{\mathrm{f}}(X')\times X}$,

$$\begin{aligned}(\langle f,g\rangle\circ\min\{h_1,h_2\})_{\rho,\langle i,z\rangle} &= (\langle f\circ\min\{h_1,h_2\}, g\circ\min\{h_1,h_2\}\rangle)_{\rho,\langle i,z\rangle} \\ &= (f\circ\min\{h_1,h_2\})_{\rho,z} \\ &= \min\{(f\circ h_1)_{\rho,z}, (f\circ h_2)_{\rho,z}\} \\ &= \min\{\langle f\circ h_1, g\circ h_1\rangle_{\rho,\langle i,z\rangle}, \langle f\circ h_2, g\circ h_2\rangle_{\rho,\langle i,z\rangle}\} \\ &= \min\{(\langle f,g\rangle\circ h_1)_{\rho,\langle i,z\rangle}, (\langle f,g\rangle\circ h_2)_{\rho,\langle i,z\rangle}\}\end{aligned}$$

and similarly if $i=1$.

Moreover, suppose $f \in Q^{\Pi}_!(X,Y+Z)$ is additive, and let us show that $\pi_1(f) \in Q^{\Pi}_!(X,Y)$, defined by $(\pi_1(f))_{\mu,y} = f_{\mu,\langle 0,y\rangle}$, is also additive: first observe that $\pi_1(f) = \pi_1\circ f$, where $\pi_1 \in Q^{\Pi}_!(Y+Z,Y)$ is given by $(\pi_1)_{\mu,y} = 0$ if $\mu = [y]\oplus\varnothing$ and is $\infty$ otherwise; moreover, $\pi_1(\min\{g,h\}) = \min\{\pi_1(g),\pi_1(h)\}$, since $(\pi_1(\min\{g,h\}))_{\mu,y} = (\min\{g,h\})_{\mu,\langle 0,y\rangle} = \min\{g_{\mu,\langle 0,y\rangle}, h_{\mu,\langle 0,y\rangle}\} = (\min\{\pi_1(g),\pi_1(h)\})_{\mu,y}$. Now, given $h_1,h_2 \in Q^{\Pi}_!(X',X)$, we have that $\pi_1(f)\circ\min\{h_1,h_2\} = (\pi_1\circ f)\circ\min\{h_1,h_2\} = \pi_1\circ(f\circ\min\{h_1,h_2\}) = \pi_1\circ\min\{f\circ h_1, f\circ h_2\} = \pi_1(\min\{f\circ h_1, f\circ h_2\}) = \min\{\pi_1(f\circ h_1), \pi_1(f\circ h_2)\} = \min\{\pi_1(f)\circ h_1, \pi_1(f)\circ h_2)$.

2. It is clear then that $\Lambda(\infty) = \infty$, and moreover

$$\Lambda(\min\{f,g\})_{\mu,\nu,y} = \min\{f,g\}_{\mu\oplus\nu,y} = \min\{f_{\mu\oplus\nu,y}, g_{\mu\oplus\nu,y}\} = \min(\Lambda(f),\Lambda(g))_{\mu,\nu,y}$$

$\square$

For any morphism $f \in Q_!^{\Pi}(X,Y)$, let us define a morphism $\mathsf{D}(f) \in Q_!^{\Pi}(X+X,Y)$, i.e. $\mathsf{D}(f) \in Q^{\mathcal{M}_{\mathrm{f}}(X+X),Y}$, by

$$\mathsf{D}(f)_{\mu,y} = \begin{cases} f_{\mu'+x,y} & \text{if } \mu = [x]\oplus\mu' \\ \infty & \text{otherwise} \end{cases}$$

**Proposition 19.** *The category $Q_!^{\Pi}$, endowed with the operator $\mathsf{D}$, is a cartesian closed differential category.*

*Proof.* We must check axioms (1)-(7) of cartesian differential categories plus axiom (D-curry).

**(1)** $\mathsf{D}(\min\{f,g\}) = \min\{\mathsf{D}(f),\mathsf{D}(g)\}$ and $\mathsf{D}(\infty) = \infty$: while the latter is obvious, for the former we have $\mathsf{D}(\min\{f,g\})_{[x]\oplus\nu,y} = \min\{f,g\}_{\nu+x,y} = \min\{f_{\nu+x,y}, g_{\nu+x,y}\} = \min\{\mathsf{D}(f),\mathsf{D}(g)\}_{[x]\oplus\nu,y}$, and if $\mu \neq [x]\oplus\nu$, $\mathsf{D}(\min\{f,g\})_{\mu,y} = \infty = \min\{\infty,\infty\} = \min\{\mathsf{D}(f),\mathsf{D}(g)\}_{\mu,y}$.

**(2)** $\mathsf{D}(f)\circ\langle\min\{h,k\},v\rangle = \min\{\mathsf{D}(f)\circ\langle h,v\rangle, \mathsf{D}(f)\circ\langle k,v\rangle\}$, and $\mathsf{D}(f)\circ\langle\infty,v\rangle = \infty$: we can compute

$$\begin{aligned}
(\mathsf{D}(f)\circ\langle\min\{h,k\},v\rangle)_{\mu,y} &= \inf\Big\{\sum_{i=1}^{n}\min\{h,k\}_{\rho_i,w_i} + \sum_{j=1}^{m} v_{\nu_j,z_j} + f_{[z_1,\dots,z_m]+w,y} \\
&\qquad \mid \mu = \sum_{i=1}^{n}\rho_i + \sum_{j=1}^{m}\nu_j, [w] = [w_1,\dots,w_n]\Big\} \\
&= \inf\Big\{\min\{h,k\}_{\rho,w} + \sum_{j=1}^{m} v_{\nu_j,z_j} + f_{[z_1,\dots,z_m]+w,y} \\
&\qquad \mid \mu = \rho + \sum_{j=1}^{m}\nu_j\Big\} \\
&= \min\Big\{\inf\big\{h_{\rho,w} + \sum_{j=1}^{m} v_{\nu_j,z_j} + f_{[z_1,\dots,z_m]+w,y} \mid \mu = \rho + \sum_{j=1}^{m}\nu_j\big\}, \\
&\qquad \inf\big\{k_{\rho,w} + \sum_{j=1}^{m} v_{\nu_j,z_j} + f_{[z_1,\dots,z_m]+w,y} \mid \mu = \rho + \sum_{j=1}^{m}\nu_j\big\}\Big\} \\
&= \min\Big\{(\mathsf{D}(f)\circ\langle h,v\rangle)_{\mu,y}, (\mathsf{D}(f)\circ\langle k,v\rangle)_{\mu,y}\Big\} \\
&= \Big(\min\big\{\mathsf{D}(f)\circ\langle h,v\rangle, \mathsf{D}(f)\circ\langle k,v\rangle\big\}\Big)_{\mu,y}
\end{aligned}$$

where, in the first equation, the condition $[w_1,\dots,w_n] = [w]$ (i.e. $n = 1$) is forced by the fact that, otherwise, the application of $\mathsf{D}(f)$ would give $\infty$. Moreover, we have

$$(\mathsf{D}(f)\circ\langle\infty,v\rangle)_{\mu,y} = \inf\Big\{\infty + \sum_{j=1}^{m} v_{\nu_j,z_j} + f_{[z_1,\dots,z_m]+w,y} \mid \mu = \rho + \sum_{j=1}^{m}\nu_j\Big\} = \infty$$

**(3)** $\mathsf{D}(\mathrm{id}) = \pi_1$, $\mathsf{D}(\pi_i) = \pi_i \circ \pi_1$: recall that $\mathrm{id}_{[x],x} = 0$ and $\mathrm{id}_{\mu,x} = \infty$, if $\mu \neq [x]$. Moreover $(\pi_1)_{\mu,x} = 0$ if $\mu = [x] \oplus \varnothing$, and is $\infty$ otherwise, and $\pi_2$ is defined similarly. Hence $\mathsf{D}(\mathrm{id})_{[x]\oplus\nu,y} = \mathrm{id}_{\nu+x,y}$ is 0 precisely when $x = y$ and $\nu = \varnothing$, and in all other cases is $\infty$. This shows that $\mathsf{D}(\mathrm{id}) = \pi_1$.

$\mathsf{D}(\pi_1) \in Q^{\mathcal{M}_{\mathrm{f}}((X+Y)+(X+Y))\times Y}$ is given by $\mathsf{D}(\pi_1)_{[x\oplus\varnothing]\oplus(\mu\oplus\nu),y} = (\pi_1)_{(\mu\oplus\nu)+\langle 0,x\rangle,y}$, which is 0 precisely when $(\mu \oplus \nu) + \langle 0, x\rangle = y \oplus \varnothing$, i.e. when $x = y$ and $\mu = \nu = \varnothing$; in all other cases one can check that $\mathsf{D}(\pi_1)_{\rho,y} = \infty$, so we conclude $\mathsf{D}(\pi_1) = \pi_1 \circ \pi_1$. One can argue similarly for $\pi_2$.

**(4)** $\mathsf{D}(\langle f, g\rangle) = \langle \mathsf{D}(f), \mathsf{D}(g)\rangle$: we have

$$\mathsf{D}(\langle f, g\rangle)_{[x]\oplus\mu,\langle 0,y\rangle} = (\langle f, g\rangle)_{\mu+x,\langle 0,y\rangle} = f_{\mu+x,y} = \mathsf{D}(f)_{[x]\oplus\mu,y}$$
$$\mathsf{D}(\langle f, g\rangle)_{[x]\oplus\mu,\langle 1,y\rangle} = (\langle f, g\rangle)_{\mu+x,\langle 1,y\rangle} = g_{\mu+x,y} = \mathsf{D}(g)_{[x]\oplus\mu,y}$$

from which we deduce $\mathsf{D}(\langle f, g\rangle)_{[x]\oplus\mu,\langle i,y\rangle} = \langle \mathsf{D}(f), \mathsf{D}(g)\rangle_{[x]\oplus\mu,\langle i,y\rangle}$ by the definition of $\langle \_, \_\rangle$. If $\rho \neq [x] \oplus \mu$, then $\mathsf{D}(\langle f, g\rangle)_{\rho,\langle i,y\rangle} = \infty = \langle \infty, \infty\rangle = \langle \mathsf{D}(f), \mathsf{D}(g)\rangle_{\rho,\langle i,y\rangle}$ (where the equation $\infty = \langle \infty, \infty\rangle$ is to be read as an equality between the functions $X + Y \longrightarrow Q$ defined by $\langle i, y\rangle \mapsto \infty$ and by $\begin{matrix}\langle 0, x\rangle \mapsto \infty \\ \langle 1, y\rangle \mapsto \infty\end{matrix}$, respectively).

**(5)** $\mathsf{D}(f \circ g) = \mathsf{D}(f) \circ \langle \mathsf{D}(g), g \circ \pi_2\rangle$: we can compute

$$\begin{aligned}
\Big(\mathsf{D}(f) \circ \langle \mathsf{D}(g), g \circ \pi_2\rangle\Big)_{[x]\oplus\mu,z} &= \inf\Big\{\mathsf{D}(g)_{[x]\oplus\mu',w} + \sum_i g_{\mu_i,w_i} + \mathsf{D}(f)_{[w]\oplus[w_1,\dots,w_n],z} \\
&\qquad \mid w, w_i \in Y, \mu = \mu' + \sum_i \mu_i, \Big\} \\
&= \inf\Big\{g_{\mu'+x,w} + \sum_i g_{\mu_i,w_i} + f_{[w_1,\dots,w_n]+w,z} \\
&\qquad \mid w, w_i \in Y, \mu = \mu' + \sum_i \mu_i\Big\} \\
&= \inf\Big\{\sum_i g_{\mu_i,w_i} + f_{[w_1,\dots,w_n],z} \mid w_1, \dots, w_n \in Y, \mu + x = \sum_i \mu_i\Big\} \\
&= (f \circ g)_{\mu+x,y} = \mathsf{D}(f \circ g)_{[x]\oplus\mu,z}
\end{aligned}$$

if $\rho \neq [x] \oplus \mu$, then $\mathsf{D}(f \circ g)_{\rho,z} = \infty$ and from the first equation above it follows that also $(\mathsf{D}(f) \circ \langle \mathsf{D}(g), g \circ \pi_2\rangle)_{\rho,z} = \infty$.

**(6)** $\mathsf{D}(\mathsf{D}(f)) \circ \langle\langle g, \infty\rangle, \langle h, k\rangle\rangle = \mathsf{D}(f) \circ \langle g, k\rangle$: observe that

$$\Big(\mathsf{D}(\mathsf{D}(f))\Big)_{[\langle 1,x'\rangle]\oplus([x]\oplus\mu),z} = \Big(\mathsf{D}(f)\Big)_{[x]\oplus(\mu+x'),z} = f_{\mu+x'+x,z}$$
$$\Big(\mathsf{D}(\mathsf{D}(f))\Big)_{[\langle 0,x\rangle]\oplus(\varnothing\oplus\mu),z} = \Big(\mathsf{D}(f)\Big)_{[x]\oplus\mu,z} = f_{\mu+x,z}$$

and in all other cases $(\mathsf{D}(\mathsf{D}(f)))_{\mu,z} = \infty$. Using this fact we can compute:

$$\Big(\mathsf{D}(\mathsf{D}(f))\circ\langle\langle g,\infty\rangle,\langle h,k\rangle\rangle\Big)_{\mu,z} = \min\left\{\begin{array}{c}\inf\left\{\begin{array}{c}\infty_{\rho_1,x'}+h_{\rho_2,x}+\sum_i k_{\mu_i,w_i}+f_{[w_1,\dots,w_n]+x'+x,z}\\ \mid x,x',w_i\in Y,\mu=\rho_1+\rho_2+\sum_i\mu_i\end{array}\right\},\\ \inf\left\{\begin{array}{c}g_{\rho,x}+\sum_i k_{\mu_i,w_i}+f_{[w_1,\dots,w_n]+x,z}\\ \mid x,w_i\in Y,\mu=\rho+\sum_i\mu_i\end{array}\right\}\end{array}\right\}$$

$$= \inf\left\{g_{\rho,x}+\sum_i k_{\mu_i,w_i}+f_{[w_1,\dots,w_n]+x,z}\mid x,w_i\in Y,\mu=\rho+\sum_i\mu_i\right\}$$

$$= \Big(\mathsf{D}(f)\circ\langle g,k\rangle\Big)_{\mu,z}$$

**(7)** $\mathsf{D}(\mathsf{D}(f))\circ\langle\langle\infty,h\rangle,\langle g,k\rangle\rangle = \mathsf{D}(\mathsf{D}(f))\circ\langle\langle\infty,g\rangle,\langle h,k\rangle\rangle$: by computations similar to the case above we obtain

$$\Big(\mathsf{D}(\mathsf{D}(f))\circ\langle\langle\infty,h\rangle,\langle g,k\rangle\rangle\Big)_{\mu,z}$$

$$= \inf\left\{h_{\rho',x'}+g_{\rho,x}+\sum_i k_{\mu_i,w_i}+f_{[w_1,\dots,w_n]+x'+x,z}\mid x,x',w_i\in Y,\mu=\rho'+\rho+\sum_i\mu_i\right\}$$

$$= \inf\left\{g_{\rho,x}+h_{\rho',x'}+\sum_i k_{\mu_i,w_i}+f_{[w_1,\dots,w_n]+x+x',z}\mid x,x',w_i\in Y,\mu=\rho+\rho'+\sum_i\mu_i\right\}$$

$$= \Big(\mathsf{D}(\mathsf{D}(f))\circ\langle\langle\infty,g\rangle,\langle h,k\rangle\rangle\Big)_{\mu,z}$$

**(D-curry)** $\mathsf{D}(\Lambda(f)) = \Lambda(\mathsf{D}(f)\circ\langle\pi_1\times\infty,\pi_2\times\mathrm{id}\rangle)$: by observing that both morphisms are in $Q_!^{\Pi}(X+X,Z^Y)=Q^{\mathcal{M}_{\mathsf{f}}(X+X)\times\mathcal{M}_{\mathsf{f}}(Y)\times Z}$, and that $\langle\pi_1\times\infty,\pi_2\times\mathrm{id}\rangle\in Q_!^{\Pi}((X+X)+Y,(X+Y)+(X+Y))$, we can compute:

$$\big(\Lambda(\mathsf{D}(f)\circ\langle\pi_1\times\infty,\pi_2\times\mathrm{id}\rangle)\big)_{[x]\oplus\mu,\nu,z}$$

$$= \big(\mathsf{D}(f)\circ\langle\pi_1\times\infty,\pi_2\times\mathrm{id}\rangle\big)_{([x]\oplus\mu)\oplus\nu,z}$$

$$= \inf\left\{(\pi_1)_{[x]\oplus\varnothing,x}+\sum_i(\pi_2)_{\varnothing\oplus[w_i],w_i}+\sum_j(\mathrm{id})_{[z_j],z_j}+\mathsf{D}(f)_{([x]\oplus\varnothing)\oplus(\mu\oplus\nu)}\mid \begin{array}{r}\mu=[w_1,\dots,w_n],\\ \nu=[z_1,\dots,z_m]\end{array}\right\}$$

$$= \inf\left\{0+0+0+\mathsf{D}(f)_{([x]\oplus\varnothing)\oplus(\mu\oplus\nu)}\mid \begin{array}{r}\mu=[w_1,\dots,w_n],\\ \nu=[z_1,\dots,z_m]\end{array}\right\}$$

$$= \big(\mathsf{D}(f)\big)_{([x]\oplus\varnothing)\oplus(\mu\oplus\nu),z}$$

$$= f_{(\mu+x)\oplus\nu,z}$$

$$= \big(\Lambda(f)\big)_{\mu+x,\nu,z} = \big(\mathsf{D}(\Lambda(f))\big)_{[x]\oplus\mu,\nu,z}$$

If $\rho\neq[x]\oplus\mu$, then $(\mathsf{D}(\Lambda(f)))_{\rho,\nu,z}=\infty$ and $(\Lambda(\mathsf{D}(f)\circ\langle\pi_1\times\infty,\pi_2\times\mathrm{id}\rangle))_{\rho,\nu,z}=(\mathsf{D}(f)\circ\langle\pi_1\times\infty,\pi_2\times\mathrm{id}\rangle)_{\rho\oplus\nu,z}$, and one can check that also this is $\infty$, using the second equation above and the fact that $(\pi_1)_{\rho\oplus\varnothing,x}=\infty$.

□