Tropical methods in Lambda-Calculus?

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Among the quantitative approaches towards which the programming language community has directed its interest in recent years, two different directions have received considerable attention. On the one hand one there is the approach of *program metrics* [**?**, **?**, **?**] and *quantitative equational theories* [**?**]: when considering probabilistic or approximate computation, rather than asking whether two programs compute *the same* function, it makes more sense to ask whether they compute functions which do not differ *too much*, and thus study denotational frameworks in which types are endowed with a metric [**?**],[**?**], [**?**, **?**, **?**]. On the other hand, there is the approach based on *differential* [**?**], [**?**], [**?**, **?**, **?**]. or *resource-aware* [**?**] extensions of the λ-calculus, which is well-connected to the so-called *relational semantics* [**?**, **?**, **?**] and has a syntactic counterpart in the study of *non-idempotent* intersection types [**?**, **?**].

In both approaches the notion of *linearity*, in the sense of linear logic [**?**] (i.e. of using inputs exactly once), plays a crucial role. In metric semantics, linear programs correspond to *non-expansive* maps, that is, to functions that do not increase distances; moreover, the possibility of duplicating inputs leads to interpret *bounded* programs (i.e. programs with a fixed duplication bound) as *Lipschitz-continuous* maps [**?**]. By contrast, in the standard semantics of the differential λ-calculus, linear programs correspond to linear maps, in the usual algebraic sense, while the possibility of duplicating inputs leads to consider functions defined as *power series*.

At a first glance, there seems to be a "logarithmic" gap between the two approaches: in metric models n times duplication results in a *linear* (hence Lipschitz) function $n \cdot x$, while in differential models this results in a *polynomial* function x^n, hence not Lipschitz. The fundamental motivation of this work is then the observation that this gap is naturally overcome once we interpret these functions in the framework of tropical mathematics where, for instance, the monomial x^n precisely reads as the linear function $n \cdot x$. Tropical mathematics [**?**] is a well established algebraic and geometrical framework, with tight connections with optimisation theory [**?**], where the usual ring structure of numbers based on addition and multiplication is replaced by the semiring structure given, respectively, by "min" and "+". For instance, the polynomial $p(x,y) = x^2 + xy^2 + y^3$, when interpreted over the tropical semiring, translates as the piecewise linear function $\varphi(x,y) = \min\{2x, x+2y, 3y\}$.

Coming back to our discussion on program semantics, tropical geometry might seem to provide precisely what look for, as it turns the monomials x^n into the Lipschitz map $n \cdot x$. Actually, a tropical variant of relational semantics has already been considered [**?**], and shown capable of capturing *best-case* quantitative properties, but has not yet been studied in detail. Furthermore, connections between tropical linear algebra and metric spaces have also been observed [**?**] within the abstract setting of *quantale-enriched* categories [**?**, **?**]. However, a thorough investigation of the interpretation of the λ-calculus within tropical mathematics has not yet been undertaken.

The viewpoint that we develop in the following sections, may suggests the application of optimization methods based on tropical mathematics to the study of the λ-calculus and its quantitative extensions, but it scales to a more abstract level, leading to introduce a differential operator for continuous functors between *generalized* metric spaces (in the sense of [**?**]).

In this contribution we only sketch some of the main results of our still work in progress analysis.

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1 Tropical mathematics in a nutshell

More rigorously, we let the *tropical semiring*, the structure at the heart of tropical mathematics, be $([0,\infty],\min,+)$. Remark that this coincides with the *Lawvere quantale* $\mathbb{L} = ([0,\infty],\geq,+)$ [**?**, **?**] (the order is the reverse order $\geq$ on $\mathbb{R}$ and the monoid action is addition), the structure at the heart of the categorical study of metric spaces initiated by Lawvere himself [**?**], and we will take this point of view in the last section of this contributions. A tropical polynomial is a piece-wise linear function $f : \mathbb{L} \to \mathbb{L}$ of the form $f(x) = \min_{i_1,\dots,i_k}\{i_j x + c_{i_j}\}$ where the i_j are natural numbers and the c_{i_j} are in $\mathbb{L}$. For example, the polynomials $\varphi_n(x) = \min_{i\leq n}\{ix + 2^{-i}\}$ are illustrated in Fig. **??** for $n \leq 4$. A *tropical root* of a tropical polynomials φ is a point $x \in \mathbb{L}$ where φ is not differentiable. In other words, the roots of φ are the points where the minimum defining φ is attained at least twice (i.e. where the slope of φ changes). For instance, the tropical roots of φ_{n+1} are of the form $2^{-(i+1)}$, for $i \leq n$. With this definition, tropical roots mimic the usual factorization property of roots: if x_0 is a root of f, this factorizes as $f(x) = \min\{x,x_0\} + g(x)$. Yet, unlike in standard algebra, tropical roots can be computed in linear time [**?**]. A *tropical Laurent series* (of one variable $x \in \mathbb{L}$), shortly a *tLs*, is a function that can be expressed as $f(x) = \inf_n\{nx + \hat{f}_n\}$, with $\hat{f}_n$ a sequence in $\mathbb{L}$. In other words, a tLs is a "limit" of tropical polynomials of higher and higher degree. For instance, the function $\varphi(x) := \inf_{n\in\mathbb{N}}\{nx + \frac{1}{2^n}\}$ (illustrated in Fig. **??**), that we will take as our running example, is the "limit" of the polynomials φ_n. Since infs are not in general mins, the behavior of tLS may be less predictable than that of tropical polynomials. For instance, tropical roots for tLs (see [**?**]) may also include limit points.

2 Tropical weighted semantics in a nutshell

The study of matrices with values over the tropical semiring can be seen as a special case of the *quantitative relational semantics* [**?**], a well-studied semantics of the λ-calculus and linear logic: for a fixed *continuous* semi-ring Q [Def. II.5, [**?**]], the category $Q\mathsf{Rel}$ has sets as objects and set-indexed matrices with coefficients in Q as morphisms, i.e. $Q\mathsf{Rel}(X,Y) = Q^{X\times Y}$ ([**?**] would call it Q^{Π}). As it is expected, Q^X is a Q-semimodule and the bijection $\hat{(\cdot)}$ identifies the set of linear maps from Q^X to Q^Y with $Q\mathsf{Rel}(X,Y)$.

The tropical relational model is thus provided by the category $\mathbb{L}\mathsf{Rel}$ of matrices with values over $\mathbb{L}$ (which, being a quantale, is indeed a continuous semi-ring). It is worth observing that the formula for composition in $\mathbb{L}\mathsf{Rel}$ reads as $(st)_{a,c} := \inf_{b\in Y}\{s_{b,c} + t_{a,b}\}$; similarly, the linear functions $f : \mathbb{L}^X \to \mathbb{L}^Y$ induced by matrices, which we call *tropical linear*, are exactly those of shape $f(x)_b = \inf_{a\in X}\{\hat{f}_{a,b} + x_a\}$, for some matrix $\hat{f}$ from X to Y. By applying more or less well-known results (taken from [**?**], CITARE LEMAYYYY), one obtains that $\mathbb{L}\mathsf{Rel}$ gives rise to denotational models of several variants of the λ-calculus:

Fact 1. $\mathbb{L}\mathsf{Rel}$ *is a SMCC, thus a model of the linear* STLC.

The coKleisli $\mathbb{L}\mathsf{Rel}_!$ *is CCC, i.e. a model of* STLC.

The exponential $!$ *can be "decomposed" into a family of* graded *exponentials* $(!_n)_{n\in\mathbb{N}}$ *turning* $\mathbb{L}\mathsf{Rel}$, *together with* $(!_n)_{n\in\mathbb{N}}$, *is a model for* bSTLC.

$\mathbb{L}\mathsf{Rel}_!$ *equipped with D, is a CC∂C and the interpretation of the Taylor expansion of a* STLC*-term M, given in* (**??**), *converges to the one of M.*

In the previous result, the coKleisli composition of $s \in \mathbb{L}^{!Y\times Z}$ and $t \in \mathbb{L}^{!X\times Y}$ is the matrix $s \circ_! t \in \mathbb{L}^{!X\times Z}$ where $(s \circ_! t)_{\mu,c}$ is: $\inf_{n\in\mathbb{N},b_1\dots,b_n\in Y,\mu=\mu_1+\dots+\mu_n}\left\{s_{[b_1,\dots,b_n],c} + \sum_{i=1}^{n} t_{\mu_i,b_i}\right\}$. and the *tropical differential operator* is the map $D : \mathbb{L}\mathsf{Rel}(!X,Y) \to \mathbb{L}\mathsf{Rel}(!(X+X),Y)$ defined as $(Dt)_{\mu\oplus\rho,b} = t_{\rho+\mu,b}$ if $\#\mu = 1$ and as ∞ otherwise (where a multiset $\nu \in !(X+X)$ is identified with a disjoint sum of $\mu,\rho \in !X$).

In particular, the Taylor formula decomposes an unbounded application as a limit of bounded ones, one might well ask whether it could be possible to see this formula as interpreting a λ-term as a limit of Lipschitz maps, in some sense, thus bridging the metric and differential approaches. Here, a natural direction to look for is the *relational semantics*, i.e. the somehow canonical "Taylor" semantics for ST∂LC. However, in this semantics, terms with bounded applications correspond to *polynomials*, i.e. to non-Lipschitz functions.

Yet, what if these polynomials were tropical ones, i.e. piecewise linear functions? This way, (**??**) could really be interpreted as a decomposition of λ-terms via limits (indeed, infs) of Lipschitz maps. In other words, unbounded term application could be seen as a limit of *more and more sensitive* operations.

3 Tropical Laurent Series

As usual, a matrix $t \in \mathbb{L}\mathsf{Rel}_!(X,Y)$ yields a linear map $\mathbb{L}^{!X} \to \mathbb{L}^Y$, but we can also "express it in the base X", i.e. see it as a map $t^! : \mathbb{L}^X \to \mathbb{L}^Y$, by setting $t^!(x) := t \circ_! x$. This is the notion of *non-linear* map generated by the CCC-structure of $\mathbb{L}\mathsf{Rel}_!$. Concretely, we have

$$t^!(x)_b = \inf_{\mu \in !X} \{\mu x + t_{\mu,b}\} \tag{1}$$

where $\mu x := \sum_{a\in X} \mu(a) x_a$. These functions correspond then to tLs with possibly infinitely many variables (in fact, as many as the elements of X).

Therefore, we study in this section the properties of those functions, from the point of view of mathematical analysis.

Notice that, by identifying $!\{*\} \simeq \mathbb{N}$ and $\mathbb{L}^{\{*\}} \simeq \mathbb{L}$, the tLs generated by the morphisms in $\mathbb{L}\mathsf{Rel}_!(\{*\},\{*\})$ are exactly the functions $f : \mathbb{L} \to \mathbb{L}$ of shape $f(x) = \inf_{n\in\mathbb{N}}\{nx + \hat{f}(n)\}$, for some $\hat{f} : \mathbb{N} \to \mathbb{L}$, i.e. usual tLs's of one variable. In a similar way, the tropical polynomials can be identified with the tLs $f : \mathbb{L}^X \to \mathbb{L}^Y$ for which the support $\mathscr{F} = \{\mu \in !X \mid \hat{f}_{\mu,b} \neq \infty\}$ is *finite*, and which have thus shape $f(x)_b := \min_{\mu\in\mathscr{F}}\{\mu x + t_{\mu,b}\}$. This is again the generalisation of usual tropical polynomials to the case of infinitely many variables.

4 Relations to quantitative properties

5 Lawvere's generalised metric spaces

We already remarked that the tropical semiring $\mathbb{L}$ coincides with the *Lawvere quantale* $\mathbb{L} = ([0,\infty], \geq, +)$. In particular, a (possibly ∞) metric on a set X is nothing but a "$\mathbb{L}$-valued square matrix" $d : X \times X \to \mathbb{L}$ satisfying axioms like e.g. the triangular law (indeed, such distance matrices correspond to $\mathbb{L}$-*enriched categories*, a viewpoint we explicitly take in Section **??**).

Remark 1. *The category QRel is (equivalent to) a subcategory of the category QMod of* complete *Q-semimodules. If QRel corresponds to considering semimodules (the Q^X's) whose vectors are given in coordinates w.r.t. a* fixed base *(the set X), QMod corresponds to considering semimodules in abstract, without fixing a base. We take this viewpoint in Section* **??**.