Definition 0.1. *Let R be a semiring. The category Rel_R has sets as objects, and the homset $\mathrm{Rel}_R(X,Y)$ is the set $R^{X\times Y}$ of matrices with rows indexed by X, columns indexed by Y and coefficients in R. The identity id_X is the usual diagonal matrix $id_{a,b} := \delta_{a,b}$ (δ is the Kronecker symbol), and the composition of $t : Y\times Z \to R$ with $s : X\times Y \to R$ is the matrix $ts : X\times Z \to R$ given by $(ts)_{a,c} := \sum_{b\in Y} t_{b,c}s_{a,b}$. We suppose for the moment that all the desired infinite series converge in R.*

Our composition formula is slightly unconventional. This is because, usually, a matrix t with raws on X and columns on Y is seen as a function from R^Y to R^X (R being here any set of coefficients); however, we are using the opposite convention, which corresponds to taking the transpose. In this way, the matrix t applies to a vector $x\in R^X$ giving rise to a vector $tx\in R^Y$ (instead of the converse), and the formula takes the (slightly unconventional) form $(tx)_b := \sum_{a\in X} t_{a,b}x_a$. We use this convention because it simplifies notations, as it allows us to see matrices $t\in \mathrm{Rel}_R(X,Y)$ as functions $\hat{t} : R^X \to R^Y$.

Remark 0.2. *It is straightforward to see that each R^X is a R-*module *w.r.t. pointwise scalar multiplication and pointwise addition. If the desired infinite series behave well in R, then the functions $\hat{t}$ are all and only the linear ones, as in the finite case.*

Given $a\in X$, let us denote by $e_a \in Q^X$ the "canonical base vector" whose coordinates are $(e_a)_a := 1$ and $(e_a)_{a'} := 0$ if $a \neq a'$.

Let us set $\overline{\mathbb{R}}_{\geq 0} := \mathbb{R}_{\geq 0} \cup \{+\infty\}$.

Terminology 0.3. *Consider the complete lattice $(\overline{\mathbb{R}}_{\geq 0}, \geq)$, with joins $\bigvee A$ given by $\inf A$, and meets $\bigwedge A$ by $\sup A$. Remark that we set $\inf\varnothing := +\infty$. It becomes a semiring if we endow it with $\inf$ as addition and $+$ as multiplication*[1]*. It is called the* non-negative (min-plus-)tropical (commutative and idempotent) semiring*. Moreover, it is a (unital, commutative, idempotent) quantale with multiplication the one of its semiring structure (that is, $+$). It is also a continuous semiring. Let us call it Q.*

Remark 0.4. *The Q-module structure of Q^X is given by scalar multiplication $(r,x)\in Q\times Q^X \to r+x\in Q^X$ and addition $(x,y)\in Q\times Q^X \to \min\{x,y\}\in Q^X$. If $t\in \mathrm{Rel}_Q(X,Y)$, than the function $\hat{t}$ is* linear.

Definition 0.5. *Let us call* Trop *the co-Kleisli category $(\mathrm{Rel}_Q)_!$ of Rel_Q w.r.t. the usual multiset comonad $!$. That is, objects are sets and morphisms from X to Y are matrices $F : \mathcal{M}_f(X)\times Y \to Q$, seen as maps $\hat{F} : Q^{\mathcal{M}_f(X)} \to Q^Y$. Moreover, the identity on X is the function $id : \mathcal{M}_f(X)\times X \to X$ s.t. $id_{[x],x} = 0$ and $id_{\mu,x} = +\infty$ if $\mu \neq [x]$, and the composition $GF : \mathcal{M}_f(X)\times Z \to Q$ of $G : \mathcal{M}_f(Y)\times Z \to R$ with $F : \mathcal{M}_f(X)\times Y \to R$ takes the form:*

$$(GF)_{\mu,c} = \inf_{\rho\in\mathcal{M}_f(Y)} \left\{ G_{\rho,c} + \sum_{b\in Y} \rho_b F_{\mu,b} \right\}.$$

The induced map $\hat{F} : Q^{\mathcal{M}_f(X)} \to Q^Y$ by $F\in \mathrm{Trop}(X,Y)$ is given by:

$$\hat{F}(x)_b = \inf_{\mu\in\mathcal{M}_f(X)} \{F_{\mu,b} + \mu\cdot x\}$$

[1] The neutral element for addition being $+\infty$ and the one for multiplication 0.

where $\mu \cdot x := \sum_{a \in X} \mu_a x_a \in Q$*. We call* tropical analytical *the functions of the shape* $\hat{F}$*.*

Remark 0.6. *Since* Q *is a continuous semiring, [Weighted relational models] assures us that* Rel_Q *is:*

- *linear and continuous category*

- $*$*-autonomous with biproducts and bilinear and continuous tensor and monoidal currying*

- *Lafont category.*

Furthermore, the same paper tells us that $(\mathrm{Rel}_Q)_!$ *is a post-linear and continuous* Q*-CCC.*

Example 0.7. *The terminal object of* Trop *is* $1 := \{*\}$*. Using the trivial bijection* $\mathcal{M}_f(\{*\}) = \mathbb{N}$*, the morphisms* $F \in \mathrm{Trop}(1,1)$ *are* exactly *the functions* $F : \mathbb{N} \to Q$*, and the induced tropical analytical map* $\hat{F} : Q = Q^{\{*\}} \to Q^{\{*\}} = Q$ *has shape* $\hat{F}(x) = \inf_{n\mathbb{N}} \{F(n) + nx\}$*. A particular case is the one of tropical polynomials on* Q*, which coincide with the tropical analytical functions from* Q *to itself for which the* inf *is always a* min*. In particular, all affine functions* $\hat{F}(x) = r + mx$ with $m \in \mathbb{N}$*,* $r \in Q$ *from* Q *to itself (and thus also all affine functions from* $\mathbb{R}$ *to itself), are of this shape: is suffices to take* $F(n) := r$ *if* $n = m$ *and* $F(n) = +\infty$ *otherwise.*

Remark 0.8. *We can endow* Q^X *with two (potentially* $+\infty$*) metrics:*

$$d^\infty(x,y) := \sup_{a \in X} |x_a - y_a|$$

$$d^1(x,y) := \sum_{a \in X} |x_a - y_a|.\textit{(Forse questa la si dovrebbe prendere tropicale, ovvero } \inf_a\textit{?)}$$

Remember the following easy property of infs and sequences.

Lemma 0.9. *If* $|a_n - b_n| \leq K$ *for all* $n \in \mathbb{N}$*, then* $|\inf_n a_n - \inf_n b_n| \leq K$*.*

Proof. We have $|\inf_n b_n - \inf_n a_n| \leq |\inf_n b_n - b_m| + |b_m - a_m| + |a_m - \inf_n a_n| \leq |\inf_n b_n - b_m| + K + |a_m - \inf_n a_n|$ for all $m \in \mathbb{N}$, so $|\inf_n b_n - \inf_n a_n| \leq \inf_m \{|\inf_n b_n - b_m| + K + |a_m - \inf_n a_n|\} = K$. □

Let us call $\widetilde{F}_{\mu,b}(x) := F_{\mu,b} + \mu \cdot x$, so that $\hat{F}(x)_b = \inf_{\mu \in \mathcal{M}_f(X)} \widetilde{F}_{\mu,b}(x)$.

Remark 0.10. *Let* $\hat{F}, \hat{G} : Q^X \to Q^Y$ *tropical analytical, whose associated matrices are* F, G *respectively. We have:*

$$d^\infty(\hat{F}(x), \hat{G}(x)) \leq d^\infty(F,G)$$

for all $x \in Q^X$*.*

Indeed, we have: $|\widetilde{F}_{\mu,b}(x) - \widetilde{G}_{\mu,b}(x)| = |F_{\mu,b} - G_{\mu,b}| \leq d^\infty(F,G)$*, so by the previous Lemma* $|\hat{F}(x)_b - \hat{G}(x)_b| \leq d^\infty(F,G)$ *and thus the desired result.*

Remark 0.11. *The functions $\widetilde{F}_{\mu,b}$ are $\#\mu$-Lipschitz in Q^X (w.r.t. both metrics). Indeed we have the following straightforward inequalities:*

$$|\widetilde{F}_{\mu,b}(x)-\widetilde{F}_{\mu,b}(y)| = |\mu\cdot(x-y)| \le \mu\cdot|x-y| = \sum_{a\in X}\mu_a|x_a-y_a| \le d(x,y)\sum_{a\in X}\mu_a = d(x,y)\#\mu.$$

Remark 0.12. *The functions $\widetilde{F}_{\mu,b}$ are always non-decreasing, in the sense that $\widetilde{F}_{\mu,b}(x)\le\widetilde{F}_{\mu,b}(x+y)$. This is clear because everything takes place in non-negative real numbers, so $\mu\cdot x\le\mu\cdot(x+y)$. Therefore, also $\hat{F}$ is non-decreasing (in the same sense: for all $b\in Y$, we have $\hat{F}(x)_b\le\hat{F}(x+y)_b$).*

Remark 0.13. *Fix $\epsilon>0$. Then there is $\mu^\epsilon\in\mathcal{M}_f(X)$ s.t. $|\hat{F}(x)_b-\widetilde{F}_{\mu^\epsilon,b}(x)|\le\epsilon$ (this is by definition of $\inf$). Furthermore, if $\hat{F}(x)_b<+\infty$ then $\sum_{a\in X}\mu^\epsilon_a x_a=\mu^\epsilon\cdot x<+\infty$. This means that $\mu^\epsilon_a x_a<+\infty$ for all $a\in X$.*

Actually, for all $a\in X$, there must exist $K_a\in\mathbb{R}$ s.t. $\mu^\epsilon_a x_a\le K_a$ for all $0<\epsilon<1$ (limitare eps serve altrimenti l'argomento sotto non funziona!). In fact, otherwise, there is an $a\in X$ s.t. for all K_a there is $0<\epsilon<1$ s.t. $\widetilde{F}_{\mu^\epsilon,b}(x)\ge\mu^\epsilon_a x_a>K_a$. But, taking K_a large enough, we obtain a contradiction with he fact that, by definition of μ^ϵ, we must have $|\hat{F}(x)_b-\widetilde{F}_{\mu^\epsilon,b}(x)|\le\epsilon<1$.

Therefore, for all $a\in X$ s.t. $x_a\ne0$, there must still exist $L_a\in\mathbb{R}$ s.t. $\mu^\epsilon_a\le L_a$ for all $0<\epsilon<1$.

Remark 0.14. *Let $q\in Q$. For all $0<\epsilon<1$, we have:*

$$|\widetilde{F}_{\mu^\epsilon,b}(x)-\widetilde{F}_{\mu^\epsilon,b}(x+qe_a)| = |\mu^\epsilon\cdot(-qe_a)| = q\mu^\epsilon_a$$

and, if $x_a\ne0$, the inequality continues as $\le qL_a$.

Putting all this together we get:

Proposition 0.15. *Let $\hat{F}$ tropical analytical. Fix $x\in Q^X$ and let $a\in X$ s.t. $x_a\ne0$. For all $0<\epsilon<1$ we have:*

$$|\hat{F}(x)_b-\hat{F}(x+\epsilon e_a)_b|\le(2+2L_a)\epsilon.$$

This immediately implies that:

$$d^\infty(\hat{F}(x),\hat{F}(x+\epsilon e_a))\le(2+2L_a)\epsilon.\text{(ma non è vero per } d^1\text{)}$$

Proof. We have:

$$|\hat{F}(x)_b-\hat{F}(x+\epsilon e_a)_b|\le|\hat{F}(x)_b-\widetilde{F}_{\mu^\epsilon,b}(x)|+|\widetilde{F}_{\mu^\epsilon,b}(x)-\widetilde{F}_{\mu^\epsilon,b}(x+\epsilon e_a)|+|\widetilde{F}_{\mu^\epsilon,b}(x+\epsilon e_a)-\hat{F}(x+\epsilon e_a)_b|$$

where $|\hat{F}(x)_b-\widetilde{F}_{\mu^\epsilon,b}(x)|\le\epsilon$ and $|\widetilde{F}_{\mu^\epsilon,b}(x)-\widetilde{F}_{\mu^\epsilon,b}(x+\epsilon e_a)|\le\epsilon L_a$. So if we show that:

$$|\widetilde{F}_{\mu^\epsilon,b}(x+\epsilon e_a)-\hat{F}(x+\epsilon e_a)_b|\le\epsilon+\epsilon L_a$$

we are done. But since $\widetilde{F}_{\mu^\epsilon,b}$ is non-decreasing, and by definition of $\inf$, we have:

$$\hat{F}(x)_b\le\hat{F}(x+\epsilon e_a)_b\le\widetilde{F}_{\mu^\epsilon,b}(x+\epsilon e_a),$$

so that:

$$|\widetilde{F}_{\mu^\epsilon,b}(x+\epsilon e_a)-\hat{F}(x+\epsilon e_a)_b|\le|\widetilde{F}_{\mu^\epsilon,b}(x+\epsilon e_a)-\hat{F}(x)_b|\le|\widetilde{F}_{\mu^\epsilon,b}(x+\epsilon e_a)-\widetilde{F}_{\mu^\epsilon,b}(x)|+|\widetilde{F}_{\mu^\epsilon,b}(x)-\hat{F}(x)_b|\le\epsilon L_a+\epsilon.$$

□

Remark that, in general, the previous results reads as follows:

Fix $x \in Q^X$, $I \subseteq \mathbb{N}$ an index set and let $a_i \in X$ s.t. $x_{a_i} \neq 0$ for all $i \in I$. For all $0 < \epsilon_i < \epsilon < 1$ we have:

$$|\hat{F}(x)_b - \hat{F}(x + \sum_{i \in I} \epsilon_i e_{a_i})_b| \leq (2 + 2 \sum_{i \in I} L_{a_i})\epsilon.$$

This is because we have:

$$|\widetilde{F}_{\mu^\epsilon, b}(x) - \widetilde{F}_{\mu^\epsilon, b}(x + \sum_{i \in I} \epsilon_i e_{a_i})| = |\mu^\epsilon \cdot (-\sum_{i \in I} \epsilon_i e_{a_i})| = \sum_{i \in I} \mu^\epsilon_{a_i} \epsilon_i$$

and, if $x_{a_i} \neq 0$ for all $i \in I$, the inequality follows as $\leq \sum_{i \in I} \epsilon_i L_{a_i} \leq \epsilon \sum_{i \in I} L_{a_i}$.

Remark 0.16. *From the previous Proposition it also immediately follows that, for $\hat{F}$ tropical analytical, $x \in Q^X$, $a \in X$ s.t. $x_a \neq 0$ and for all $0 < \epsilon < 1$, we have:*

$$d^\infty(\hat{F}(x), \widetilde{F}_{\mu^\epsilon}(x + \epsilon e_a)) \leq (3 + 2L_a)\epsilon,$$

where we put $\widetilde{F}_\rho(.) := (\widetilde{F}_{\rho, b}(.))_{b \in Y} : Q^X \to Q^X$, which is clearly still $\#\rho$-Lipschitz. That is, we can approximate $\hat{F}(x)$, with arbitrary precision ϵ, by using the Lipschitz functions $\widetilde{F}_{\mu^\epsilon}(x + \epsilon e_a)$. Significa veramente che la posso approssimare con robe lipschitz LOCALMENTE *??*

We say that a function $f : \overline{\mathbb{R}}_{\geq 0} \to \overline{\mathbb{R}}_{\geq 0}$ is *tropical analytical with matrix* $\hat{f} : \mathbb{N} \to \overline{\mathbb{R}}_{\geq 0}$ iff $\hat{f} \in \mathrm{Trop}(1,1)$ and f is the tropical analytical function associated with it, under the identification of $\overline{\mathbb{R}}_{\geq 0}$ and Q^1).

Remark that, since $0 \cdot +\infty = 0$, we have: $f(+\infty) = \min\{\hat{f}(0), +\infty\} = \hat{f}(0) = \widetilde{f}(0, +\infty) = \min_{n \in \mathbb{N}} \widetilde{f}(n, +\infty)$. Here we set $\widetilde{f}(n, x) := nx + \hat{f}(n)$.

Example 0.17. *Let us study the tropical analytical function with matrix $\hat{f}(n) = \frac{1}{2^n}$, that is:*

$$f(x) = \inf_{n \in \mathbb{N}} \{nx + \frac{1}{2^n}\},$$

where $x \in [0, +\infty]$.

We immediately see that $f(+\infty) = 1$. We also immediately see that $f(0) = \inf_{n \in \mathbb{N}} \frac{1}{2^n} = 0$. In particular, remark that $f(0)$ is not *a* min *but only an* inf. *In order to see what happens in $(0, +\infty)$, let us treat n as a real variable and, for $0 < x < +\infty$, let us compute: $\frac{\partial}{\partial n} \widetilde{f}(n, x) = x - \frac{log_e 2}{2^n} \geq 0$ iff $n \geq \log_2 \left(\frac{log_e 2}{x} \right)$. So $\widetilde{f}(n, x)$ (as a function of n) has exactly one minimum and it is reached at $n_x = \log_2 \left(\frac{log_e 2}{x} \right)$. Remember now that we are only interested in $n \geq 0$, and $n_x \geq 0$ iff $x \leq log_e 2$. Putting all this together we get:*

$$f(x) = \begin{cases} \widetilde{f}(0, x) = 1 \ \ \text{if } x \in [log_e 2, +\infty) \\ \widetilde{f}(\widetilde{n}_x, x) = (\widetilde{n}_x + \frac{1}{log_e 2})x \ \ \text{if } x \in (0, log_e 2] \end{cases}$$

where $\widetilde{n}_x \in \mathbb{N}$ is either $\lceil n_x \rceil$ or $\lfloor n_x \rfloor$. In particular, remark that thus $f(x) = \min_{n \in \mathbb{N}} \{nx + \frac{1}{2^n}\}$ on $(0, +\infty)$. The only point in which $f(x)$ is not *a* min *is thus $x = 0$.*

We just proved that f is almost always a min, *but in order to find such min we have, à priori, to consider all $n \in \mathbb{N}$. But we can prove more: on intervals of shape $x \in [\frac{\log_e(2)}{2^{n+1}}, \frac{\log_e(2)}{2^n})$ (we could also have chosen $(\frac{\log_e(2)}{2^{n+1}}, \frac{\log_e(2)}{2^n}]$), f coincides in fact with a unique tropical polynomial P_n. To see this, fix $n \in \mathbb{N}$ and remark that if $\dfrac{\log_e(2)}{2^{n+1}} \leq x < \dfrac{\log_e(2)}{2^n}$ then $\underline{f}(n+1,m) \leq \widetilde{f}(m,x) < \underline{f}(n,m)$ for all $m \in \mathbb{N}$, where we set $\underline{f}(k,m) := \frac{m}{2^k}\log_e(2) + \frac{1}{2^m}$. By considering m as a real parameter and by computing the derivative $\frac{\partial}{\partial m}\underline{f}(k,m) = (\frac{1}{2^k} - \frac{1}{2^m})\log_e(2)$ we see that, as a function of m, $\underline{f}(k,m)$ has a minimum (at $m = k$). Now, remark that $\lim\limits_{m\to+\infty} \underline{f}(k,m) = +\infty$ for all $k \in \mathbb{N}$, and therefore it must exist $N_n \in \mathbb{N}$ s.t. $\underline{f}(n+1,m) \geq \min\limits_{i\in\mathbb{N}} \underline{f}(n,i)$ for all $m \geq N_n$. Putting all this together we have:*

$$\widetilde{f}(m,x) \geq \underline{f}(n+1,m) \geq \min_{i\in\mathbb{N}} \underline{f}(n,i) > \inf_{i\in\mathbb{N}} \widetilde{f}(i,x) = f(x)$$

for all $m \geq N_n$. The last inequality[2] holds because if a function admitting a minimum is always strictly greater than another function, then the same strict inequality holds for the respective min and inf. This means that all the $m \geq N_n$ are useless in order to compute $f(x)$, i.e. $f(x) = \min\limits_{m\leq N_n} \widetilde{f}(m,x) =: P_n(x)$, and now this is really a tropical polynomial since the minimum is taken on a finite *number of indices.*

The property that we just proved can be immediately strengthened. In fact, for all $0 < \epsilon < \log_e(2)$, we can consider the unique $n_\epsilon \in \mathbb{N}$ s.t. $\epsilon \in [\frac{\log_e(2)}{2^{n_\epsilon+1}}, \frac{\log_e(2)}{2^{n_\epsilon}})$, and then we have a finite *number of intervals $\{[\frac{\log_e(2)}{2^{i+1}}, \frac{\log_e(2)}{2^i})\}_{i=0,\dots,n_\epsilon}$ partitioning $[\frac{\log_e(2)}{2^{n_\epsilon+1}}, \log_e(2))$. Now, remembering the definition of P_i, for all $x \in [\epsilon, \log_e(2))$ we actually have that:*

$$f(x) = \min_{i=0,\dots,n_\epsilon} P_i(x) = \min_{m\leq N_{n_\epsilon}} \widetilde{f}(m,x) =: P_\epsilon(x)$$

where P_ϵ is still a tropical polynomial, because $N_{n_\epsilon} < +\infty$. Actually, there is no reason to limit x to be smaller than $\log_e(2)$. In fact, remembering that f is non-decreasing and that $f(x) = 1$ for $\log_e(2) \leq x < +\infty$*, with $f(+\infty) = 0$, we immediately have that for all $0 < \epsilon < +\infty$ and for all $x \in [\epsilon, +\infty]$:*

$$f(x) = \min\{1, P_\epsilon(x)\} =: \overline{P}_\epsilon(x)$$

and $\overline{P}_\epsilon(.) = \min\limits_{m\leq N_{n_\epsilon}} \{1, \widetilde{f}(m,.)\}$ is again a tropical polynomial.

Remark that this immediately implies that f is continuous on its domain. It is continuous on $(0,+\infty]$ *because for all $\epsilon > 0$, it coincides with a tropical polynomial (which is a continuous function) on $[\epsilon, +\infty]$. And, since we know that f is non-decreasing, in order to show that $\lim\limits_{x\to0^+} f(x) = f(0) = 0$, it sufficies to show that for all $K > 0$ there is $x > 0$ s.t.$f(x) < K$. This is true because we can certainly find $m,n \in \mathbb{N}$ s.t. $\underline{f}(n,m) = \frac{m}{2^n}\log_e(2) + \frac{1}{2^m} < K$, and then we have $f(x) \leq \widetilde{f}(m,x) < \underline{f}(n,m) < K$ for $x \in [\dfrac{\log_e(2)}{2^{n+1}}, \dfrac{\log_e(2)}{2^n})$.*

As a last remark, let us point out that we used that fact that $\lim\limits_{m\to+\infty} \underline{f}(k,m) = +\infty$ in order to show that an appropriate N_n always exists. But we can actually give a particular N_n doing the

[2]The crucial fact is that this inequality is strict. In fact we already knew, by definition of f, that $\widetilde{f}(m,x) \geq f(x)$.

job. In fact we already said that, as a function of m*,* $\underline{f}(k,m)$ *has a minimum at* $m=k$*, and its value for* $k=n$ *is* $\frac{1+n\log_e(2)}{2^n}$*. Now a perfectly fine* N_n *is given by any solution* m *of the inequality:* $\underline{f}(n+1,m) \geq \min_{i\in\mathbb{N}} \underline{f}(n,i)$*, that is,* $\frac{m}{2}\log_e(2) \geq 1+n\log_e(2) - \frac{2^n}{2^m}$*. A plot in WolframAlpha (or do it by studying the function) shows that a possible solution, for* $n \geq 4$*, is to take* $m = \frac{2n}{log_e(2)}$*. Since we require* m *to be natural, we can take the ceiling of that value, which turns out to be lower bounded by* $3n =: N_n$*. For* $n = 0,1,2,3$ *we can find a particular solution by hand. For* $n=0$ *the inequality becomes* $m \geq \frac{2}{\log_e(2)} - \frac{2\cdot 2^m}{\log_e(2)}$*. Now if* $m \geq 1-\log_2(\log_e(2))$*, i.e.* $\frac{2\cdot 2^m}{\log_e(2)} \leq 1$*, the desired inequality is satisfied if we take* $m \geq \frac{2}{\log_e(2)}$*. Therefore we are done taking* $m \geq \max\{1-\log_2(\log_e(2)), \frac{2}{\log_e(2}\} = \frac{2}{\log_e(2)}$*, and we can take the ceiling in order to have a natural value, which turns out to be* $3 =: N_0$*. Analogously one can see that we can take* $N_1 = 5$*,* $N_2 = 7$*,* $N_3 = 9$*.*

The fact that approaching 0 arbitrarily close (from the right) we can always find a tropical polynomial (which depends on how much we approached 0) coinciding with f from that point on, is not specific to the previous example. As the following proposition shows, this is always the case. In particular, remark that the point $x=0$ is always the only one in which $f(x)$ can fail to be a min.

Proposition 0.18. *Let* $f:\overline{\mathbb{R}}_{\geq 0} \to \overline{\mathbb{R}}_{\geq 0}$ *tropical analytical, with matrix* $\hat{f}:\mathbb{N}\to\overline{\mathbb{R}}_{\geq 0}$*. Then, for all* $0<\epsilon<+\infty$*, there is* $\mathcal{F}_\epsilon \subseteq \mathbb{N}$ *s.t.*

1. $\mathcal{F}_\epsilon$ *finite*
2. *If* $\mathcal{F}_\epsilon = \varnothing$ *then* $f(x) = +\infty$ *for all* $x \in \overline{\mathbb{R}}_{\geq 0}$
3. *If* $f(x_0) = +\infty$ *for some* $x_0 < +\infty$ *then* $\mathcal{F}_\epsilon = \varnothing$
4. *For all* $x \in [\epsilon, +\infty]$ *we have* $f(x) = P_\epsilon(x)$*, where* P_ϵ *is the* tropical polynomial*:*

$$P_\epsilon(x) = \min_{n\in\mathcal{F}_\epsilon}\{\hat{f}(n) + nx\}.$$

Proof. Let us set $\mathcal{F}_\epsilon$ to be the complementary in $\mathbb{N}$ of the set:

$$\{n\in\mathbb{N} \mid \textit{either } \hat{f}(n) = +\infty \textit{ or there is } m<n \textit{ s.t. } \hat{f}(m) \leq \hat{f}(n)+\epsilon\}.$$

In other words, $n\in\mathcal{F}_\epsilon$ iff $\hat{f}(n) < +\infty$ and for all $m<n$, one has $\hat{f}(m) > \hat{f}(n)+\epsilon$.

1). Suppose that $\mathcal{F}_\epsilon$ is infinite, and let us enumerate its elements as:

$$\mathcal{F}_\epsilon =: \{m_0 < m_1 < \cdots\}.$$

By definition of $\mathcal{F}_\epsilon$ we have then:

$$+\infty > \hat{f}(m_0) > \hat{f}(m_1) + \epsilon > \hat{f}(m_2) + 2\epsilon > \cdots$$

so that $+\infty > \hat{f}(m_0) > \hat{f}(m_i) + i\epsilon \geq i\epsilon$ for all $i\in\mathbb{N}$. This contradicts the Archimedean property of $\mathbb{R}$.

2). We show that if $\mathcal{F}_\epsilon = \varnothing$, then $\hat{f}(n) = +\infty$ for all $n\in\mathbb{N}$. This immediately entails the desired result. We go by strong induction on $n\in\mathbb{N}$:

- if $n = 0 \notin \mathcal{F}_\epsilon$, then $\hat{f}(n) = +\infty$, because there is no $m < n$.

- if $n \geq 1 \notin \mathcal{F}_\epsilon$, then either $\hat{f}(n) = +\infty$ and we are done, or there is $m < n$ s.t. $\hat{f}(m) \leq \hat{f}(n) + \epsilon$. By strong induction $\hat{f}(m) = +\infty$ and, since $\epsilon < +\infty$, this entails $\hat{f}(n) = +\infty$.

3). If $f(x_0) = +\infty$ with $x_0 < +\infty$, then necessary $\hat{f}(n) = +\infty$ for all $n \in \mathbb{N}$. Therefore, no $n \in \mathbb{N}$ belongs to $\mathcal{F}_\epsilon$.

4). By 1), it suffices to show that we can compute $f(x)$ by taking the inf, that is therefore a min, only in $\mathcal{F}_\epsilon$ (instead of all $\mathbb{N}$). If $\mathcal{F}_\epsilon = \varnothing$ then by 2) we are done (remember that $\min\varnothing := +\infty$). If $\mathcal{F}_\epsilon \neq \varnothing$, we show that for all $n \in \mathbb{N} - \mathcal{F}_\epsilon$, there is $m \in \mathcal{F}_\epsilon$ s.t. $\hat{f}(m) + mx \leq \hat{f}(n) + nx$. We do it by induction on $n \in \mathbb{N}$:

- if $n = 0$, then by definition of $\mathcal{F}_\epsilon$, we have $\hat{f}(n) = +\infty$ (because there is no $n' < n$). So any element of $\mathcal{F}_\epsilon \neq \varnothing$ works.

- if $n \geq 1$, then we have two cases: either $\hat{f}(n) = +\infty$, in which case we are done as before by taking any element of $\mathcal{F}_\epsilon \neq \varnothing$. Or $\hat{f}(n) < +\infty$, in which case (again by definition of $\mathcal{F}_\epsilon$) there is $n' < n$ s.t.

$$\hat{f}(n') \leq \hat{f}(n) + \epsilon. \tag{1}$$

Therefore we have (remark that the following inequalities hold also for the case $x = +\infty$):

$$\begin{array}{rcll} \hat{f}(n') + n'x & \leq & \hat{f}(n) + \epsilon + n'x & \textit{by } (1) \\ & \leq & \hat{f}(n) + (n - n')x + n'x & \textit{because } \epsilon \leq x \textit{ and } n \geq n' \\ & = & \hat{f}(n) + nx. & \end{array}$$

Now, if $n' \in \mathcal{F}_\epsilon$ we are done. Otherwise $n' \notin \mathcal{F}_\epsilon$ and we can apply the induction hypothesis on it, obtaining an $m \in \mathcal{F}_\epsilon$ s.t. $\hat{f}(m) + mx \leq \hat{f}(n') + n'x$. Therefore this m works. □

Che fare quando y non è della forma x+qe_a ??

Continuità a tratti ??

Derivate ??

Diff lambda cat ??

Remember that on the $\overline{\mathbb{R}}_{\geq 0}$-module $\overline{\mathbb{R}}_{\geq 0}^X$ (we mean $\overline{\mathbb{R}}_{\geq 0}$ with the usual + and $\cdot$ operations, which is also called the *probability semiring*), we can define the subtraction $(x_a)_{a\in X} - (y_a)_{a\in X} := (x_a - y_a)_{a\in X}$ if $x_a - y_a \geq 0$ and $:= 0$ otherwise.

Proposition 0.19. *Let $f : \overline{\mathbb{R}}_{\geq 0}^X \to \overline{\mathbb{R}}_{\geq 0}$ and $x \in \overline{\mathbb{R}}_{>0}^X$ (so $x_a \neq 0$ for all $a \in X$). If there is a convex open $V \subseteq \overline{\mathbb{R}}_{\geq 0}^X$ with $x \in V$ and s.t. $f_{|V}$ is concave and lower bounded by a constant $K \geq 0$, then f is continuous at x.*

Proof. Fix now and for all the proof an $\epsilon \in (0,1)$.

If $f(x) = +\infty$ then for all $y \in V$ we have: $f((1-\epsilon)x + \epsilon y) \geq (1-\epsilon)f(x) + \epsilon f(y) = +\infty$, so $f = +\infty$ on an open neighbourhood of x (actually, on all V), and in particular it is continuous at $x \in W$.

Suppose now $f(x) < +\infty$. Since $x_a \neq 0$ for all $a \in X$, there certainly is a convex open neighbourhood $W \subseteq V$ of x s.t. $W \subseteq \overline{\mathbb{R}}_{>0}$ and W is symmetrical w.r.t. x, meaning that if $w \in W$ then $2x - w \in W$. Said differently, W is of shape $x + \hat{W}$, for some open $\hat{W} \subseteq \overline{\mathbb{R}}^X$ s.t. if $y \in \hat{W}$ then $-y \in \hat{W}$ (that is, symmetrical w.r.t. 0). Now fix $y \in \hat{W}$, that is, s.t. $x + y \in W$.

Since W is convex, $x + \epsilon y = (1-\epsilon)x + \epsilon(x+y) \in W$, and therefore we have:

$$f(x+\epsilon y) \geq (1-\epsilon)f(x) + \epsilon f(x+y) \geq f(x) - \epsilon f(x) + \epsilon K = f(x) + \epsilon(K - f(x))$$

so:

$$\epsilon(f(x) - K) \geq f(x) - f(x+\epsilon y). \tag{2}$$

Now remark that $x = \frac{1}{1+\epsilon}(x+\epsilon y) + \frac{\epsilon}{1+\epsilon}(x-y)$ where $\frac{1}{1+\epsilon} = 1 - \frac{\epsilon}{1+\epsilon}$, $\frac{\epsilon}{1+\epsilon} \in (0,1)$ and $x - y = 2x - (x+y) \in W$. Therefore we have:

$$f(x) \geq \frac{1}{1+\epsilon} f(x+\epsilon y) + \frac{\epsilon}{1+\epsilon} f(x-y) \geq \frac{1}{1+\epsilon}(f(x+\epsilon y) + \epsilon K)$$

so:

$$f(x) - f(x+\epsilon y) \geq -\epsilon(f(x) - K). \tag{3}$$

Putting 2 and 3 together and remembering that y was arbitrary in $\hat{W}$, we have shown that for all points $w \in x + \epsilon\hat{W} \subset W$, $|f(x) - f(w)| \leq \epsilon(f(x) - K)$ (and $f(x) - K \geq 0$). Since ϵ was arbitrary in $(0,1)$, we have shown that $\exists \lim_{w\to x} f(w) = f(x)$. □

Definition 0.20. *A* semifield *is a semiring R s.t. $(R-\{0\},\cdot)$ is a group. A semifield is commutative iff it is commutative as a semiring. Remark that if the semiring R is actually a ring (that is, if $(R,+)$ is a group), then the semifield is a division ring.*

$\overline{\mathbb{R}}_{\geq 0}$ with the usual $+$ and $\cdot$ operations is a semiring, also called the *probability semiring*. It is a semifield under the usual inverse operation.

Definition 0.21. *A* topological field *is the data of a field R together with a topology on it s.t. $+ : R\times R \to R$, $-(.) : R \to R$, $\cdot : R\times R \to R$ and $(.)^{-1} : R \to R$ are continuous (where $R\times R$ is endowed with the product topology). Similarly for the definitions of topological division ring, semifield, ring and semiring.*

Definition 0.22. *Let R be a topological semifield. A* topological R-module *is the data of an R-module A together with a topology on it s.t. $+ : A\times A \to A$ and $\cdot : R\times A \to A$ are continuous (where R is considered with its topology of topological semifield and the product spaces are endowed with the product topologies). As a particular case, we have the definition of topological R-module for R a semiring. If R is a field, a topological R-module is called a* topological R-vector space*, and as a particular case we have the definition of topological module on a ring.*

Remark 0.23. *If A is a topological module on a semiring R then the translation maps $T_x : A \to A$, $z \mapsto z + x$, are continuous for all $x \in A$. Remark that the set $\{T_x \mid x \in A\}$ forms a monoid under composition. If R is a ring, then the translation maps are homeomorphisms and they form a group under composition.*

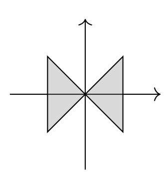
Definition 0.24. *Let Φ be a semigroup (under composition) of continuous maps from a topological module A on a semiring R to itself. The topology τ of A is* weakly invariant[3] under Φ *iff ϕO is open for all O open and $\phi \in \Phi$. We can write this as $\Phi\tau \subseteq \tau$. It is* invariant under Φ *iff it is weakly invariant and every open is of shape ϕO for some open O and $\phi \in \Phi$. We can write this as $\Phi\tau = \tau$.*

Remark 0.25. *The topology of a topological module A on a semiring R is weakly translation invariant. This means that for all $x \in A$ and open O, the set $\{z + x \mid z \in O\} =: O + x$ is open. If R is a ring, then the topology is translation invariant (that is, $\tau + x = \tau$ for all $x \in A$).*

Definition 0.26. *A* normed semiring *is the data of a semiring R together with a* norm *on it, that is, a function $|\,.\,| : R \to \mathbb{R}$ satisfying the usual axioms of the absolute value function*[4]*. One can immediately check that if R is a ring, then any norm on it induces a metric $d(r,s) := |r - s|$ on it (and thus also a topology). A* topological normed ring *is a normed ring which is a topological ring w.r.t. the topology induced by the norm distance.*

Definition 0.27. *Let R be normed semiring, let A be a topological R-module and let $C \subseteq A$. We say that C is* convex *iff for all $x, y \in C$ and for all $r, s \in R$ s.t. $|r| + |s| = 1$, we have $rx + sy \in C$. We say that C is* balanced *iff for all $x \in C$ and for all $r, s \in R$ s.t. $|r| + |s| = 1$, we have $rx, sx \in C$. We say that C is a* disk at $x \in A$ *iff it is of shape $C = x + D$ for some D both convex and balanced.*

Example 0.28. *An example of a balanced but not convex set in (the normed field) $\mathbb{R}^2$.*



[3] Non penso che esista come terminologia nella letteratura, la uso qui giusto per capirci.

[4] I.e.: $|x| \geq 0$, $|x| = 0$ *iff* $x = 0$, $|xy| = |x||y|$ and $|x + y| \leq |x| + |y|$.

Definition 0.29. *Let R be a normed field and let $\mathbb{V}$ be a topological R-vector space. We say that $\mathbb{V}$ is* locally convex *iff every $x \in \mathbb{V}$ admits a local basis consisting of disks at x. Since the topology of $\mathbb{V}$ is necessarily translation invariant, it is equivalent to only ask that 0 has such a local basis.*

Proposition 0.30. *Let $\mathbb{V}$ be a locally convex topological $\mathbb{R}$-vector space (where $\mathbb{R}$ is endowed with its usual absolute value and the induced topology by it). Fix $x \in \mathbb{V}$, a neighbourhood $V \subseteq \mathbb{V}$ of x and $f : V \to \mathbb{R} \cup \{+\infty, -\infty\}$. If V is convex, f is concave and lower bounded by a finite constant K, then f is continuous at x (w.r.t. the subspace topology on V).*

Proof. Since V is a neighbourhood of x, x admits an open neighbourhood $U \subseteq V$, and by local convexity of $\mathbb{V}$ there is a (non necessary open) disk $D \subseteq U$ at x. Now fix $\epsilon \in (0,1)$ and $r, s \in \mathbb{R}$ with $0 \leq r \leq \epsilon$, $s = 1 - r$. Fix also $w \in D$. By the convexity of D we have $sx + rw \in D$, and thus the convexity of f entails that:

$$f(sx+rw) \geq sf(x) + rf(w) \geq sf(x) + rK = (1-r)f(x) + rK$$

that is,

$$f(x) - f(sx+rw) \leq r(f(x) - K).$$

Now remark that $x = \frac{1}{s+2r}(sx+rw) + \frac{r}{s+2r}(2x-w)$, where $\frac{1}{s+2r} + \frac{r}{s+2r} = 1$, $\frac{1}{s+2r} < 1$ and $2x - w \in D$ (because D is a disk at x). Therefore by the convexity of f we have:

$$f(x) \geq \frac{1}{s+2r} f(sx+rw) + \frac{r}{s+2r} f(2x-w) \geq \frac{1}{s+2r}(f(sx+rw) + rK)$$

that is,

$$f(sx+rw) - f(x) \leq r(f(x) - K).$$

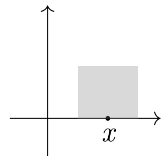
We have thus shown that $|f(sx+rw) - f(x)| \leq r(f(x) - K)$ for all $w \in D$. Since this holds also for all $\epsilon \in (0,1)$, and due to the choice of r, s, the points of shape $sx + rw$ span D when w spans D and ϵ spans $(0,1)$. That is, we have shown that $|f(w) - f(x)| \leq r(f(x) - K)$ for all $w \in W$. Since $f(x) - K \geq 0$ and $r \leq \epsilon$, we have that $\exists \lim_{w \to x} f(w) = f(x)$. □

Corollary 0.31. *Let $f : (\mathbb{R}^X_{>0}, \|.\|_\infty) \to (\overline{\mathbb{R}}_{\geq 0}, |\,.\,|)$ and $x \in \mathbb{R}^X_{>0}$. If there is a convex neighbourhood $V \subseteq \overline{\mathbb{R}}^X_{>0}$ of x s.t. $f_{|V}$ is concave, then f is continuous at x.*

Proof. The $\mathbb{R}$-vector space $\mathbb{R}^X$ is topological w.r.t. the topology τ_∞ induced on it by the norm $\|.\|_\infty$, and it is clearly locally convex. Call τ^+_∞ the topology induced by $\|.\|_\infty$ on $\mathbb{R}^X_{>0}$. It is clear that it coincides with τ^+_∞. Since moreover $\mathbb{R}^X_{>0}$ is open in $(\mathbb{R}^X, \tau_\infty)$, the neighbourhood V of x in $(\mathbb{R}^X_{>0}, \|.\|_\infty)$ is also a neighbourhood of x in $(\mathbb{R}^X, \tau_\infty)$. We can therefore apply Proposition 0.30 to $f_{|V}$ (which is lower bounded by 0 by definition of f) and obtain that $f_{|V}$ is continuous at x w.r.t. the subspace topology τ_V induced by τ_∞ on V. But since V is contained in $\mathbb{R}^X_{>0}$, the topology τ_V coincides with the subspace topology induced on V by τ^+_∞. So f is continuous at x w.r.t. τ^+_∞. □

One may wonder if the same proof as above makes it possible to state the previous corollary replacing $\mathbb{R}^X_{>0}$ with $\mathbb{R}^X_{\geq 0}$ (so, in particular, taking $x \in \mathbb{R}^X_{\geq 0}$). This is not possible because in the proof we crucially use that $\mathbb{R}^X_{>0}$ is open in $(\mathbb{R}^X, \tau_\infty)$, which is not the case of $\mathbb{R}^X_{\geq 0}$. In fact, this allowed us to say that V is a neighbourhood of x in $(\mathbb{R}^X, \tau_\infty)$, and therefore to be able to apply Proposition 0.30. Taking $\mathbb{R}^X_{\geq 0}$ instead, this is in general not true: we could have a neighbourhood of x w.r.t. the subspace topology on $\mathbb{R}^X_{\geq 0}$ induced by τ_∞, which does not contain any open neighbourhood of x w.r.t. τ_∞ (i.e. it is not a neighbourhood w.r.t. τ_∞).

Example 0.32. *An example of a neighbourhood of x w.r.t. the subspace topology on $\mathbb{R}^X_{\geq 0}$ induced by τ_∞, which is not a neighbourhood w.r.t. τ_∞.*



Corollary 0.33. *Tropical functions $f : (\mathbb{R}^X_{\geq 0}, \|.\|_\infty) \to (\overline{\mathbb{R}}_{\geq 0}, |\,.\,|)$ are continuous on $\mathbb{R}^X_{>0}$.*

Proof. We know that tropical functions are concave on all their domain. Therefore the result immediate follows by Corollary 0.31, since the topology induced by $\|.\|_\infty$ on $\mathbb{R}^X_{>0}$ coincides with the subspace topology induced by $(\mathbb{R}^X_{\geq 0}, \|.\|_\infty)$ on it. $\square$

Due to the previous discussion about the impossibility of stating Corollary 0.31 in the case where one of the coordinates of x is 0, the continuity of tropical functions on the hyperplanes $\mathcal{H}_a := \{x \in \mathbb{R}^X_{\geq 0} \mid x_a = 0\}$, for $a \in X$, must be treated separately. Similarly, the continuity at points with infinite coordinates must be treated separately as well.

Discussione sui coni ? (Vedi altra pagina)

Definition 0.34. *An* $\overline{\mathbb{R}}_{\geq 0}$-cone *is a commutative* $\overline{\mathbb{R}}_{\geq 0}$*-semimodule with cancellative addition*[5]*.*

In [Selinger] cones are required to also have "strict addition", meaning that $x+y=0 \Rightarrow x=y=0$. We do not add this requirement since it will automatic hold when considering normed cones.

Remark 0.35. *The addition of a cone P (which forms a commutative monoid) turns P into a poset by setting:*

$$x \leq y \text{ iff } y = x+z, \text{ for some } z \in P.$$

By the cancellative property, when such z exists it is unique, and we denote it by $y-x$. Such order is called the cone-order *on P.*

Definition 0.36. *A* normed $\overline{\mathbb{R}}_{\geq 0}$-cone *is the data of a* $\overline{\mathbb{R}}_{\geq 0}$*-cone together with a* $\leq$*-monotone*[6] *norm*[7] *on it.*

In [Ehrhard-Pagani-Tasson, Crubillie] a normed $\overline{\mathbb{R}}_{\geq 0}$-cone is simply called a cone.

Remark that in a normed $\overline{\mathbb{R}}_{\geq 0}$-cone, by monotonicity of the norm, we have: $\|x+y\| = 0 \Rightarrow x = y = 0$. Therefore, as already mentioned, in a normed cone we have: $x+y=0 \Rightarrow \|x+y\| = 0 \Rightarrow x=y=0$, that is, addition is strict.

Example 0.37. $\overline{\mathbb{R}}_{\geq 0}^X$ *is a normed cone with the norm* $\|x\|_\infty := \sup_{a \in X} x_a \in \overline{\mathbb{R}}_{\geq 0}$*.*

Notation 0.38. *We call* $\overline{\mathscr{B}}_1^{\|.\|}(P)$ *the closed unit ball* $\{x \in \overline{\mathbb{R}}_{\geq 0} \mid \|x\| \leq 1\}$ *of a normed* $\overline{\mathbb{R}}_{\geq 0}$*-cone P. We will also simply call it* $\overline{\mathscr{B}}$ *when the other informations are clear from the context. Similarly,* $\mathscr{B}$ *denotes the open unit ball.*

Remark 0.39. *The cone-order on* $\overline{\mathbb{R}}_{\geq 0}^X$ *is the pointwise usual order on* $\overline{\mathbb{R}}_{\geq 0}$*.*

Remark also that tropical functions have no reason to be linear nor sublinear.

Remark 0.40. *Tropical functions are monotone w.r.t. to the cone order on its domain and codomain. This is clear, using the definition of tropical functions, because $\mu x \leq \mu y$ if $x \leq y$.*

Remark 0.41. *The image fD of a directed subset D of a poset P under a monotone function $f : P \to P'$ is always directed. This is clear because for all $x,y \in P$, we have $f(x) \leq f(z) \geq f(y)$, where $z \in D$ is s.t. $x \leq z \geq y$ and it exists in D since D is directed.*

Remark 0.42. *The cone-order on* $\overline{\mathbb{R}}_{\geq 0}^X$ *makes it into a dcpo with least element* 0*.*

Lemma 0.43. *Let P be a poset and X,I sets. Fix the pointwise order on P^X.*

1. *Let $x^i \in P^X$ for $i \in I$. If $\bigvee_{i\in I} x^i$ exists P^X, then $\bigvee_{i \in I} x_a^i$ exists in P for all $a \in X$, and we have:*

$$\bigvee_{i\in I} x_a^i = \left(\bigvee_{i\in I} x^i\right)_a .$$

[5] I.e.: $x+y = x+y' \Rightarrow y = y'$.

[6] I.e.: $x \leq y \Rightarrow \|x\| \leq \|y\|$. Remark that requiring this property (for all x,y) is equivalent to requiring that $\|x\| \leq \|x+y\|$ for all x,y.

[7] A *norm* on a $\overline{\mathbb{R}}_{\geq 0}$-abstract cone P is a map $\|.\| : P \to \overline{\mathbb{R}}$ satisfying the usual axioms of norms: $\|x\| \geq 0$, $\|x\| = 0 \Rightarrow x = 0$, $\|rx\| = r\|x\|$ and $\|x+y\| \leq \|x\| + \|y\|$.

2. Let $x^i_a \in P$ for $i \in I, a \in X$. If $\bigvee_{i\in I} x^i_a$ exists P for all $a \in X$, then $\bigvee_{i\in I} x^i$ exists in P^X, and we have:

$$\left(\bigvee_{i\in I} x^i\right)_a = \bigvee_{i\in I} x^i_a.$$

Proof. (1). Fix $a \in X$. Let us show that $\left(\bigvee_{i\in I} x^i\right)_a$ is indeed the sup of $\{x^i_a \mid i \in I\}$. It is an upper bound because $\bigvee_{i\in I} x^i \geq x^i$. Now let $d \in P$ be an upper bound of $\{x^i_a \mid i \in I\}$. In order to show that $d \geq \left(\bigvee_{i\in I} x^i\right)_a$, let us consider $h \in P^X$ defined by $h_c := \left(\bigvee_{i\in I} x^i\right)_c$ if $c \neq a$ and $h_a := d$. By construction h is an upper bound of $\{x^i \mid i \in I\}$. So, by definition of sup, $h \geq \bigvee_{i\in I} x^i$. Hence $d = h_a \geq \left(\bigvee_{i\in I} x^i\right)_a$.

(2). Analogue. □

A *directed net* in a poset P with indices in a set I is a function $s : I \to P$, denoted by $(s_i)_{i\in I}$, s.t. its image is directed. We say that a directed net in P *admits a sup* iff its image admits a sup in P. We say that a directed net s in a normed cone is *bounded* iff the set $\{\|s_i\| \mid i \in I\}$ is bounded in $\mathbb{R}_{\geq 0}$. In the study of dcpo's the role played by sequences in usual topology is often played by directed nets.

Remark 0.44. *Monotone functions between posets transport directed nets to directed nets*[8].

Definition 0.45. *A function $f : P \to P'$ between posets is* Scott-continuous *iff for all directed net $(s_i)_i$ in P admitting a sup, we have $\exists \bigvee_i f(s_i) = f(\bigvee_i s_i)$ in P'.*

As it is well known, the order theoretic notion of Scott-continuity coincides with the topological continuity w.r.t. to the *Scott-topology* on P, whose opens are exactly the upward closed $O \subseteq P$ s.t. for all directed $D \subseteq P$ with $\bigvee D \in O$, one has $D \cap O \neq \varnothing$.

Proposition 0.46. *Let P be a normed $\overline{\mathbb{R}}_{\geq 0}$-cone s.t. every bounded directed net in P admits a sup. Let $(v_i)_{i\in I}$ be a directed net in P with an upper bound $v \in P$. Then $\exists \bigvee_{i\in I} v_i \in P$ and, if $\inf_{i\in I} \|v - v_i\| = 0$, one has: $\bigvee_{i\in I} v_i = v$.*

Proof. Remark that $v - v_i$ exists in P by hypothesis and so does $\bigvee_{i\in I} v_i$, thanks to the monotonicity of the norm. Now, since $v \geq v_i$ for all i, we have that $v \geq \bigvee_{i\in I} v_i$, and so $v - \bigvee_{i\in I} v_i$ exists in P. Fix $i \in I$. Since $v_i \leq \bigvee_{i\in I} v_i$, then $v - \bigvee_{i\in I} v_i \leq v - v_i$ and, by monotonicity of the norm, $\|v - \bigvee_{i\in I} v_i\| \leq \|v - v_i\|$. Since this holds for all $i \in I$, we have: $0 \leq \|v - \bigvee_{i\in I} v_i\| \leq \inf_{i\in I} \|v - v_i\| = 0$, where the last equality holds by hypothesis. Thus $\|v - \bigvee_{i\in I} v_i\| = 0$, i.e. $v = \bigvee_{i\in I} v_i$. □

[8] We simply mean that if $(s_i)_i$ is a directed net in the domain, then $(f(s_i))_i$ is a directed net in the codomain.

Definition 0.47. *A normed $\overline{\mathbb{R}}_{\geq 0}$-cone P is* Scott-complete *iff its norm is Scott-continuous (where the codomain $\overline{\mathbb{R}}_{\geq 0}$ is endowed with its usual order) and every bounded directed net in P admits a sup. This is equivalent to asking that its closed unit ball is a dcpo.*

Proposition 0.48. *The normed cone $\overline{\mathbb{R}}_{\geq 0}^X$ is Scott-complete.*

Proof. $\overline{\mathbb{R}}_{\geq 0}^X$ being a dcpo, all the existences of sup's that we should check do automatically hold, so we only have to show that $\bigvee$ and $\|.\|_\infty$ commute: $\|\bigvee_i x_i\|_\infty = \sup_a \sup_i (x_i)_a = \sup_i \sup_a (x_i)_a = \bigvee_i \|x_i\|_\infty$. □

Proposition 0.49. *All tropical functions $f : \overline{\mathbb{R}}_{\geq 0}^X \to \overline{\mathbb{R}}_{\geq 0}^Y$ are Scott-continuous on $\mathbb{R}_{>0}^X$ w.r.t. the cone-orders on its domain and codomain.*

Proof. Let $(x_i)_i$ a directed net in $\mathbb{R}_{>0}^X$ s.t. $\bigvee_i x^i$ exists in $\mathbb{R}_{>0}^X$. Then $\inf_i \|\bigvee_i x^i - x^i\|_\infty = 0$, where $\bigvee_i x^i - x^i$ exists because $\bigvee_i x^i \geq x^i$ for all i. Since f is $\|.\|_\infty$-continuous on $\mathbb{R}_{>0}^X$ (Corollary 0.33), then $\inf_i \|f(\bigvee_i x^i) - f(x^i)\|_\infty = 0$, where $f(\bigvee_i x^i) - f(x^i)$ exists because $f(\bigvee_i x^i) \geq f(x^i)$ for all i being f monotone (Remark 0.40). We can therefore apply Proposition 0.46 to the directed net $(f(x^i))_i$ in $\overline{\mathbb{R}}_{\geq 0}^Y$, obtaining that $\bigvee_i f(x^i)$ exists in $\overline{\mathbb{R}}_{\geq 0}^Y$ and it coincides with $f(\bigvee_i x^i)$. □

Definition 0.50. *Let P be a dcpo. We define a relation by: $x << y$ iff for all directed $D \subseteq P$, if $y \leq \bigvee D$, then $x \leq d$, for some $d \in D$. Call $\twoheaddownarrow y := \{x \in P \mid x << y\}$. A dcpo P is* Scott-continuous *iff $\twoheaddownarrow x$ is directed and $x = \bigvee \twoheaddownarrow x$ for all $x \in P$.*

Remark 0.51. *In any dcpo we have: $\twoheaddownarrow x \subseteq \downarrow x$. This immediately follows by considering the directed set $\{x\}$.*

Lemma 0.52. *In $\overline{\mathbb{R}}_{\geq 0}^X$, every set $\twoheaddownarrow x$ is directed.*

Proof. It is immediate that $0 \in \twoheaddownarrow x$, so it is non-empty. Now let $y, y' << x$. Since we are in $\overline{\mathbb{R}}_{\geq 0}^X$, there is $y \vee y' \in \overline{\mathbb{R}}_{\geq 0}^X$. So we only have to show that $y \vee y' << x$. For that, let D be a directed set in $\overline{\mathbb{R}}_{\geq 0}^X$ s.t. $x \leq \bigvee D$. Since $y, y' << x$ we find $d, d' \in D$ s.t. $y \leq d$, $y' \leq d'$. Since D is directed, there is $\hat{d} \in D$ s.t. $y \leq d \leq \hat{d} \geq d' \geq y'$. But then, by definition of sup, it must be $\hat{d} \geq y \vee y'$ and we are done. □

Remark 0.53. *The following is a known property (that we will not use): let P be a complete normed cone which is a dcpo w.r.t. its cone-order. Then P is Scott-continuous as a dcpo iff its closed unit ball is Scott-continuous as a dcpo.*

Remark 0.54. *Consider $\overline{\mathbb{R}}_{\geq 0}$ with its usual order (which coincides with its cone-order). It is easily seen (and well known) that $y << x$ iff either $y = 0$ or $y < x$. This immediately implies that $\overline{\mathbb{R}}_{\geq 0}$ is Scott-continuous, since $x = \sup_{y<x} y$.*

Lemma 0.55. *Let $x, y \in \overline{\mathbb{R}}_{\geq 0}^X$ (considered as a dcpo with its cone order, which is the pointwise one). Fix $a \in X$. If $y_a << x_a$ and $y_c = 0$ for all $c \neq a$, then $y << x$.*

Proof. Towards a contradiction, assume that there is a directed set D in $\overline{\mathbb{R}}_{\geq 0}^X$ with $x \leq \bigvee D$ and s.t. $y \nleq d$ for all $d \in D$. Call $D_a := \{d_a \mid d \in D\}$ and remark that it is directed, because D is. Also, by Lemma 0.43, $\bigvee D_a = (\bigvee D)_a$. Therefore from $y_a << x_a \leq \bigvee D_a$ we obtain a $d \in D$ s.t. $y_a \leq d_a$. By the absurd hypothesis we have $y \nleq d$, so there must be $c \in X$ s.t. $y_c \nleq d_c$, i.e. (because real numbers are totally ordered) $y_c > d_c$. Therefore it must be $c \neq a$. But then we have $0 = y_c > d_c \geq 0$, contradiction. □

Lemma 0.56. *In $\overline{\mathbb{R}}_{\geq 0}^X$ we have: if $y << x$ then $y_a << x_a$ for all $a \in X$.*

Proof. Fix $a \in X$. Let D directed set in $\overline{\mathbb{R}}_{\geq 0}$ s.t. $x_a \leq \bigvee D$. We look for a $d \in D$ s.t. $y_a \leq d$. For $d \in D$, let $x^{a,d} \in \overline{\mathbb{R}}_{\geq 0}^X$ defined by $x_c^{a,d} := x_c$ if $c \neq a$ and $x_a^{a,d} := d$. Let $D_a^x := \{x^{a,d} \mid d \in D\}$. By Lemma 0.43 D_a^x admits sup in $\overline{\mathbb{R}}_{\geq 0}^X$ and it is $(\bigvee D_a^x)_c = x_c$ if $c \neq a$ and $(\bigvee D_a^x)_a = \bigvee D$. Hence, $x \leq \bigvee D_a^x$. If we prove that D_a^x is directed, we are done: indeed, since $y << x$, there is $d \in D$ s.t. $y \leq x^{a,d}$ and thus, in particular, $y_a \leq x_a^{a,d} = d$. Let us finally prove that D_a^x is directed: it is clearly non-empty, since D is. Let now $d, d' \in D$. We want to show that there is $\hat{d} \in D$ s.t. $x^{a,d} \leq x^{a,\hat{d}} \geq x^{a,d'}$. But since D is directed, there is $\hat{d} \in D$ s.t. $d \leq \hat{d} \geq d'$, and therefore $x_c^{a,d} = x_c = x_c^{a,\hat{d}} = x_c = x_c^{a,d'}$ for all $c \neq a$, and $x_a^{a,d} = d \leq \hat{d} = x_a^{a,\hat{d}} = \hat{d} \geq d' = x_a^{a,d'}$. □

Lemma 0.57. *In $\overline{\mathbb{R}}_{\geq 0}^X$ we have: $\bigvee_{y << x_a} y = \bigvee_{y << x} y_a$.*

Proof. If we show that $\twoheaddownarrow x_a = \{y_a \mid y \in \twoheaddownarrow x\}$, then we are done because in the statement we are taking their sup's. The inclusion ($\supseteq$) immediately follows from Lemma 0.56. For ($\subseteq$), let $d << x_a$. Then the $y \in \overline{\mathbb{R}}_{\geq 0}^X$ defined by $y_c := 0$ if $c \neq a$ and $y_a := d$, is s.t. $y \in \twoheaddownarrow x$ by Lemma 0.55. Thus, $d \in \{y_a \mid y \in \twoheaddownarrow x\}$. □

Corollary 0.58. *The dcpo $\overline{\mathbb{R}}_{\geq 0}^X$ is Scott-continuous.*

Proof. The fact that $\twoheaddownarrow x$ is directed is given by Lemma 0.52. The fact that $x = \bigvee \twoheaddownarrow x$ is given by the following equalities: $x_a = \bigvee_{y << x_a} y = \bigvee_{y << x} y_a = \left(\bigvee_{y << x} y\right)_a = (\bigvee \twoheaddownarrow x)_a$. The first equality follows from Remark 0.54, the second one from Lemma 0.57, the third one from Lemma 0.43. □