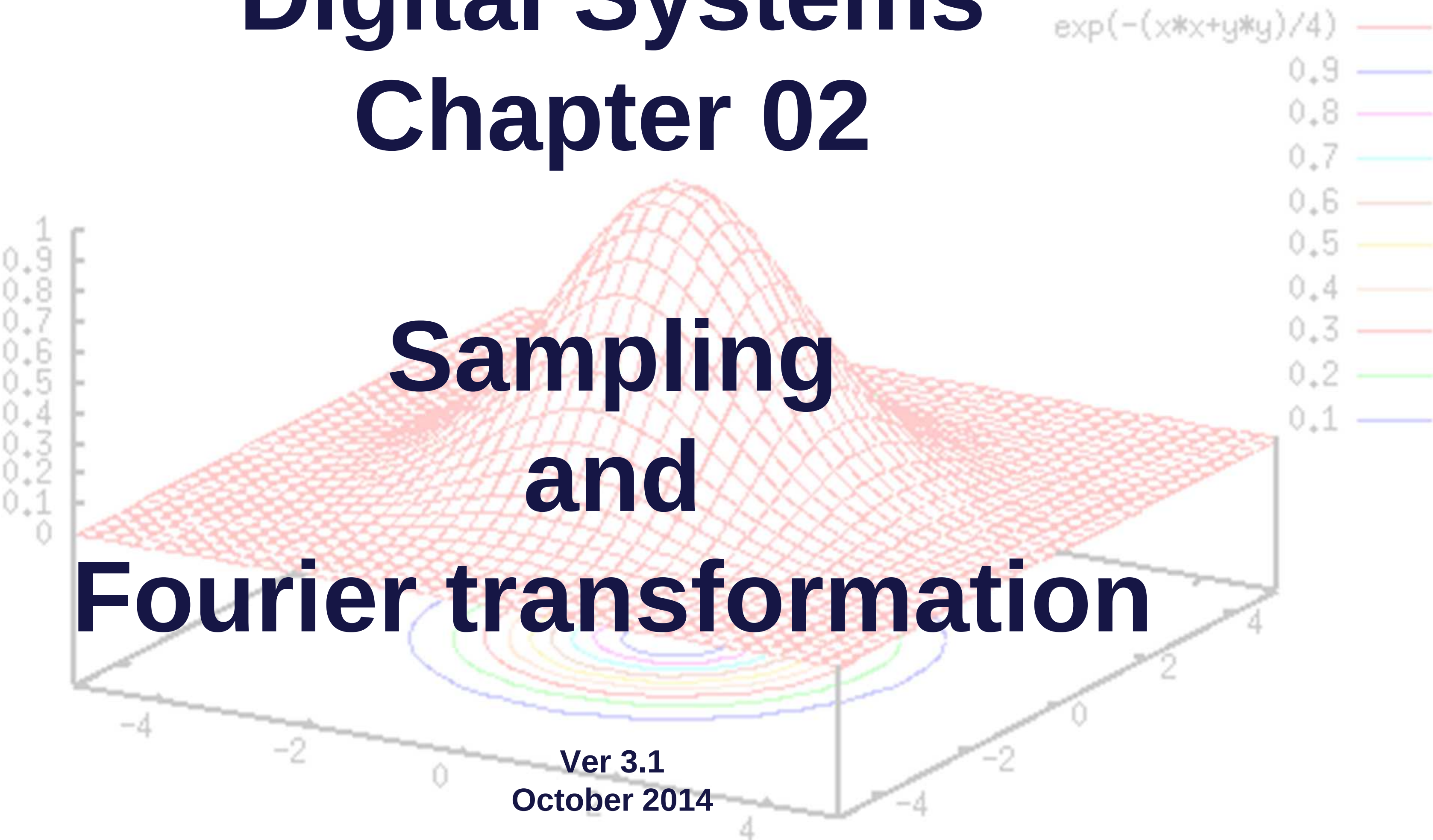
Digital Systems
Chapter 02

Sampling
and
Fourier transformation



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Ch 2 Sampling and Fourier Transformation

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2.1.1 One dimensional sampling

- **2.1.1.1 Sampling theorem**
- We assume that the analog input signal is s(t). After low-pass filtering to avoid aliasing the sampled signal is given:

$$s_T(t) = s(t) \cdot \sum_{n=-\infty}^{+\infty} \delta(t - n \cdot T_s) = \sum_{n=-\infty}^{+\infty} s(t - n \cdot T_s)$$

- Applying the Fourier transformation the one dimensional signal description in the frequency domain (spectrum) is obtained:

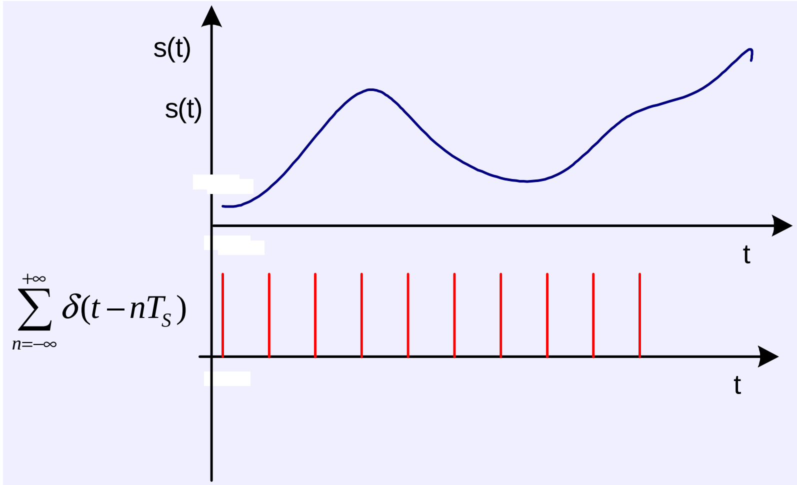
$$f_S = \frac{1}{T_s}$$

$$S_\perp(f) = S(f) * \sum_{n=-\infty}^{+\infty} \delta\left(f - n\frac{1}{T_s}\right) = S(f) * \sum_{n=-\infty}^{+\infty} \delta(f - n \cdot f_S)$$

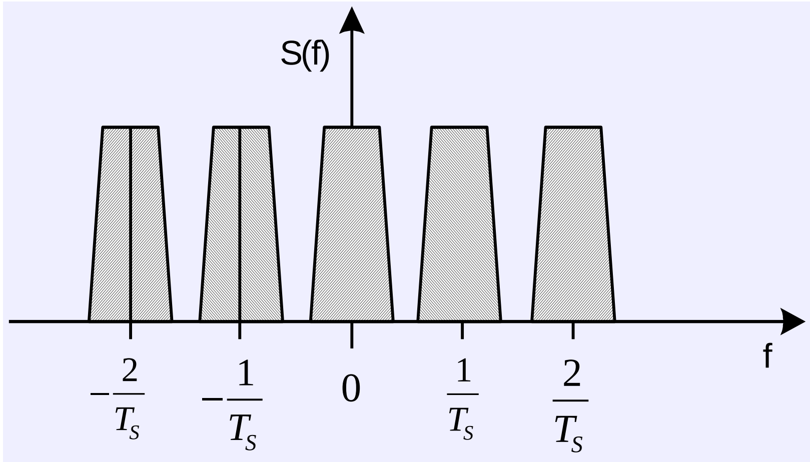
- As far as the input signal s(t) consists of frequency components higher than half the sampling frequency $f_S/2$ the signal spectrums overlap → **alias.**
- According to the Nyquist criteria this alias is avoided when:

$$f_s = \frac{1}{T_S} \geq 2 \cdot f_g$$

- Note: Alias once generated can't be eliminated anymore.

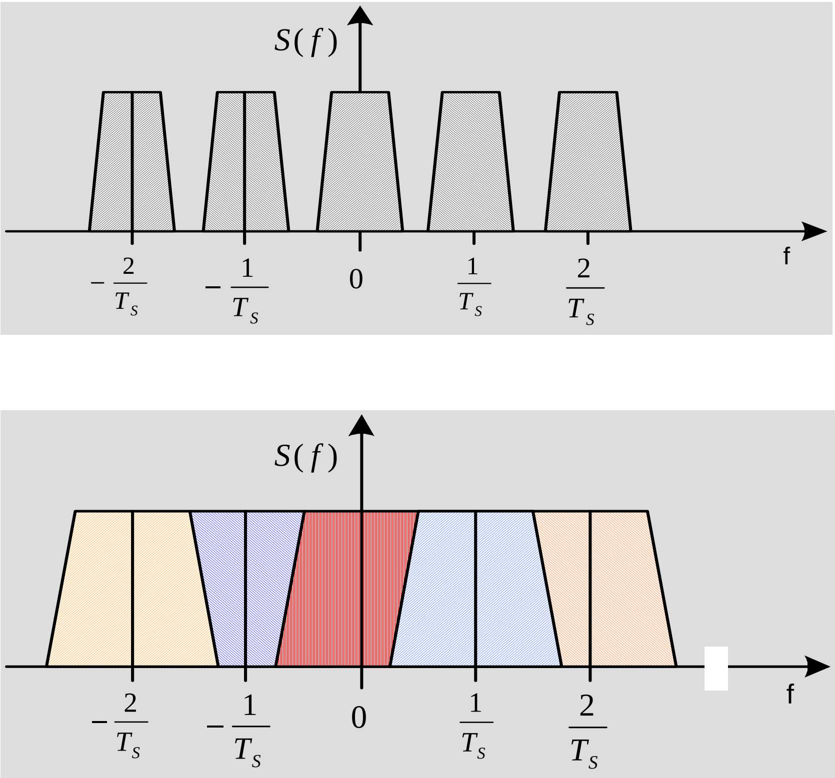


Sampling in time domain

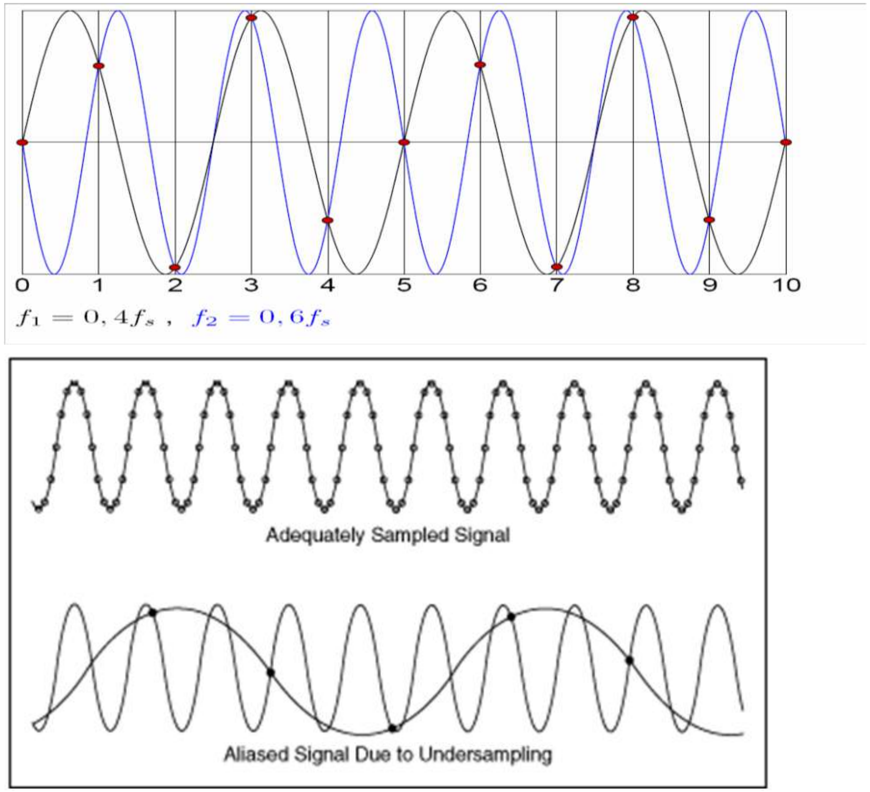


Spectrum of the sampled signal

- **2.1.1.2 Alias vs. down sampling**
- Generated by sampling frequencies which don‘t fulfill the sampling criteria, see before.
  - We talk about alias, when spectral components overlap
  - When the sampling theorem is not fulfilled but no overlapping spectrums are generated we talk about **down sampling**



Spectrum of sampled signal with alias



Sampling without pre-filtering

2.1.1.3 Sample and Hold amplifier (SH amplifier)

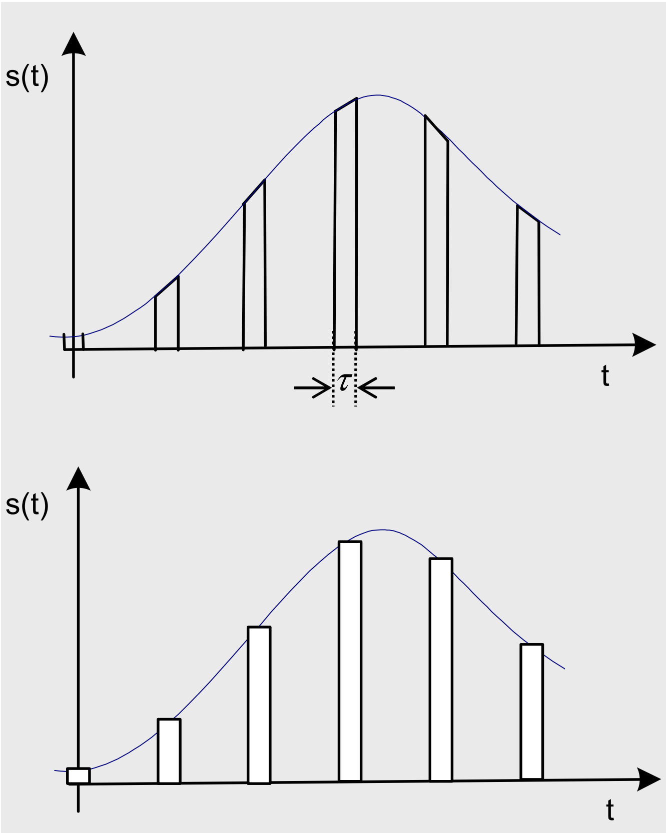
- The previous considerations were based on an ideal sampling unit, by means with an infinitesimal small aperture time.
- Real ADCs have a finite aperture, that means that the input signal is integrated over that aperture time.

$$s_{\perp}(t) = s(t) \cdot \left\{ \frac{1}{\tau} \cdot rect\left(\frac{t}{\tau}\right) * \sum_{n=-\infty}^{+\infty} \delta(t - n \cdot T_S) \right\}$$

$$\updownarrow$$

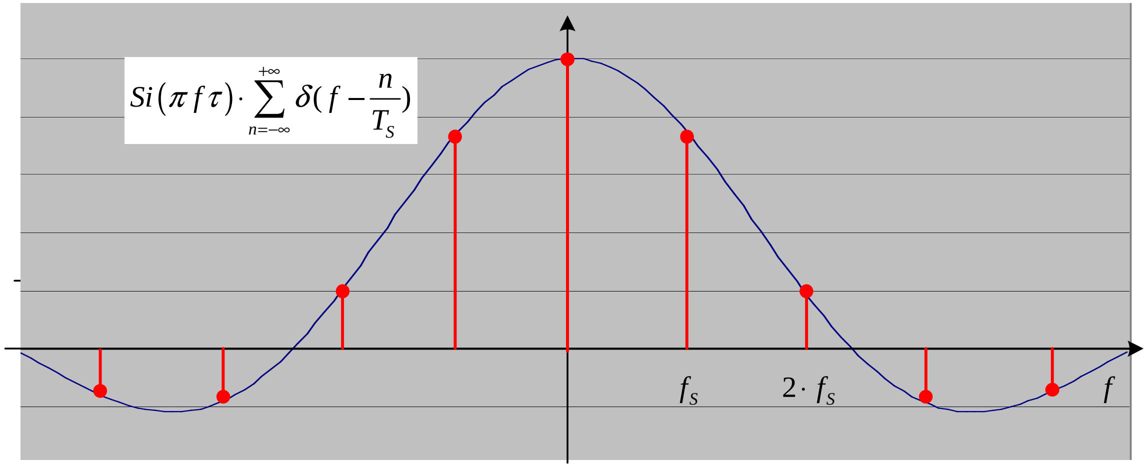
$$S_{\perp}(f) = S(f) * \left\{ Si(\pi f \tau) \cdot \sum_{n=-\infty}^{+\infty} \delta(f - f_S) \right\}$$

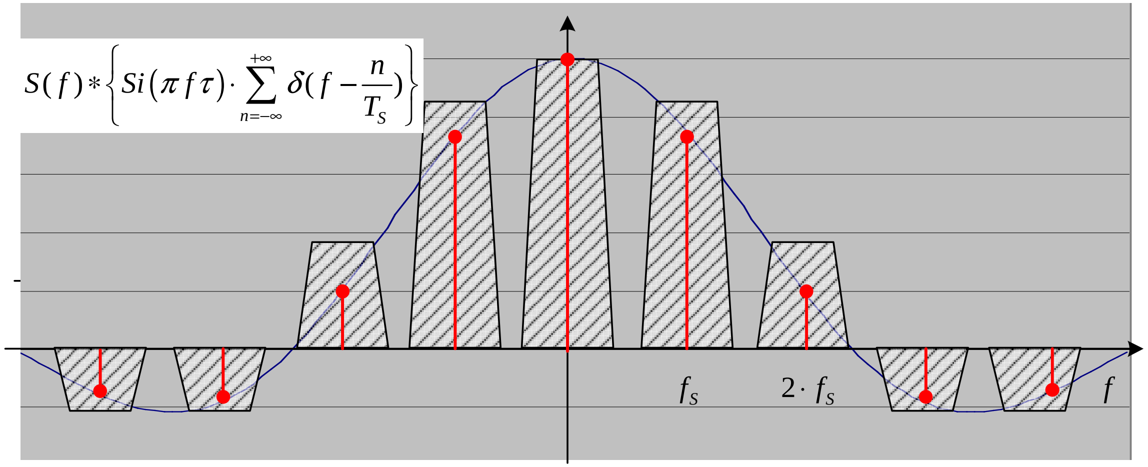
Derivation

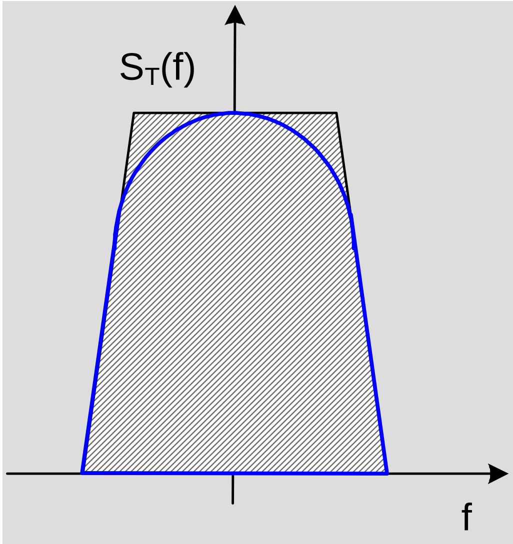


Sampling a) linear gate
b) Sample and Hold

$$S_T(f) = S(f) * \left\{ Si(\pi f \tau) \cdot \sum_{n=-\infty}^{+\infty} \delta(f - \frac{n}{T_S}) \right\}$$



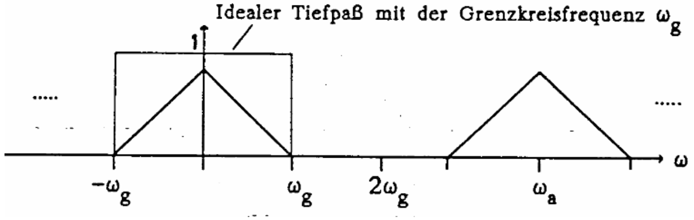




Spectrum is weightend with si-function when sampled with finite aperture time

2.1.1.4 Reconstruction of the original signal from the spectrum of the sampled signal

- The continuous signal is completely described by the sampled values, because the time-discrete function carries the same information as the continuous function
- s(t) is expressed as the sum of Si-functions that are weighted with the corresponding sampled value



Application of a low pass filter to spectrum of sampled signal

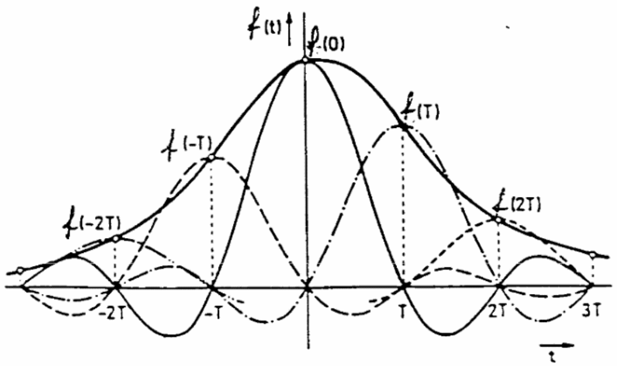
$$S(f) = S_{\perp}(f) \cdot T_s \cdot rect\left(\frac{f}{2 \cdot f_g}\right)$$

$$\updownarrow$$

$$s(t) = s_{\perp}(t) * \left(T_s \cdot 2 \cdot f_g \cdot Si\left(2\pi f_g t\right)\right); \quad \text{Nyquist: } T_s = \frac{1}{2 \cdot f_g}$$

$$s(t) = \sum_{n=-\infty}^{+\infty} s(n \cdot T_S) \cdot \delta(t - n \cdot T_S) * \left(Si\left(\pi \frac{t}{T_S}\right)\right) \quad \text{mask property of dirac}$$

$$s(t) = \sum_{n=-\infty}^{+\infty} s(n \cdot T_S) \cdot Si\left(\pi \frac{(t - n \cdot T_S)}{T_S}\right)$$



Reconstructed signal

Images from http://prof.beuth-hochschule.de/uploads/media/Abtastung_N.pdf

What did we learn in the previous lecture?

1. Which mathematic “tool/function” is used to describe sampling? What is an important property of this “tool/function”?

2. What happens with the spectrum of a signal when it is sampled?

3. What happens in frequency domain if the signal is sampled with less than the frequency specified by the Nyquist criterion?

4. What changes in the spectrum of a signal that was sampled using a real AD converter instead of a ideal converter?

5. How can the original signal be reconstructed from the spectrum of the sampled signal?

6. Is the original signal s(t) fully described by the sampled values $s(nT_s)$?

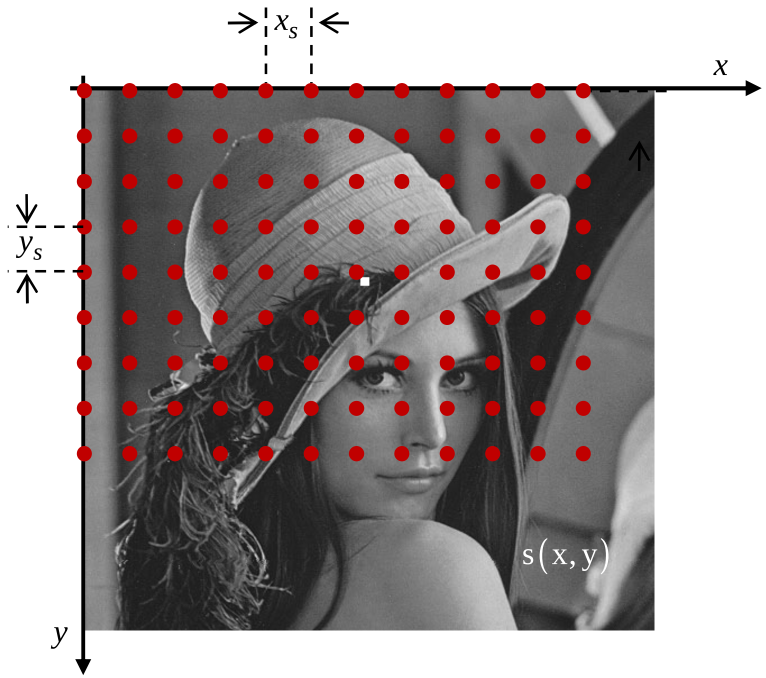
2.1.2 Two dimensional sampling in x-and y-direction

- Two-dimensional description of video signals allows a by far better understanding of video filtering and the structure of color television:

$$s_{\perp}(x,y) = s(x,y) \cdot \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m \cdot x_s) \cdot \delta(y - n \cdot y_s)$$

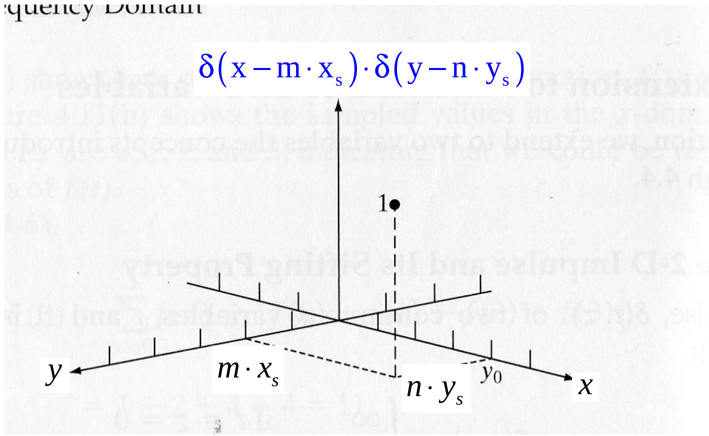
$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s(x - m \cdot x_s, y - n \cdot y_s)$$

  – x_s and y_s are the distance between two sampling points



- Here an ideal sampling with dirac impulses is considered.
- We also have to take into account the samples in horizontal and vertical direction are limited:

  – 720 x 560 standard TV
  – 640 x 480 VGA
  – 1920x1080 HD



- Nevertheless the above mentioned mathematical description is correct when we assume that the sample points to ±infinity are zero padded.

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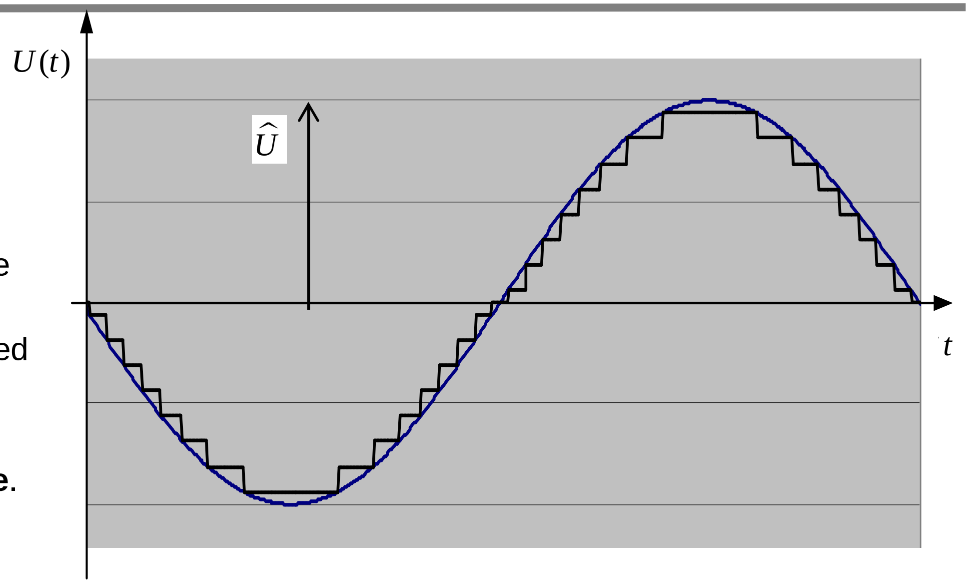
2.2.1 Quantization error



Quantization of the input signal has the effect that the max possible signal range is divided in a discrete number of amplitude ranges

- In case of linear quantization these amplitude ranges are equal

- The correct input signal is more or less correctly represented by its digital value.
- This quantization error can be expressed as noise.
  - That's the reason why we talk about **quantization noise**.

- We assume an ADC with N bit (e.g. 8 bit) resolution
  - We ignore all kind of additional errors like thermal noise, differential and integral errors
  - Considering just linear quantization, by means equidistant quantization levels, the number of quantization steps is given:

$$m = 2^N - 1$$

- The minimum resolution (LSB):

$$LSB = \frac{2\widehat{U}}{2^N - 1} = \widehat{U}_q$$

- Then the maximum quantization error is given:

$$U_{q-error,\max} = \pm \frac{\widehat{U}_q}{2} = \frac{\widehat{U}}{2^N - 1} = \pm \frac{1}{2} LSB$$

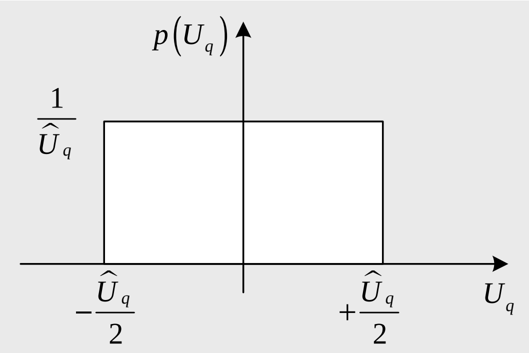
2.2.2 Signal-to-noise ratio of a sinusoidal signal

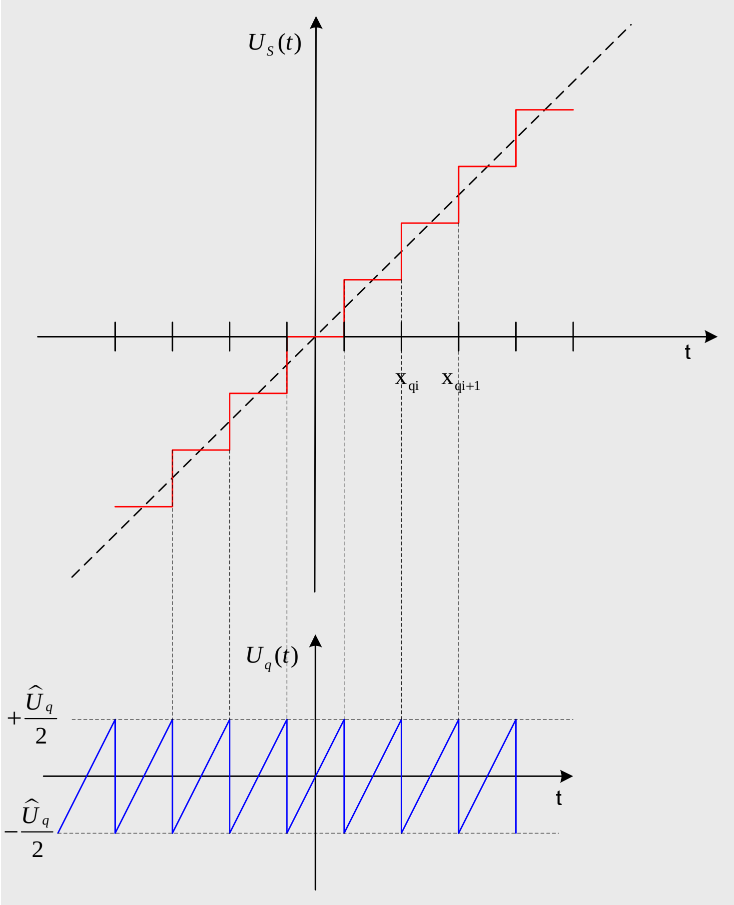
- That means that from an ADC code the correct input signal can no more be reconstructed, even with a perfect DAC (digital to analog converter).
- To define the quantization noise, we assume that the input signal is a saw tooth signal. Even that’s not the case the input signal can be considered as linear between two quantization levels when the number of quantization levels is high enough in other words, that the positive and negative error maxima can be connected be a nearly straight line.
- The difference between the input signal and the quantized signal is considered as quantization noise:

$$-\frac{\widehat{U}_q}{2} \leq U_q \leq \frac{\widehat{U}_q}{2}$$

- The saw tooth like behavior of the quantization error allows to describe the quantization error as equally distributed between the maximum und minimum quantization error:

$$\boxed{p(U_q) = \frac{1}{\widehat{U}_q} \cdot \mathrm{rect}\left(\frac{U_q}{\widehat{U}_q}\right)}$$





Power distribution spectrum of quantization error

- The power of the quantization noise N_q is calculated (normalized to a resistance of 1 Ohm):

$$N_q = \int_{-\infty}^{+\infty} \Bigg(\underbrace{U_{\text{signal}} - U_{\text{quantized signal}}}_{U_q} \Bigg)^2 p(U_q) \cdot dU_q = \int_{-\infty}^{+\infty} U_q^{\,2} \cdot \frac{1}{\hat{U}_q} \cdot \operatorname{rect}\left(\frac{U_q}{\hat{U}_q}\right) \cdot dU_q$$

$$= \frac{1}{\hat{U}_q} \cdot \int_{-\frac{\hat{U}_q}{2}}^{+\frac{\hat{U}_q}{2}} U_q^{\,2}\, dU_q = \frac{1}{\hat{U}_q} \cdot \frac{U_q^{\,3}}{3} \Bigg|_{-\frac{\hat{U}_q}{2}}^{+\frac{\hat{U}_q}{2}} = \frac{1}{3 \cdot \hat{U}_q} \left(\frac{\hat{U}_q^{\,3}}{8} + \frac{\hat{U}_q^{\,3}}{8} \right) = \frac{1}{12} \cdot \hat{U}_q^{\,2}$$

- The quantization noise of linear quantized signals is:

$$\boxed{N_q = \frac{1}{12} \cdot \hat{U}_q^{\,2}}$$

- The signal power S is given (in case of sinusoidal signals):

$$\boxed{S = \frac{1}{2} \cdot \hat{U}^{\,2}}$$

- So the signal to noise ratio is calculated as (n: number of Bits of the ADC):

$$\boxed{\left.\frac{S \sim}{N_q}\right|_{dB} = 10\log\left(2^{2n}\right) + 10\log(1,5) = 6 \cdot n + 1,76}$$

2.3.1 One dimensional Fourier transformation

2.3.1.1 DFT

$$X(f) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j \cdot 2 \cdot \pi \cdot f \cdot t} dt$$

- Starting point for the DFT is the Fourier transformation of time-continuous signals:

$$X(f) = \sum_{n=-\infty}^{+\infty} x(nT) \cdot e^{-j \cdot 2 \cdot \pi \cdot f \cdot n \cdot T} \boxed{\cdot T}$$

- The time-continuous signal x(t) is sampled at equidistant points n·T ; the integral is approximated by a sum.
  - An infinite number of samples is obtained.

$$X_w(f) = \sum_{n=0}^{N-1} x(nT) \cdot e^{-j \cdot 2 \cdot \pi \cdot n \cdot \frac{f}{f_S}} \boxed{\cdot T} \qquad T = \frac{1}{f_s}$$

  - We limit the DFT on an finite sample number (N). This way a kind of windowing of the sampled input signal takes place (index w).

$$f = 0, \frac{f_S}{N}, 2 \cdot \frac{f_S}{N}\ 3 \cdot \frac{f_S}{N}, \ldots (N-1) \cdot \frac{f_S}{N}$$

  - Furthermore the sample period $\boxed{T}$ is considered to be 1, by means T can be omitted:
- The function $X_w(f)$ is a periodic function with frequency f_S= 1/T and N independent, equidistant values:

$$X_w\left(k \frac{f_S}{N}\right) = \sum_{n=0}^{N-1} x(nT) \cdot e^{-j \cdot 2 \cdot \pi \cdot n \cdot \frac{k \frac{f_S}{N}}{f_S}} \Leftrightarrow X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j \cdot 2 \cdot \pi \cdot n \cdot \frac{k}{N}}; \quad k = 0,1,2 \ldots N-1$$

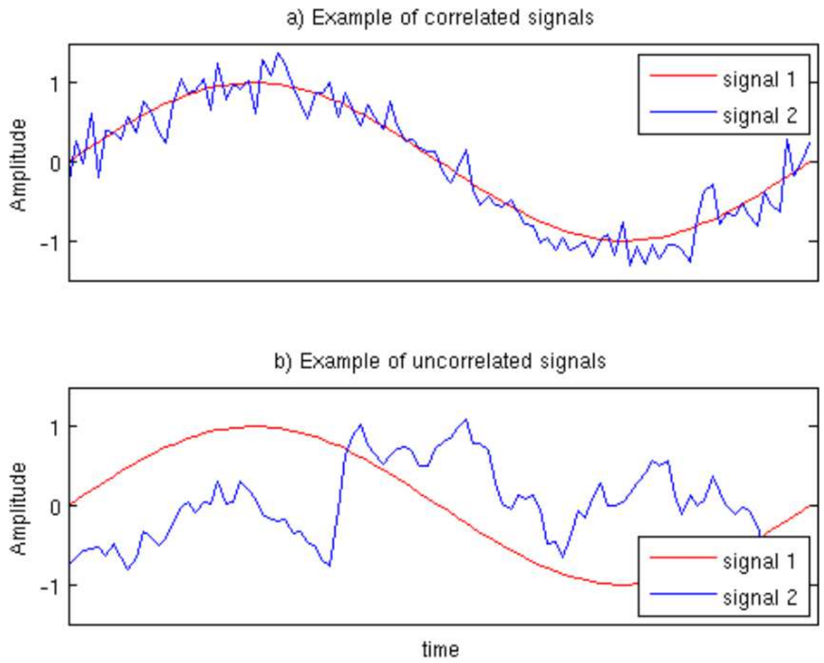
DFT base functions:

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi n \frac{k}{N}} = \sum_{n=0}^{N-1} x(n) \cdot \left(\cos\left(2\pi n \frac{k}{N}\right) - j \cdot \sin\left(2\pi n \frac{k}{N}\right)\right)$$

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot \cos\left(2\pi n \frac{k}{N}\right) - j \cdot \sum_{n=0}^{N-1} x(n) \cdot \sin\left(2\pi n \frac{k}{N}\right)$$

$$k = 0 \ \dots \ (N-1)$$

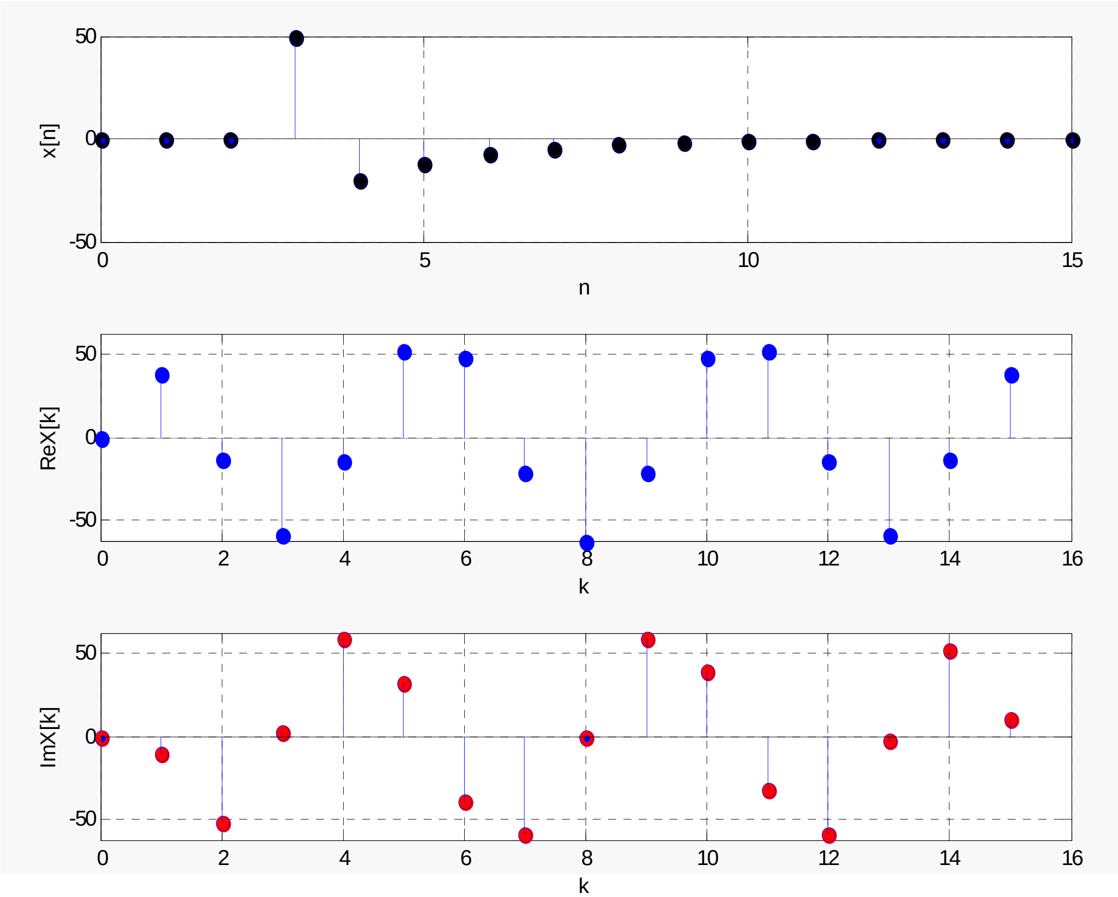
- The parameter k indicates how many sine or cosine-periods are within the range of N samples.
- k=1 means 1 period.
- k=N means N periods.
- k=0 corresponds to the DC value.

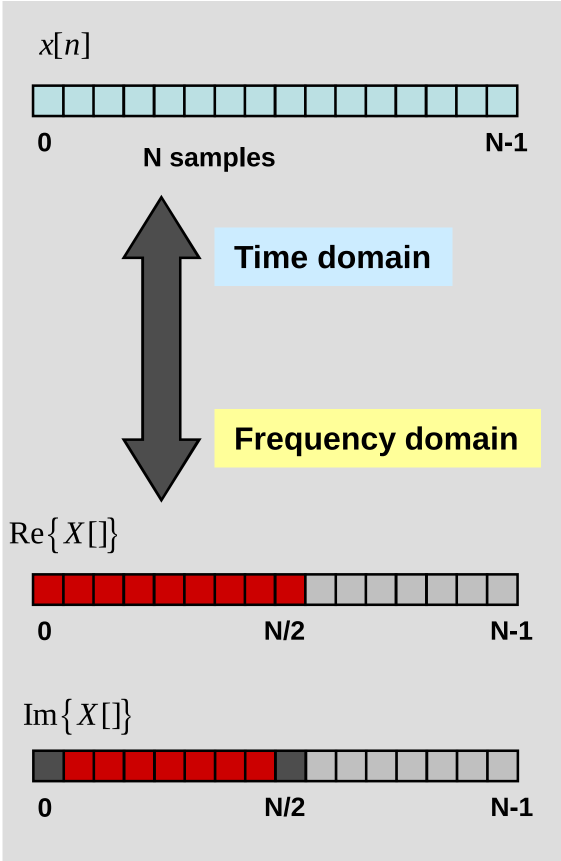


http://practicalcryptography.com/miscellaneous/machine-learning/intuitive-guide-discrete-fourier-transform/

- The DFT coefficients are repeated with respect to N/2 (Example:8) (Aliasing)
- The imaginary coefficients k=0 and k=N/2 are always zero.
- That means that N samples always result in N coefficients.

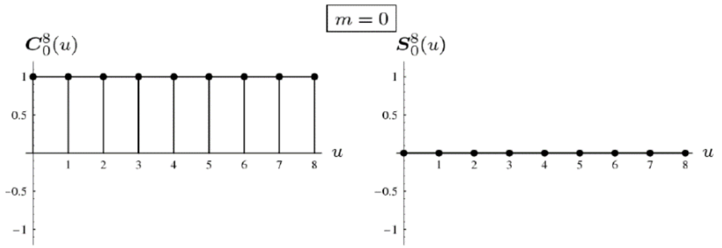
$$x[n] = \mathrm{Re}\{x[n]\} + j\,\mathrm{Im}\{x[n]\} \rightleftarrows X[k] = \mathrm{Re}\{X[k]\} + j\,\mathrm{Im}\{X[k]\}$$

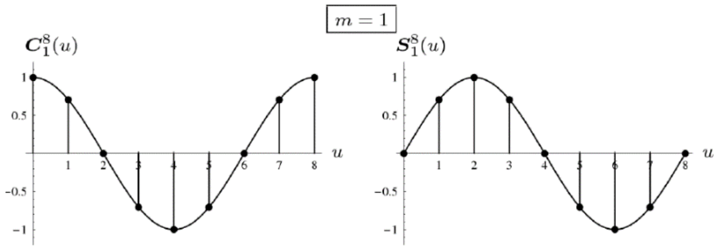


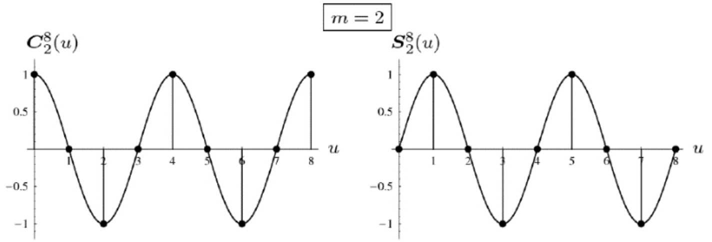


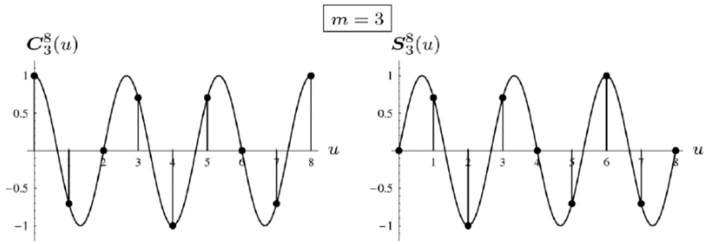
DFT of an input sequence x[n]

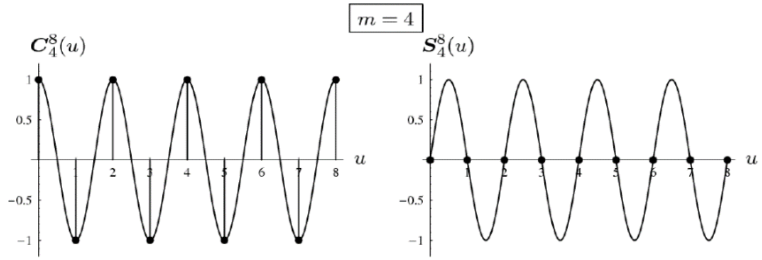
http://www.fourier-series.com/fourierseries2/flash_programs/Discrete_Basis_Functions/index.html
http://practicalcryptography.com/miscellaneous/machine-learning/intuitive-guide-discrete-fourier-transform/

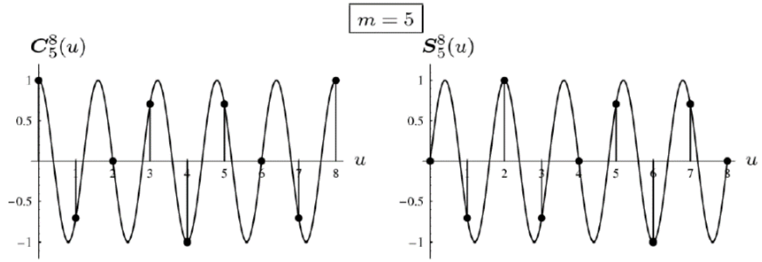


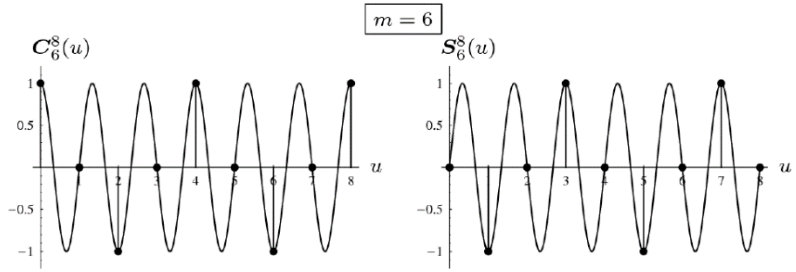


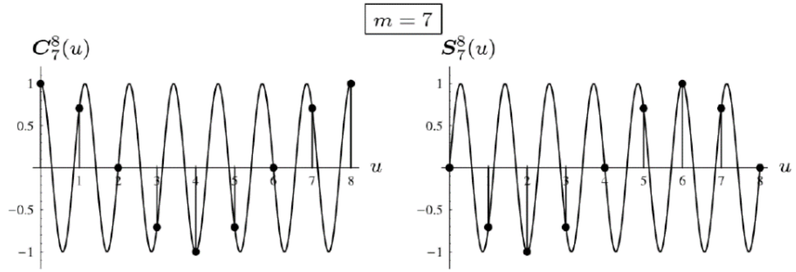












Burger & Burge, 2005

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2.3.1.2 Real- und imaginary part of DFT

- The similarity of the real part of x[n] with a cosine is represented by the DFT as real-part of X[k] (axis symmetry).
- The similarity of the real part of x[n] with a sine is represented by the DFT as imaginary part of X[k] (point symmetry).

- The similarity of the imaginary part of x[n] with a cosine is represented by the DFT as imaginary part of X[k] (axis symmetry).
- The similarity of the imaginary part of x[n] with a sine is represented by the DFT as real part of X[k] (point symmetry).

$$
\begin{array}{lccccccc}
x[n]= & x_{e,r}[n] & + & x_{o,r}[n] & + & j\cdot x_{e,i}[n] & + & j\cdot x_{o,i}[n] \\
 & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
X[k]= & X_{e,r}[k] & + & j\cdot X_{o,i}[k] & + & j\cdot X_{e,i}[k] & + & X_{o,r}[k]
\end{array}
$$

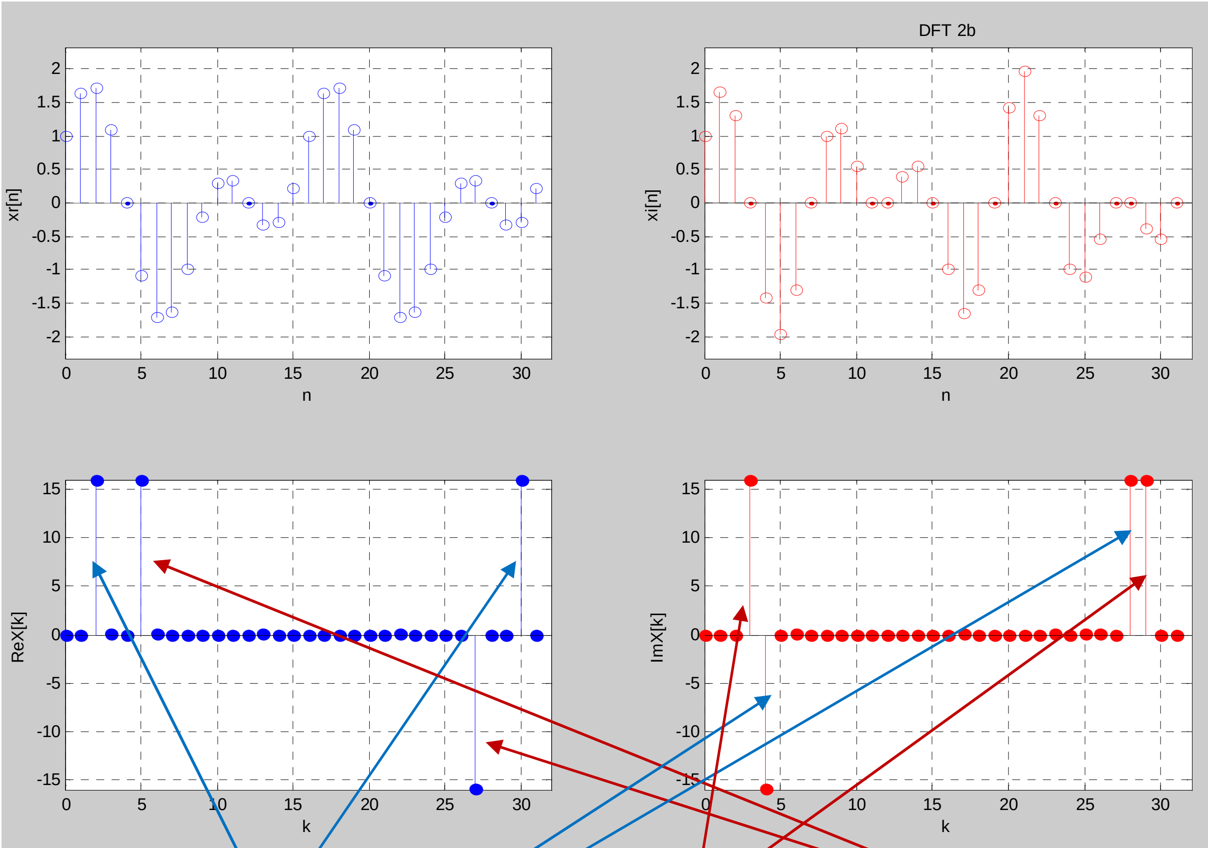
$$X(k)=\sum_{n=0}^{N-1} x(n)\cdot e^{-j2\pi n\frac{k}{N}}=\sum_{n=0}^{N-1} x(n)\cdot\left(\cos\left(2\pi n\frac{k}{N}\right)-j\cdot\sin\left(2\pi n\frac{k}{N}\right)\right)$$

$$X(k)=\sum_{n=0}^{N-1} x(n)\cdot\cos\left(2\pi n\frac{k}{N}\right)-j\cdot\sum_{n=0}^{N-1} x(n)\cdot\sin\left(2\pi n\frac{k}{N}\right)$$

$$k=0\ \ldots\ (N-1)$$

$$\begin{array}{lccccccc} x[n]= & x_{e,r}[n] & + & x_{o,r}[n] & + & j\cdot x_{e,i}[n] & + & j\cdot x_{o,i}[n] \\ & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ X[k]= & X_{e,r}[k] & + & j\cdot X_{o,i}[k] & + & j\cdot X_{e,i}[k] & + & X_{o,r}[k] \end{array}$$

$$X(k)=\sum_{n=0}^{N-1}x(n)\cdot\cos\left(2\pi n\frac{k}{N}\right)-j\cdot\sum_{n=0}^{N-1}x(n)\cdot\sin\left(2\pi n\frac{k}{N}\right)$$



$$x[n]=\cos\left(\frac{2\pi\cdot 2}{32}\cdot n\right)+\sin\left(\frac{2\pi\cdot 4}{32}\cdot n\right)+j\cdot\cos\left(\frac{2\pi\cdot 3}{32}\cdot n\right)+j\cdot\sin\left(\frac{2\pi\cdot 5}{32}\cdot n\right)$$

2.3.1.3 Inverse DFT

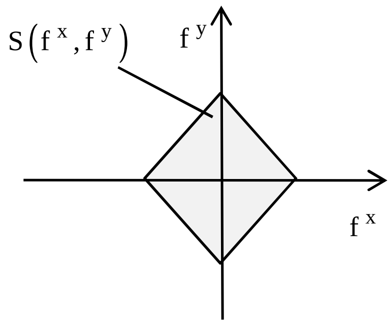
- The inverse discrete Fourier transformation (IDTF) is described by:

$$x[n] = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X[f] \cdot e^{+j \cdot 2 \cdot \pi \cdot k \cdot \frac{n}{N}}; \quad n = 0,1,2 \ldots N-1$$

$$x[n] \quad \rightleftarrows \quad X[k]$$

The values X[k] are complex ; they are also designated as DFT-coefficients.
The DFT is an unambiguous (in both directions) image between complex input sequence x[n] and complex spectrum X[k].

2.3.2 Two dimensional Fourier Transformation of a continuous signal



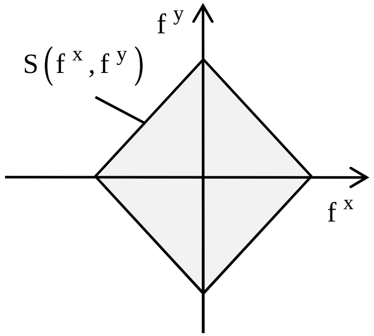
- We assume an image s(x,y) with infinite extension (from -∞ to ∞) in x and y direction
- The two-dimensional Fourier transformation is specified as:

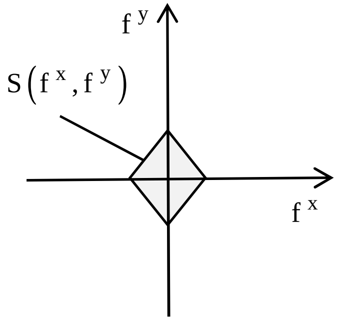
$$s(x,y) \overset{2}{\rightleftarrows} S(f^x, f^y)$$

$$S(f^x, f^y) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} s(x,y) e^{-j2\pi(f^x \cdot x + f^y \cdot y)} dx \cdot dy$$

  - Note s(x,y) is a continuous image and not sampled so far

- $S(f^x,f^y)$ is the two-dimensional spectrum of the scene.
- When the image has a low resolution in x or y direction, the spectrum $S(f^x,f^y)$ is more or less extended in f^x and f^y direction.
  - This spectrum is influenced by optical resolution of the camera the lenses or the filter function etc.





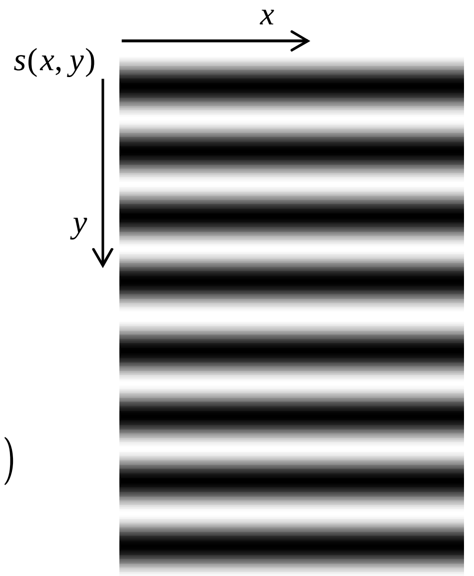
- **2.3.2.1 Example: vertical sinusoidal**
- An infinite extended vertical, sinusoidal template with spatial frequency f^y_0 can be described as:

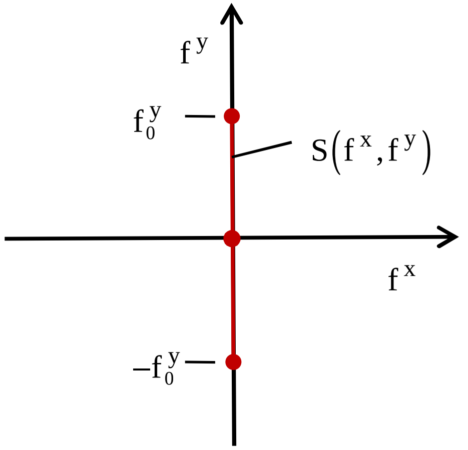
$$s(x,y)=\frac{1}{2}\cdot\left[1+\cos\left(2\pi\cdot f_0^y\cdot y\right)\right]\cdot 1(x)$$

- In frequency domain:

$$s(x,y)\overset{2}{\rightleftarrows}S\left(f^x,f^y\right)$$

$$S\left(f^x,f^y\right)=\left\{\frac{1}{2}\delta\left(f^y\right)+\frac{1}{4}\delta\left(f^y+f_0^y\right)+\frac{1}{4}\delta\left(f^y-f_0^y\right)\right\}\cdot\delta\left(f^x\right)$$

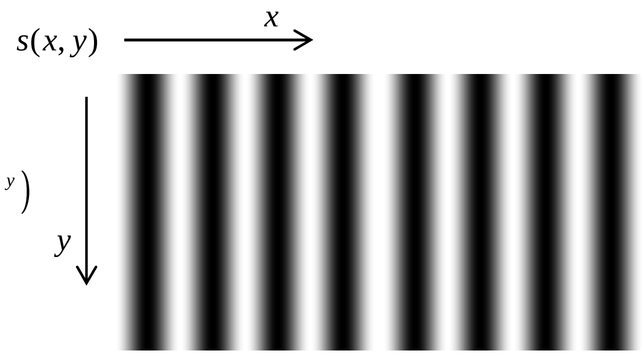


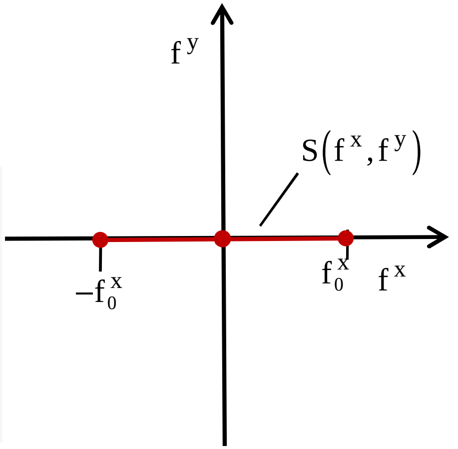


- **2.3.2.2 Example: horizontal sinusoidal:**
- For a horizontal, sinusoidal template with spatial frequency f^x_0 :

$$s(x,y)=\frac{1}{2}\cdot\left[1+\cos\left(2\pi\cdot f_0^x\cdot x\right)\right]\cdot 1(y)$$

$$S\left(f^x,f^y\right)=\left\{\frac{1}{2}\delta\left(f^x\right)+\frac{1}{4}\delta\left(f^x+f_0^x\right)+\frac{1}{4}\delta\left(f^x-f_0^x\right)\right\}\delta\left(f^y\right)$$





2.3.3 Two dimensional Fourier transformation of a sampled continuous signal

- Starting from the two dimensional Fourier spectrum of a continuous signal

$$S\left(f^x,f^y\right)=\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} s(x,y)\,e^{-j2\pi\left(f^x\cdot x+f^y\cdot y\right)}dx\cdot dy$$

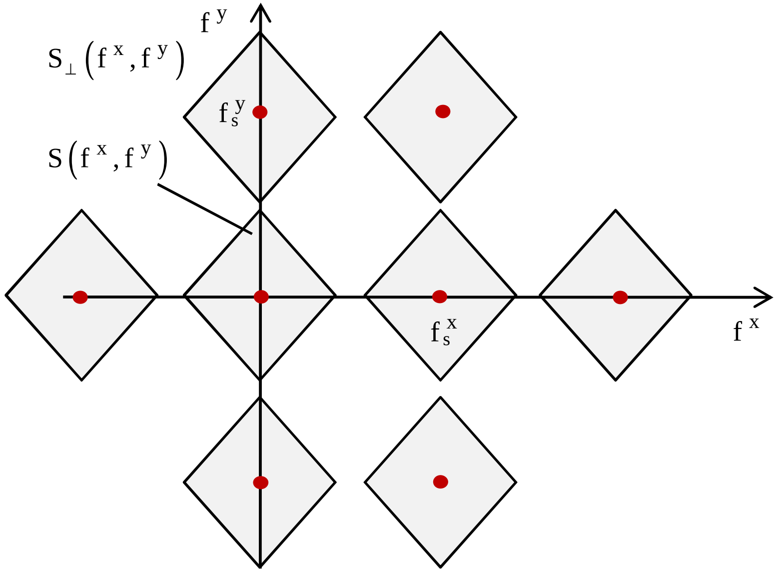
- Now the two dimensional image is sampled in x and y direction:

$$x=m\cdot x_s;\,y=n\cdot y_s$$

$$S_\perp\left(f^x,f^y\right)=\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} s_\perp(x,y)\cdot e^{-j2\pi\left(f^x\cdot x+f^y\cdot y\right)}\cdot dx\cdot dy$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} s\left(x-m\cdot x_s,y-n\cdot y_s\right)\cdot e^{-j2\pi\left(f^x\cdot x+f^y\cdot y\right)}\cdot x_s\cdot y_s$$

$$=S\left(f^x,f^y\right)\overset{2}{*}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta\left(f^x-m\cdot f_s^x\right)\cdot\delta\left(f^y-n\cdot f_s^y\right)$$



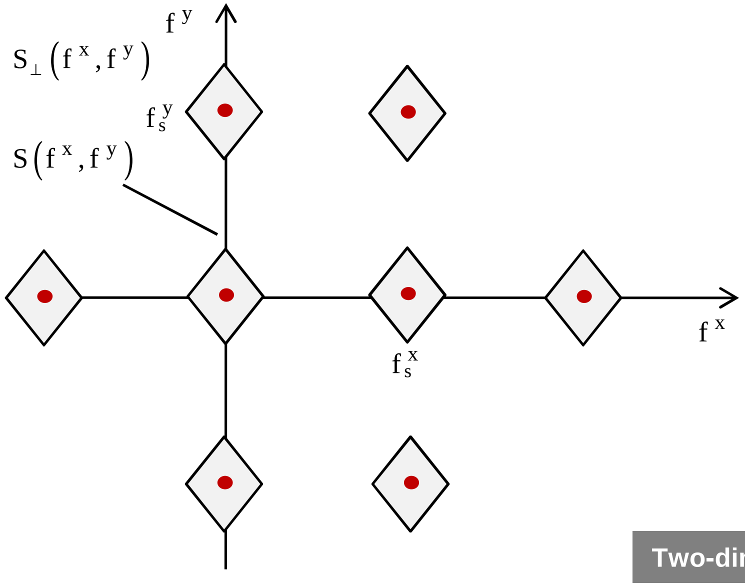
- The integrals become sums and dx and dy are the distance between two sample points: $dy=x_s$ and $dy=y_s$:

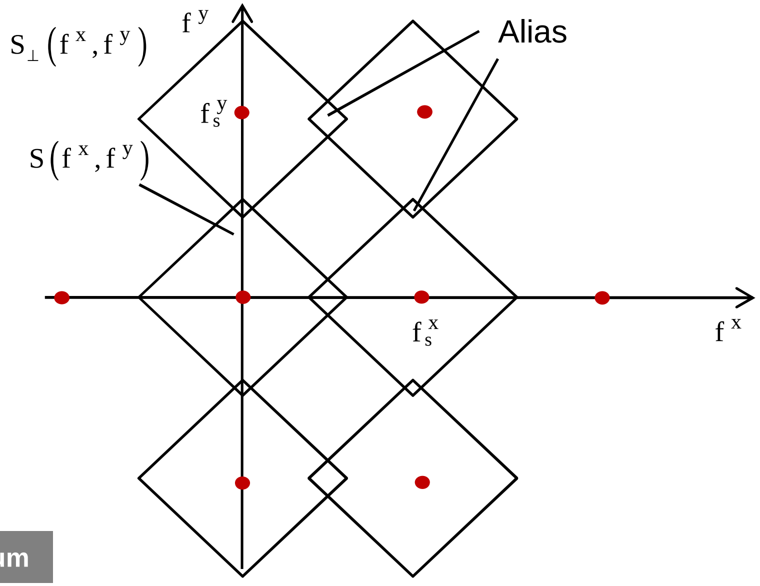
- f_s^x and f_s^y are the sampling frequency in horizontal and vertical direction: pw: picture width, ph: picture-height

Scanning in time-domain: $\left[f\right]=\frac{1}{s}=Hz$

Scanning in spatial-domain: $\left[f^x,f^y\right]=\frac{samples}{pw},\frac{samples}{ph}$

- $S(f^x,f^y)$ is the two-dimensional spectrum of the scene.
  - This spectrum is influenced by optical resolution of the camera etc.
- When the spectrum $S(f^x,f^y)$ of the image has more details → alias
- When an image is sampled with less points then x_s and y_s become larger→ f_s^x and f_s^y are smaller → alias.





Two-dimensional spectrum

2.3.4 Two dimensional Discrete Fourier transformation

- It is clear that it's not possible to sample an infinite pixels in both direction.
- A real picture is of size M x N (M pixels in horizontal, N in vertical direction)
- s(x,y) should already be the sampled signal (furthermore we just consider sampled signals)

$$S\left(f^x, f^y\right) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} s(x,y) \cdot e^{-j2\pi\left(f^x \cdot x + f^y \cdot y\right)} \cdot x_s \cdot y_s;$$

$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} s(x,y) \cdot e^{-j2\pi\left(\frac{u \cdot f_s^x}{M} \cdot x + \frac{v \cdot f_s^y}{N} \cdot y\right)} \cdot x_s \cdot y_s \qquad f^x = u \cdot \frac{f_s^x}{M}, f^y = v \cdot \frac{f_s^y}{N}$$

With $x_s = 1, y_s = 1$

$$S(u,v) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} s(x,y) \cdot e^{-j2\pi\left(\frac{u \cdot m}{M} + \frac{v \cdot n}{N}\right)}; \qquad f_s^x = \frac{1}{x_s} = 1, f_s^y = \frac{1}{y_s} = 1$$

- For the inverse two-dimensional discrete Fourier transformation:

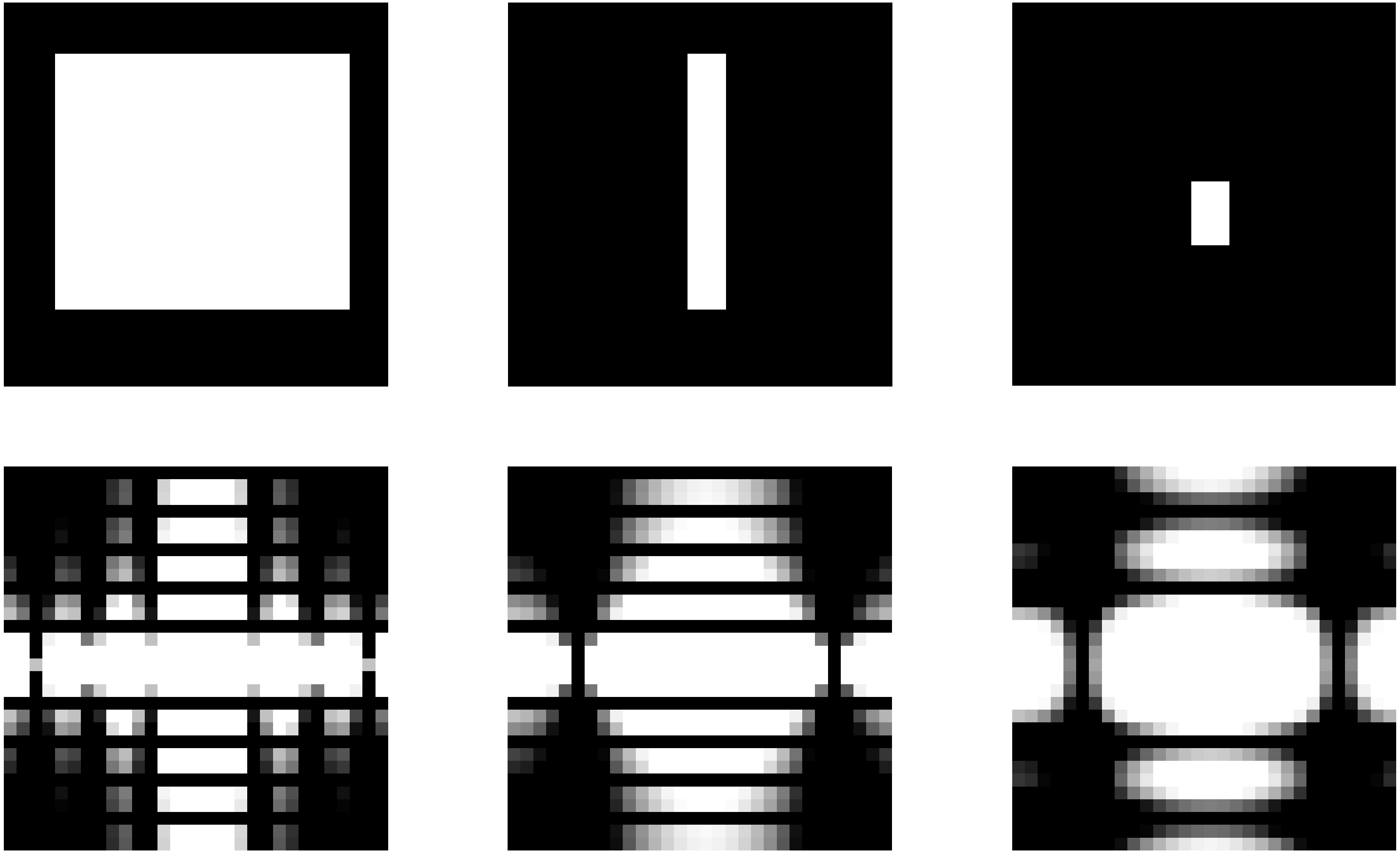
$$h(x,y) = \frac{1}{M \cdot N} \cdot \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} H(u,v) \cdot e^{+j2\pi\left(\frac{u \cdot x}{M} + \frac{v \cdot y}{N}\right)}$$

- Similar to the one dimensional DFT also the two-dimensional DFT is just defined At discrete frequency components:

$$S\left(f^x, f^y\right) = \begin{cases} \neq 0 & f^x = \{0....M-1\} \cdot \frac{f_s^x}{M} \\ 0 & \text{else} \end{cases}$$

$$S\left(f^x, f^y\right) = \begin{cases} \neq 0 & f^y = \{0....N-1\} \cdot \frac{f_s^y}{N} \\ 0 & \text{else} \end{cases}$$

– In horizontal direction: In vertical direction:



University Library:

https://katalog.bibliothek.tu-chemnitz.de/Search/Advanced

- Digital image processing; an algorithmic introduction using Java (Wilhelm Burger; Mark James Burge), Springer
- Principles of digital image processing (Wilhelm Burger; Mark James Burge), Springer
- Digital Image Processing Using MATLAB (Gonzalez, Woods, Eddins), Gatesmark Publishing
- Digital signal and image processing using MATLAB (Blanchet, Gerard, Charbit, Maurice)

Tutorials:

http://www.cs.otago.ac.nz/cosc451/Resources/matlab_ipt_tutorial.pdf

http://www.mathworks.com/products/image/index.html?s_tid=gn_loc_drop

http://www.fourier-series.com/fourierseries2/DFT_tutorial.html

Matlab is available in the PC pools at the university (https://www.tu-chemnitz.de/urz/pools/index.html)

End of chapter