Design of Software for Embedded Systems

## Chapter 3

# Mathematical Theory of Systems and Control

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**Chair of Operating Systems**

# Purpose of this Extended Lecture

- Control system engineering is a complex task
- What (I hope) you will take away from this lecture:
  - Necessary basics to:
    - Be able to understand simple systems
    - Read further literature (if necessary)
    - Communicate with engineers
    - Build better systems
  - Motivation to use math and modelling more frequently
  - Appreciate another view on systems

# Contents

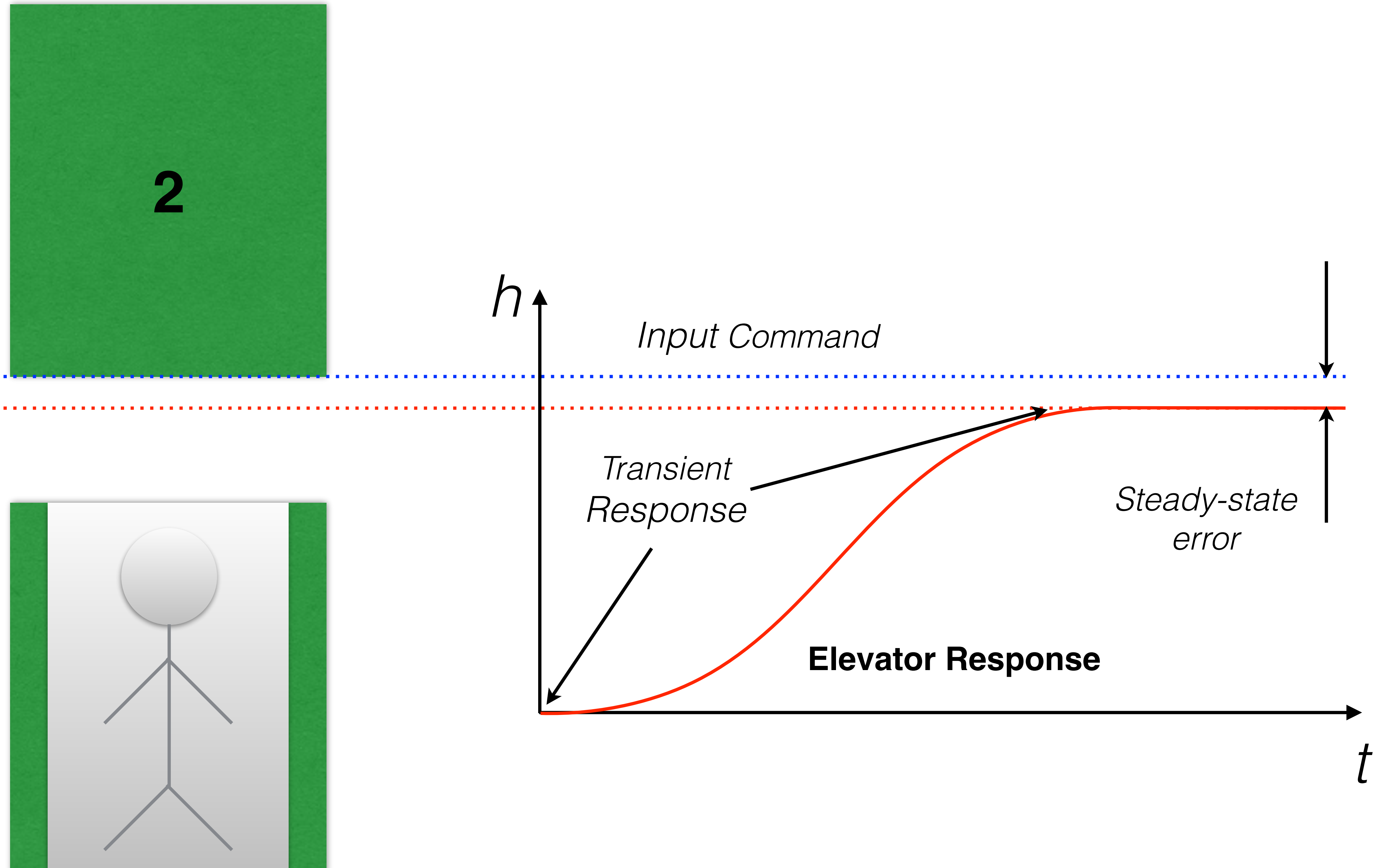
# 1.1 Control System Definition (refreshment)

**Definition**: A control system consists of *subsystems* and *processes* (or *plants*) assembled for the purpose of obtaining a desired *output* with desired *performance*, given a specified *input*.

- Two *major goals* of performance:
  - **Transient response**
  - **Steady-State Error**

Source: [2]

# Example: Elevator

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# 1.2 Open-Loop vs. Closed-Loop (refreshment)

| Open-Loop | Closed-Loop |
|---|---|
| Stable | Can become unstable |
| No feedback | Has a feedback loop |
| Not precise | Precise |
| Cannot compensate disturbances | Compensates disturbances |
| Simple | Complex |
| Cheap | Expensive |

# 1.3 Objectives of Analysis and Design

- Three major goals:
  - Producing the desired transient response
  - Reducing steady-state error
  - Achieving Stability

**Definition**: Control system is *stable* if its natural response a) eventually approaches zero or b) oscillates. If the natural response of the system grows without bound and becomes much greater than forced response, the system is considered *unstable*.

- Other goals:
  - Finances
  - Robust design

Source: [2]

# Control System Design Process: Overview

Determine a physical system and specifications from the requirements.

Draw a functional block diagram.

Transform the physical system into a schematic.

Use the schematic to obtain a block diagram, signal-flow diagram, or state-space representation.

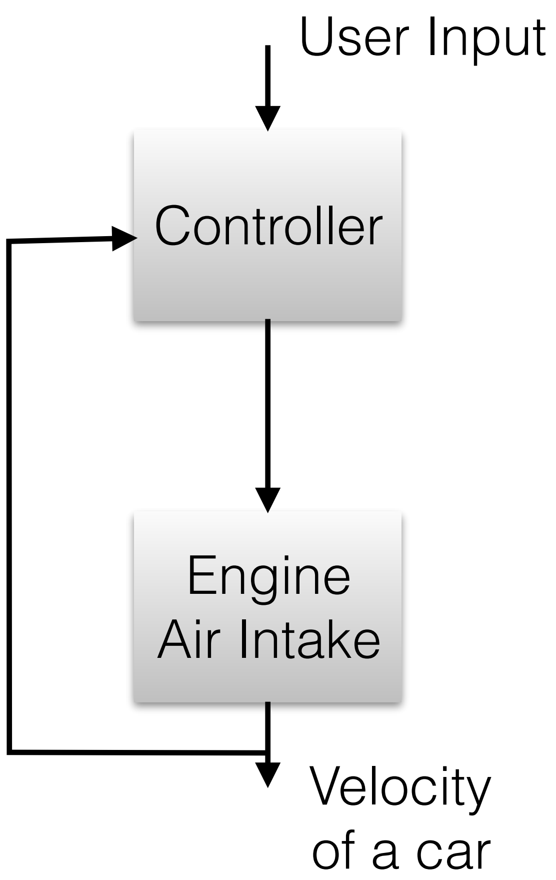
If multiple blocks, reduce the block diagram to a single block or closed-loop system.

Analyze, design, and test to see that requirements and specifications are met.

Source: [2]

# Control System Design Process: Stages

1. Transform requirements into a physical system
    1. Determine physical dimensions, e.g. position, mass
    2. Using the requirements, define specifications for the design, e.g. transient response, steady-state accuracy
2. Functional block diagram
    1. Translate qualitative description into functional subsystems
    2. Define interconnections between subsystems
3. Create schematic
    1. Make approximations of the system
    2. Make assumptions about subsystems
    3. Simplify, but not oversimplify
    4. Instantiate the modules from functional block diagram

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# Control System Design Process: Stages (2)

4. Develop a mathematical model (block diagram)
    1. From schematic and physical laws (Kirchhoff, Newton's)
    2. Many systems can be described by a linear ODE which relates input with the output

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + b_0 r(t)$$

    3. Assumptions and approximations may simplify this ODE
    4. Often other representations are used instead:
        1. For *linear, time-invariant (LTI)* systems, Laplace transform derives a *transfer function*
        2. Alternatively, one can-use state-space representation (possible for non-LTI systems)
5. Reduce Block Diagram
    1. Single block
    2. Only system input and output present


Source: [2]

# Control System Design Process: Stages (3)

6. Analyze and Design
   1. Mostly testing and verification
   2. Redesign if necessary
   3. Test signals commonly used:

| Input | Function | Use |
|---|---|---|
| Impulse | $\delta(t)$ | Transient response |
| Step | $u(t)$ | Transient response<br>Steady-state error |
| Ramp | $u(t)t$ | Steady-state error |
| Parabola | $\frac{1}{2}u(t)t^2$ | Steady-state error |
| Sinusoid | $\sin \omega t$ | Transient response<br>Steady-state error |

Source: [2]

# 2.1. Complex Numbers (refreshment)

- Complex numbers were introduced to solve equations like:

$$x^2 = -1$$

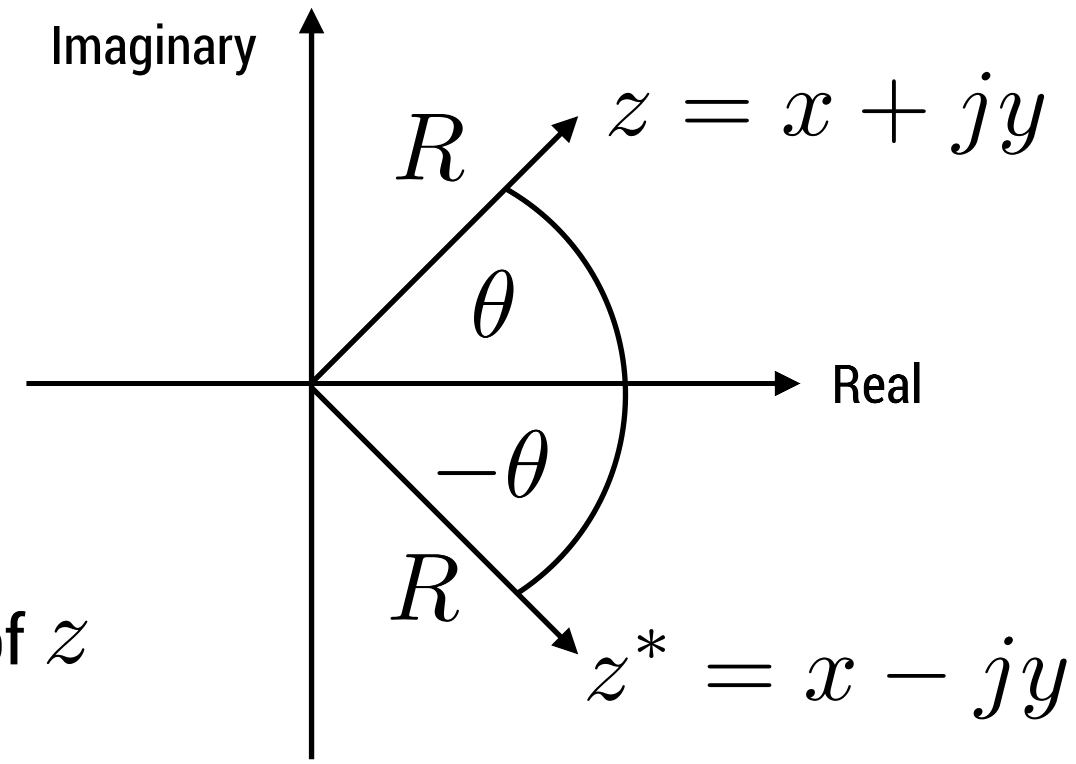
- A complex number written in **rectangular form:**

$$z = x + jy, j = \sqrt{-1}$$

  - *j* is called the imaginary unit.
- Alternatively,

$$x = R\cos\theta$$
$$y = R\sin\theta$$

  - R is the magnitude and $\theta$ is the phase of $z$




Source: [1]

# Complex Numbers Properties (refreshment)

- Complex numbers written in the Euler form:

$$\begin{aligned} z &= R\cos\theta + jR\sin\theta \\ &= R(\cos\theta + j\sin\theta) \\ &= Re^{j\theta} \end{aligned}$$

- Addition/Subtraction is easier to do in the rectangular form:

$$z_1 \pm z2 = (x_1 \pm x_2) \pm j(y_1 \pm y_2)$$

- Multiplication/Division are more convenient in the Euler form:

$$z_1 z_2 = (R_1 R_2)e^{j(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = (\frac{R_1}{R_2})e^{j(\theta_1 - \theta_2)}$$

Source: [1]

# 2.2 Differential Equations (refreshment)

**Definition**: *Differential equation* is any equation which contains derivatives, either ordinary derivatives or partial.

**Definition**: *Solution* to a differential equation on an interval $\alpha < t < \beta$ is any function $y(t)$ which satisfies the equation on this interval.

**Definition**: *Order* of differential equation is the larges derivative present in it.

Source: [3]

# 2.2 Differential Equations (refreshment)

- Example (Newton's Second Law):

$$F = ma$$

$$F = m\frac{d^2x}{dt^2}$$

**Definition**: *Linear differential equation* is any differential equation that can be written in the following form:

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + ... + a_1(t)y'(t) + a_0(t)y(t) = g(t)$$

Note: there are no products of the function and its derivatives and neither the the function or its derivatives occur to any power other than the first power.

Source: [3]

# 2.2 Differential Equations (refreshment)

**Definition**: Differential equation is *time-invariant* if it does not depend explicitly on time.

- Many techniques for solving ODEs:
  - Linear 1st order ODEs $\frac{dy}{dt} + p(t)y = g(t)$
  - Separable ODEs $N(y)\frac{dy}{dx} = M(x)$
  - Exact ODEs $M(x,y) + N(x,y)\frac{dy}{dx} = 0$
  - Bernoulli ODEs $y' + p(x)y = q(x)y^n$

  ...

Source: [3]

# 2.3 Laplace Transform

- Motivation:
  - Differential equation relates input to output, yes
  - But: All system parameters, input and output are mixed up in the equation
  - Hence: it is not a satisfying representation from a system perspective

**Definition**: The *Laplace transform* is defined as

$$\mathcal{L}|f(t)| = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

where $s = \delta + j\omega$ is a complex variable. Condition for existence:

$$\int_{0}^{\infty} \left|f(t)e^{-kt}\right|dt < \infty, k \in \mathbb{R}$$

Note: the notation for the lower limit means that even if the function is discontinuous at *t=0, it is possible to start integration prior to discontinuity.*

# 2.3 Laplace Transform Simple Example

- Problem: Find the Laplace transform of

$$f(t) = Ae^{-at}u(t)$$

- Solution:

$$\begin{aligned} F(s) &= \int_0^{\infty} f(t)e^{-st}dt \\ &= \int_0^{\infty} Ae^{-at}e^{-st}dt \\ &= A\int_0^{\infty} e^{-(s+a)t}dt \\ &= -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty} \\ &= \frac{A}{s+a} \end{aligned}$$

This integral converges only if

$$-(s+a) < 0$$
$$s+a > 0$$

Source: [2]

# 2.3 Laplace Transform table

| Function | Transform |
|---|---|
| $\delta(t)$ | $1$ |
| $u(t)$ | $\frac{1}{s}$ |
| $u(t)t$ | $\frac{1}{s^2}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $e^{-at}u(t)$ | $\frac{1}{s+a}$ |
| $\sin(\omega t)u(t)$ | $\frac{\omega}{s^2+\omega^2}$ |
| $\cos(\omega t)u(t)$ | $\frac{s}{s^2+\omega^2}$ |

...

# 2.3 Laplace Transform Theorems

| | |
|---|---|
| $\mathcal{L}\|kf(t)\| = kF(s)$ <br> $\mathcal{L}\|f_1(t) + f_2(t)\| = F_1(s) + F_2(s)$ | **Linearity Theorem** |
| $\mathcal{L}\|e^{-at}f(t)\| = F(s+a)$ | **Frequency Shift Theorem** |
| $\mathcal{L}\|f(t-T)\| = e^{-sT}F(s)$ | **Time Shift Theorem** |
| $\mathcal{L}\|f(at)\| = \frac{1}{a}F(\frac{s}{a})$ | **Scaling Theorem** |
| $\mathcal{L}\|\frac{df}{dt}\| = sF(s) - f(-0)$ <br> $\mathcal{L}\|\frac{d^2f}{dt^2}\| = s^2F(s) - sf(-0) - f'(-0)$ | **Differentiation Theorem** |

...

# 2.3 Laplace Transform Another Example

- Problem: Find the Laplace transform of

$$f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$$

- Solution:

$$F(s) = 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s}$$

$$= \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$$

Source: [3]

# 2.3 Laplace Transform Complex Example

- Problem: Find the Laplace transform of

$$f(t) = t^2 sin(2t)$$

- Solution:

  Using the fact that $\mathcal{L}|tf(t)| = -F'(s)$ and $\mathcal{L}|tsin(at)| = \frac{2as}{(s^2+a^2)^2}$

$$F(s) = \frac{4s}{(s^2+4)^2}$$

$$F'(s) = -\frac{12s^2-16}{(s^2+4)^3}$$

$$H(s) = \frac{12s^2-16}{(s^2+4)^3}$$

Source: [3]

# 2.3 Inverse Laplace Transform

**Definition**: The *inverse Laplace transform* is defined as

$$\mathcal{L}^{-1}|F(s)| = \frac{1}{2\pi j}\int_{\delta-j\infty}^{\delta+j\infty} F(s)e^{st}ds = f(t)u(t)$$

where $s = \delta + j\omega$ is a complex variable, $\delta$ is real and *u(t)* is a unit-step function.

## Example

Problem: Find inverse Laplace transform of $F(s) = \frac{1}{(s+3)^2}$

Solution: We use frequency shift theorem and the transform of $\mathcal{L}|u(t)t| = \frac{1}{s^2}$

So, $\mathcal{L}^{-1}|\frac{1}{(s+a)^2}| = e^{-at}u(t)t$ and $f(t) = e^{-3t}u(t)t$

Source: [2]

# 2.3 Partial-Fraction Expansion

- **Problem:** how to find an ILT of a complex function?
- **Solution:** we can *convert the function into a sum of simpler terms*, for which we know the ILT.
- This method is called *partial fraction expansion*
- Assume that $F(s) = \dfrac{N(s)}{D(s)}$, where the order of $N(s)$ is lower than that of $D(s)$
- If this is not the case, divide polynomials until this condition is fulfilled
- Three cases are possible:
  - Roots of the denominator $D(s)$ are real and distinct
  - Roots of the denominator are real and repeated
  - Roots of the denominator are complex or imaginary (*not considered here*)

Source: [2]

# 2.3 Partial-Fraction Expansion (Case 1)

- Roots of denominator are real and distinct
- It is possible to write the PFE as a sum of terms
- Constants, called *residues*, form numerators
- For example,

$$F(S) = \frac{2}{(s+1)(s+2)} = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

- To find $K_1$ , we multiply this equation by *(s+1)* which isolates $K_1$

$$\frac{2}{s+2} = K_1 + \frac{(s+1)K_2}{s+2}$$

- Then we let s approach -1 which eliminates last term and we get $K_1 = 2$
- Similarly solving this for the second residue, we get $K_2 = -2$
- The resulting function can be transformed now: $f(t) = (2e^{-t} - 2e^{-2t})u(t)$

Source: [2]

# 2.3 Partial-Fraction Expansion (Case 2)

- Case 2: Roots of denominator are real and repeated

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)^r(s+p_2)...(s+p_n)}$$

$$\frac{K_1}{(s+p_1)^r} + \frac{K_2}{(s+p_1)^{r-1}} + ... + \frac{K_r}{s+p_1} + \frac{K_{r+1}}{s+p_2} + ... + \frac{K_n}{s+p_n}$$

- To find the residues for the roots of multiplicity greater than unity:

$$K_i = \frac{1}{(i-1)!} \frac{d^{i-1}F(s)}{ds^{i-1}}\Big|_{s \to -p_1}, i = 1, .., r$$

- For all other residues, use the method of PFE for the case 1

Source: [2]

# 2.3 Using Laplace Transform To Solve Linear ODE

- Problem: Solve this differential equation if all initial conditions are zero.

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t)$$

- Solution:
  - Using differentiation properties and Laplace transform of unit-step response, we get:

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}$$

$$Y(s) = \frac{32}{s(s^2+12s+32)} = \frac{32}{s(s+4)(s+8)}$$

$$Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{s+4} + \frac{K_3}{s+8}$$

$$K_1 = \frac{32}{(s+4)(s+8)}\Big|_{s\to 0} = 1$$

$$K_2 = \frac{32}{s(s+8)}\Big|_{s\to -4} = -2$$

$$K_3 = \frac{32}{s(s+4)}\Big|_{s\to -8} = 1$$

$$Y(s) = \frac{1}{s} - \frac{2}{s+4} + \frac{1}{s+8}$$

$$Y(s) = (1 - 2e^{-4t} + e^{-8t})u(t)$$

Source: [2]

# 2.3 Using Laplace Transform To Solve Linear ODE

## In general:

1. Having a **linear** ODE, take a Laplace transform of the ODE using Laplace transform rules and initial conditions
2. Algebraically solve the resulting equation
3. Take the inverse Laplace transform using PFE

Source: [2]

# 3.1 Transfer Functions

- It is now possible for us to give a definition of a function that algebraically relates output of the system to its input
- Take a general n-th order linear time-invariant ODE:

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + ... + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + ... + b_0 r(t)$$

- Here, *a, b* are constants and *c(t)* and *r(t)* are output and input, respectively
- Assuming that all initial conditions are =0 and taking Laplace transform of both sides:

$$(a_n s^n + a_{n-1} s^{n-1} + ... + a_0) C(s) = (b_m s^m + b_{m-1} s^{m-1} + ... + b_0) R(s)$$

- The ratio

$$\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + ... + b_0)}{a_n s^n + a_{n-1} s^{n-1} + ... + a_0}$$

is called the *transfer function.* It is evaluated with zero initial conditions.The output can be found by:

$$C(s) = G(s)R(s)$$

Source: [2]

# 3.1 Transfer Functions - Example

- Problem: Find the transfer function represented by ODE:

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

- Solution:
  - Taking Laplace transform of both sides and assuming zero initial conditions,

$$sC(s) + 2C(s) = R(s)$$

$$G(s) = \frac{1}{s+2}$$

  - If a system response is needed for a unit-step input:

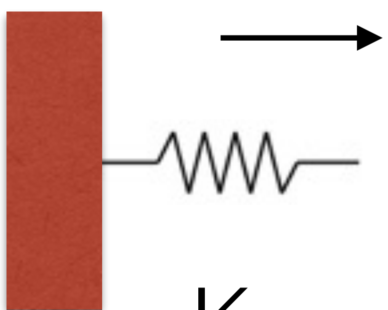
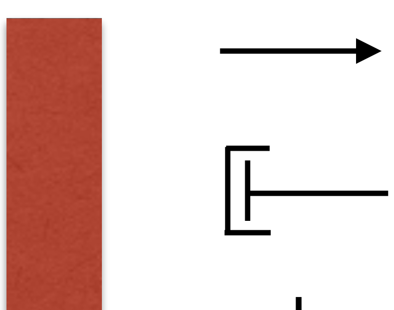
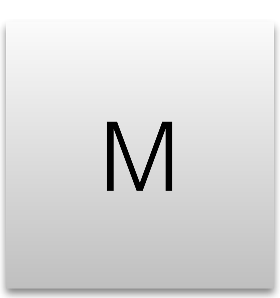
$$C(s) = R(s)G(s) = \frac{1}{s(s+2)}$$

  - Using PFE and taking inverse Laplace transform:

$$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2} \qquad c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

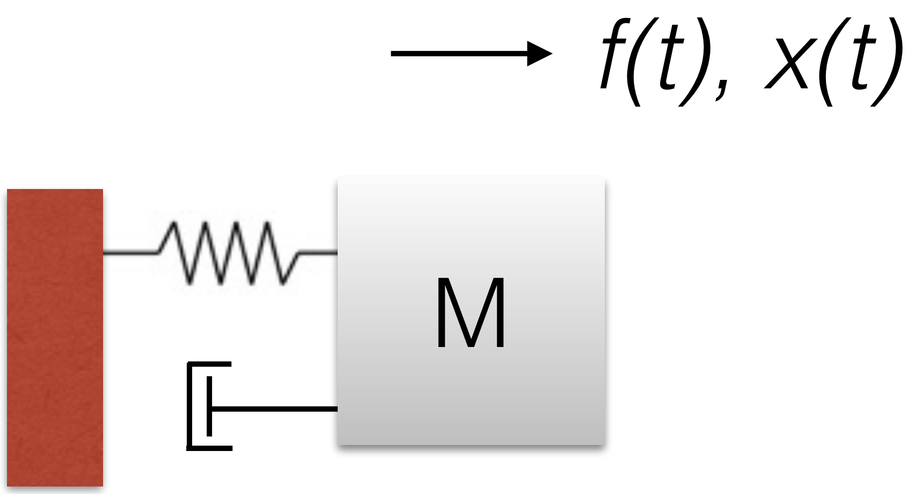
Source: [2]

# 3.2 Mechanical Translational Transfer Functions

| Component | | Force-Displacement | Transfer Function |
|---|---|---|---|
| Spring  K | $f(t), x(t)$ | $f(t) = Kx(t)$ | $K$ |
| Viscous damper  b | $f(t), x(t)$ | $f(t) = b\frac{dx(t)}{dt}$ | $bs$ |
| Mass  | $f(t), x(t)$ | $f(t) = M\frac{d^2x(t)}{dt^2}$ | $Ms^2$ |

Source: [2]

# 3.2 Mechanical Translational Transfer Functions Example




**Problem**: Derive a transfer function for this system

**Solution**: First, write down the differential equation describing the system

$$M\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + Kx(t) = f(t)$$

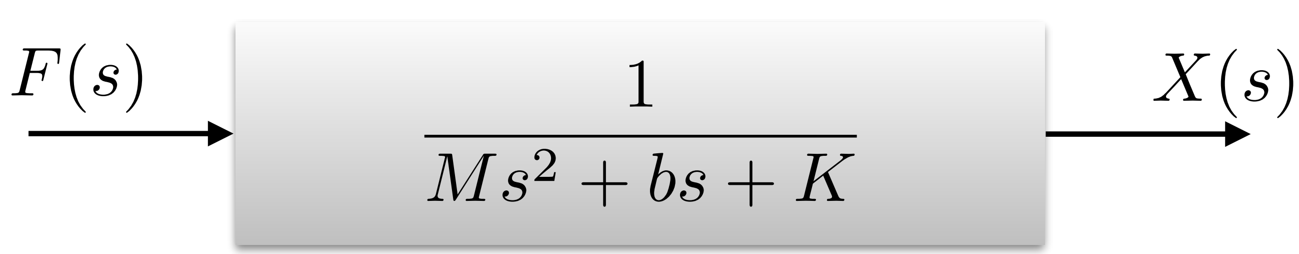
Taking the Laplace Transform:

$$Ms^2X(s) + bsX(s) + KX(s) = F(s)$$

After some algebraic manipulations:

$$(Ms^2 + bs + K)X(s) = F(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + bs + K}$$

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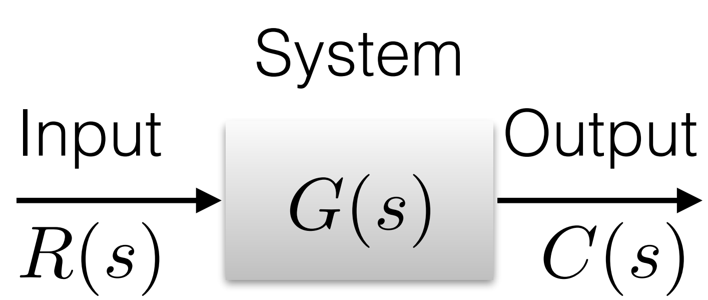
# Summary of the last lecture

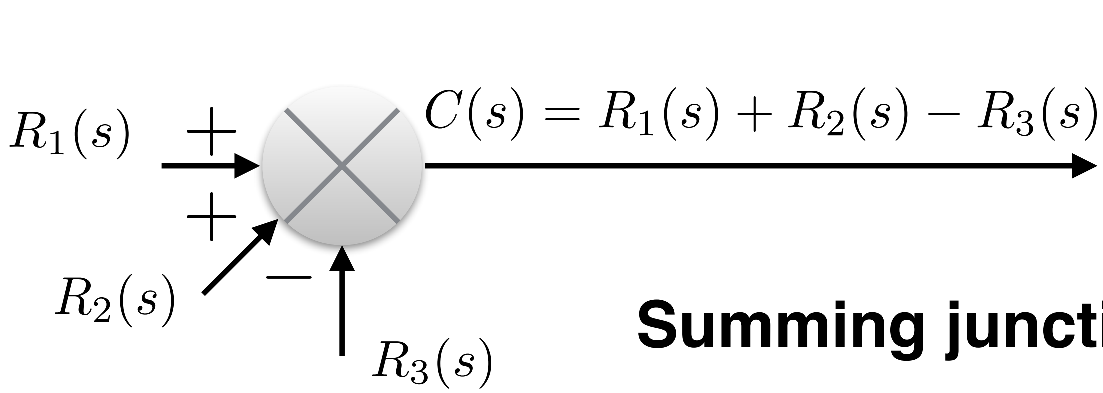
- Previous lecture was about *step 4* of control design: creating a mathematical model
  - How do we do that?
    - Describe a system as a differential equation
    - If the differential equation is ordinary, time-invariant and linear, we use Laplace transform
    - As a result, we get representation of the system in complex domain (transfer function)
  - Why do we do that?
    - Easy solution of differential equations
    - Easy reduction of complex systems (*step 5*): **discussed in this lecture**
    - Easy reasoning about time response, stability (*step 6*): **discussed in this lecture**

# 3.3 Block diagrams

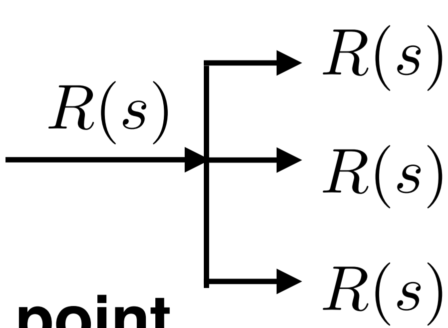
- Complex systems are represented by interconnections of many subsystems
- Our goal: represent response of the *whole system* as a **single** transfer function
- **Block diagrams** are used for frequency-domain analysis and design
- **Signal flow graphs** are used for state-space methods
- Subsystem is represented as a block with: *input, output* and *transfer function*




- For building more complex systems, two more elements are needed:




**Summing junction**




**Pickoff point**

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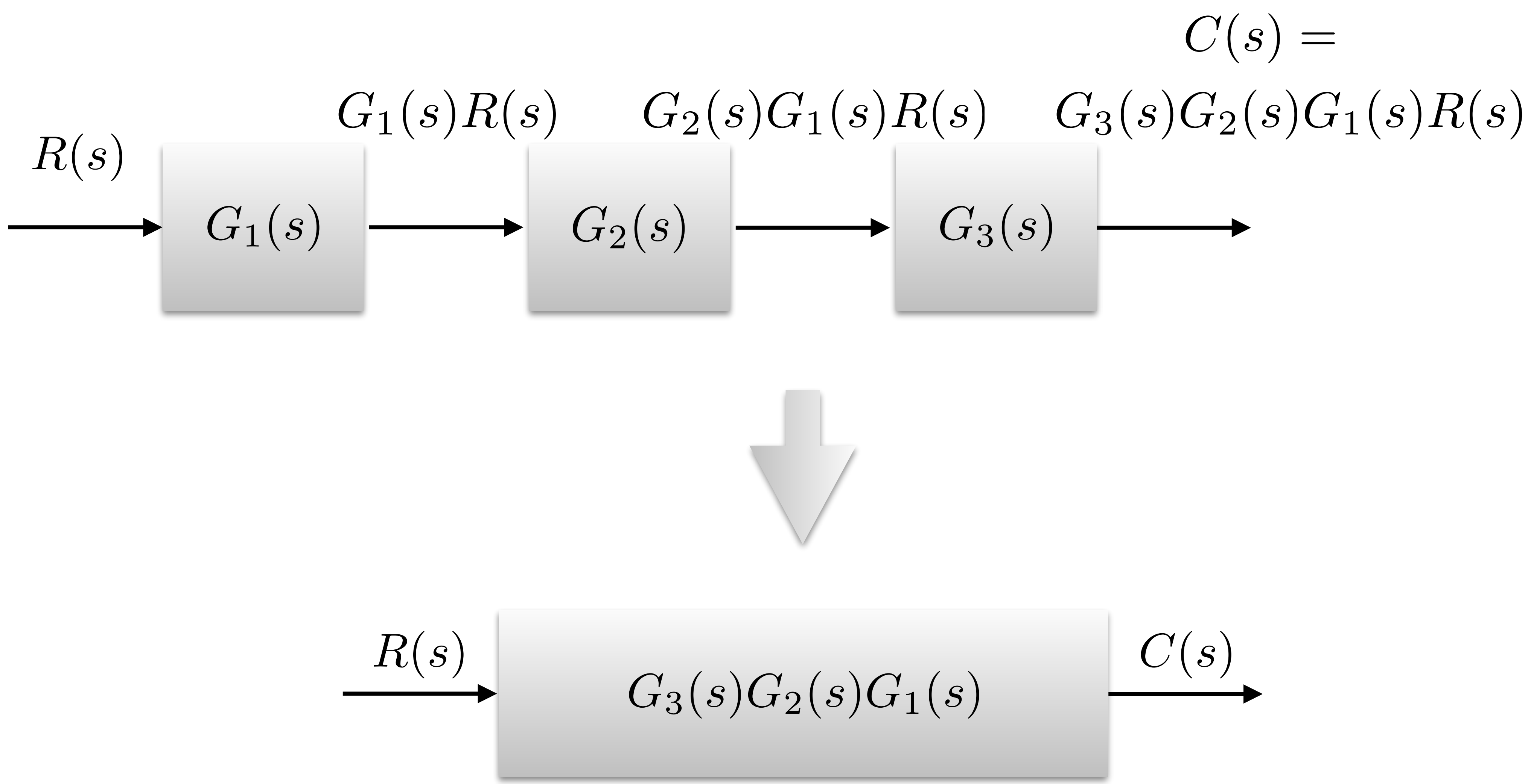
# 3.3 Cascade Form of Transfer Functions

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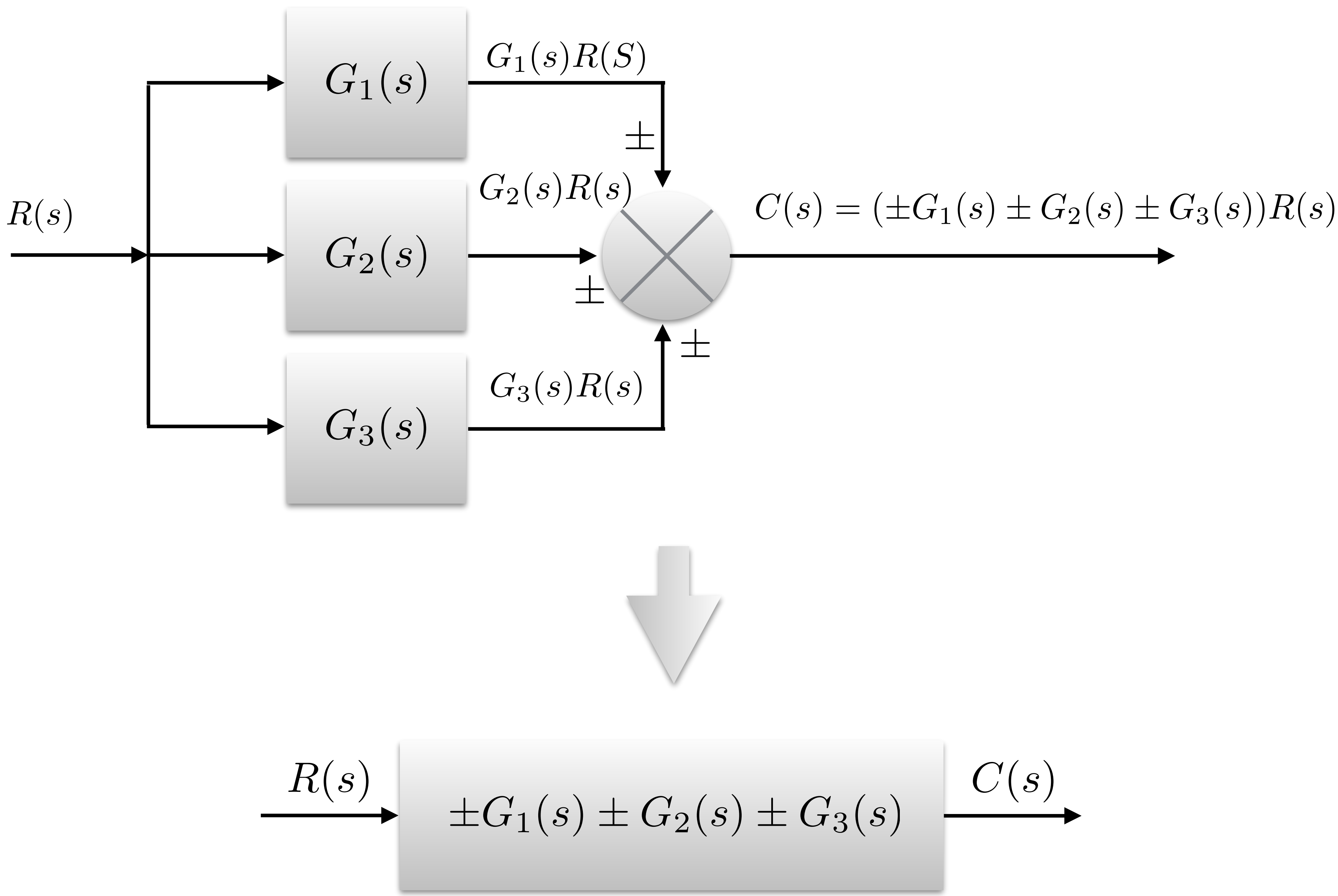
# 3.3 Parallel Form of Transfer Functions

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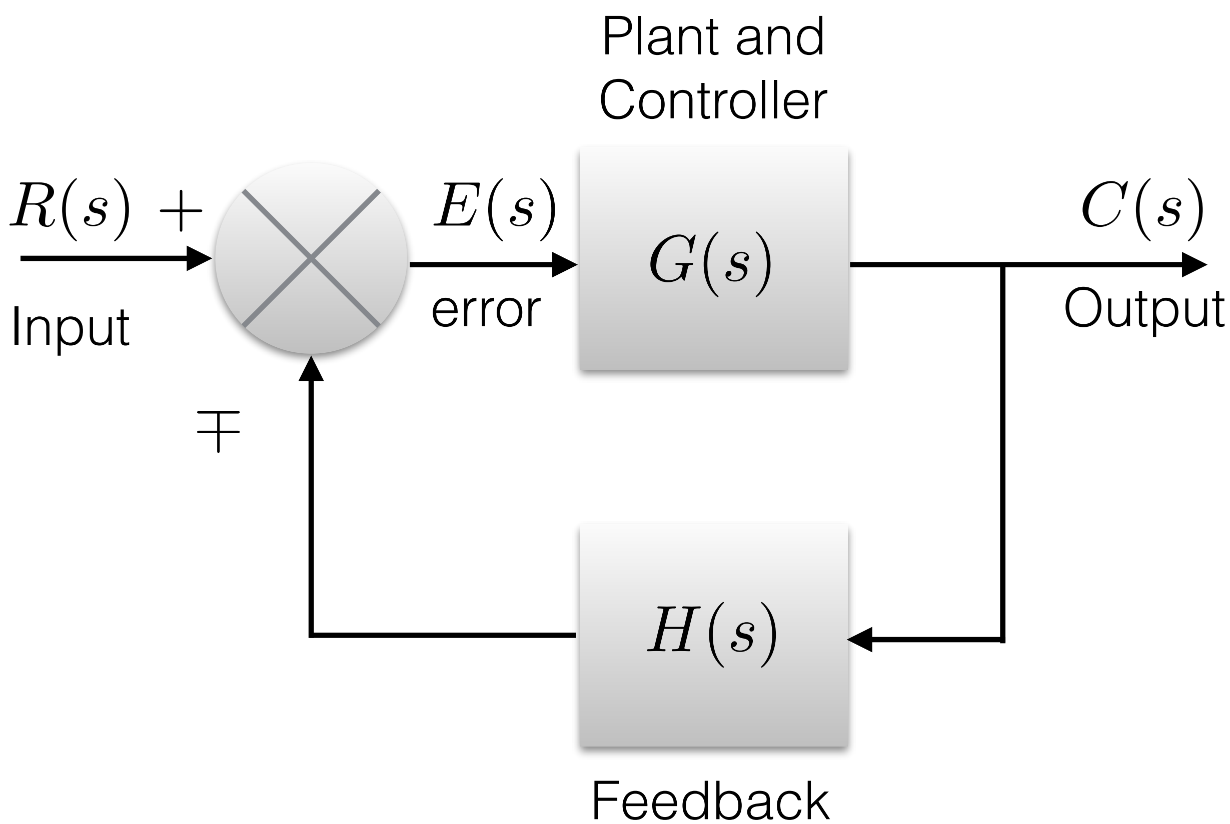
# 3.3 Feedback Form of Transfer Functions




Using

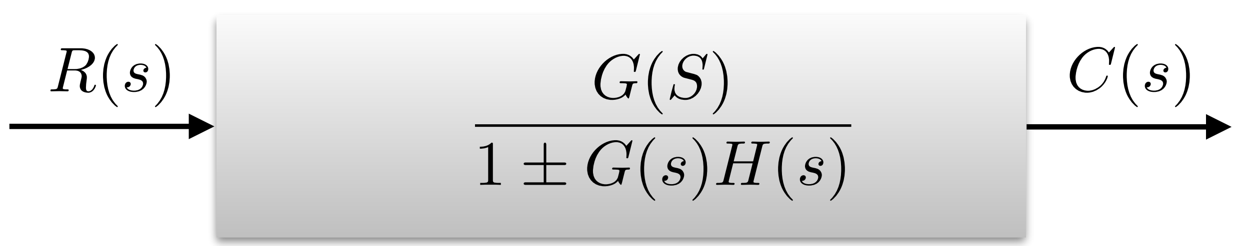
$$E(s) = R(s) \mp C(s)H(s)$$

and

$$C(s) = E(s)G(s) \Rightarrow E(s) = \frac{C(s)}{G(s)}$$

and solving for $\dfrac{C(s)}{R(s)} = G_e(s)$

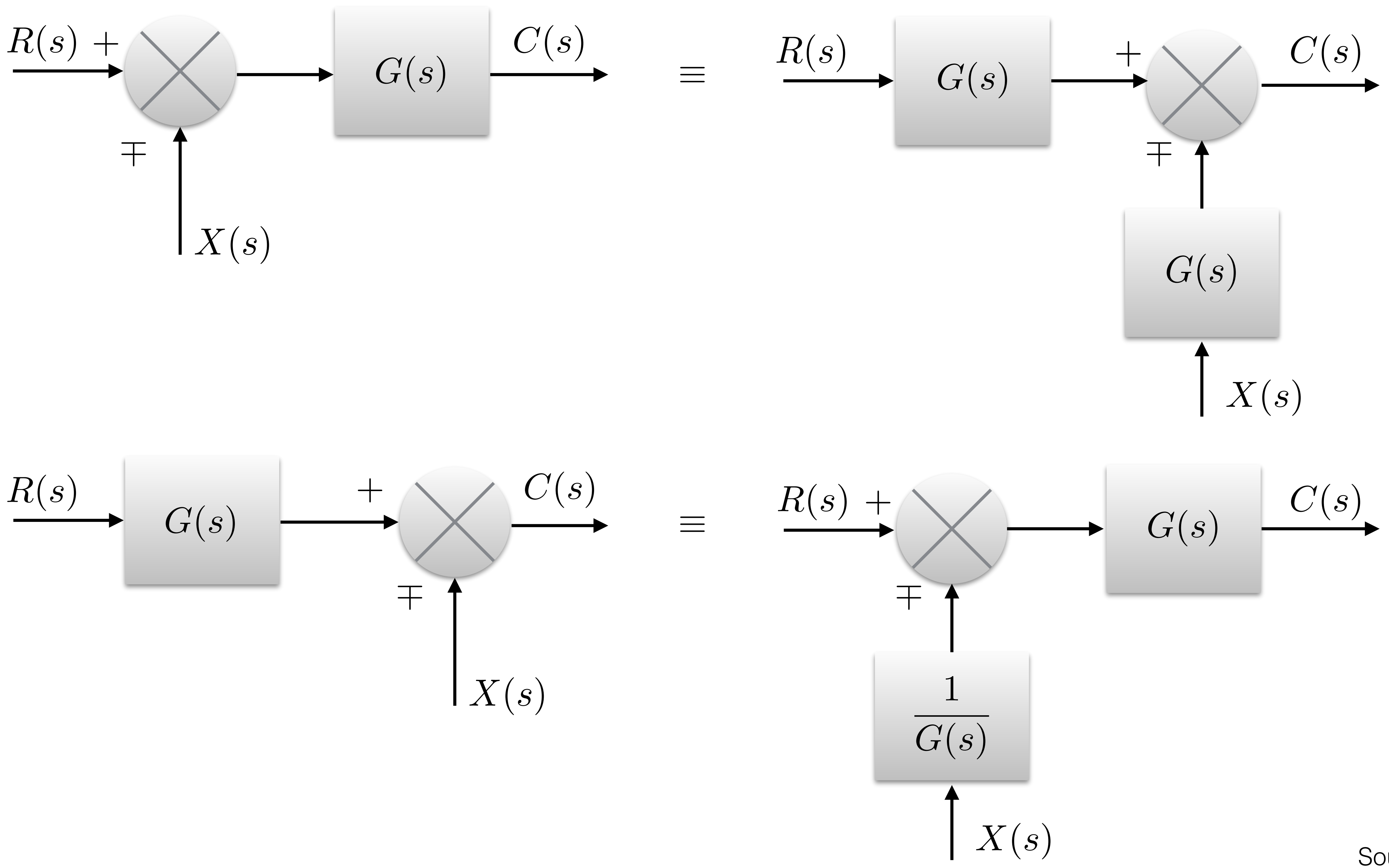
we get: $\dfrac{G(S)}{1 \pm G(s)H(s)}$




$G(s)H(s)$ is called loop gain or open loop transfer function

Source: [2]

# 3.3 Restructuring Topologies (Summing Junction)

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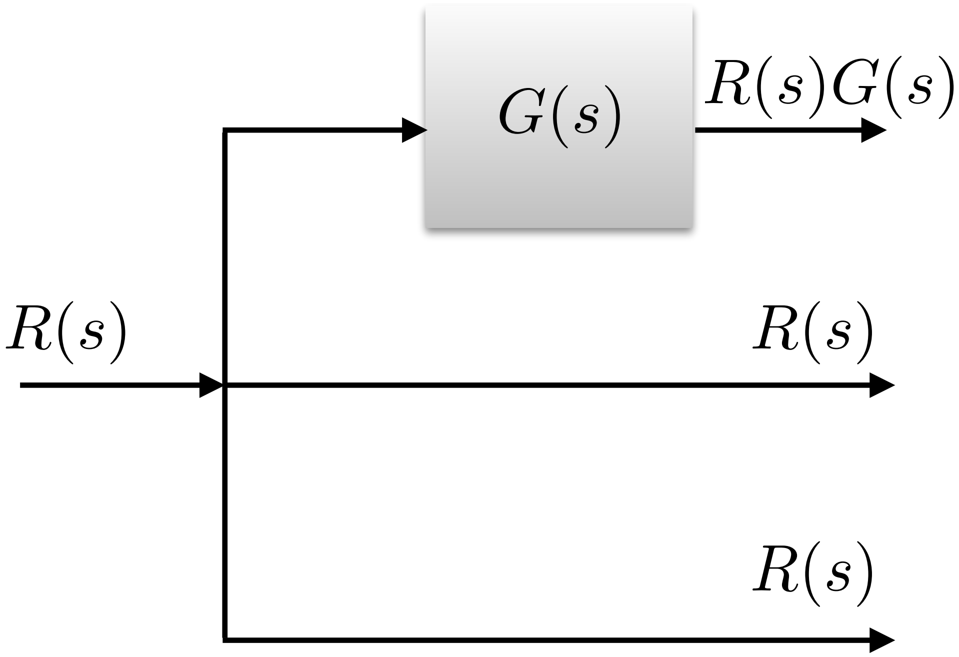
# 3.3 Restructuring Topologies (Summing Junction)

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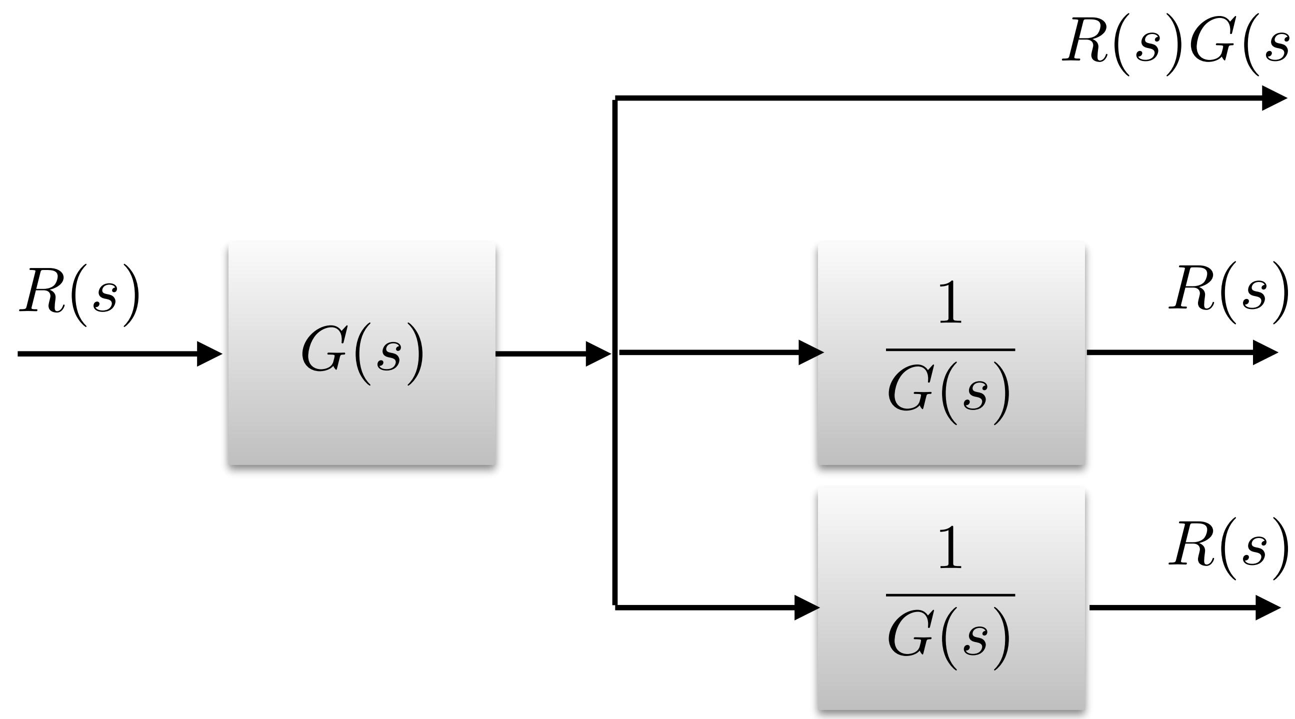


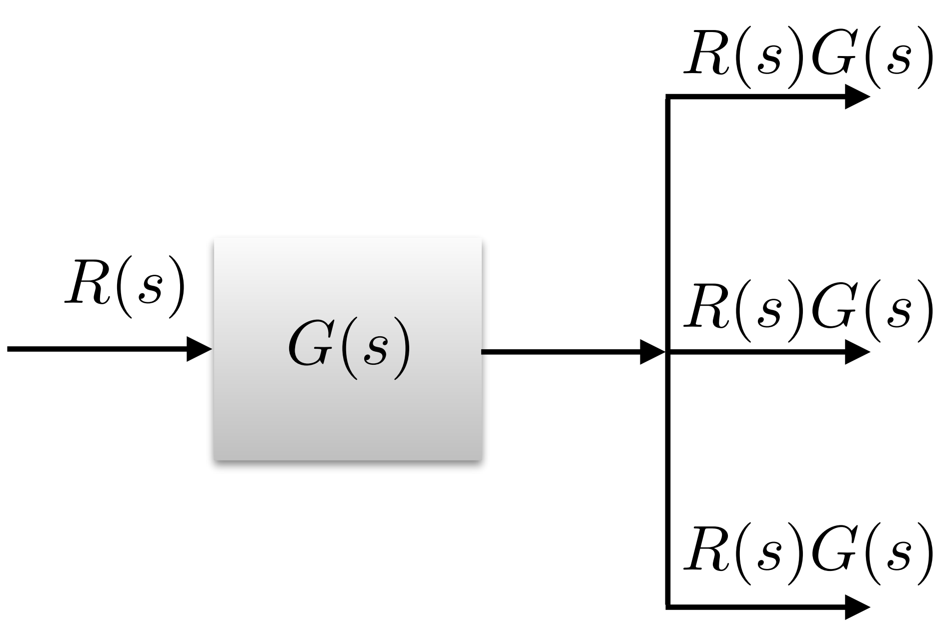


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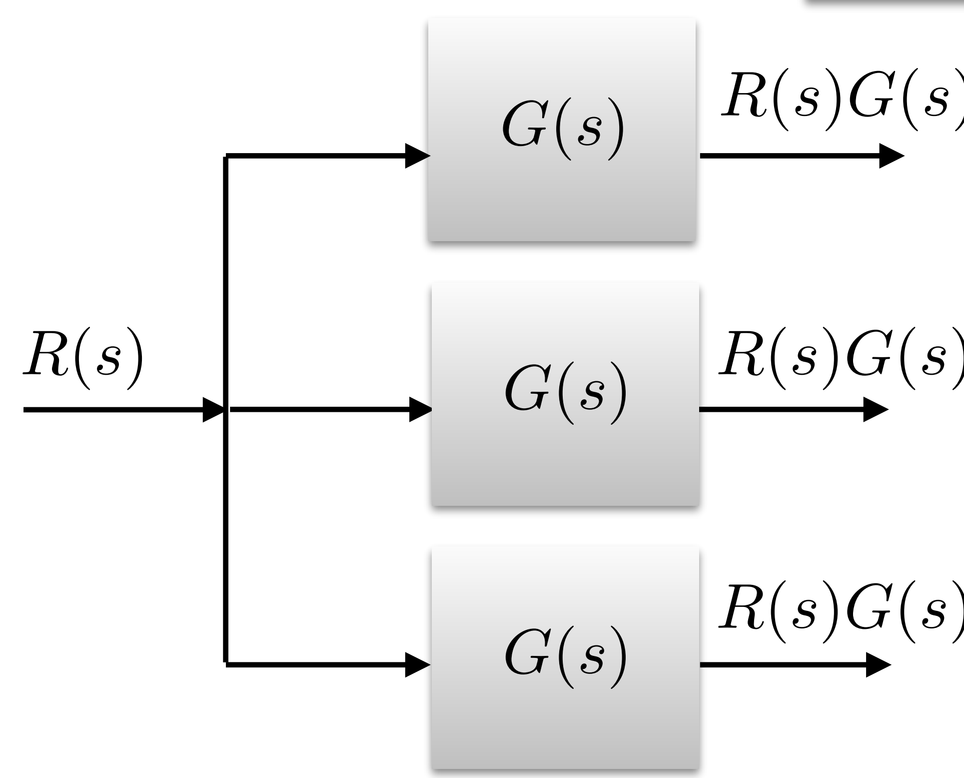


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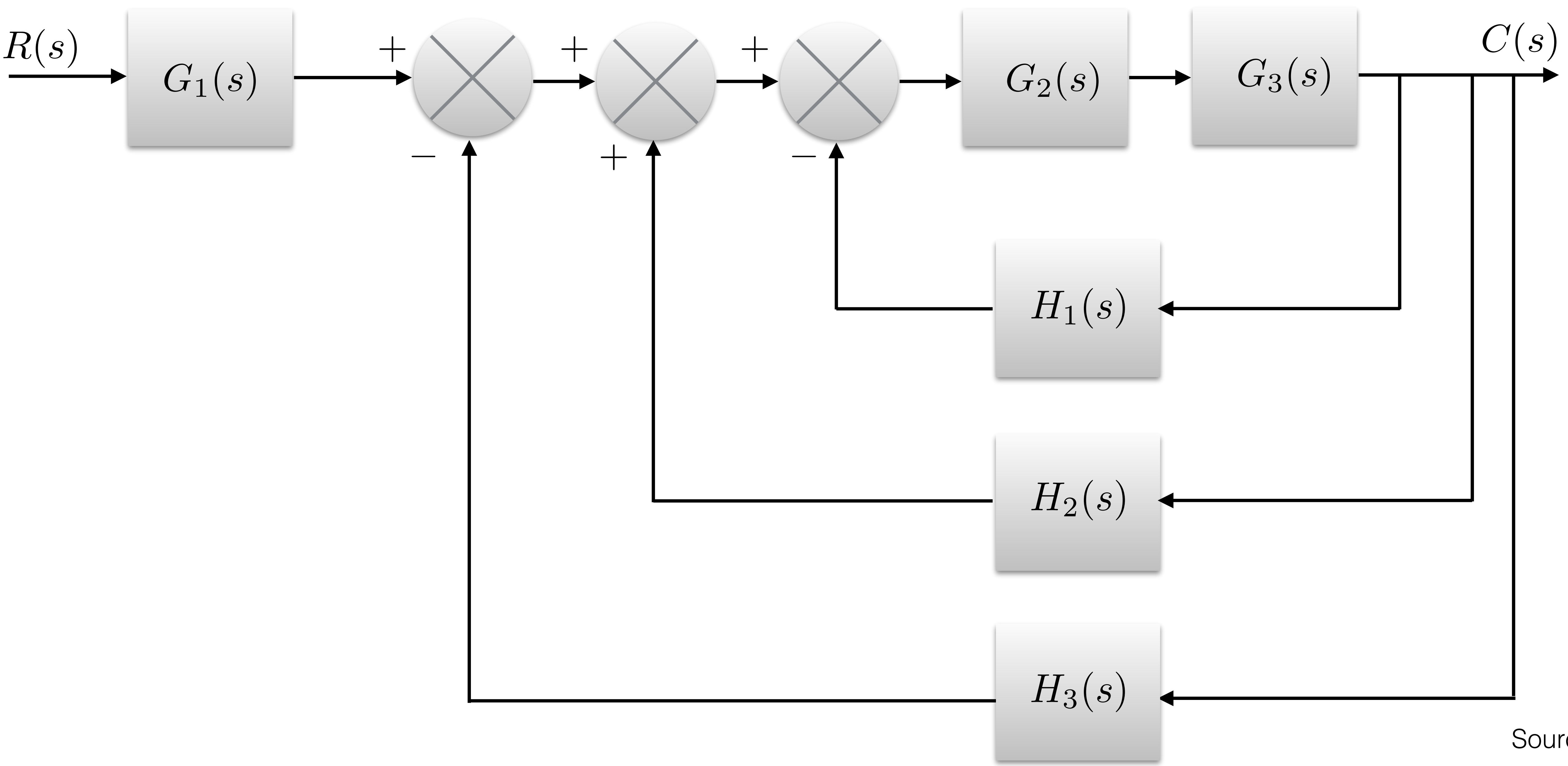
# 3.3 Example

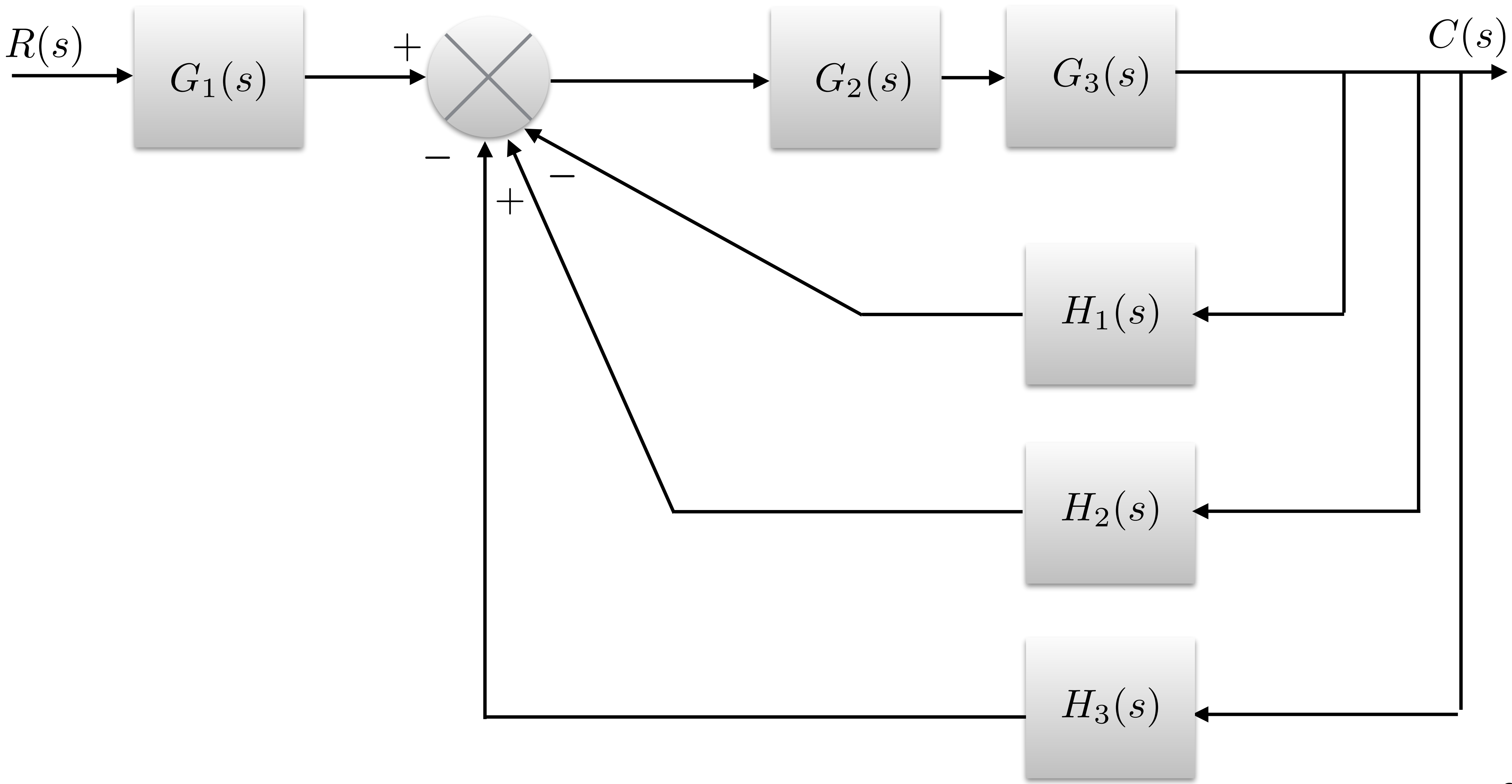
- Problem: reduce the block diagram and find the transfer function of the system:




Source: [2]

# 3.3 Example

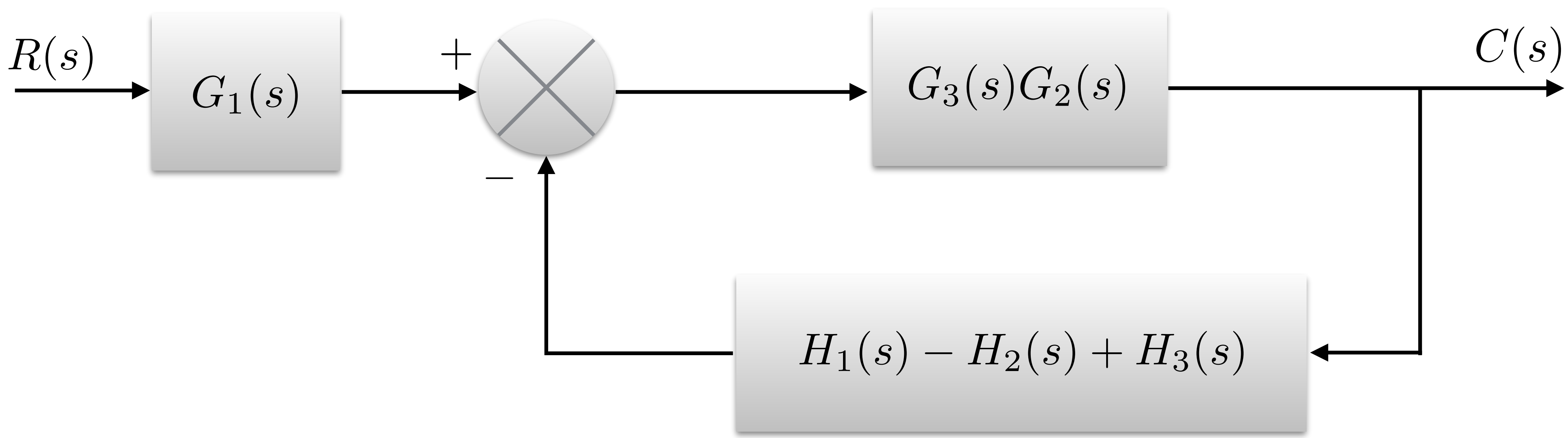
- Step 1: collapse 3 summing junctions into a single one

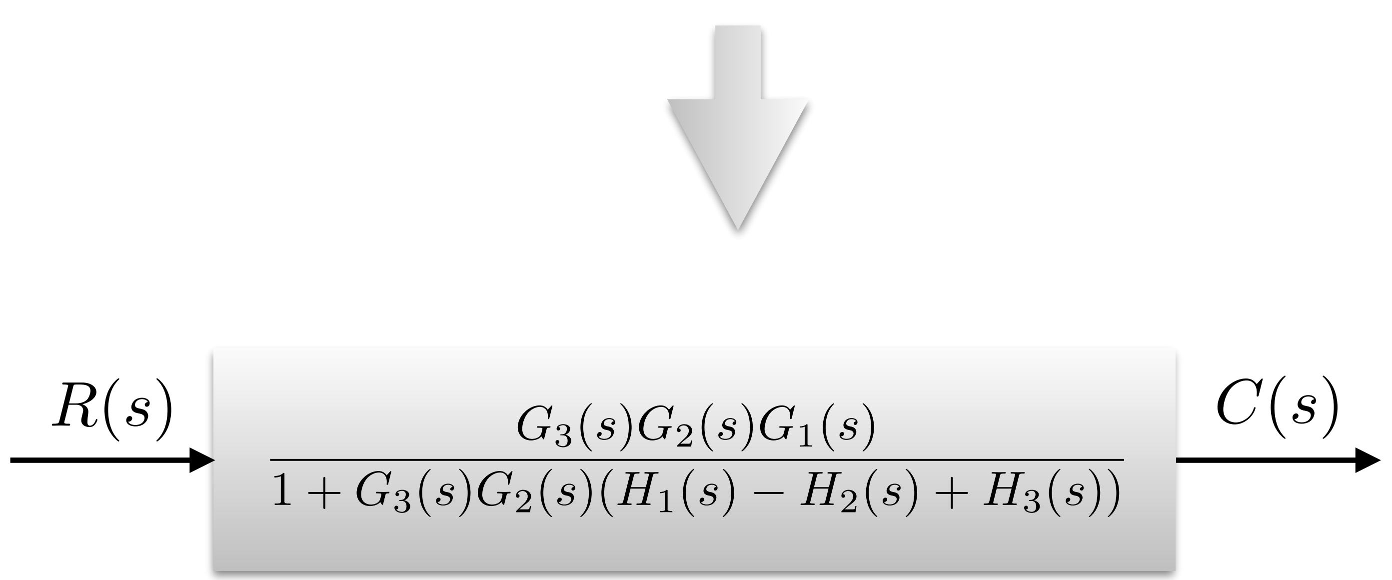



Source: [2]

# 3.3 Example

- Steps 2&3: use cascade, parallel and feedback forms to get the final result




Source: [2]

# 4. Time Response

- We can now turn to analysis of the system
- Last stage of the control system design process
- Here only
  - Time Response
  - Stability Analysis
- The output of the system is the sum of two responses
  - Forced Response
  - Natural Response
- **Problem:** We want to be able to *rapidly evaluate output response* without solving DE or taking inverse Laplace transform
- **Solution:** Using *poles* and *zeros* of the transfer function

Source: [2]

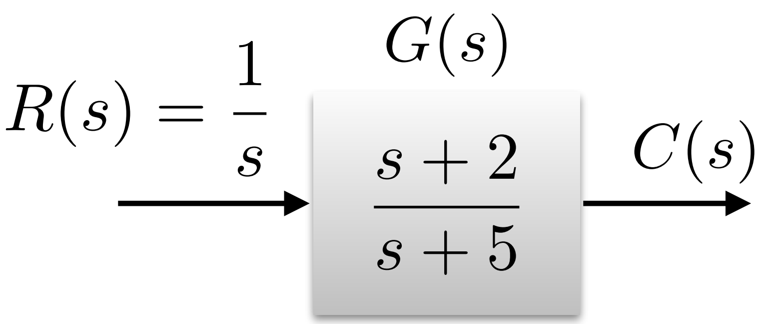
# 4.1 Poles and Zeros

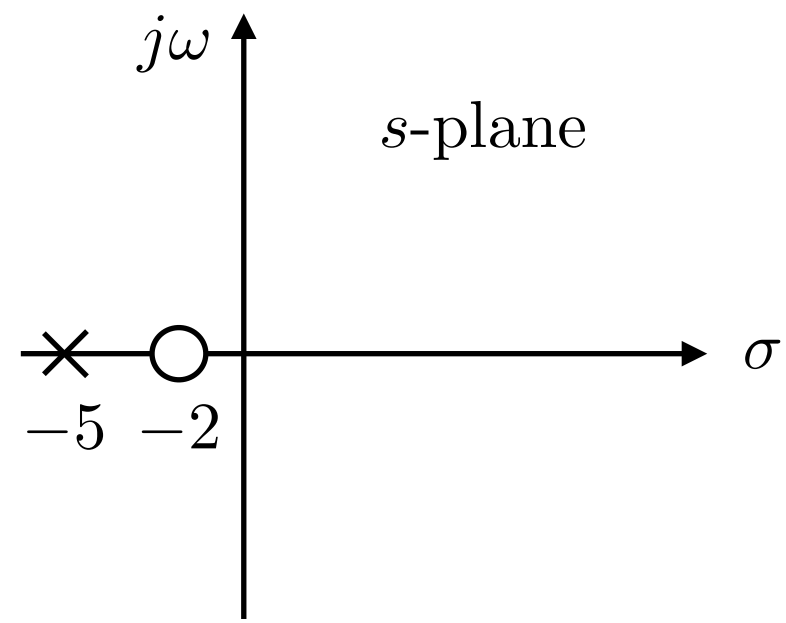
**Definition**: The *poles* of a transfer function $\frac{N(s)}{D(s)}$ are
(1) the values of the complex Laplace transform variable *s* that make the transfer function infinite
(2) roots of denominator that are also roots of the numerator

**Definition**: The *zeros* of a transfer function $\frac{N(s)}{D(s)}$ are
(1) the values of the complex Laplace transform variable *s* that cause the transfer function to become zero
(2) roots of numerator that are also roots of the denominator

Source: [2]

# 4.1 Poles and Zeros Example




$$C(s) = G(s)R(s) = \frac{s+2}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}$$

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

$$A = \frac{s+2}{s+5}\bigg|_{s\to 0} = \frac{2}{5}$$

Forced response

Natural response

$$B = \frac{s+2}{s}\bigg|_{s\to -5} = \frac{3}{5}$$

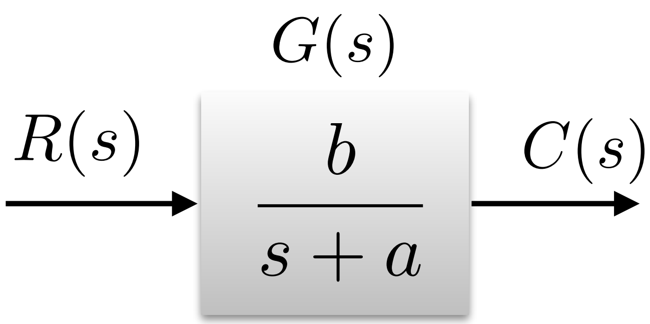
Pole of the input function generates forced response

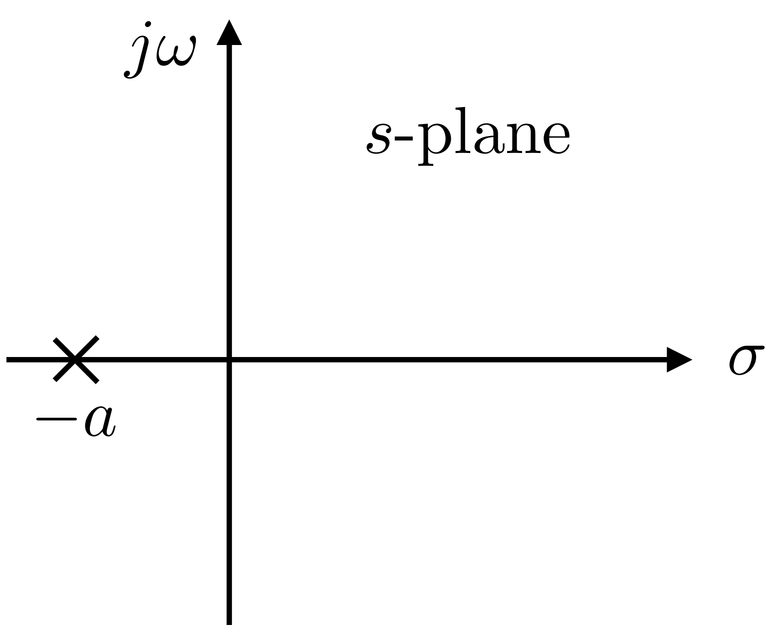
Pole of the transfer function generates natural response

Zeros and poles generate amplitudes for both natural and forced responses.

Source: [2]

# 4.2 First-Order Systems




$$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$

$$C(s) = \frac{K_1}{s} + \frac{K_2}{s+a}$$

$$K_1 = \frac{b}{a} \qquad K_2 = -\frac{b}{a}$$

$$\alpha = \frac{b}{a} \quad \Rightarrow \quad C(S) = \alpha(\frac{1}{s} - \frac{1}{s+a})$$

$$c(t) = \alpha(1 - e^{-at})$$

For simplicity: $b = a \Rightarrow \alpha = 1$

$$c(t) = 1 - e^{-at}|_{t=1/a} = 1 - e^{-1} = 0.63$$

Source: [2]

# 4.2 First-Order Systems Characteristics

**Definition**: $\frac{1}{a}$ is called *time constant,* time needed for the step response to reach 63% of its final value

**Definition**: *Rise time* is defined as the time for the response to go from 0.1 to 0.9 of its final value:

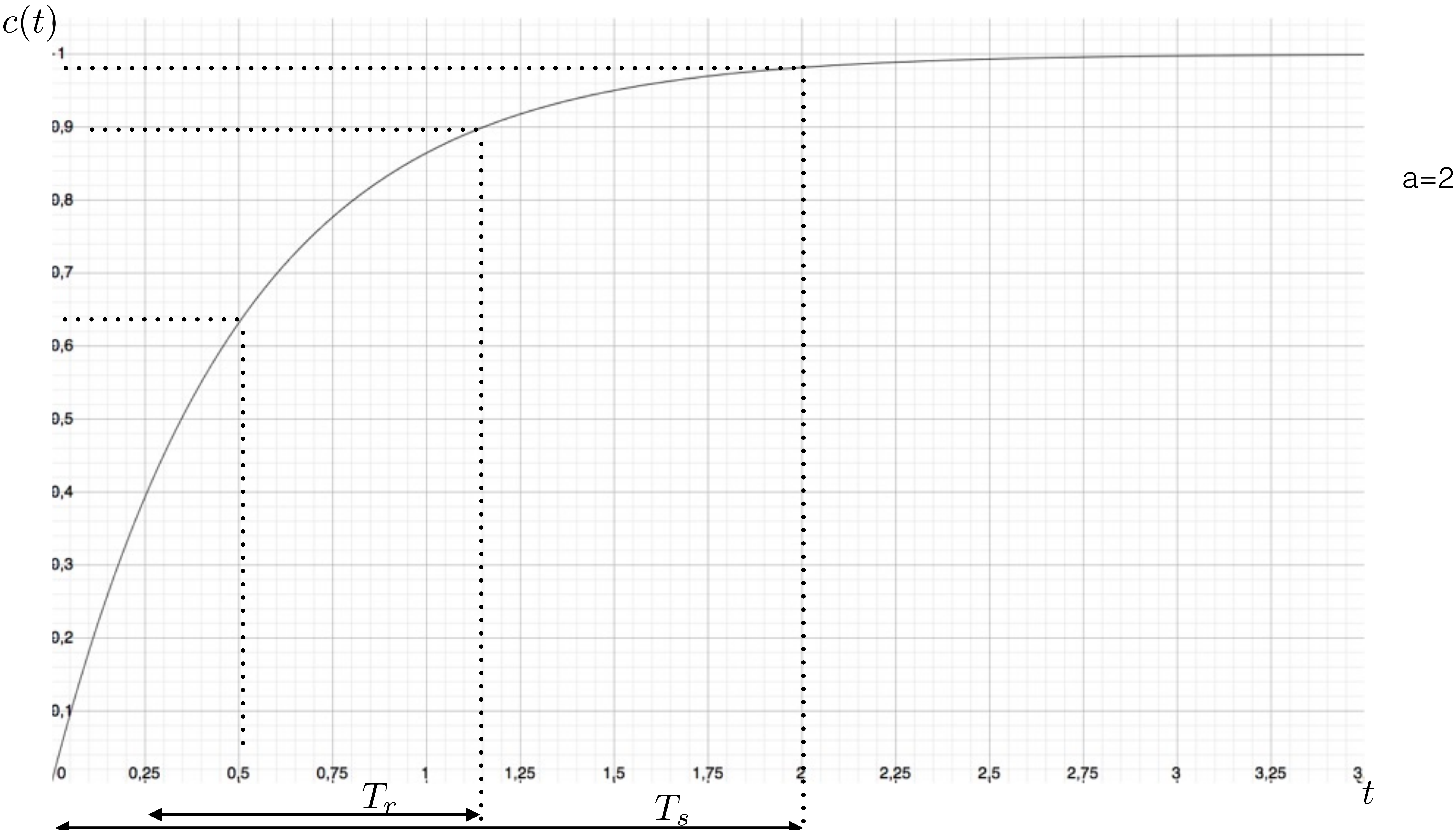
$$T_r = \frac{2.2}{a}$$

**Definition**: *Settling time* is defined as the time for the response to reach and stay within 2% of its final value:

$$T_s = \frac{4}{a}$$

Source: [2]

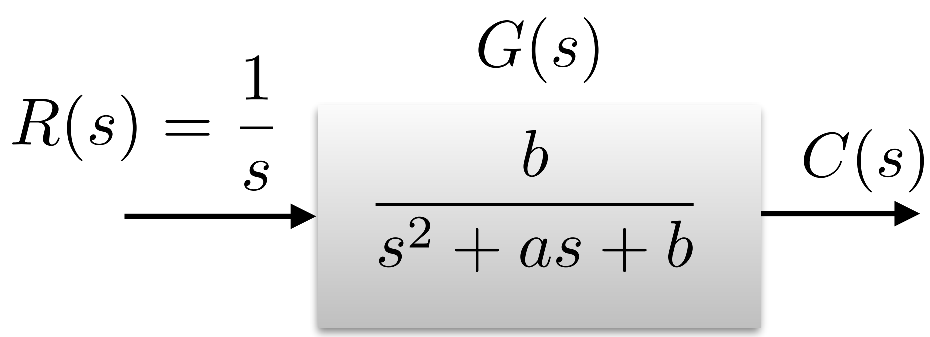
# 4.2 First-Order Systems Characteristics




Source: [2]

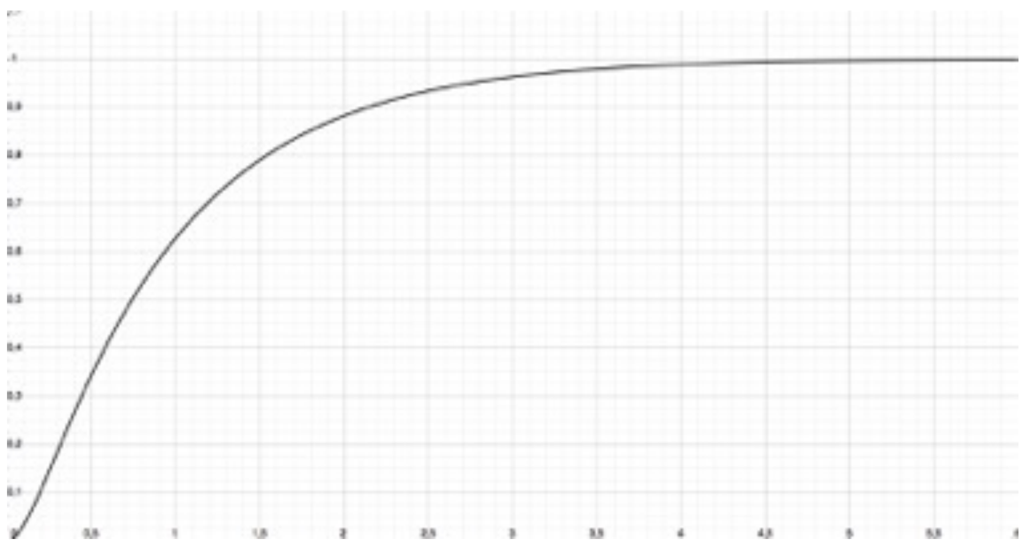
# 4.3 Second-Order Systems

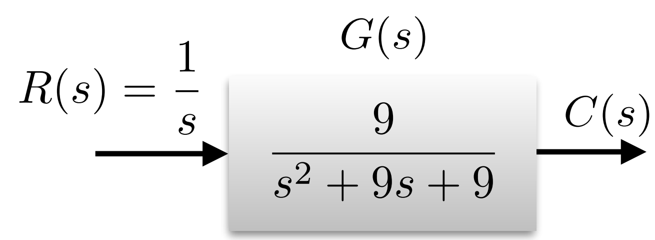
- 1st-order systems are easy to analyse and handle:
  - single parameter
  - the *a*-parameter is responsible for the speed of response
- 2nd-order systems parameters change the form of the response as well
- General form:

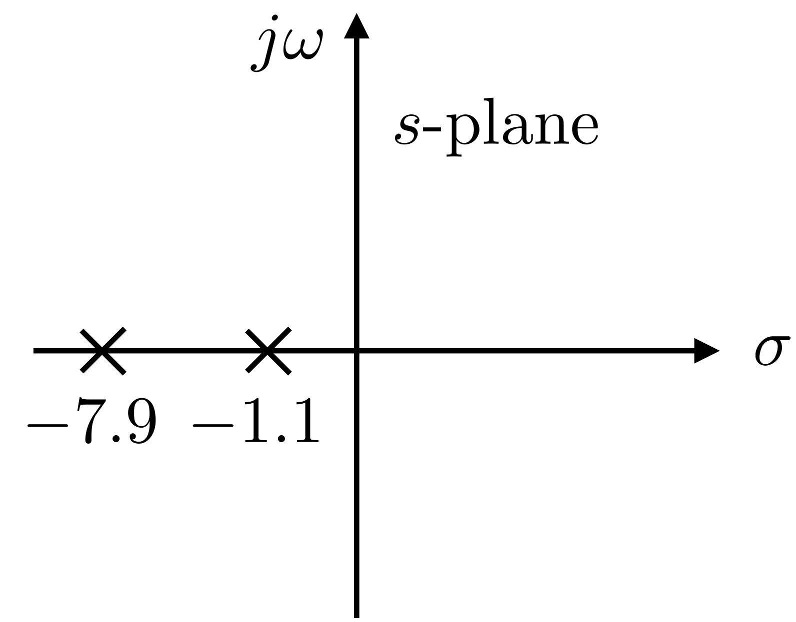



Source: [2]

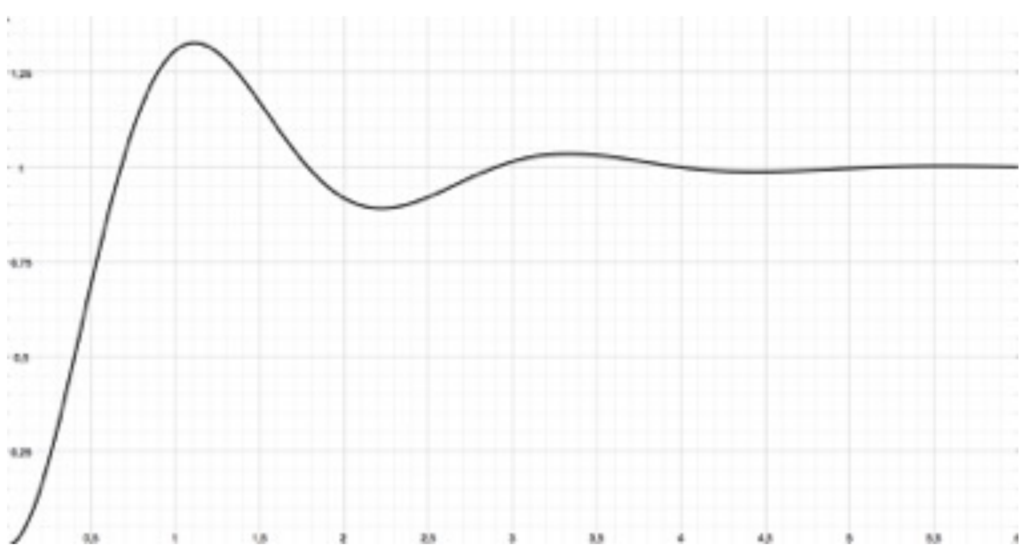
# 4.3 Second-Order Systems Types (1)

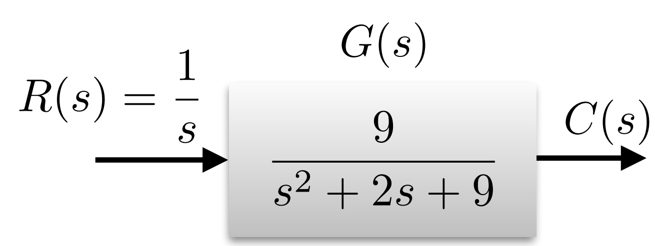


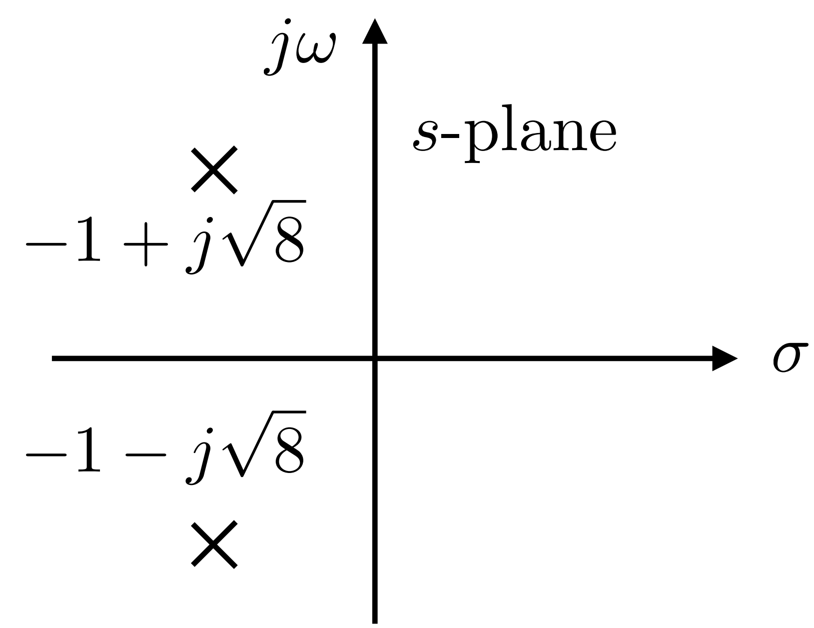



**Overdamped**



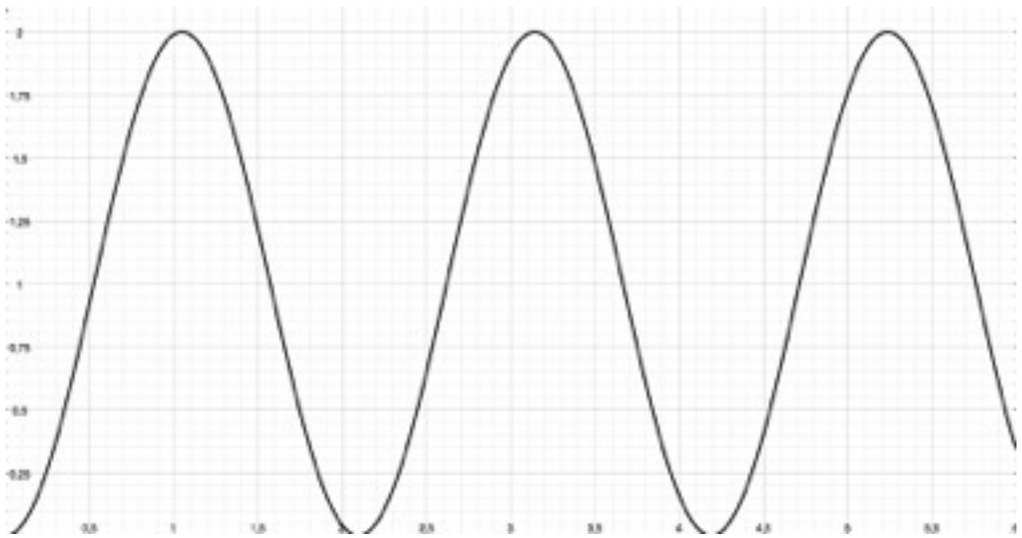


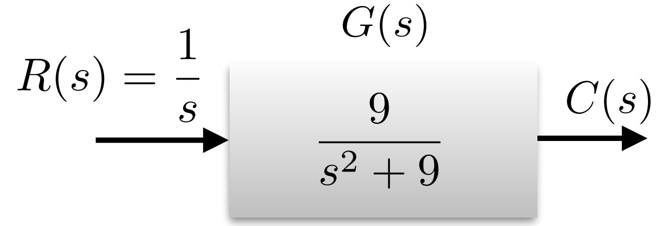




**Underdamped**

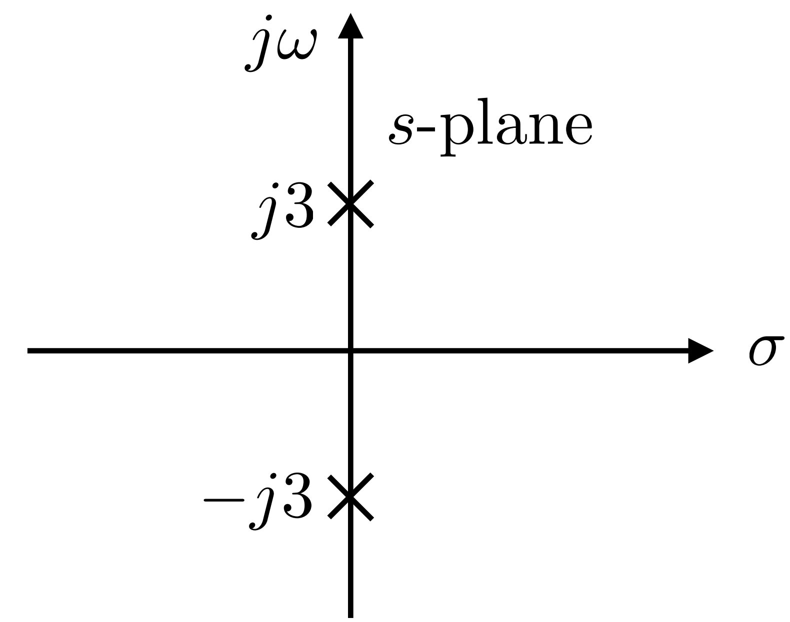
Source: [2]

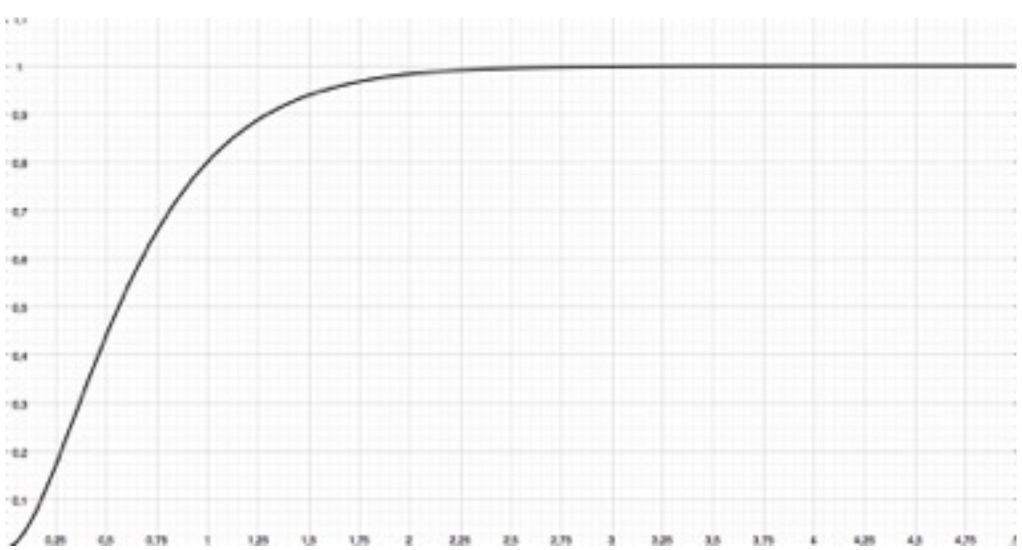
# 4.3 Second-Order Systems Types (2)

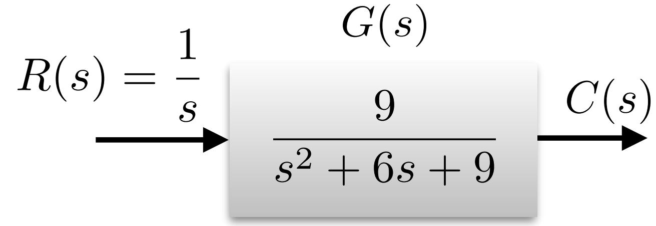




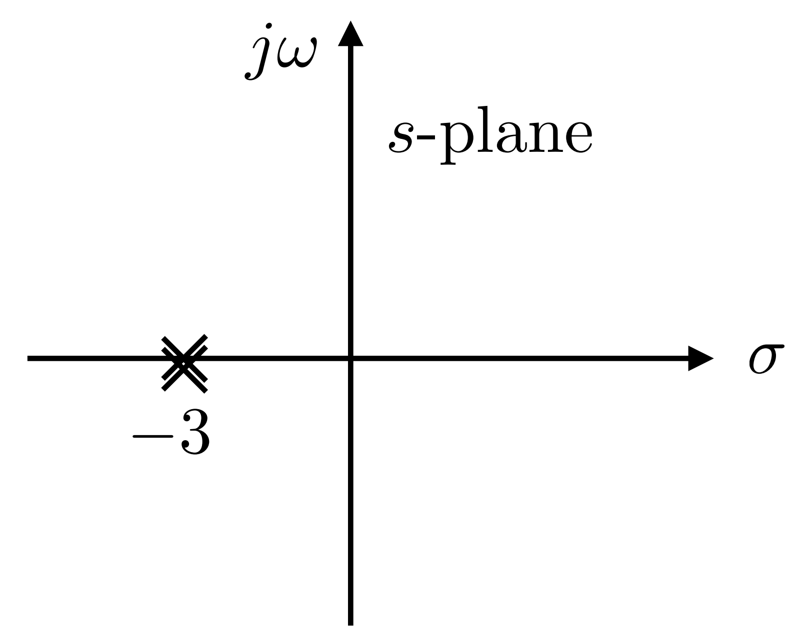

**Undamped**






**Critically Damped**




Source: [2]

# 4.3 Second-Order Systems Types (3)

| Type of Response | Poles | Response |
|---|---|---|
| Overdamped | Two real at $-\delta_1, \delta_2$ | $c(t) = K_1 e^{-\delta_1 t} + K_2 e^{-\delta_2 t}$ |
| Underdamped | Two complex at $-\delta_d \pm j\omega_d$ | $c(t) = Ae^{-\delta_d t} \cos(\omega_d t - \phi)$ |
| Undamped | Two imaginary at $\pm j\omega_1$ | $c(t) = A\cos(\omega_1 t - \phi)$ |
| Critically Damped | Two real at $-\delta_1$ | $c(t) = K_1 e^{-\delta_1 t} + K_2 t e^{-\delta_1 t}$ |

Source: [2]

# 4.3 General Second-Order System

- It is possible to describe a 1st-order system behaviour with time constant *a*
- We want to do the same for 2nd-order systems:
  - This time two parameters are needed
  - *Natural frequency* $\omega_n$ = frequency without damping
  - *Damping ratio* $\xi$ which quantitatively describes damped oscillation regardless of time scale

$$G(s) = \frac{b}{s^2 + as + b}$$

$$G(s) = \frac{b}{s^2 + b}$$

$$\boxed{\omega_n = \sqrt{b}}$$

$$\boxed{\xi = \frac{\text{Exponential decay frequency}}{\text{Natural frequency}} = \frac{a/2}{\omega_n}}$$

Rewriting in general form:

$$\boxed{G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega s + \omega_n^2}}$$

Source: [2]

# 4.3 General Second-Order System

| $\xi$ | **Response Type** |
|---|---|
| $\xi = 0$ | Undamped |
| $0 < \xi < 1$ | Underdamped |
| $\xi = 1$ | Critically Damped |
| $\xi > 1$ | Overdamped |

Source: [2]

# 4.3 Second-Order System Characteristics

**Definition**: *Peak time* is the time required to for the response waveform to reach the first, or maximum, peak.

**Definition**: *Rise time* is defined as the time for the response to go from 0.1 to 0.9 of its final value.

**Definition**: *Settling time* is defined as the time for the response to reach and stay within 2% of its final value.

**Definition**: *Percent overshoot* is the amount the response waveform overshoots the steady-state, expressed as a percentage of steady-state value.

Source: [2]

# 5. System Stability Analysis

- Stability is the one of the most important properties of the system
- If a system is unstable, transient response and steady-state errors are meaningless

**Definition**: Control system is *stable* if its natural response a) eventually approaches zero or b) oscillates. If the natural response of the system grows without bound and becomes much greater than forced response, the system is considered *unstable*.

**Definition(BIBO)**: A system is *stable* if **every** bounded input produces a bounded output.

Alternatively, a system is *unstable* if **any** bounded input produces an unbounded output.

Source: [2]

# 5. System Stability Analysis

- How to check if the system is stable? Many possibilities
  - Informally, by inspecting closely the transfer function
  - Formally, e.g. Routh-Hurwitz Analysis

- Informal analysis:
  - Recall:

Observation 1:

- Poles in the left-half complex plane produce either exponential decay or damped sinusoidal response
- Thus, *stable systems have closed-loop transfer functions with poles exclusively in the lhp.*

Source: [2]

# 5. System Stability Analysis

## Observation 2:

- Poles in the right half-plane yield
  - pure exponentially increasing responses
  - exponentially increasing sinusoidal responses
- Thus, *unstable systems have closed-loop transfer functions with at least one pole in the rhp and/or poles of multiplicity greater than 1 on the imaginary axis*

## Observation 3:

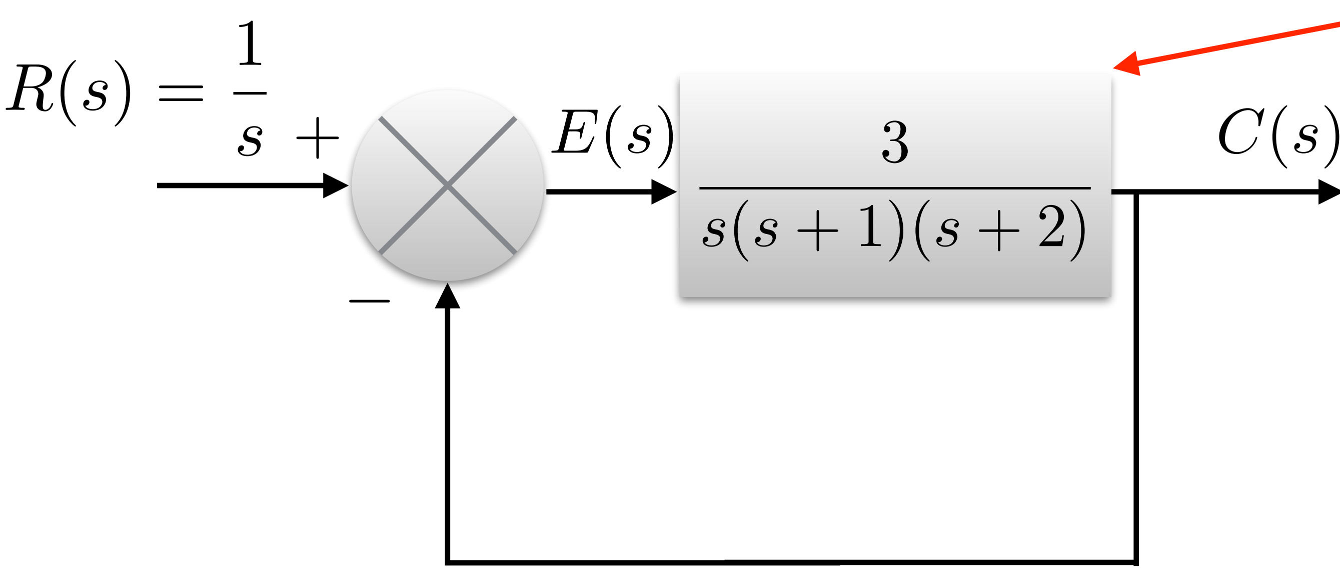
- Poles which are purely imaginary with multiplicity exactly 1 produce pure oscillations (sinusoidal) which neither increasing nor decreasing amplitude
- Thus, *marginally stable systems have closed-loop transfer functions with purely imaginary poles of multiplicity 1 and poles in the lhp*

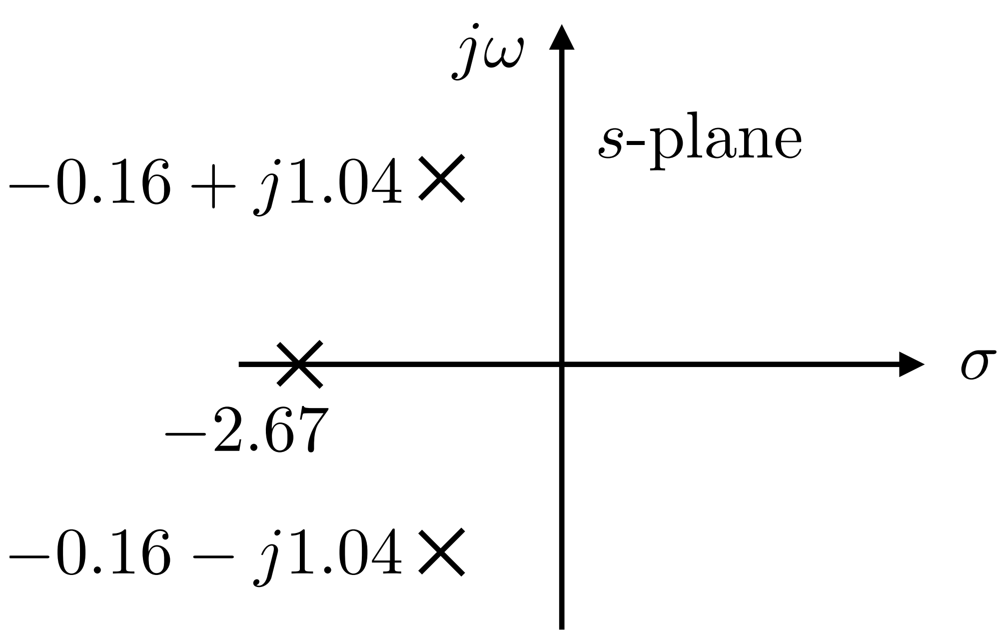
Source: [2]

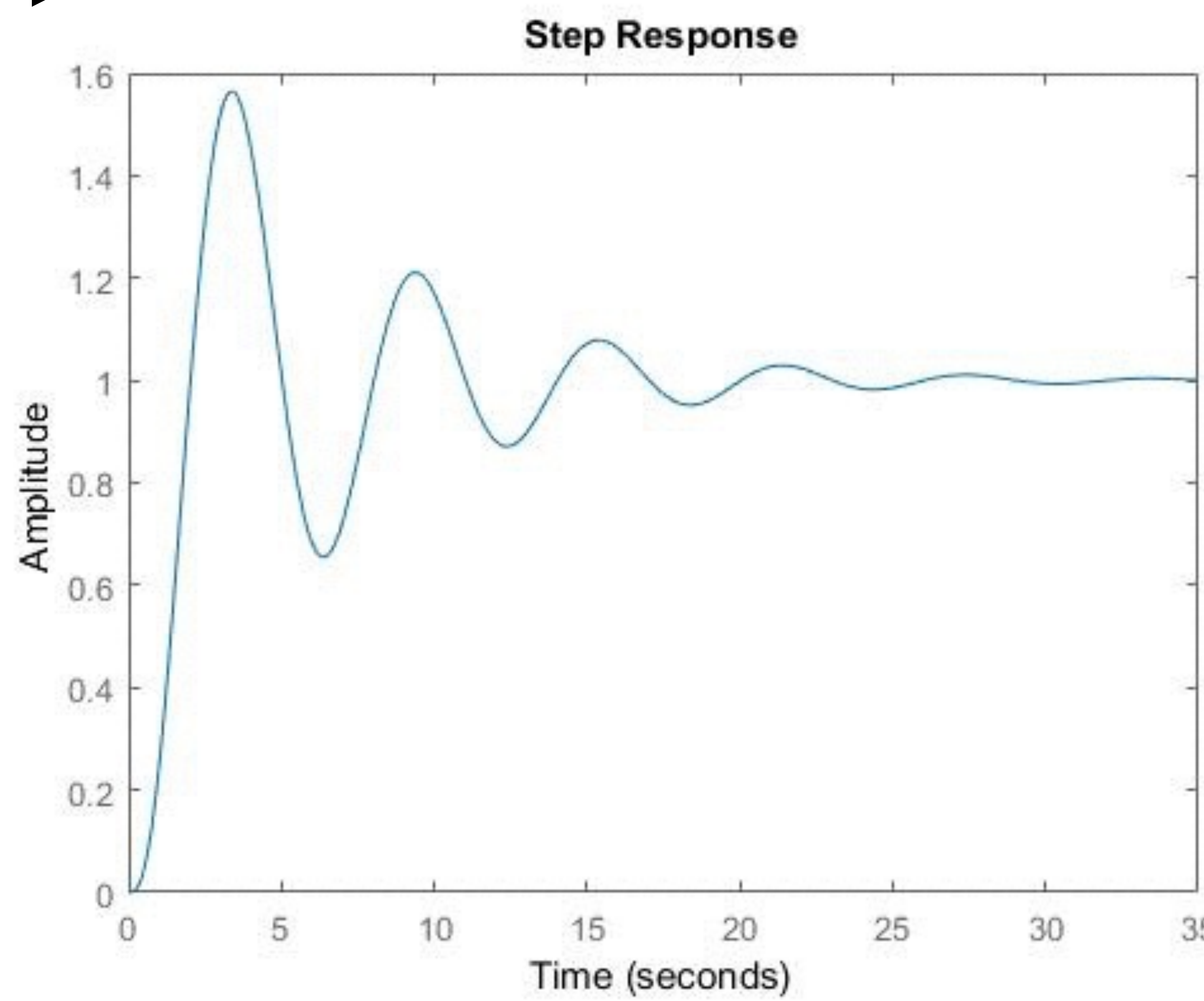
# 5. Informal System Stability Analysis: Example 1

Careful: THIS is NOT the transfer function of the system!

Apply the feedback rule to find TF and solve!



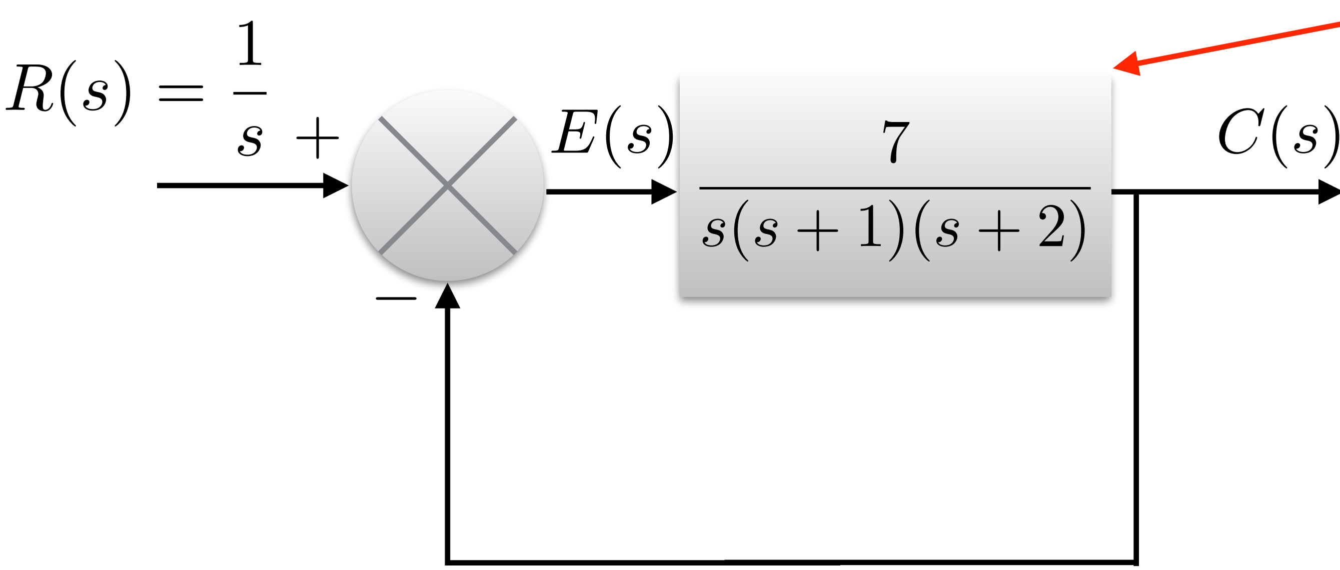



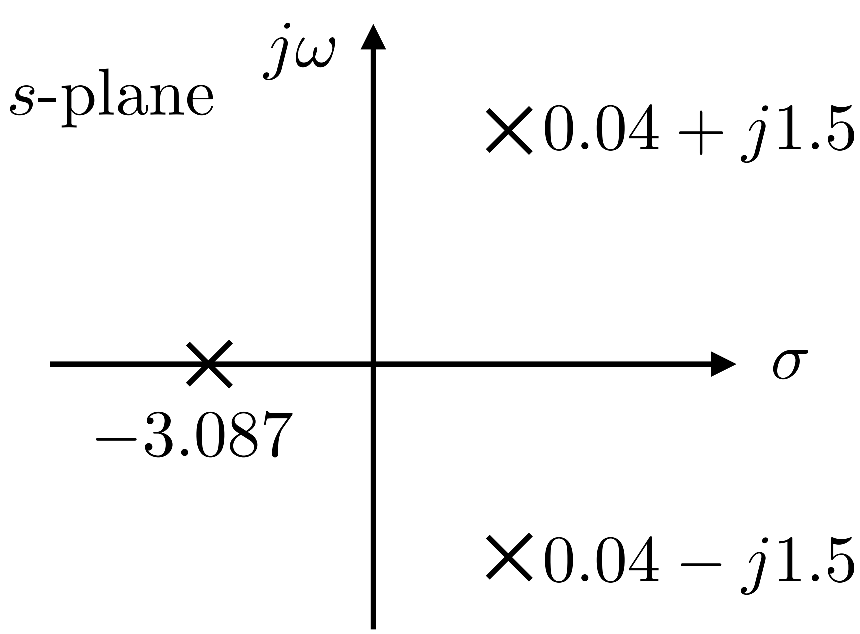

Source: [2]

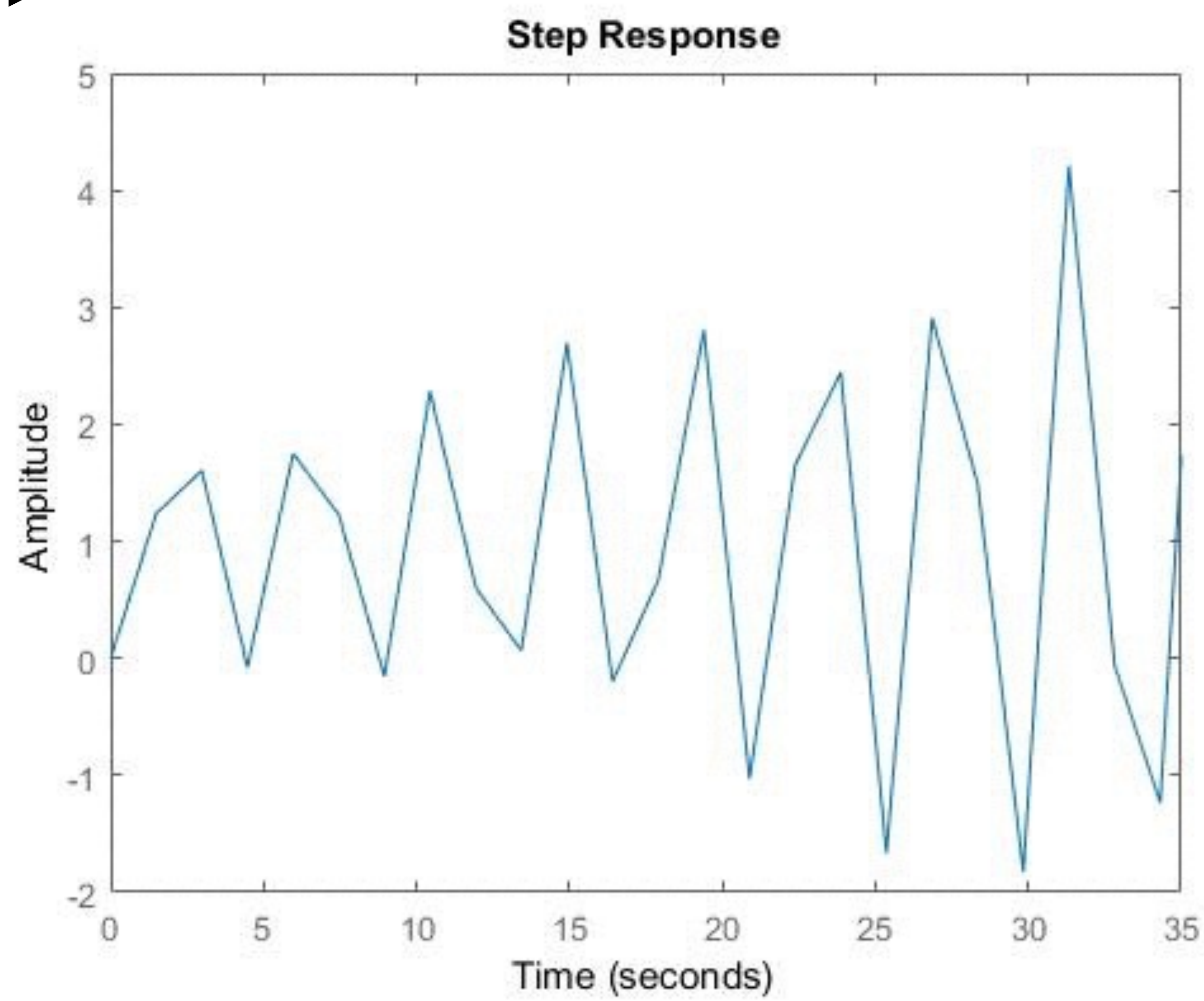
# 5. Informal System Stability Analysis: Example 2

Careful: THIS is NOT the transfer function of the system!

Apply the feedback rule to find TF and solve!




Source: [2]

# 5. Informal System Stability Analysis: Problem

- A problem arises when a transfer function is given in an unfactored way

$$\frac{10(s+2)}{s^5 + 28s^4 + 284s^3 + 1232s^2 + 1930s + 20} = \frac{10(s+2)}{s(s+4)(s+6)(s+8)(s+10)}$$

- In such a case it is possible to use calculator or computer to find the roots
- Or use other techniques to analyse for stability, e.g. Routh-Hurwitz method

Source: [2]

# 6. PID-Tuning

- How to determine *optimal* values for PID-controller?
  - Depends on definition of optimality
  - Possible goals:
    - Stability
    - Minimum overshoot
    - Damped noise
    - Minimum steady-state error
    - Short settling time

# 6. Ziegler-Nichols Method for PID-Tuning

- Several possible methods exist
- One of the simplest: Ziegler-Nichols Method
- First, the *ultimate gain* $K_u$ parameter has to be found
  - Ultimate gain factor describes the value of the P-controller which makes the system oscillate with undamped constant amplitude indefinitely with period $T_u$

| **Control Type** | $K_p$ | $T_i$ | $T_d$ |
|---|---|---|---|
| **PI** | $0.45K_u$ | $T_u/1.2$ | - |
| **Classic PID** | $0.6K_u$ | $T_u/2$ | $T_u/8$ |
| **No Overshoot PID** | $0.2K_u$ | $T_u/2$ | $T_u/3$ |

# References

- [1] Benjamin C. Kuo and Farid Golnaraghi. 2002. Automatic Control Systems (8th ed.). John Wiley & Sons, Inc., New York, NY, USA.
- [2] Norman S. Nise. 2011. Control Systems Engineering (6th ed.). John Wiley & Sons, Inc., New York, NY, USA.
- [3] Paul Dawkins. Paul's Online Math Notes - Differential Equations, 2007, http://tutorial.math.lamar.edu/Classes/DE/DE.aspx
- [4] Joseph J. Distefano, Allen J. Stubberud, and I. J. Williams. 1997. Schaum's Outline of Feedback and Control Systems (2nd ed.). McGraw-Hill Professional.