For the autoregressive model the decision variable are the five coefficients that go in the estimate of y values. The estimates are constrained to be a function of the previous five values weighted by the coefficients, except for the first y estimate, which is based on the first actual y value. The model finds the weights such that sum of squared difference between actual and estimated y values is minimized. An estimate by the moving average model is also calculated here. The error of the two models are calculated and displayed before.

In [19]:
```
data = readdlm("uy_data.csv", ',')   #Get data from file
x = data[:, 1]          #X values in 1-D array
y = data[:, 2]          #Y values in 1-D array
n = length(x)           #Number of data points
width = 5               #Width of estimate
;
```

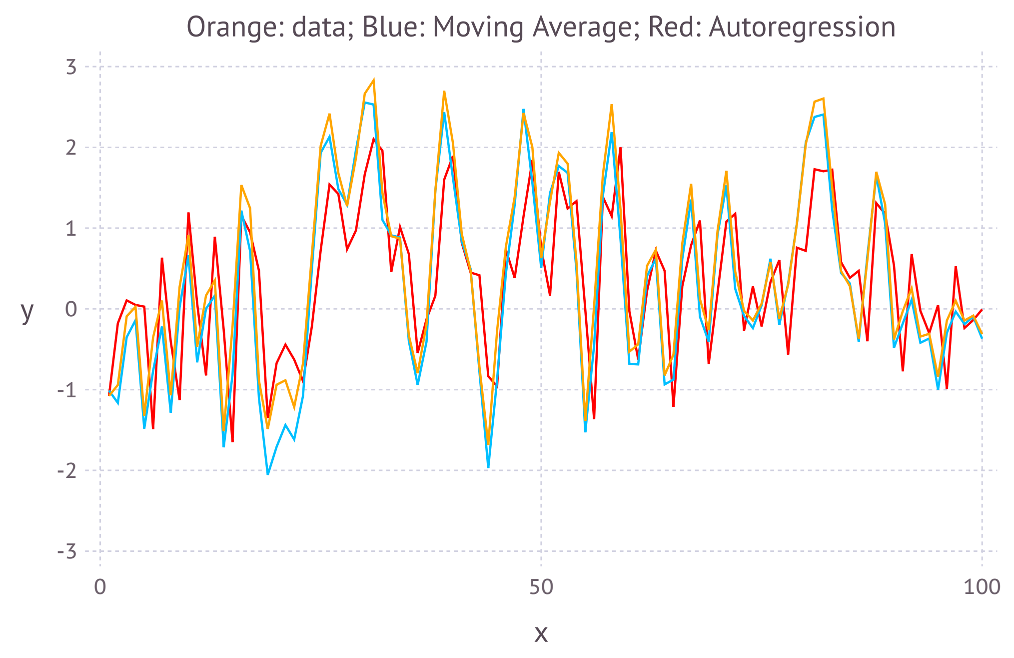
In [24]:
```
#The moving average model
function movingAverage(width)
    A = zeros(n,width)    #Matrix of input values

    for i = 1:width
        A[i:n,i] = x[1:n - i + 1]
    end

    weight = A\y          #Weights on inputs
    yEst = A*weight       #Estimated y values

    return yEst
end;
```

In [25]:
```
#The autoregressive model
using JuMP
m = Model()

function autoregression(width)
    @defVar(m, weight[1:width])   #Weights on previous outputs
    @defVar(m, yEst[1:n])         #Estimates of y values

    #The first y estimate is set to the first y
    @addConstraint(m, yEst[1] == y[1])
    #Estimate of current y values based on previous y's
    for i in 2:n
        if(i > width)
            @addConstraint(m, yEst[i] == sum(y[(i - width):(i-1)].*weight))
        else
            @addConstraint(m, yEst[i] == sum(y[1:(i-1)].*weight[1:i-1]))
        end
    end

    #Minimize error
    @setObjective(m, Min, sum((y - yEst).^2))
    solve(m)
    return getValue(yEst)
end;
```

In [26]:
```
#Get results and plot
movAve = movingAverage(width)
autReg = autoregression(width)

using Gadfly
plot(
layer(x = collect(1:n), y = y, Geom.line, Theme(default_color = colorant"orange")),
layer(x = collect(1:n), y = movAve, Geom.line),
layer(x = collect(1:n), y = autReg, Geom.line, Theme(default_color = colorant"red")),
Guide.title("Orange: data;  Blue: Moving Average;  Red: Autoregression")
)
```

```
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:      585
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:      100

Total number of variables............................:      105
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:      100
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 1.08e+00 4.13e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -9.2011953e+01 6.66e-16 3.36e-15  -1.0 2.10e+00    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 1

                                   (scaled)                 (unscaled)
Objective...............:  -9.2011953129639650e+01   -9.2011953129639650e+01
Dual infeasibility......:   3.3584246494910985e-15    3.3584246494910985e-15
Constraint violation....:   6.6613381477509392e-16    6.6613381477509392e-16
Complementarity.........:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error.......:   3.3584246494910985e-15    3.3584246494910985e-15


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 2
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 2
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.004
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
```

Out[26]:



In [28]:
```
#Display error of the models
println("Error: ")
println("Moving average model: ", sum((y - movAve).^2))
println("Autoregressive model: ", sum((y - autReg).^2));
```

```
Error:
Moving average model: 6.0558043202672796
Autoregressive model: 55.17919840036031
```

The ARMA model chooses the weight on x values and the weight on y values. The prediction on y is constrained to be the sum of the current x value and the previous y value, each weighted by their respective coefficients. The first y is estimated based on the first x value alone. The model finds the two weights such that the error of the estimate is minimized.

In [1]:
```
data = readdlm("uy_data.csv", ',')   #Get data from file
x = data[:, 1]        #X values in 1-D array
y = data[:, 2]        #Y values in 1-D array
n = length(x)         #Number of data points
width = 5             #Width of estimate for MA and AR
;
```

In [2]:
```
#The moving average model
function movingAverage(width)
    A = zeros(n,width)    #Matrix of input values

    for i = 1:width
        A[i:n,i] = x[1:n - i + 1]
    end

    weight = A\y          #Weights on inputs
    yEst = A*weight       #Estimated y values

    return yEst
end;
```

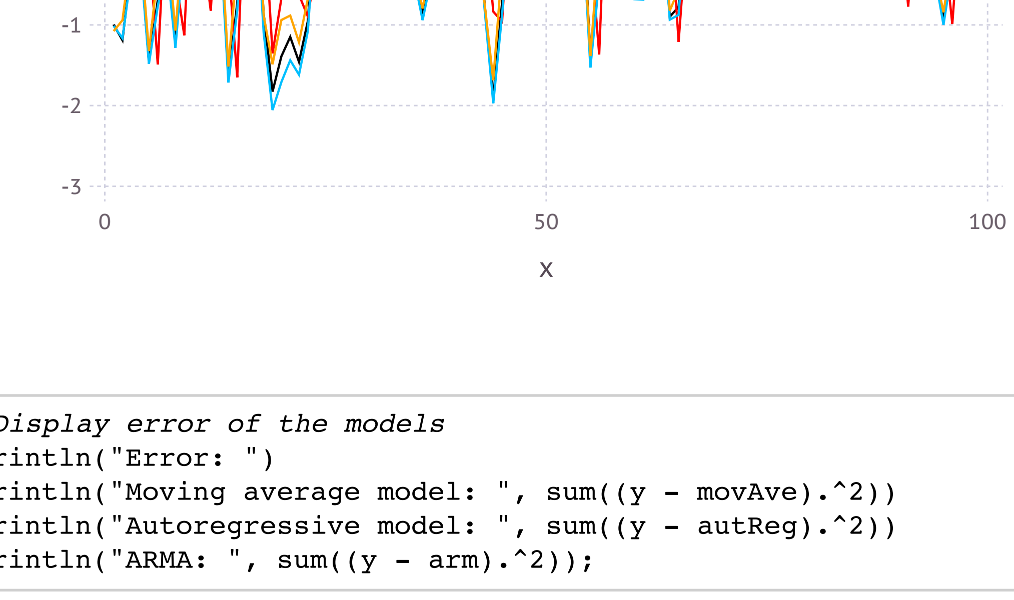
In [3]:
```
#The autoregression model
using JuMP
m = Model()

function autoregression(width)
    @defVar(m, weight[1:width])   #Weights on previous outputs
    @defVar(m, yEst[1:n])         #Estimates of y values

    #Estimate of current y values based on previous ones
    for i in 1:n
        if(i > width)
            @addConstraint(m, yEst[i] == sum(y[(i - width):(i-1)].*weight))
        else
            @addConstraint(m, yEst[i] == sum(y[1:(i-1)].*weight[1:i-1]))
        end
    end

    #Minimize error
    @setObjective(m, Min, sum((y - yEst).^2))
    solve(m)
    return getValue(yEst)
end;
```

In [4]:
```
#The autoregressive moving average model
m2 = Model()
function arma()
    @defVar(m2, weightY)  #Weight on y terms
    @defVar(m2, weightX)  #Weight on x terms
    @defVar(m2, yEst[1:n]) #Estimates on y

    #Estimate current y value based on current x and previous y value.
    @addConstraint(m2, yEst[1] == weightX*x[1])
    @addConstraint(m2, est[i in 2:n], yEst[i] == weightX*x[i] +weightY*y[i - 1])

    #Minimize error
    @setObjective(m2, Min, sum((y - yEst).^2))
    solve(m2)
    return getValue(yEst)
end;
```

In [5]:
```
#Get results and plot
movAve = movingAverage(width)
autReg = autoregression(width)
arm = arma()

using Gadfly
plot(
layer(x = collect(1:n), y = y, Geom.line, Theme(default_color = colorant"orange")),
layer(x = collect(1:n), y = movAve, Geom.line),
layer(x = collect(1:n), y = autReg, Geom.line, Theme(default_color = colorant"red")),
layer(x = collect(1:n), y = arm, Geom.line, Theme(default_color = colorant"black")),
Guide.title("Orange: data;  Blue: Moving Average;  Red: Autoregression; Black: ARMA")
)
```

```
******************************************************************************
This program contains Ipopt, a library for large-scale nonlinear optimization.
 Ipopt is released as open source code under the Eclipse Public License (EPL).
         For more information visit http://projects.coin-or.org/Ipopt (http://projects.coin-or.org/Ipopt)
******************************************************************************

This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:      585
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:      100

Total number of variables............................:      105
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:      100
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 0.00e+00 4.13e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -9.0851378e+01 6.66e-16 3.36e-15  -1.0 2.10e+00    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 1

                                   (scaled)                 (unscaled)
Objective...............:  -9.0851377839639639e+01   -9.0851377839639639e+01
Dual infeasibility......:   3.3584246494910985e-15    3.3584246494910985e-15
Constraint violation....:   6.6613381477509392e-16    6.6613381477509392e-16
Complementarity.........:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error.......:   3.3584246494910985e-15    3.3584246494910985e-15


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 2
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 2
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.093
Total CPU secs in NLP function evaluations           =      0.060

EXIT: Optimal Solution Found.
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:      299
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:      100

Total number of variables............................:      102
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:      100
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 0.00e+00 5.21e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -1.4374425e+02 4.44e-16 1.84e-15  -1.0 2.62e+00    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 1

                                   (scaled)                 (unscaled)
Objective...............:  -1.4374425178151643e+02   -1.4374425178151643e+02
Dual infeasibility......:   1.8396742462734039e-15    1.8396742462734039e-15
Constraint violation....:   4.4408920985006262e-16    4.4408920985006262e-16
Complementarity.........:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error.......:   1.8396742462734039e-15    1.8396742462734039e-15


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 2
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 2
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.001
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
```

Out[5]:



In [6]:
```
#Display error of the models
println("Error: ")
println("Moving average model: ", sum((y - movAve).^2))
println("Autoregressive model: ", sum((y - autReg).^2))
println("ARMA: ", sum((y - arm).^2));
```

```
Error:
Moving average model: 6.0558043202672796
Autoregressive model: 56.33977369036031
ARMA: 3.4468997484834625
```

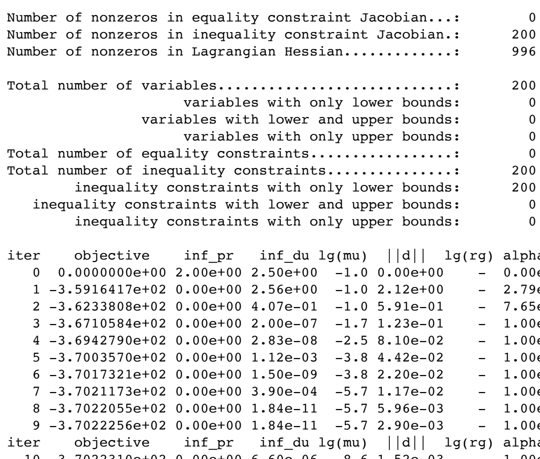
The decision variable of the model is the voltage supply over time. The smoothness function is defined as an expression in terms of the supply variable. It is stipulated that for every time unit the supply of voltage must not be less than the demand. The model finds the supply profile that minimizes the difference between voltage supplied and demanded, and smoothness, which is weighted by a tradeoff parameter lambda. The lambda is varied and the resulting voltage gap and smoothness is plotted. Then the version of the model with the lambda that minimizes total cost is chosen for displaying the smoothed voltage against the required voltage.

In [55]:

```
#Voltage demand data
rawData = readdlm("voltages.csv")

#Turn raw data into a 1-D array
volt = []

for i in 1:length(rawData)
    push!(volt, rawData[i, 1])
end

n = length(volt)        #Number of voltage data

#---Model---#
using JuMP
function voltageSmoothing(λ)

    m = Model()

    @defVar(m, supply[1:n])       #Voltage supplied

    #Relationship between smoothness and voltage supplied
    @defExpr(smoothness[i in 1:(n-1)], (supply[i+1] - supply[i]))
    @addConstraint(m, fulfill[i in 1:n], supply[i] >= volt[i])

    #Minimize the sum of energy supplied and smoothness measurement
    @setObjective(m, Min, sum((supply - volt).^2) + λ*sum(smoothness.^2))

    solve(m)
    return(getValue(sum((supply - volt).^2)), getValue(sum(smoothness.^2)), getValue(supply))
end

#Vary tradeoff and display results
gp = []   #Store gaps
sm = []   #Store smoothness
lm = 0

for i in 1:11
    res = voltageSmoothing(lm)
    push!(gp, res[1])
    push!(sm, res[2])
    lm = lm +0.1
end

using Gadfly
plot(x = gp, y = sm, Guide.xlabel("Signal smoothness"), Guide.ylabel("Total signal difference"))
```

```
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:      200
Number of nonzeros in Lagrangian Hessian.............:      200

Total number of variables............................:      200
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:      200
        inequality constraints with only lower bounds:      200
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 2.00e+00 2.50e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -3.8182141e+02 0.00e+00 1.00e-06  -1.0 2.12e+00    -  1.00e+00 1.00e+00f  1
   2 -3.8662134e+02 0.00e+00 2.00e-07  -1.7 1.98e-01    -  1.00e+00 1.00e+00f  1
   3 -3.8900554e+02 0.00e+00 2.83e-08  -2.5 1.08e-01    -  1.00e+00 1.00e+00f  1
   4 -3.8974379e+02 0.00e+00 1.50e-09  -3.8 5.67e-02    -  1.00e+00 1.00e+00f  1
   5 -3.8992814e+02 0.00e+00 1.50e-09  -3.8 2.80e-02    -  1.00e+00 1.00e+00f  1
   6 -3.8997355e+02 0.00e+00 1.50e-09  -3.8 1.34e-02    -  1.00e+00 1.00e+00f  1
   7 -3.8999330e+02 0.00e+00 1.84e-11  -5.7 7.95e-03    -  1.00e+00 1.00e+00f  1
   8 -3.8999823e+02 0.00e+00 1.84e-11  -5.7 3.95e-03    -  1.00e+00 1.00e+00f  1
   9 -3.8999946e+02 0.00e+00 1.84e-11  -5.7 1.92e-03    -  1.00e+00 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 -3.8999986e+02 0.00e+00 2.51e-14  -8.6 1.07e-03    -  1.00e+00 1.00e+00f  1
  11 -3.8999997e+02 0.00e+00 2.51e-14  -8.6 5.36e-04    -  1.00e+00 1.00e+00f  1
  12 -3.8999999e+02 0.00e+00 2.51e-14  -8.6 2.67e-04    -  1.00e+00 1.00e+00f  1
  13 -3.9000000e+02 0.00e+00 2.51e-14  -8.6 1.32e-04    -  1.00e+00 1.00e+00f  1
  14 -3.9000000e+02 0.00e+00 2.51e-14  -8.6 6.40e-05    -  1.00e+00 1.00e+00f  1
  15 -3.9000000e+02 0.00e+00 9.09e-15  -9.0 3.35e-05    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 15

                                   (scaled)                 (unscaled)
Objective...............:  -3.8999999977607968e+02   -3.8999999977607968e+02
Dual infeasibility......:   9.0909132435465501e-15    9.0909132435465501e-15
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   3.1475633856745632e-09    3.1475633856745632e-09
Overall NLP error.......:   3.1475633856745632e-09    3.1475633856745632e-09


Number of objective function evaluations             = 16
Number of objective gradient evaluations             = 16
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 16
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 16
Number of Lagrangian Hessian evaluations             = 15
Total CPU secs in IPOPT (w/o function evaluations)   =      0.019
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:      200
Number of nonzeros in Lagrangian Hessian.............:      996

Total number of variables............................:      200
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:      200
        inequality constraints with only lower bounds:      200
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 2.00e+00 2.50e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -3.7882091e+02 0.00e+00 8.76e-02  -1.0 2.12e+00    -  9.19e-01 1.00e+00f  1
   2 -3.8362694e+02 0.00e+00 2.00e-07  -1.7 2.09e-01    -  1.00e+00 1.00e+00f  1
   3 -3.8608915e+02 0.00e+00 2.83e-08  -2.5 1.10e-01    -  1.00e+00 1.00e+00f  1
   4 -3.8683068e+02 0.00e+00 1.50e-09  -3.8 5.72e-02    -  1.00e+00 1.00e+00f  1
   5 -3.8700482e+02 0.00e+00 1.50e-09  -3.8 2.82e-02    -  1.00e+00 1.00e+00f  1
   6 -3.8704462e+02 0.00e+00 1.50e-09  -3.8 1.35e-02    -  1.00e+00 1.00e+00f  1
   7 -3.8706311e+02 0.00e+00 1.84e-11  -5.7 7.99e-03    -  1.00e+00 1.00e+00f  1
   8 -3.8706713e+02 0.00e+00 1.84e-11  -5.7 3.97e-03    -  1.00e+00 1.00e+00f  1
   9 -3.8706810e+02 0.00e+00 1.84e-11  -5.7 1.93e-03    -  1.00e+00 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 -3.8706844e+02 0.00e+00 2.51e-14  -8.6 1.08e-03    -  1.00e+00 1.00e+00f  1
  11 -3.8706852e+02 0.00e+00 2.51e-14  -8.6 5.38e-04    -  1.00e+00 1.00e+00f  1
  12 -3.8706854e+02 0.00e+00 2.51e-14  -8.6 2.68e-04    -  1.00e+00 1.00e+00f  1
  13 -3.8706854e+02 0.00e+00 2.51e-14  -8.6 1.33e-04    -  1.00e+00 1.00e+00f  1
  14 -3.8706854e+02 0.00e+00 2.51e-14  -8.6 6.43e-05    -  1.00e+00 1.00e+00f  1
  15 -3.8706854e+02 0.00e+00 9.10e-15  -9.0 3.36e-05    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 15

                                   (scaled)                 (unscaled)
Objective...............:  -3.8706854458522594e+02   -3.8706854458522594e+02
Dual infeasibility......:   9.1038288019262836e-15    9.1038288019262836e-15
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   3.1672864711046893e-09    3.1672864711046893e-09
Overall NLP error.......:   3.1672864711046893e-09    3.1672864711046893e-09


Number of objective function evaluations             = 16
Number of objective gradient evaluations             = 16
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 16
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 16
Number of Lagrangian Hessian evaluations             = 15
Total CPU secs in IPOPT (w/o function evaluations)   =      0.022
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:      200
Number of nonzeros in Lagrangian Hessian.............:      996

Total number of variables............................:      200
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:      200
        inequality constraints with only lower bounds:      200
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 2.00e+00 2.50e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -3.7610472e+02 0.00e+00 3.76e-01  -1.0 2.12e+00    -  7.24e-01 1.00e+00f  1
   2 -3.8098511e+02 0.00e+00 2.00e-07  -1.7 2.01e-01    -  1.00e+00 1.00e+00f  1
   3 -3.8355663e+02 0.00e+00 2.83e-08  -2.5 1.11e-01    -  1.00e+00 1.00e+00f  1
   4 -3.8430499e+02 0.00e+00 1.50e-09  -3.8 5.80e-02    -  1.00e+00 1.00e+00f  1
   5 -3.8447235e+02 0.00e+00 1.50e-09  -3.8 2.87e-02    -  1.00e+00 1.00e+00f  1
   6 -3.8451066e+02 0.00e+00 1.50e-09  -3.8 1.37e-02    -  1.00e+00 1.00e+00f  1
   7 -3.8452857e+02 0.00e+00 1.84e-11  -5.7 8.09e-03    -  1.00e+00 1.00e+00f  1
   8 -3.8453241e+02 0.00e+00 1.84e-11  -5.7 4.02e-03    -  1.00e+00 1.00e+00f  1
   9 -3.8453334e+02 0.00e+00 1.84e-11  -5.7 1.95e-03    -  1.00e+00 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 -3.8453365e+02 0.00e+00 4.34e-07  -8.6 1.09e-03    -  1.00e+00 1.00e+00f  1
  11 -3.8453372e+02 0.00e+00 2.51e-14  -8.6 5.44e-04    -  1.00e+00 1.00e+00f  1
  12 -3.8453374e+02 0.00e+00 2.51e-14  -8.6 2.71e-04    -  1.00e+00 1.00e+00f  1
  13 -3.8453375e+02 0.00e+00 2.51e-14  -8.6 1.34e-04    -  1.00e+00 1.00e+00f  1
  14 -3.8453375e+02 0.00e+00 2.51e-14  -8.6 6.50e-05    -  1.00e+00 1.00e+00f  1
  15 -3.8453375e+02 0.00e+00 9.10e-15  -9.0 3.39e-05    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 15

                                   (scaled)                 (unscaled)
Objective...............:  -3.8453374737451253e+02   -3.8453374737451253e+02
Dual infeasibility......:   9.1038288019262836e-15    9.1038288019262836e-15
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   3.2141860140706342e-09    3.2141860140706342e-09
Overall NLP error.......:   3.2141860140706342e-09    3.2141860140706342e-09


Number of objective function evaluations             = 16
Number of objective gradient evaluations             = 16
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 16
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 16
Number of Lagrangian Hessian evaluations             = 15
Total CPU secs in IPOPT (w/o function evaluations)   =      0.023
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:      200
Number of nonzeros in Lagrangian Hessian.............:      996

Total number of variables............................:      200
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:      200
        inequality constraints with only lower bounds:      200
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 2.00e+00 2.50e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -3.7359182e+02 0.00e+00 6.61e-01  -1.0 2.12e+00    -  6.00e-01 1.00e+00f  1
   2 -3.7871345e+02 0.00e+00 2.00e-07  -1.7 2.43e-01    -  1.00e+00 1.00e+00f  1
   3 -3.8130879e+02 0.00e+00 2.83e-08  -2.5 1.13e-01    -  1.00e+00 1.00e+00f  1
   4 -3.8204776e+02 0.00e+00 1.50e-09  -3.8 5.85e-02    -  1.00e+00 1.00e+00f  1
   5 -3.8220852e+02 0.00e+00 1.50e-09  -3.8 2.89e-02    -  1.00e+00 1.00e+00f  1
   6 -3.8224585e+02 0.00e+00 1.50e-09  -3.8 1.39e-02    -  1.00e+00 1.00e+00f  1
   7 -3.8226323e+02 0.00e+00 1.84e-11  -5.7 8.15e-03    -  1.00e+00 1.00e+00f  1
   8 -3.8226690e+02 0.00e+00 1.84e-11  -5.7 4.05e-03    -  1.00e+00 1.00e+00f  1
   9 -3.8226778e+02 0.00e+00 1.84e-11  -5.7 1.97e-03    -  1.00e+00 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 -3.8226808e+02 0.00e+00 7.15e-07  -8.6 1.09e-03    -  1.00e+00 1.00e+00f  1
  11 -3.8226814e+02 0.00e+00 2.51e-14  -8.6 5.47e-04    -  1.00e+00 1.00e+00f  1
  12 -3.8226816e+02 0.00e+00 2.51e-14  -8.6 2.73e-04    -  1.00e+00 1.00e+00f  1
  13 -3.8226816e+02 0.00e+00 2.51e-14  -8.6 1.35e-04    -  1.00e+00 1.00e+00f  1
  14 -3.8226816e+02 0.00e+00 2.51e-14  -8.6 6.55e-05    -  1.00e+00 1.00e+00f  1
  15 -3.8226816e+02 0.00e+00 9.10e-15  -9.0 3.42e-05    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 15

                                   (scaled)                 (unscaled)
Objective...............:  -3.8226816498185906e+02   -3.8226816498185906e+02
Dual infeasibility......:   9.1038288019262836e-15    9.1038288019262836e-15
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   3.2438610265677239e-09    3.2438610265677239e-09
Overall NLP error.......:   3.2438610265677239e-09    3.2438610265677239e-09


Number of objective function evaluations             = 16
Number of objective gradient evaluations             = 16
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 16
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 16
Number of Lagrangian Hessian evaluations             = 15
Total CPU secs in IPOPT (w/o function evaluations)   =      0.022
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:      200
Number of nonzeros in Lagrangian Hessian.............:      996

Total number of variables............................:      200
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:      200
        inequality constraints with only lower bounds:      200
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 2.00e+00 2.50e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -3.7123502e+02 0.00e+00 9.43e-01  -1.0 2.12e+00    -  5.12e-01 1.00e+00f  1
   2 -3.7659509e+02 0.00e+00 1.24e-01  -1.7 2.91e-01    -  7.81e-01 9.74e-01f  1
   3 -3.7787994e+02 0.00e+00 2.00e-07  -1.7 8.99e-02    -  1.00e+00 1.00e+00f  1
   4 -3.7954181e+02 0.00e+00 2.83e-08  -2.5 6.90e-02    -  1.00e+00 1.00e+00f  1
   5 -3.8004754e+02 0.00e+00 1.50e-09  -3.8 3.88e-02    -  1.00e+00 1.00e+00f  1
   6 -3.8015663e+02 0.00e+00 1.50e-09  -3.8 1.89e-02    -  1.00e+00 1.00e+00f  1
   7 -3.8019056e+02 0.00e+00 9.04e-05  -5.7 1.04e-02    -  1.00e+00 9.96e-01f  1
   8 -3.8019827e+02 0.00e+00 1.84e-11  -5.7 5.20e-03    -  1.00e+00 1.00e+00f  1
   9 -3.8020010e+02 0.00e+00 1.84e-11  -5.7 2.55e-03    -  1.00e+00 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 -3.8020062e+02 0.00e+00 3.26e-06  -8.6 1.36e-03    -  1.00e+00 9.99e-01f  1
  11 -3.8020074e+02 0.00e+00 2.51e-14  -8.6 6.81e-04    -  1.00e+00 1.00e+00f  1
  12 -3.8020077e+02 0.00e+00 2.51e-14  -8.6 3.40e-04    -  1.00e+00 1.00e+00f  1
  13 -3.8020077e+02 0.00e+00 2.51e-14  -8.6 1.69e-04    -  1.00e+00 1.00e+00f  1
  14 -3.8020078e+02 0.00e+00 2.51e-14  -8.6 8.27e-05    -  1.00e+00 1.00e+00f  1
  15 -3.8020078e+02 0.00e+00 9.10e-15  -9.0 4.24e-05    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 15

                                   (scaled)                 (unscaled)
Objective...............:  -3.8020077633907778e+02   -3.8020077633907778e+02
Dual infeasibility......:   9.1038288019262836e-15    9.1038288019262836e-15
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   4.5153621185011507e-09    4.5153621185011507e-09
Overall NLP error.......:   4.5153621185011507e-09    4.5153621185011507e-09


Number of objective function evaluations             = 16
Number of objective gradient evaluations             = 16
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 16
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 16
Number of Lagrangian Hessian evaluations             = 15
Total CPU secs in IPOPT (w/o function evaluations)   =      0.023
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:      200
Number of nonzeros in Lagrangian Hessian.............:      996

Total number of variables............................:      200
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:      200
        inequality constraints with only lower bounds:      200
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 2.00e+00 2.50e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -3.6900394e+02 0.00e+00 1.22e+00  -1.0 2.12e+00    -  4.48e-01 1.00e+00f  1
   2 -3.6904579e+02 0.00e+00 1.00e-06  -1.0 5.31e-01    -  1.00e+00 1.00e+00f  1
   3 -3.7510706e+02 0.00e+00 2.00e-07  -1.7 1.69e-01    -  1.00e+00 1.00e+00f  1
   4 -3.7744792e+02 0.00e+00 2.83e-08  -2.5 7.38e-02    -  1.00e+00 1.00e+00f  1
   5 -3.7809115e+02 0.00e+00 2.06e-04  -3.8 4.09e-02    -  1.00e+00 9.98e-01f  1
   6 -3.7823314e+02 0.00e+00 1.50e-09  -3.8 2.01e-02    -  1.00e+00 1.00e+00f  1
   7 -3.7827453e+02 0.00e+00 2.09e-04  -5.7 1.09e-02    -  1.00e+00 9.92e-01f  1
   8 -3.7828414e+02 0.00e+00 1.84e-11  -5.7 5.50e-03    -  1.00e+00 1.00e+00f  1
   9 -3.7828639e+02 0.00e+00 1.84e-11  -5.7 2.69e-03    -  1.00e+00 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 -3.7828701e+02 0.00e+00 4.85e-06  -8.6 1.43e-03    -  1.00e+00 9.99e-01f  1
  11 -3.7828715e+02 0.00e+00 2.53e-14  -8.6 7.14e-04    -  1.00e+00 1.00e+00f  1
  12 -3.7828718e+02 0.00e+00 2.53e-14  -8.6 3.56e-04    -  1.00e+00 1.00e+00f  1
  13 -3.7828719e+02 0.00e+00 2.53e-14  -8.6 1.77e-04    -  1.00e+00 1.00e+00f  1
  14 -3.7828719e+02 0.00e+00 2.53e-14  -8.6 8.68e-05    -  1.00e+00 1.00e+00f  1
  15 -3.7828719e+02 0.00e+00 9.33e-15  -9.0 4.44e-05    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 15

                                   (scaled)                 (unscaled)
Objective...............:  -3.7828719240630738e+02   -3.7828719240630738e+02
Dual infeasibility......:   9.3258734068513149e-15    9.3258734068513149e-15
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   4.8672585239217965e-09    4.8672585239217965e-09
Overall NLP error.......:   4.8672585239217965e-09    4.8672585239217965e-09


Number of objective function evaluations             = 16
Number of objective gradient evaluations             = 16
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 16
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 16
Number of Lagrangian Hessian evaluations             = 15
Total CPU secs in IPOPT (w/o function evaluations)   =      0.020
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:      200
Number of nonzeros in Lagrangian Hessian.............:      996

Total number of variables............................:      200
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:      200
        inequality constraints with only lower bounds:      200
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 2.00e+00 2.50e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -3.6687761e+02 0.00e+00 1.49e+00  -1.0 2.12e+00    -  3.99e-01 1.00e+00f  1
   2 -3.6758562e+02 0.00e+00 1.00e-06  -1.0 5.48e-01    -  1.00e+00 1.00e+00f  1
   3 -3.7341897e+02 0.00e+00 2.00e-07  -1.7 1.57e-01    -  1.00e+00 1.00e+00f  1
   4 -3.7568940e+02 0.00e+00 2.83e-08  -2.5 7.39e-02    -  1.00e+00 1.00e+00f  1
   5 -3.7630821e+02 0.00e+00 3.55e-04  -3.8 4.10e-02    -  1.00e+00 9.96e-01f  1
   6 -3.7644543e+02 0.00e+00 1.50e-09  -3.8 2.02e-02    -  1.00e+00 1.00e+00f  1
   7 -3.7648534e+02 0.00e+00 2.42e-04  -5.7 1.09e-02    -  1.00e+00 9.90e-01f  1
   8 -3.7649456e+02 0.00e+00 1.84e-11  -5.7 5.52e-03    -  1.00e+00 1.00e+00f  1
   9 -3.7649671e+02 0.00e+00 1.84e-11  -5.7 2.69e-03    -  1.00e+00 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 -3.7649730e+02 0.00e+00 5.09e-06  -8.6 1.43e-03    -  1.00e+00 9.98e-01f  1
  11 -3.7649743e+02 0.00e+00 2.51e-14  -8.6 7.16e-04    -  1.00e+00 1.00e+00f  1
  12 -3.7649747e+02 0.00e+00 2.53e-14  -8.6 3.57e-04    -  1.00e+00 1.00e+00f  1
  13 -3.7649747e+02 0.00e+00 2.53e-14  -8.6 1.77e-04    -  1.00e+00 1.00e+00f  1
  14 -3.7649747e+02 0.00e+00 2.53e-14  -8.6 8.67e-05    -  1.00e+00 1.00e+00f  1
  15 -3.7649748e+02 0.00e+00 9.33e-15  -9.0 4.41e-05    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 15

                                   (scaled)                 (unscaled)
Objective...............:  -3.7649747517940318e+02   -3.7649747517940318e+02
Dual infeasibility......:   9.3258734068513149e-15    9.3258734068513149e-15
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   4.8819545942114297e-09    4.8819545942114297e-09
Overall NLP error.......:   4.8819545942114297e-09    4.8819545942114297e-09


Number of objective function evaluations             = 16
Number of objective gradient evaluations             = 16
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 16
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 16
Number of Lagrangian Hessian evaluations             = 15
Total CPU secs in IPOPT (w/o function evaluations)   =      0.025
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:      200
Number of nonzeros in Lagrangian Hessian.............:      996

Total number of variables............................:      200
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:      200
        inequality constraints with only lower bounds:      200
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 2.00e+00 2.50e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -3.6484079e+02 0.00e+00 1.76e+00  -1.0 2.12e+00    -  3.59e-01 1.00e+00f  1
   2 -3.6616667e+02 0.00e+00 4.29e-02  -1.0 5.61e-01    -  9.55e-01 1.00e+00f  1
   3 -3.7174863e+02 0.00e+00 2.00e-07  -1.7 1.46e-01    -  1.00e+00 1.00e+00f  1
   4 -3.7401273e+02 0.00e+00 2.83e-08  -2.5 7.50e-02    -  1.00e+00 1.00e+00f  1
   5 -3.7462536e+02 0.00e+00 1.50e-09  -3.8 4.15e-02    -  1.00e+00 1.00e+00f  1
   6 -3.7475939e+02 0.00e+00 1.50e-09  -3.8 2.03e-02    -  1.00e+00 1.00e+00f  1
   7 -3.7479848e+02 0.00e+00 2.80e-04  -5.7 1.10e-02    -  1.00e+00 9.89e-01f  1
   8 -3.7480748e+02 0.00e+00 1.84e-11  -5.7 5.58e-03    -  1.00e+00 1.00e+00f  1
   9 -3.7480957e+02 0.00e+00 1.84e-11  -5.7 2.72e-03    -  1.00e+00 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 -3.7481014e+02 0.00e+00 5.40e-06  -8.6 1.44e-03    -  1.00e+00 9.98e-01f  1
  11 -3.7481026e+02 0.00e+00 2.51e-14  -8.6 7.22e-04    -  1.00e+00 1.00e+00f  1
  12 -3.7481029e+02 0.00e+00 2.53e-14  -8.6 3.59e-04    -  1.00e+00 1.00e+00f  1
  13 -3.7481030e+02 0.00e+00 2.53e-14  -8.6 1.78e-04    -  1.00e+00 1.00e+00f  1
  14 -3.7481030e+02 0.00e+00 2.53e-14  -8.6 8.65e-05    -  1.00e+00 1.00e+00f  1
  15 -3.7481030e+02 0.00e+00 9.33e-15  -9.0 4.32e-05    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 15

                                   (scaled)                 (unscaled)
Objective...............:  -3.7481030315363591e+02   -3.7481030315363591e+02
Dual infeasibility......:   9.3258734068513149e-15    9.3258734068513149e-15
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   4.9094140061050139e-09    4.9094140061050139e-09
Overall NLP error.......:   4.9094140061050139e-09    4.9094140061050139e-09


Number of objective function evaluations             = 16
Number of objective gradient evaluations             = 16
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 16
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 16
Number of Lagrangian Hessian evaluations             = 15
Total CPU secs in IPOPT (w/o function evaluations)   =      0.023
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:      200
Number of nonzeros in Lagrangian Hessian.............:      996

Total number of variables............................:      200
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:      200
        inequality constraints with only lower bounds:      200
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 2.00e+00 2.50e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -3.6288197e+02 0.00e+00 2.03e+00  -1.0 2.12e+00    -  3.28e-01 1.00e+00f  1
   2 -3.6479288e+02 0.00e+00 1.45e-01  -1.0 5.73e-01    -  8.77e-01 1.00e+00f  1
   3 -3.7010152e+02 0.00e+00 2.00e-07  -1.7 1.36e-01    -  1.00e+00 1.00e+00f  1
   4 -3.7240781e+02 0.00e+00 2.83e-08  -2.5 7.68e-02    -  1.00e+00 1.00e+00f  1
   5 -3.7302416e+02 0.00e+00 1.50e-09  -3.8 4.23e-02    -  1.00e+00 1.00e+00f  1
   6 -3.7315908e+02 0.00e+00 1.50e-09  -3.8 2.07e-02    -  1.00e+00 1.00e+00f  1
   7 -3.7319809e+02 0.00e+00 3.23e-04  -5.7 1.12e-02    -  1.00e+00 9.88e-01f  1
   8 -3.7320707e+02 0.00e+00 1.84e-11  -5.7 5.69e-03    -  1.00e+00 1.00e+00f  1
   9 -3.7320913e+02 0.00e+00 1.84e-11  -5.7 2.77e-03    -  1.00e+00 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 -3.7320969e+02 0.00e+00 5.81e-06  -8.6 1.46e-03    -  1.00e+00 9.98e-01f  1
  11 -3.7320982e+02 0.00e+00 2.53e-14  -8.6 7.33e-04    -  1.00e+00 1.00e+00f  1
  12 -3.7320985e+02 0.00e+00 2.53e-14  -8.6 3.64e-04    -  1.00e+00 1.00e+00f  1
  13 -3.7320985e+02 0.00e+00 2.53e-14  -8.6 1.79e-04    -  1.00e+00 1.00e+00f  1
  14 -3.7320986e+02 0.00e+00 2.53e-14  -8.6 8.58e-05    -  1.00e+00 1.00e+00f  1
  15 -3.7320986e+02 0.00e+00 9.33e-15  -9.0 4.09e-05    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 15

                                   (scaled)                 (unscaled)
Objective...............:  -3.7320985625292047e+02   -3.7320985625292047e+02
Dual infeasibility......:   9.3258734068513149e-15    9.3258734068513149e-15
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   4.8125262341397169e-09    4.8125262341397169e-09
Overall NLP error.......:   4.8125262341397169e-09    4.8125262341397169e-09


Number of objective function evaluations             = 16
Number of objective gradient evaluations             = 16
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 16
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 16
Number of Lagrangian Hessian evaluations             = 15
Total CPU secs in IPOPT (w/o function evaluations)   =      0.023
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:      200
Number of nonzeros in Lagrangian Hessian.............:      996

Total number of variables............................:      200
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:      200
        inequality constraints with only lower bounds:      200
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 2.00e+00 2.50e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -3.6099216e+02 0.00e+00 2.30e+00  -1.0 2.12e+00    -  3.01e-01 1.00e+00f  1
   2 -3.6349824e+02 0.00e+00 2.62e-01  -1.0 5.83e-01    -  8.15e-01 9.85e-01f  1
   3 -3.6855013e+02 0.00e+00 2.00e-07  -1.7 1.17e-01    -  1.00e+00 1.00e+00f  1
   4 -3.7088168e+02 0.00e+00 2.83e-08  -2.5 7.84e-02    -  1.00e+00 1.00e+00f  1
   5 -3.7149858e+02 0.00e+00 1.50e-09  -3.8 4.30e-02    -  1.00e+00 1.00e+00f  1
   6 -3.7163360e+02 0.00e+00 1.50e-09  -3.8 2.11e-02    -  1.00e+00 1.00e+00f  1
   7 -3.7167236e+02 0.00e+00 3.60e-04  -5.7 1.14e-02    -  1.00e+00 9.87e-01f  1
   8 -3.7168127e+02 0.00e+00 1.84e-11  -5.7 5.78e-03    -  1.00e+00 1.00e+00f  1
   9 -3.7168330e+02 0.00e+00 1.84e-11  -5.7 2.81e-03    -  1.00e+00 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 -3.7168385e+02 0.00e+00 6.19e-06  -8.6 1.48e-03    -  1.00e+00 9.98e-01f  1
  11 -3.7168398e+02 0.00e+00 2.53e-14  -8.6 7.41e-04    -  1.00e+00 1.00e+00f  1
  12 -3.7168401e+02 0.00e+00 2.53e-14  -8.6 3.67e-04    -  1.00e+00 1.00e+00f  1
  13 -3.7168401e+02 0.00e+00 2.51e-14  -8.6 1.78e-04    -  1.00e+00 1.00e+00f  1
  14 -3.7168401e+02 0.00e+00 2.51e-14  -8.6 8.23e-05    -  1.00e+00 1.00e+00f  1
  15 -3.7168401e+02 0.00e+00 9.33e-15  -9.0 3.84e-05    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 15

                                   (scaled)                 (unscaled)
Objective...............:  -3.7168401461168992e+02   -3.7168401461168992e+02
Dual infeasibility......:   9.3258734068513149e-15    9.3258734068513149e-15
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   4.1742880194784315e-09    4.1742880194784315e-09
Overall NLP error.......:   4.1742880194784315e-09    4.1742880194784315e-09


Number of objective function evaluations             = 16
Number of objective gradient evaluations             = 16
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 16
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 16
Number of Lagrangian Hessian evaluations             = 15
Total CPU secs in IPOPT (w/o function evaluations)   =      0.024
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:      200
Number of nonzeros in Lagrangian Hessian.............:      996

Total number of variables............................:      200
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:      200
        inequality constraints with only lower bounds:      200
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 2.00e+00 2.50e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -3.5916417e+02 0.00e+00 2.56e+00  -1.0 2.12e+00    -  2.79e-01 1.00e+00f  1
   2 -3.6233808e+02 0.00e+00 4.07e-01  -1.0 5.91e-01    -  7.65e-01 9.08e-01f  1
   3 -3.6710584e+02 0.00e+00 2.00e-07  -1.7 1.23e-01    -  1.00e+00 1.00e+00f  1
   4 -3.6942790e+02 0.00e+00 2.83e-08  -2.5 8.10e-02    -  1.00e+00 1.00e+00f  1
   5 -3.7003570e+02 0.00e+00 1.12e-03  -3.8 4.42e-02    -  1.00e+00 9.90e-01f  1
   6 -3.7017321e+02 0.00e+00 1.50e-09  -3.8 2.20e-02    -  1.00e+00 1.00e+00f  1
   7 -3.7021173e+02 0.00e+00 3.90e-04  -5.7 1.17e-02    -  1.00e+00 9.86e-01f  1
   8 -3.7022055e+02 0.00e+00 1.84e-11  -5.7 5.96e-03    -  1.00e+00 1.00e+00f  1
   9 -3.7022256e+02 0.00e+00 1.84e-11  -5.7 2.90e-03    -  1.00e+00 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 -3.7022310e+02 0.00e+00 6.60e-06  -8.6 1.52e-03    -  1.00e+00 9.98e-01f  1
  11 -3.7022322e+02 0.00e+00 2.53e-14  -8.6 7.58e-04    -  1.00e+00 1.00e+00f  1
  12 -3.7022324e+02 0.00e+00 2.53e-14  -8.6 3.72e-04    -  1.00e+00 1.00e+00f  1
  13 -3.7022325e+02 0.00e+00 2.51e-14  -8.6 1.77e-04    -  1.00e+00 1.00e+00f  1
  14 -3.7022325e+02 0.00e+00 2.53e-14  -8.6 7.57e-05    -  1.00e+00 1.00e+00f  1
  15 -3.7022325e+02 0.00e+00 9.10e-15  -9.0 3.86e-05    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 15

                                   (scaled)                 (unscaled)
Objective...............:  -3.7022325130842626e+02   -3.7022325130842626e+02
Dual infeasibility......:   9.1038288019262836e-15    9.1038288019262836e-15
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   4.0509247813131655e-09    4.0509247813131655e-09
Overall NLP error.......:   4.0509247813131655e-09    4.0509247813131655e-09


Number of objective function evaluations             = 16
Number of objective gradient evaluations             = 16
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 16
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 16
Number of Lagrangian Hessian evaluations             = 15
Total CPU secs in IPOPT (w/o function evaluations)   =      0.025
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
```

Out[55]:



In [56]:

```
#Display desired voltage and smoothed voltage with the minimal cost lambda

minR = findmin(gp + sm)[2]      #Index of the version of the model with the minimum cost
minL = (minR - 1)/10            #The lambda at that version
sup = voltageSmoothing(minL)[3] #The voltage supplied in that model

#Display results
plot( layer(x = collect(1:n), y = volt, Geom.line),
      layer(x = collect(1:n), y = sup, Geom.line, Theme(default_color = colorant"orange")),
      Guide.xlabel("Time"), Guide.ylabel("Voltage"),
      Guide.title("Blue: Desired Voltage, Orange: Smoothed Voltage")
)
```

```
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:      200
Number of nonzeros in Lagrangian Hessian.............:      996

Total number of variables............................:      200
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:      200
        inequality constraints with only lower bounds:      200
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 2.00e+00 2.50e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -3.5916417e+02 0.00e+00 2.56e+00  -1.0 2.12e+00    -  2.79e-01 1.00e+00f  1
   2 -3.6233808e+02 0.00e+00 4.07e-01  -1.0 5.91e-01    -  7.65e-01 9.08e-01f  1
   3 -3.6710584e+02 0.00e+00 2.00e-07  -1.7 1.23e-01    -  1.00e+00 1.00e+00f  1
   4 -3.6942790e+02 0.00e+00 2.83e-08  -2.5 8.10e-02    -  1.00e+00 1.00e+00f  1
   5 -3.7003570e+02 0.00e+00 1.12e-03  -3.8 4.42e-02    -  1.00e+00 9.90e-01f  1
   6 -3.7017321e+02 0.00e+00 1.50e-09  -3.8 2.20e-02    -  1.00e+00 1.00e+00f  1
   7 -3.7021173e+02 0.00e+00 3.90e-04  -5.7 1.17e-02    -  1.00e+00 9.86e-01f  1
   8 -3.7022055e+02 0.00e+00 1.84e-11  -5.7 5.96e-03    -  1.00e+00 1.00e+00f  1
   9 -3.7022256e+02 0.00e+00 1.84e-11  -5.7 2.90e-03    -  1.00e+00 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 -3.7022310e+02 0.00e+00 6.60e-06  -8.6 1.52e-03    -  1.00e+00 9.98e-01f  1
  11 -3.7022322e+02 0.00e+00 2.53e-14  -8.6 7.58e-04    -  1.00e+00 1.00e+00f  1
  12 -3.7022324e+02 0.00e+00 2.53e-14  -8.6 3.72e-04    -  1.00e+00 1.00e+00f  1
  13 -3.7022325e+02 0.00e+00 2.51e-14  -8.6 1.77e-04    -  1.00e+00 1.00e+00f  1
  14 -3.7022325e+02 0.00e+00 2.53e-14  -8.6 7.57e-05    -  1.00e+00 1.00e+00f  1
  15 -3.7022325e+02 0.00e+00 9.33e-15  -9.0 3.86e-05    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 15

                                   (scaled)                 (unscaled)
Objective...............:  -3.7022325130842643e+02   -3.7022325130842643e+02
Dual infeasibility......:   9.3258734068513149e-15    9.3258734068513149e-15
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   4.0509247813047919e-09    4.0509247813047919e-09
Overall NLP error.......:   4.0509247813047919e-09    4.0509247813047919e-09


Number of objective function evaluations             = 16
Number of objective gradient evaluations             = 16
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 16
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 16
Number of Lagrangian Hessian evaluations             = 15
Total CPU secs in IPOPT (w/o function evaluations)   =      0.023
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
```

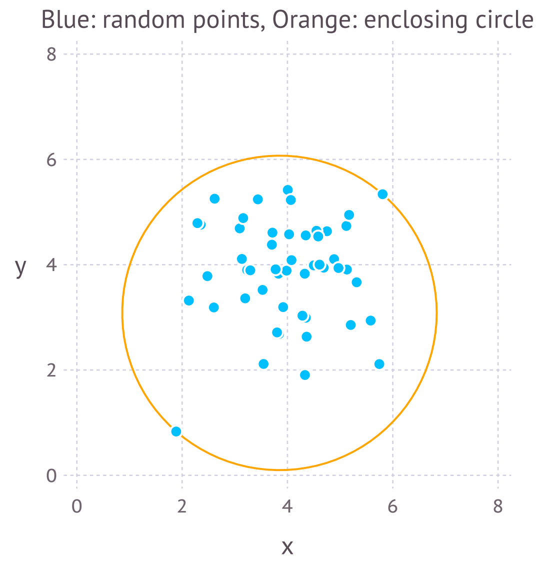
Out[56]:

The choice variables of this model are the coordinates of the center of the circle and the radius of the circle. To ensure that all points are enclosed by the circle it is required that the distance between the center and every point is no more than the radius. The model finds this circle that has the minimum area.

In [8]:
```
#Matrix of random coordinates
#(for X[a,b] a and b is the horizontal and vertical positions, respectively)
X = 4 + randn(2,50)

#-----Model-----#
using JuMP
m = Model()

#Coordinates of the center of the circle
@defVar(m, xCenter)
@defVar(m, yCenter)
#Radius of the circle
@defVar(m, radius >= 0)

#All points must be covered by the circle
@addConstraint(m, inclusion[i in 1:50], (X[1,i] - xCenter)^2 + (X[2,i] - yCenter)^2 <= radius^2)

#Minimize circle area
@setObjective(m, Min, pi*radius*radius)
solve(m)

#-----Display results-----#
using Gadfly

#The enclosing circle
circ = linspace(0, 2*pi, 100)

#Display points and circle
plot(   layer(x = X[1,:], y = X[2,:], Geom.point),
        layer(x = getValue(radius)*cos(circ) + getValue(xCenter) , y = getValue(radius)*sin(circ) + getValue(yCenter),
              Geom.PolygonGeometry, Theme(default_color = colorant"orange")),
        Coord.cartesian(fixed = true), Guide.title("Blue: random points,  Orange: enclosing circle")
)
```

```
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:      500
Number of nonzeros in Lagrangian Hessian.............:      151

Total number of variables............................:        3
                     variables with only lower bounds:        1
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:       50
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:       50

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  3.1415864e-04 6.22e+01 1.37e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  4.5429242e-04 6.06e+01 4.87e+00  -1.0 1.71e+01    -  6.45e-03 3.60e-02h  1
   2  4.5413626e-04 6.05e+01 2.13e+01  -1.0 1.24e+01    -  2.77e-02 1.49e-03h  1
   3  1.5790608e-03 4.89e+01 3.13e+01  -1.0 1.93e+01    -  1.28e-02 2.06e-01f  1
   4  1.0505003e-02 1.38e+01 8.26e+01  -1.0 9.47e+00    -  2.13e-01 1.00e+00h  1
   5  1.2284572e-02 9.26e+00 5.20e+01  -1.0 1.81e+01    -  5.21e-01 3.69e-01h  1
   6  1.2294289e-02 9.16e+00 5.14e+01  -1.0 2.59e+01    -  4.29e-03 1.15e-02h  1
   7  8.5814658e-04 1.03e+01 8.65e+01  -1.0 5.22e+01    -  7.88e-04 1.45e-01f  1
   8  1.3134177e-03 9.91e+00 8.51e+01  -1.0 1.54e+01   0.0 1.22e-01 4.13e-02h  1
   9  7.4854792e-04 1.37e+01 5.75e+01  -1.0 9.20e+01    -  3.49e-04 1.38e-01f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10  4.5094166e-03 1.14e+01 1.24e+02  -1.0 1.79e+01   0.4 1.09e-01 1.97e-01h  1
  11  2.7946388e-01 1.13e+01 1.24e+02  -1.0 7.64e+01   1.8 4.91e-04 3.41e-03h  1
  12  6.3568118e+01 0.00e+00 5.62e+03  -1.0 5.53e+01   1.3 2.47e-05 7.60e-02h  2
  13  4.3876935e+01 0.00e+00 8.23e+02  -1.0 7.61e-01   1.7 1.00e+00 1.00e+00f  1
  14  4.2071478e+01 0.00e+00 1.80e+01  -1.0 7.77e-02   1.2 1.00e+00 1.00e+00h  1
  15  3.1712173e+01 6.32e-02 1.47e+01  -1.0 2.42e+01    -  6.30e-01 1.91e-01f  1
  16  2.7643155e+01 1.86e-01 1.20e+01  -1.0 1.37e+01    -  6.33e-01 1.95e-01f  1
  17  2.8001399e+01 3.16e-01 1.83e+00  -1.0 3.08e+00    -  5.92e-01 1.00e+00h  1
  18  2.7896424e+01 8.41e-02 5.96e-03  -1.0 1.68e+00    -  1.00e+00 1.00e+00h  1
  19  2.7948290e+01 2.77e-02 6.78e-02  -2.5 5.18e-01    -  9.42e-01 7.90e-01h  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  20  2.8022862e+01 0.00e+00 5.88e-05  -2.5 1.59e-01    -  1.00e+00 1.00e+00h  1
  21  2.8020866e+01 0.00e+00 9.67e-07  -3.8 2.21e-02    -  1.00e+00 1.00e+00h  1
  22  2.8020665e+01 0.00e+00 2.99e-09  -5.7 1.20e-03    -  1.00e+00 1.00e+00h  1
  23  2.8020662e+01 0.00e+00 4.33e-13  -8.6 1.39e-05    -  1.00e+00 1.00e+00h  1

Number of Iterations....: 23

                                   (scaled)                 (unscaled)
Objective...............:   2.8020661736293746e+01    2.8020661736293746e+01
Dual infeasibility......:   4.3292690745933229e-13    4.3292690745933229e-13
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   2.5157652740136198e-09    2.5157652740136198e-09
Overall NLP error.......:   2.5157652740136198e-09    2.5157652740136198e-09


Number of objective function evaluations             = 27
Number of objective gradient evaluations             = 24
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 27
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 24
Number of Lagrangian Hessian evaluations             = 23
Total CPU secs in IPOPT (w/o function evaluations)   =      0.018
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
```

Out[8]:



The choice variables of the model are the four coefficients of the cubic polynomial fit. The estimate of fluorescence as a function of light intensity is stored in a list. There is a constraint stipulating that y(0) = 0. The model finds the coefficients in the way that minimizes the sum of the squared difference between real and estimated fluorescence.

In [34]:
```
data = readdlm("xy_data.csv", ',')    #Get data from file
intensity = data[:, 1]                #Light intensity values
fluorescence = data[:, 2]             #Fluorescence values
n = length(intensity)                 #Number of data points

using JuMP
m = Model()
@defVar(m, coeff[1:4])    #Coefficients of the function

#Fluorescence estimate as a function of intensity
@defExpr(fEst[i in 1:n], coeff[1]*intensity[i]^3 + coeff[2]*intensity[i]^2
                + coeff[3]*intensity[i] + coeff[4])
#Fluorescence is zero when intensity is zero
@addConstraint(m, coeff[1]*0^3 + coeff[2]*0^2 + coeff[3]*0 + coeff[4] == 0)

#Minimize error
@setObjective(m, Min, sum((fluorescence - fEst).^2))
solve(m);
```

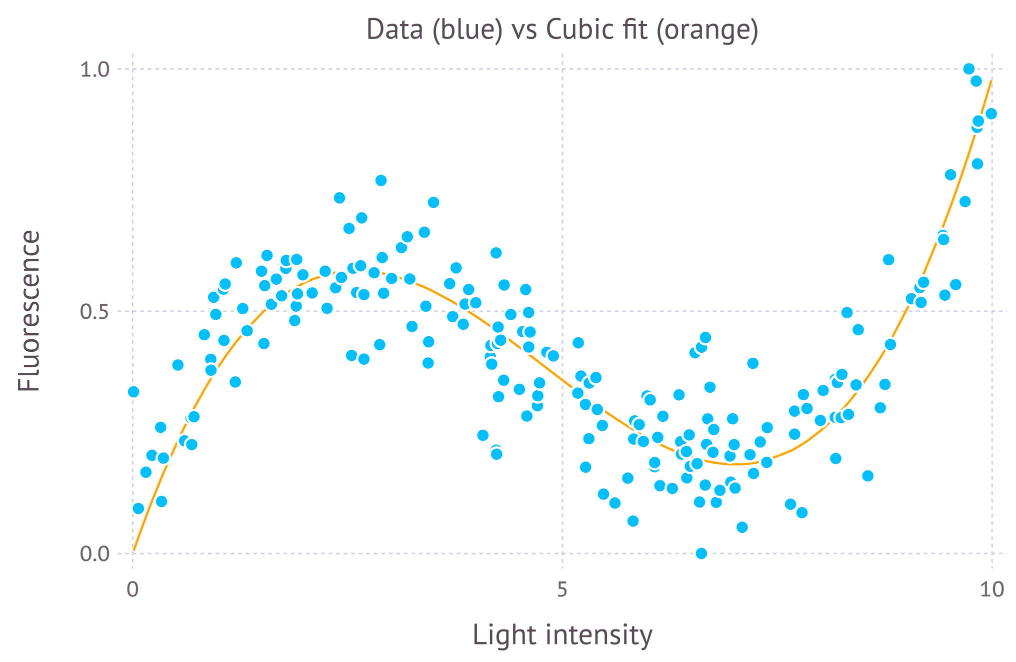
```
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        1
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:     3200

Total number of variables............................:        4
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        1
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 0.00e+00 1.00e+02  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -3.7547282e+01 5.05e-29 2.23e-13  -1.0 5.11e-01    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 1

                                   (scaled)                 (unscaled)
Objective...............:  -9.6933032063723765e-02   -3.7547282387722049e+01
Dual infeasibility......:   2.2305762177262658e-13    8.6401996668428234e-11
Constraint violation....:   5.0487097934144756e-29    5.0487097934144756e-29
Complementarity.........:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error.......:   2.2305762177262658e-13    8.6401996668428234e-11


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 2
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 2
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.003
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
```

In [35]:
```
#Display form of the function
println()
cf = getValue(coeff)
println("Polynomial fit: y = ", cf[1], "x^3 + ", cf[2], "x^2 + ", cf[3], "x + ", cf[4])
```

```
Polynomial fit: y = 0.009325012593407019x^3 + -0.13454611181527108x^2 + 0.5111547756872178x + 5.0487097934
14476e-29
```

In [36]:
```
#Plot data points and best fit cubic function
using Gadfly
plot(  layer(x = intensity, y = fluorescence, Geom.point),
layer(x = intensity, y = getValue(fEst), Geom.line, Theme(default_color = colorant"orange")),
       Guide.xlabel("Light intensity"), Guide.ylabel("Fluorescence"),
       Guide.title("Data (blue) vs Cubic fit (orange)")
)
```

Out[36]:



This model is similar to the one in 4(a). However, now the decision variables are the coefficients of the two pieces of the function.
It is stipulated that the estimated fluorescence are calculated using different sets of coefficients depending on the range.
That both pieces must have the same value at x = 4 is encoded as an equality constraint between calculating estimated fluorescence using the two sets of of coefficients. That the slope of the two pieces must agree at x = 4 is encoded as an equality constraint on the derivative of the pieces at that point.

In [12]:
```
data = readdlm("xy_data.csv", ',')    #Get data from file
intensity = data[:, 1]                #Light intensity values
fluorescence = data[:, 2]             #Fluorescence values
n = length(intensity)                 #Number of data points
splitNo = 4

#Fluorescence fit function of intensity
function fit(i, cf)
    return cf[1]*(i^2) + cf[2]*i + cf[3]
end

#Derivative of the fit function
function fitDer(i, cf)
    return 2*cf[1]*i + cf[2]
end


using JuMP
m = Model()
@defVar(m, coeff1[1:3])    #Coefficients of the first part of the function
@defVar(m, coeff2[1:3])    #Coefficients of the second part of the function
@defVar(m, fEst[1:n])      #Estimates of fluorescence

#Match sets of coefficients to the right ranges in the data
@addConstraint(m, approx1[i in 1:76], fEst[i] == fit(intensity[i], coeff1))
@addConstraint(m, approx2[i in 77:n], fEst[i] == fit(intensity[i], coeff2))

#Fluorescence is zero when intensity is zero
@addConstraint(m, fit(0, coeff1) == 0)
#Output at x=4 must agree
@addConstraint(m, fit(splitNo, coeff1) == fit(splitNo, coeff2))
#Slope at x=4 must agree
@addConstraint(m, fitDer(splitNo, coeff1) == fitDer(splitNo, coeff2))

#Minimize error
@setObjective(m, Min, sum((fluorescence - fEst).^2))
solve(m);
```

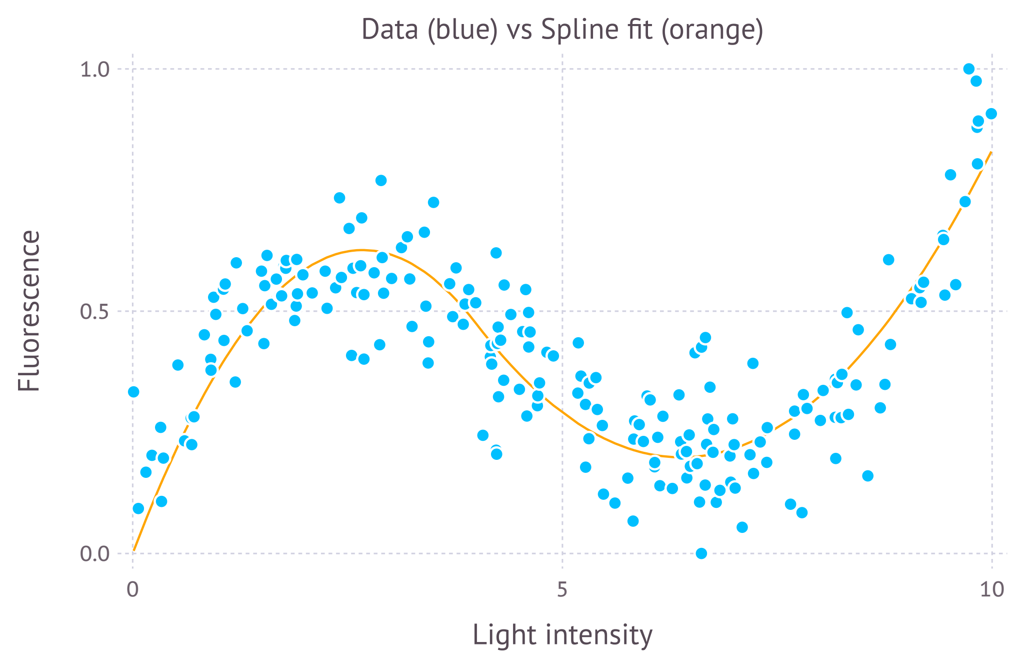
```
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:      811
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:      200

Total number of variables............................:      206
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:      203
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 0.00e+00 2.94e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -3.7369529e+01 1.55e-15 3.02e-14  -1.0 2.17e+00    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 1

                                   (scaled)                 (unscaled)
Objective...............:  -3.7369528913507295e+01   -3.7369528913507295e+01
Dual infeasibility......:   3.0198066269804258e-14    3.0198066269804258e-14
Constraint violation....:   1.5543122344752192e-15    1.5543122344752192e-15
Complementarity.........:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error.......:   3.0198066269804258e-14    3.0198066269804258e-14


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 2
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 2
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.005
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
```

In [13]:
```
#Display form of the function
println()
cf1 = getValue(coeff1)
cf2 = getValue(coeff2)
println("Spline fit: y = {", cf1[1], "x^2 + ", cf1[2], "x + ", cf1[3], ", 0 <= x < 4")
println("                {", cf2[1], "x^2 + ", cf2[2], "x + ", cf2[3], ", 4 <= x < 10")
```

```
Spline fit: y = {-0.08732605915558489x^2 + 0.46768203922332197x + -2.2559572996574326e-29, 0 <= x < 4
                {0.048468267047463175x^2 + -0.6186725704010627x + 2.1727092192487696, 4 <= x < 10
```

In [14]:
```
using Gadfly
#Plot data points and spline fit
plot(  layer(x = intensity, y = fluorescence, Geom.point),
       layer(x = intensity, y = getValue(fEst), Geom.line, Theme(default_color = colorant"orange")),
Guide.xlabel("Light intensity"), Guide.ylabel("Fluorescence"), Guide.title("Data (blue) vs Spline fit (oran
)
```

Out[14]:



hw3-q5(a)

The model represents their thruster inputs, velocity and position as pairs of values, vertical and horizontal. The dynamics on position and velocity are encoded as constraints on position and velocity as related to other variables at each time after t = 1. Their initial position and velocity are encoded as equality constraints on the first values of their velocity and position. That they must rendezvous at the end means that their position variables must be equal at t = 60. The model finds the path that minimizes total energy used as represented as the sum of squared norm of the thruster inputs.

In [15]:
```
T = 60      #Number of seconds available

using JuMP
m = Model()

@defVar(m, xA[1:2, 1:T])       #Alice's position
@defVar(m, xB[1:2, 1:T])       #Bob's position
@defVar(m, vA[1:2, 1:T])       #Alice's velocity
@defVar(m, vB[1:2, 1:T])       #Bob's velocity
@defVar(m, uA[1:2, 1:T])       #Alice's thruster inputs
@defVar(m, uB[1:2, 1:T])       #Bob's thruster inputs

#Dynamics on both of their position and velocity
for t in 1:(T - 1)
    @addConstraint(m, xA[:,t+1] .== xA[:,t] + vA[:,t]/3600)
    @addConstraint(m, xB[:,t+1] .== xB[:,t] + vB[:,t]/3600)
    @addConstraint(m, vA[:,t+1] .== vA[:,t] + uA[:,t])
    @addConstraint(m, vB[:,t+1] .== vB[:,t] + uB[:,t])
end

#Their initial velocities
@addConstraint(m, vA[:, 1] .== [0, 20])
@addConstraint(m, vB[:, 1] .== [30, 0])
#Bob's relative position at start time
@addConstraint(m, xB[:, 1] .== xA[:, 1] + [0.5, 0])
#Both have the same position at the end
@addConstraint(m, xA[:, T] .== xB[:, T])

#Minimize energy used
@setObjective(m, Min, sum(uA.^2) + sum(uB.^2))
solve(m);
```

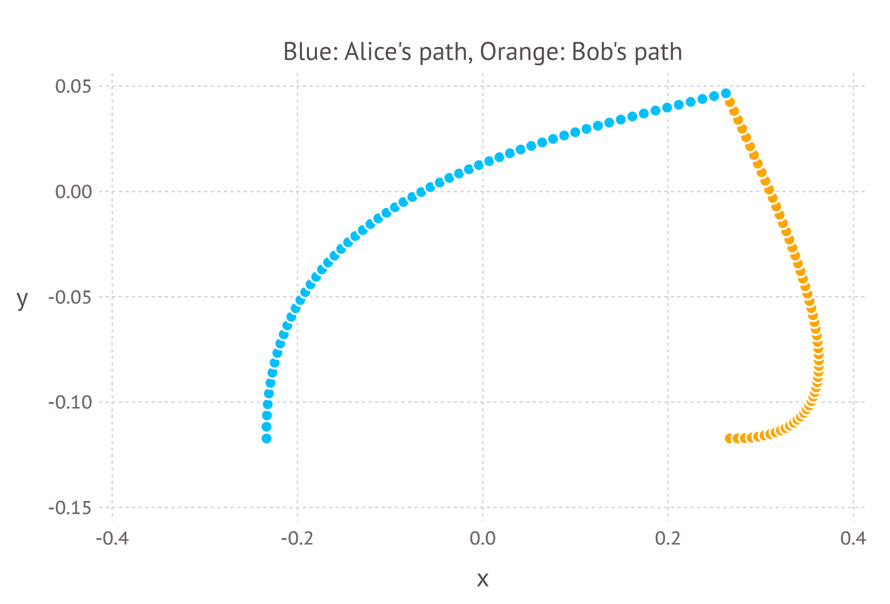
```
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:     1428
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:      240

Total number of variables............................:      720
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:      480
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 3.00e+01 0.00e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  1.0593070e+02 6.61e-15 4.44e-16  -1.0 4.58e+01    -  1.00e+00 1.00e+00h  1

Number of Iterations....: 1

                                   (scaled)                 (unscaled)
Objective...............:   1.0593070479102037e+02    1.0593070479102037e+02
Dual infeasibility......:   4.4408920985006262e-16    4.4408920985006262e-16
Constraint violation....:   6.6058269965196814e-15    6.6058269965196814e-15
Complementarity.........:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error.......:   6.6058269965196814e-15    6.6058269965196814e-15


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 2
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 2
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.007
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
```

In [29]:
```
#Display thruster inputs
thrA = getValue(uA)
println("Alice's thruster inputs:")
for i in 1:T
    println("Time $i: [ ", thrA[1,i], ", ", thrA[2,i], "]")
end

println()
thrB = getValue(uB)
println("Bob's thruster inputs:")
for i in 1:T
    println("Time $i: [ ", thrB[1,i], ", ", thrB[2,i], "]")
end
```

```
Alice's thruster inputs:
Time 1: [ 1.5514993481095176, -0.512820512820513]
Time 2: [ 1.5247493593490087, -0.5039787798408489]
Time 3: [ 1.4979993705884997, -0.495137046861185]
Time 4: [ 1.471249381827991, -0.4862953138815209]
Time 5: [ 1.444499393067482, -0.4774535809018569]
Time 6: [ 1.4177494043069732, -0.4686118479221929]
Time 7: [ 1.3909994155464642, -0.4597701149425289]
Time 8: [ 1.3642494267859553, -0.45092838196286483]
Time 9: [ 1.3374994380254461, -0.4420866489832008]
Time 10: [ 1.3107494492649372, -0.4332449160035368]
Time 11: [ 1.2839994605044283, -0.42440318302387275]
Time 12: [ 1.2572494717439193, -0.41556145004420875]
Time 13: [ 1.2304994829834104, -0.40671971706454474]
Time 14: [ 1.2037494942229017, -0.39787798408488073]
Time 15: [ 1.1769995054623927, -0.3890362511052167]
Time 16: [ 1.1502495167018838, -0.3801945181255527]
Time 17: [ 1.1234995279413746, -0.37135278514588865]
Time 18: [ 1.0967495391808657, -0.3625110521662247]
Time 19: [ 1.0699995504203568, -0.3536693191865607]
Time 20: [ 1.0432495616598478, -0.3448275862068967]
Time 21: [ 1.016499572899339, -0.3359858532272326]
Time 22: [ 0.98974958413883, -0.3271441202475686]
Time 23: [ 0.9629995953783211, -0.31830238726790455]
Time 24: [ 0.9362496066178122, -0.30946065428824054]
Time 25: [ 0.9094996178573033, -0.3006189213085766]
Time 26: [ 0.8827496290967943, -0.2917771883289126]
Time 27: [ 0.8559996403362855, -0.2829354553492486]
Time 28: [ 0.8292496515757765, -0.27409372236958457]
Time 29: [ 0.8024996628152676, -0.26525198938992056]
Time 30: [ 0.7757496740547586, -0.25641025641025655]
Time 31: [ 0.7489996852942498, -0.24756852343059252]
Time 32: [ 0.7222496965337409, -0.2387267904509285]
Time 33: [ 0.695499707773232, -0.22988505747126448]
Time 34: [ 0.6687497190127231, -0.22104332449160047]
Time 35: [ 0.6419997302522141, -0.21220159151193643]
Time 36: [ 0.6152497414917052, -0.20335985853227243]
Time 37: [ 0.5884997527311963, -0.1945181255526084]
Time 38: [ 0.5617497639706873, -0.18567639257294438]
Time 39: [ 0.5349997752101784, -0.17683465959328035]
Time 40: [ 0.5082497864496694, -0.16799292661361634]
Time 41: [ 0.4814997976891605, -0.15915119363395233]
Time 42: [ 0.4547498089286516, -0.1503094606542883]
Time 43: [ 0.42799982016814264, -0.1414677276746243]
Time 44: [ 0.4012498314076337, -0.13262599469496028]
Time 45: [ 0.3744998426471248, -0.12378426171529626]
Time 46: [ 0.3477498538866159, -0.11494252873563224]
Time 47: [ 0.320999865126107, -0.10610079575596822]
Time 48: [ 0.29424987636559813, -0.0972590627763042]
Time 49: [ 0.2674998876050892, -0.08841732979664019]
Time 50: [ 0.24074989884458028, -0.07957559681697617]
Time 51: [ 0.21399991008407135, -0.07073386383731214]
Time 52: [ 0.18724992132356244, -0.06189213085764813]
Time 53: [ 0.1604999325630535, -0.05305039787798411]
Time 54: [ 0.1337499438025446, -0.044208664898320094]
Time 55: [ 0.10699995504203567, -0.03536693191865607]
Time 56: [ 0.08024996628152675, -0.026525198938992054]
Time 57: [ 0.05349997752101784, -0.017683465959328036]
Time 58: [ 0.02674998876050892, -0.008841732979664018]
Time 59: [ 0.0, 0.0]
Time 60: [ 0.0, 0.0]

Bob's thruster inputs:
Time 1: [ -1.551499348109519, 0.5128205128205126]
Time 2: [ -1.5247493593490098, 0.5039787798408486]
Time 3: [ -1.4979993705885009, 0.4951370468611846]
Time 4: [ -1.471249381827992, 0.4862953138815206]
Time 5: [ -1.4444993930674828, 0.4774535809018566]
Time 6: [ -1.4177494043069738, 0.46861184792219257]
Time 7: [ -1.390999415546465, 0.4597701149425856]
Time 8: [ -1.3642494267859557, 0.45092838196286456]
Time 9: [ -1.3374994380254468, 0.44208664898320055]
Time 10: [ -1.3107494492649376, 0.43324491600353654]
Time 11: [ -1.2839994605044287, 0.42440318302387253]
Time 12: [ -1.2572494717439198, 0.4155614500442085]
Time 13: [ -1.2304994829834106, 0.4067197170645445]
Time 14: [ -1.2037494942229017, 0.397877984084805]
Time 15: [ -1.1769995054623925, 0.3890362511052165]
Time 16: [ -1.1502495167018836, 0.3801945181255525]
Time 17: [ -1.1234995279413746, 0.3713527851458885]
Time 18: [ -1.0967495391808655, 0.3625110521662245]
Time 19: [ -1.0699995504203565, 0.3536693191865604]
Time 20: [ -1.0432495616598474, 0.3448275862068964]
Time 21: [ -1.0164995728993385, 0.3359858532272324]
Time 22: [ -0.9897495841388295, 0.327144120247568]
Time 23: [ -0.9629995953783206, 0.318302387267904]
Time 24: [ -0.9362496066178118, 0.309460654288240]
Time 25: [ -0.9094996178573028, 0.30061892130857637]
Time 26: [ -0.8827496290967939, 0.29177718832891236]
Time 27: [ -0.855999640336285, 0.28293545534924835]
Time 28: [ -0.8292496515757761, 0.27409372236958435]
Time 29: [ -0.8024996628152672, 0.26525198938992034]
Time 30: [ -0.7757496740547583, 0.25641025641025633]
Time 31: [ -0.7489996852942493, 0.24756852343059235]
Time 32: [ -0.7222496965337404, 0.23872679045092834]
Time 33: [ -0.6954997077732316, 0.22988505747126434]
Time 34: [ -0.6687497190127226, 0.22104332449160033]
Time 35: [ -0.6419997302522137, 0.21220159151193632]
Time 36: [ -0.6152497414917049, 0.20335985853227231]
Time 37: [ -0.5884997527311959, 0.19451812555260833]
Time 38: [ -0.561749763970687, 0.18567639257294433]
Time 39: [ -0.5349997752101782, 0.17683465959328032]
Time 40: [ -0.5082497864496692, 0.1679929266136163]
Time 41: [ -0.48149979768916035, 0.1591511936339523]
Time 42: [ -0.45474980892865147, 0.1503094606542883]
Time 43: [ -0.42799982016814253, 0.1414677276746243]
Time 44: [ -0.40124983140763365, 0.13262599469496028]
Time 45: [ -0.37449984264712477, 0.12378426171529627]
Time 46: [ -0.34774985388661583, 0.11494252873563225]
Time 47: [ -0.32099986512610695, 0.10610079575596823]
Time 48: [ -0.294249876365598, 0.09725906277630421]
Time 49: [ -0.26749988760508914, 0.0884173297966402]
Time 50: [ -0.24074989884458023, 0.07957559681697618]
Time 51: [ -0.2139999100840713, 0.07073386383731216]
Time 52: [ -0.18724992132356238, 0.06189213085764813]
Time 53: [ -0.16049993256305348, 0.05305039787798411]
Time 54: [ -0.13374994380254457, 0.04420866489832009]
Time 55: [ -0.10699995504203566, 0.03536693191865607]
Time 56: [ -0.08024996628152675, 0.02652519893899205]
Time 57: [ -0.05349997752101783, 0.017683465959328036]
Time 58: [ -0.026749988760508915, 0.008841732979664018]
Time 59: [ 0.0, 0.0]
Time 60: [ 0.0, 0.0]
```

In [30]:
```
#Display paths
using Gadfly
plot(layer(x = getValue(xA[1,:]), y = getValue(xA[2,:]), Geom.point),
layer(x = getValue(xB[1,:]), y = getValue(xB[2,:]), Geom.point, Theme(default_color = colorant"orange")),
Guide.title("Blue: Alice's path,  Orange: Bob's path")
)
```

Out[30]:



In [32]:
```
println("Rendezvous location: ", getValue(xA[:, T]))   #Display rendezvous location

#Get both of their maximum speed
speedA = []
speedB = []

for i in 1:T
    push!(speedA, sqrt(sum(getValue(vA[:, i]).^2)))
    push!(speedB, sqrt(sum(getValue(vB[:, i]).^2)))
end

println("Alice's Maximum speed: ", findmax(speedA)[1], " mph")
println("Bob's Maximum speed: ", findmax(speedB)[1], " mph")
```

```
Rendezvous location: [0.26244806418496114,0.0466231086432428]
Alice's Maximum speed: 46.027783679859574 mph
Bob's Maximum speed: 30.0 mph
```

The model is similar to the one in in 5(a) but with an added constraint that their velocity variables must be equal to each other at t = 60. The optimal rendezvous location in this case is different than in 5(a). In 5(a) it was at [0.262, 0.046] while here it is at [0.244, 0.062]

In [7]:

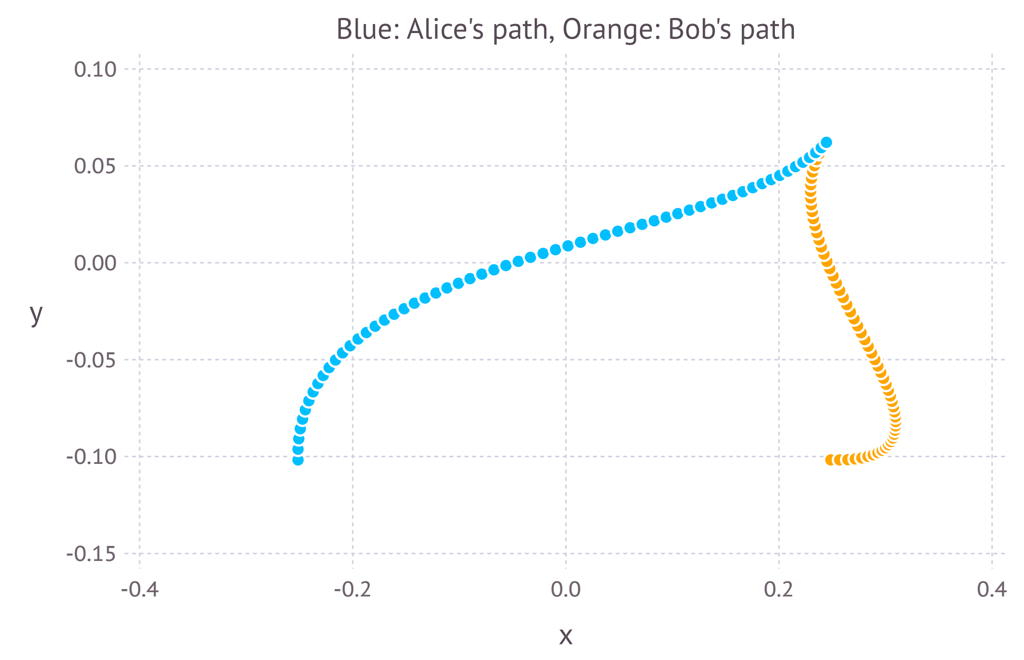
```
T = 60      #Number of seconds available

using JuMP
m = Model()

@defVar(m, xA[1:2, 1:T])       #Alice's position
@defVar(m, xB[1:2, 1:T])       #Bob's position
@defVar(m, vA[1:2, 1:T])       #Alice's velocity
@defVar(m, vB[1:2, 1:T])       #Bob's velocity
@defVar(m, uA[1:2, 1:T])       #Alice's thruster inputs
@defVar(m, uB[1:2, 1:T])       #Bob's thruster inputs

#Dynamics on both of their position and velocity
for t in 1:(T - 1)
    @addConstraint(m, xA[:,t+1] .== xA[:,t] + vA[:,t]/3600)
    @addConstraint(m, xB[:,t+1] .== xB[:,t] + vB[:,t]/3600)
    @addConstraint(m, vA[:,t+1] .== vA[:,t] + uA[:,t])
    @addConstraint(m, vB[:,t+1] .== vB[:,t] + uB[:,t])
end

#Their initial velocities
@addConstraint(m, vA[:, 1] .== [0, 20])
@addConstraint(m, vB[:, 1] .== [30, 0])
#Bob's relative position at start time
@addConstraint(m, xB[:, 1] .== xA[:, 1] + [0.5, 0])
#Both have the same position at the end
@addConstraint(m, xA[:, T] .== xB[:, T])
#Their velocity must be the same at the rendezvous time
@addConstraint(m, vA[:, T] .== vB[:, T])

#Minimize energy used
@setObjective(m, Min, sum(uA.^2) + sum(uB.^2))
solve(m);
```

```
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:     1432
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:      240

Total number of variables............................:      720
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:      482
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 3.00e+01 0.00e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  2.3457043e+02 5.44e-15 8.88e-16  -1.0 4.22e+01    -  1.00e+00 1.00e+00h  1

Number of Iterations....: 1

                                   (scaled)                 (unscaled)
Objective...............:   2.3457042665108142e+02    2.3457042665108142e+02
Dual infeasibility......:   8.8817841970012523e-16    8.8817841970012523e-16
Constraint violation....:   5.4400928206632670e-15    5.4400928206632670e-15
Complementarity.........:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error.......:   5.4400928206632670e-15    5.4400928206632670e-15


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 2
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 2
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.006
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
```

In [8]:

```
#Display paths
using Gadfly
plot(layer(x = getValue(xA[1,:]), y = getValue(xA[2,:]), Geom.point),
layer(x = getValue(xB[1,:]), y = getValue(xB[2,:]), Geom.point, Theme(default_color = colorant"orange")),
Guide.title("Blue: Alice's path,  Orange: Bob's path")
)
```

Out[8]:



In [9]:

```
println("Rendezvous location: ", getValue(xA[:, T]))  #Display rendezvous location

#Get both of their maximum speed
speedA = []
speedB = []

for i in 1:T
    push!(speedA, sqrt(sum(getValue(vA[:, i]).^2)))
    push!(speedB, sqrt(sum(getValue(vB[:, i]).^2)))
end

println("Alice's Maximum speed: ", findmax(speedA)[1], " mph")
println("Bob's Maximum speed: ", findmax(speedB)[1], " mph")
```

```
Rendezvous location: [0.2445487531657683,0.06209742349144542]
Alice's Maximum speed: 42.79755747657692 mph
Bob's Maximum speed: 30.0 mph
```

In [ ]:

The previous models in 5(a) and 5(b) did violate this speed limit as there it is displayed that Alice's maximum speed was 46 mph and 42 mph respectively. To account for the speed limit the model now calculates the speed at each time t using the respective velocity. Then the speed values are constrained not to exceed the speed limit. The solution this time does respect the speed limit as below it is displayed that Alice and Bob's maximum speeds achieved were 35 mph and 30 mph respectively.

In [1]:

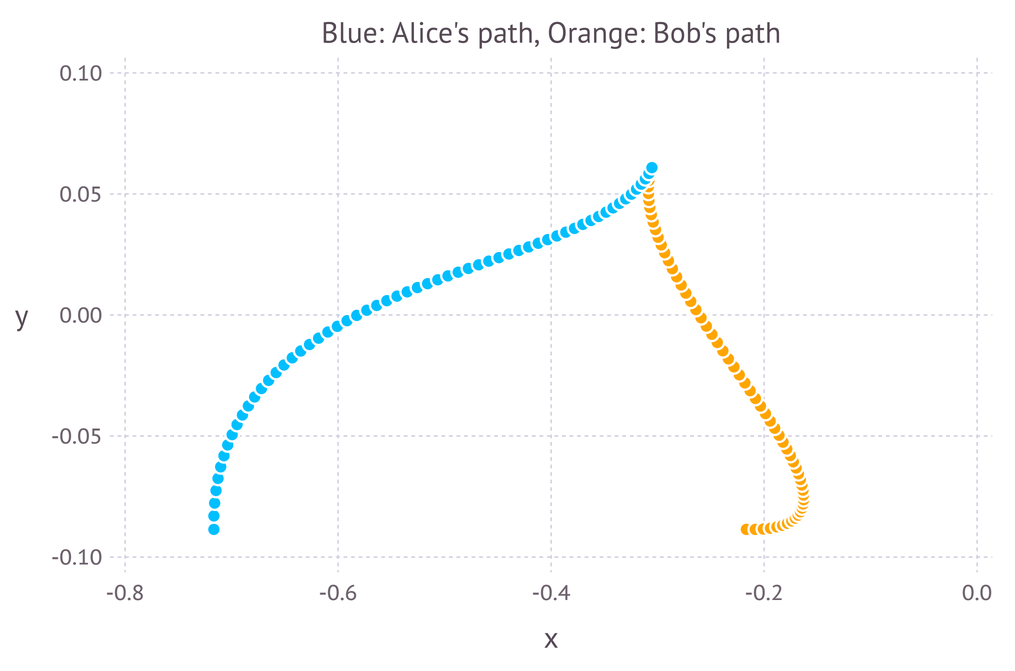
```
T = 60       #Number of seconds available
topSpeed = 35    #Top speed of the hovercrafts

using JuMP
m = Model()

@defVar(m, xA[1:2, 1:T])        #Alice's position
@defVar(m, xB[1:2, 1:T])        #Bob's position
@defVar(m, vA[1:2, 1:T])        #Alice's velocity
@defVar(m, vB[1:2, 1:T])        #Bob's velocity
@defVar(m, uA[1:2, 1:T])        #Alice's thruster inputs
@defVar(m, uB[1:2, 1:T])        #Bob's thruster inputs

#Dynamics on both of their position and velocity
for t in 1:(T - 1)
    @addConstraint(m, xA[:,t+1] .== xA[:,t] + vA[:,t]/3600)
    @addConstraint(m, xB[:,t+1] .== xB[:,t] + vB[:,t]/3600)
    @addConstraint(m, vA[:,t+1] .== vA[:,t] + uA[:,t])
    @addConstraint(m, vB[:,t+1] .== vB[:,t] + uB[:,t])
end

#Their initial velocities
@addConstraint(m, vA[:, 1] .== [0, 20])
@addConstraint(m, vB[:, 1] .== [30, 0])
#Bob's relative position at start time
@addConstraint(m, xB[:, 1] .== xA[:, 1] + [0.5, 0])
#Both have the same position at the end
@addConstraint(m, xA[:, T] .== xB[:, T])
#Their velocity must be the same at the rendezvous time
@addConstraint(m, vA[:, T] .== vB[:, T])

#Top speed constraint
@addConstraint(m, topSpeedA[i in 1:T], sum(vA[:, i].^2) <= topSpeed^2)
@addConstraint(m, topSpeedB[i in 1:T], sum(vB[:, i].^2) <= topSpeed^2)

#Minimize energy used
@setObjective(m, Min, sum(uA.^2) + sum(uB.^2))
solve(m);
```

```
******************************************************************************
This program contains Ipopt, a library for large-scale nonlinear optimization.
 Ipopt is released as open source code under the Eclipse Public License (EPL).
         For more information visit http://projects.coin-or.org/Ipopt (http://projects.coin-or.org/Ipopt)
******************************************************************************

This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:     1432
Number of nonzeros in inequality constraint Jacobian.:      480
Number of nonzeros in Lagrangian Hessian.............:      480

Total number of variables............................:      720
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:      482
Total number of inequality constraints...............:      120
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:      120

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 3.00e+01 0.00e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  1.4977469e+03 6.14e-07 2.53e+01  -1.0 3.00e+01  -4.0 9.90e-01 1.00e+00H  1
   2  6.7455898e+02 4.44e-15 4.78e+00  -1.0 3.15e+02    -  8.34e-01 1.00e+00F  1
   3  3.0189887e+02 6.00e-15 1.79e+00  -1.0 6.13e+02    -  7.71e-01 1.00e+00F  1
   4  2.6913261e+02 6.34e-09 1.10e-01  -1.0 1.22e+02  -4.5 9.05e-01 1.00e+00F  1
   5  2.5328551e+02 6.31e-09 2.71e-02  -1.0 1.33e+02  -5.0 9.86e-01 1.00e+00F  1
   6  2.4510191e+02 2.89e-15 1.05e-02  -1.7 1.77e+02  -5.4 1.00e+00 1.00e+00f  1
   7  2.3941900e+02 5.16e-15 2.74e-03  -1.7 2.55e+02  -5.9 1.00e+00 1.00e+00h  1
   8  2.3825289e+02 9.67e+00 1.46e-03  -1.7 1.88e+02    -  1.00e+00 4.75e-01h  1
   9  2.3842310e+02 6.16e-15 1.18e-03  -1.7 2.96e+01    -  1.00e+00 1.00e+00h  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10  2.3813020e+02 5.53e-02 8.60e-05  -2.5 2.72e+01  -6.4 1.00e+00 9.63e-01h  1
  11  2.3813289e+02 5.22e-15 3.05e-05  -2.5 6.37e+00    -  1.00e+00 1.00e+00h  1
  12  2.3810729e+02 1.34e-03 2.61e-05  -3.8 6.08e+00  -6.9 1.00e+00 8.99e-01h  1
  13  2.3810584e+02 6.83e-15 3.99e-06  -3.8 1.54e+00  -7.3 1.00e+00 1.00e+00h  1
  14  2.3810477e+02 6.95e-06 6.44e-06  -5.7 5.83e-01  -7.8 1.00e+00 9.35e-01h  1
  15  2.3810472e+02 5.11e-15 2.55e-08  -5.7 5.79e-02    -  1.00e+00 1.00e+00h  1
  16  2.3810470e+02 5.36e-15 1.27e-10  -8.6 6.06e-03  -8.3 1.00e+00 1.00e+00h  1

Number of Iterations....: 16

                                   (scaled)                 (unscaled)
Objective...............:   2.3810470469367411e+02    2.3810470469367411e+02
Dual infeasibility......:   1.2679086163687157e-10    1.2679086163687157e-10
Constraint violation....:   5.3568260938163803e-15    5.3568260938163803e-15
Complementarity.........:   7.3419014064403857e-09    7.3419014064403857e-09
Overall NLP error.......:   7.3419014064403857e-09    7.3419014064403857e-09


Number of objective function evaluations             = 27
Number of objective gradient evaluations             = 17
Number of equality constraint evaluations            = 27
Number of inequality constraint evaluations          = 27
Number of equality constraint Jacobian evaluations   = 17
Number of inequality constraint Jacobian evaluations = 17
Number of Lagrangian Hessian evaluations             = 16
Total CPU secs in IPOPT (w/o function evaluations)   =      0.162
Total CPU secs in NLP function evaluations           =      0.076

EXIT: Optimal Solution Found.
```

In [2]:

```
#Display paths
using Gadfly
plot(layer(x = getValue(xA[1,:]), y = getValue(xA[2,:]), Geom.point),
layer(x = getValue(xB[1,:]), y = getValue(xB[2,:]), Geom.point, Theme(default_color = colorant"orange")),
Guide.title("Blue: Alice's path,  Orange: Bob's path")
)
```

Out[2]:



In [3]:

```
println("Rendezvous location: ", getValue(xA[:, T]))  #Display rendezvous location

#Get both of their maximum speed
speedA = []
speedB = []

for i in 1:T
    push!(speedA, sqrt(sum(getValue(vA[:, i]).^2)))
    push!(speedB, sqrt(sum(getValue(vB[:, i]).^2)))
end

println("Alice's Maximum speed: ", findmax(speedA)[1], " mph")
println("Bob's Maximum speed: ", findmax(speedB)[1], " mph")
```

```
Rendezvous location: [-0.3053064205055817,0.06084046370428334]
Alice's Maximum speed: 35.0000001639357 mph
Bob's Maximum speed: 30.0 mph
```