Hamiltonian approach to plane colorings

Sami Heinäsmäki

Abstract

I propose a fixed-range interaction multicomponent spin model, to be used as a physical analog to problems in plane geometry. Specifically, the model is applied to the open problem of the *chromatic number of the plane*. When spin values are interpreted as colors, the lowest energy configurations of the lattice spin system can be interpreted as approximations to plane colorings. I introduce the concept of optimal coloring, corresponding to minimum probability of any color realizing distance one. Approximate optimal lattice colorings with two to seven colors towards the continuum limit suggest that a true coloring of the plane cannot be achieved with less than seven colors.

1 Introduction

Imagine throwing a stick of unit length randomly on an infinite Euclidean plane, colored with q colors. If the endpoints of the stick always fall on different colors, the plane is said to exhibit a *q–coloring*[1]. The smallest such number q is the *chromatic number of the plane*.

The chromatic number of the plane is currently known to be either 5, 6, or 7. The upper bound is based on explicit colorings of the plane, satisfying the unit-distance condition. The exclusion of the lower values 2–4 can be shown using special unit distance graphs. In the simplest case, consider an equilateral triangle with side length 1 on a plane. Two of the vertices are necessarily on a same color, thus proving a two–coloring impossible. Likewise, a slightly more complicated seven–vertex *Moser spindle* [1] shows a three–coloring impossible. This much was known from the problem's inception in the 1950's, until de Grey in 2018 constructed a 1581–vertex graph [2], which proved the impossibility of a four–coloring[2]. The history of the problem prior to de Grey's finding is described in great detail in the book by Soifer [4], which includes discussions of known results and related problems.

In this note I propose a method not directly aiming at a proof, but rather to provide an alternative approach to the problem. It is in a sense related to the aforementioned statistical stick–throwing view of the problem. The method is based on explicit, but necessarily approximate, attempted colorings of the plane. They are obtained as lowest energy configurations of

[1] In other words: none of the colors realizes distance 1.

[2] The graph has since been reduced to 553 vertices [3].

a spin model, when the spin values are interpreted as colors. The model is defined on a lattice, and the colorings in a mathematical sense correspond to the continuum limit of the model. The behavior of the computed ground states as the lattice approaches continuum is taken to indicate whether the corresponding plane coloring is possible.

The model proposed here also gives rise to a natural generalization: one may ask not only when none of the colors realizes distance one, but when the probability to find equal colors separated by unit distance attains a minumum value for a given number of colors, even when zero value is not possible. The minimum energy principle automatically points to such *optimal colorings* as ground states of the model. Approximations to these types of colorings for small number of colors are described here.

The numerical approach comes with many caveats, discussed below, but to the extent computations have been performed here, they point to some general features. The computed optimal results for 2–4 colors are rather simple map–type colorings. For 5–6 colors the optimal results are not purely map–type but contain diffuse areas. Whether the plane can be colored with five or six colors is not resolved here, but only in the case of seven colors there was convergence to zero probability in finding equal colors separated by unit distance. However, even in the seven color case the computed finite lattice coloring was not strictly periodic, so it cannot be directly generalized to obtain a coloring for the infinite plane. Another argument for the existence of colorings can be based on *frustrations* in the lattice, defined as locations where spins have high energy. As discussed below, these correspond to same–color neighbors unit–distance apart within the proposed model. The best–converged lattice colorings for five and six colors showed periodic structures of frustrations which are unlike numerical noise and whose existence indicates that there are obstacles for coloring the lattice with less than seven colors. If the trend persists in further analysis, it suggests that the chromatic number of the plane is seven.

Spin models based on long–range interactions are not commonly met, but there is at least one previous study [5], where Goulko and Kent used a two–spin model with a fixed total magnetization and variable interaction length to study another problem in plane geometry. The discussion in their paper about the validity of the numerical approach is also relevant here.

2 The model

The model proposed here is technically a fixed–range interaction Potts model, with integer-valued spins $\boldsymbol{s}_i$ ($1 \leq \boldsymbol{s}_i \leq q$) located at lattice sites $\boldsymbol{r}_i$. It is described by the Hamiltonian

$$H = \sum_{ij} \delta_{\boldsymbol{s}_i \boldsymbol{s}_j} \delta(|\boldsymbol{r}_i - \boldsymbol{r}_j| - 1) \tag{1}$$

where the summation extends over all pairs of lattice sites. This form immediately points to the main idea: the energy (the value of the classical Hamiltonian for a given configuration of the spins) is lowest when there is a minimum number of equal spins located at lattice sites separated by unit distance. If the energy is zero, all the spins separated by unit distance are different, and when the individual spin values are interpreted as colors, we have a coloring of the lattice. The main hypothesis here is that one may search for the smallest number of spin

values realizing the zero energy in the continuum limit, and this should equal the chromatic number of the plane.

Since we are working in a lattice, a potential problem with the Hamiltonian (1) is evident: it is a mixture of discrete and continuum concepts. The Kronecker delta is appropriate for computing the interaction energy of the spin pairs in the lattice, but the delta function determining the interaction distance must be interpreted suitably in order to apply it in the discretized setting. Fortunately, this problem has been analyzed thoroughly by the fluid mechanics community [6]. The proper approach is based on the relation

$$\delta(x) = \lim_{a\to 0} \frac{1}{a}\phi\left(\frac{x}{a}\right) \tag{2}$$

where ϕ is a distribution with unit integral. When ϕ satisfies certain additional conditions discussed in [6], and a is taken to be the lattice constant, our Hamiltonian can be approximated with a function having a proper interpretation in the lattice and possessing (1) as an appropriate continuum limit.

Taking into account the definition of the delta function in the lattice, a suitably normalized lattice constant–dependent energy functional can be defined by taking the ratio of the energies of the current spin configuration and one where all spins are equal. It is the energy value targeted in the simulations, and can be computed by taking the average of the corresponding single–site quantitites:

$$\epsilon(a) = \frac{1}{N}\sum_{k=1}^{N} \epsilon_k(a) \tag{3}$$

where N is the number of lattice sites and

$$\epsilon_k(a) = \frac{1}{D_\phi(a)}\sum_{i=1}^{N} \delta_{\boldsymbol{s}_i \boldsymbol{s}_k}\phi\left(\frac{|\boldsymbol{r}_i - \boldsymbol{r}_k| - 1}{a}\right). \tag{4}$$

The normalization factor $D_\phi(a)$ is the maximum energy of a single site. It is a constant factor for each choice of the distribution ϕ and the lattice constant a:

$$D_\phi(a) = \sum_{i=1}^{N} \phi\left(\frac{|\boldsymbol{r}_i - \boldsymbol{r}_j| - 1}{a}\right) \tag{5}$$

which is independent of the reference site j. A common factor of a^{-1} has been omitted above. The functional $\epsilon(a)$ stays finite in the limit of an infinite lattice $N \to \infty$ and further when the continuum is approached in the additional limit $a \to 0$ (in the simulations below Na^2 was kept fixed, but of course this merely tends towards a finite region of the plane).

It follows from (3) that always $0 \leq \epsilon(a) \leq 1$ and thus one can also view $\epsilon(a)$ as the probability of finding equal spins separated by unit distance in the lattice. This is utilized below, when we search for *optimal colorings* in the cases $q < 7$, defined by $\epsilon(a)$ attaining a minimum value. Note that there exists an alternative concept in the literature, which can be used as a criterion of goodness of a given coloring [7, 4]. This is the *coloring type*, which is a coloring of a plane with n colors and denoted by $(d_1, d_2, \ldots, d_n)$ such that color i $(1 \leq i \leq n)$ does not realize distance d_i. The goodness is then measured by how many of the d_i can be made equal to one. While this

concept is useful when explicit colorings are constructed, I use the probabilistic measure here, as it is natural in the context of the present spin model analogy where all colors are treated on equal footing.

There is also an another criterion which can be used to judge the quality of the results within the present model: besides aiming at minimum value of $\epsilon(a)$ in (3), one can look at the distribution of the single–site relative energies $\epsilon_k(a)$ over the lattice. A regular distribution of nonzero values is suggestive of a converged local minimum in the spin system. This can be contrasted to an irregular distribution, which is natural to identify with numerical noise. The distributions of the values $\epsilon_k(a)$ can be analyzed together with the convergence of the whole–lattice average $\epsilon(a)$, and they are useful in visualizing the results of the lattice computations below.

An essential feature of the model is that the fixed–range interaction becomes dependent on the lattice constant a. This forms the basis for the study of the model's behavior when approaching the continuum limit. The dependence on a also brings in a major complicating feature, namely that the number of interacting neighbours of each lattice site grows towards the continuum limit. This is in stark contrast to spin models with nearest–neighbour interactions. As an example, for a lattice constant $a = 0.1$ each site has 240 interacting neighbours, whereas for $a = 0.01$ the number is 2520 (these numbers are almost independent on the choice of the distribution ϕ).

3 Lattice computations

Six different spin systems were studied in several sets of computations. The number of spin values (referred also as colors in what follows) varied from 2 to 7. A default set of computations used a lattice of 10×10 unit areas, with four different lattice constants 0.1, 0.05, 0.025, and 0.0125. Also additional computations with larger and smaller lattices were performed, as were computations with smaller lattice constants. This was done to analyze the convergence properties of the computations as a function of a, as well as the finite–size effects on the generated colorings.

The search for the lowest energy configuration, measured by the quantity $\epsilon(a)$ in (3), was performed by simulated annealing. In this approach one starts from a random configuration of spins, and performs a series of Monte Carlo updates of the lattice. The probability to accept updates increasing the individual site energies is gradually lowered (effectively, lowering the temperature of the system), until the spins approach a fixed configuration. The Metropolis algorithm was used for updating the spins. This amounts to generating a new candidate spin for the lattice site, computing the energies ϵ_{new} and ϵ_{old} according to (4), and finally updating the spin with probability $\min[1, \exp((\epsilon_{old} - \epsilon_{new})/T)]$. Here T is an effective temperature–like quantity, whose initial value is proportional to the site energy in a random spin system. An essential part of simulated annealing is the scheme to lower T in the course of simulation. In this work a simple scaling was employed, where a new T value was computed as $T_{i+1} = cT_i$, using values $0.83 \leq c \leq 0.95$. For each T value the lattice was repeatedly updated so that $\epsilon(a)$ converged to a roughly constant value, before lowering T. In all computations a square lattice

was used, and the spins were updated with typewriter ordering. There is no canonical way to perform the simulated annealing process, and I encourage interested readers to write their own simulators and test different schemes. The simulation code used in this work is available as free software [8], including the input files used to generate the results presented here[3].

The lattice serves as an analog to the plane, but an additional approximation is necessary in simulations: the lattice must of course be finite. This potentially causes boundary effects, which were in in some forms present, as seen later. Choosing the boundary conditions is essential in the spin update process. Here free boundaries were used, meaning that the lattice spin values $1 \cdots q$ were treated as being surrounded by zero spins, which were never updated. They thus didn't give contribution to the energy. The zero value played the role of a constant extra color. This was judged to have the least effect on the lattice spins. Boundary effects could be minimized by cyclic boundary conditions, but this would change the underlying topology, and the model would not be a plane analog anymore. Another potentially problematic consequence of the finite lattice is that even a solution that converged to a global minimum is not necessarily periodic, so it cannot be extended to obtain a coloring for an infinite lattice.

The following distribution was used here for approximating the lattice delta function:

$$\phi(x) = \begin{cases} \frac{1}{4}\left(1 + \cos\left(\frac{\pi x}{2}\right)\right), & |x| \leq 2 \\ 0, & |x| > 2 \end{cases} \tag{6}$$

Other possible forms are discussed e.g. by Yang *et al* [9], but were not investigated here. Discretization effects on the lattice computations were also studied in [5], where (6) was used as a default distribution. Notwithstanding the particular approximation used, the most important factor affecting $\epsilon(a)$ is that the finiteness of the lattice makes the computed $\epsilon(a)$ always smaller than the "true" value. This follows from the long range of the interaction: for a lattice size of $L \times L$ unit areas, only a fraction of $(1 - 2/L)^2$ of the sites have all interacting neighbours inside the lattice, the rest being "border spins" having less neighbours and thus contributing less to $\epsilon(a)$ (remember that all individual contributions to $\epsilon(a)$ are always positive). For $L = 10$ this fraction is 0.64 and for $L = 6$ it is only 0.44. Nevertheless, as seen later, the effects resulting from this are not dramatic and can be analyzed by computing $\epsilon(a)$ cumulatively for the lattice, starting from the interior. The fact that lowering a increases the number of interacting neighbours each lattice site has, is the most important limiting factor in the computations. As a rough rule, halving a (while keeping Na^2 constant) increases the computation time tenfold.

A summary of the computed energies is given here, and a more comprehensive discussion of the computations and the associated lattices is given in the following sections. The convergencies of the energies $\epsilon(a)$ towards the continuum limit are shown in Fig. 1. For all numbers of colors except seven the convergence slowed or halted for small a. While only a limited set of computations was performed in this work, and very small lattice constants were not used in simulations, the computed values were rather consistent across simulations and the quoted results probably give reliable order–of–magnitude estimates. The values given in Fig. 1 were obtained in computations using several different lattice sizes, and they are not corrected for finite–size effects. A collection of best estimates for the energies $\epsilon(a)$ are given in Table 1.

[3]Exact reproduction of the results would require a known initial state of the pseudorandom number generator. This is not utilized here, but the best–converged lattices are available for download.

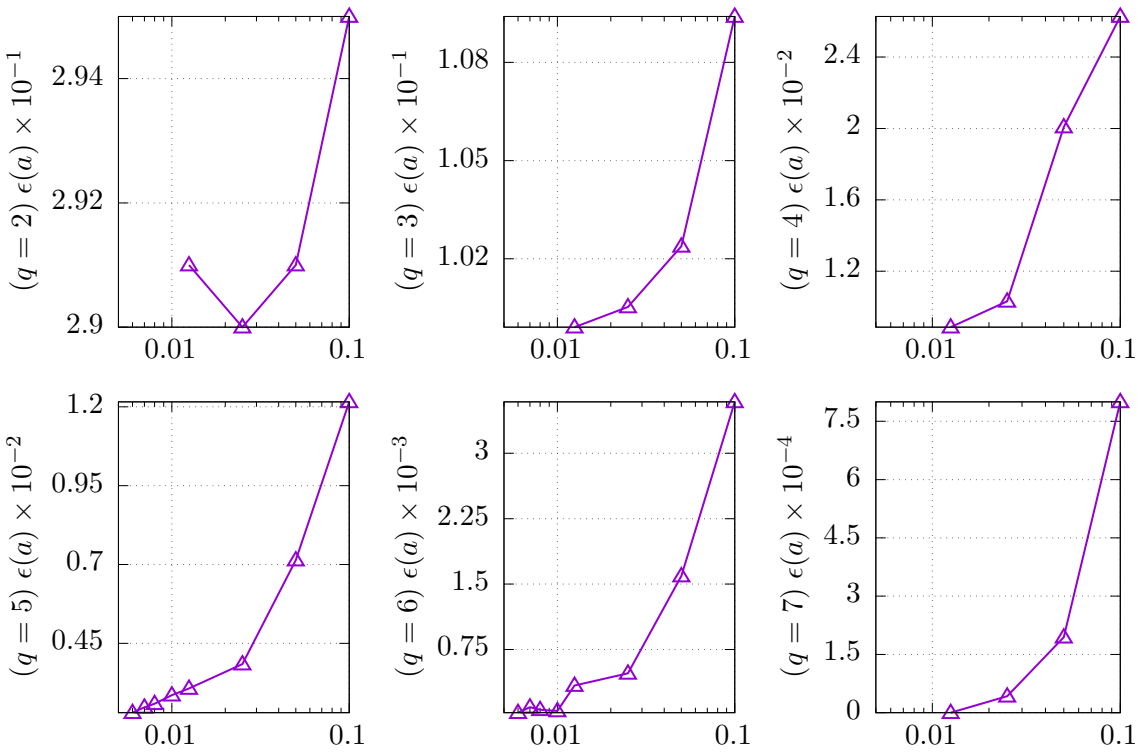

Figure 1: Convergence of the parameter $\epsilon(a)$ for the six cases. Note the logarithmic scale of the x axis denoting the value of the lattice constant a. The case $q = 7$ was the only one which converged to zero. For the cases $q = 5, 6$ there was apparent convergence towards small but finite values. Estimates for the final $\epsilon(a)$ values are given in Table 1.

Besides the values $\epsilon(a)$ for the six cases studied, the lattice parameters employed in those simulations are given. The energy values were approximately corrected to take into account the finite size of the lattice using a method described in the end of Sec. 3.1, but they still represent only order–of–magnitude estimates.

3.1 Two colors

There is no true two–coloring of the plane, but performing the annealing procedure for two (as well as three and four below) colors gives important feedback of the performance of the proposed scheme. For the model to be a useful analog to the original problem, $\epsilon(a)$ should converge to a finite value at least for all $q < 5$, when a tends towards zero. Especially we hope to answer the interesting question of what could be the optimal coloring of the plane with few colors, realizing the lowest probability to find points of same color separated by unit distance.

We look at the two–color case more closely, since here we can compare the numerical results to simple analytically solvable reference configurations. We can estimate the goodness of the solutions, and visualize some general features characteristic to the current approach. It turns out that the the numerical results point to some general potential problems related to finite lattices, but also, that the optimal coloring with two colors might be rather simple.

colors	$\epsilon(a)$	Lattice area	a
2	1/3	20×20	0.0125
3	0.12	10×10	0.0125
4	0.01	10×10	0.0125
5	0.004	8×8	0.0080
6	0.0002	6×6	0.0060
7	0.0	6×6	0.0125

Table 1: Estimates for the optimal values for energies $\epsilon(a)$ in (3), and the associated lattice parameters used in those calculations. The result for two colors is exact for the configuration obtained in a simulation, and the result for even colors was zero within the double precision arithmetic, but is known to be exactly zero in the continuum limit. In other cases $\epsilon(a)$ values are rough estimates obtained using a method described in the end of Sec. 3.1.

The results of the lattice computations for the 10×10 unit areas lattice are shown in Fig. 2. The figures show the evolution of the coloring, as the lattice constant is reduced from 0.1 to 0.0125. In all cases the configurations correspond to "stripes", which straighten and consist of approximately hexagonal elements, joined by short straight sections, towards smaller a. Boundary effects can be seen in all cases. The value $\epsilon(a)$ quickly settles to around 0.29. As noted earlier, the border regions of the lattice have less interacting neighbors inside the lattice, causing the computed $\epsilon(a)$ to underestimate the true probability. The value of $\epsilon(a)$ may also vary between lattices simply due to location of the boundaries with respect to the stripes. Visual estimation of the results suggests that the width of the stripes is below 1. One might think that the roughly hexagonal form of the stripes in the $a = 0.0125$ lattice emerges due to its optimal properties. The other option is that the shape is a boundary effect. To investigate that further, another computation for the same lattice constant, but this time with a 20×20 area (1600×1600 sites) lattice, was performed. The central parts of the 10×10 and 20×20 area lattices are shown in Fig. 3. The larger lattice has approximately straight stripes covering 69 lattice sites, corresponding to width $69a \approx 0.863$. The result suggests that the hexagonal shape in the smaller lattice might arise due to the boundaries, and the larger lattice is needed to reveal the true behavior.

At this point one may note that among the configurations consisting of straight stripes, the simulation result for the 20×20 area lattice is optimal within the accuracy permitted by the lattice. To show that, I analytically determine the probability that points unit distance apart in the plane have the same color for a plane colored with alternatingly colored straight stripes of width h. For comparison I also present the corresponding result for another configuration which easily springs into mind: a checkerboard–colored plane. The checkerboard is taken to consist of squares with side length h. Denote the probabilities for the striped plane and the checkerboard–coloring by $p_s(h)$ and $p_c(h)$, respectively. If the stripes are horizontal, one may choose a reference stripe, whose lower edge serves as the origin of the vertical coordinate z. Introduce the indicator function

$$f(z, \phi, h) = 1 - \mathrm{mod}\left[\mathrm{floor}\left(\frac{z + \sin\phi}{h}\right), 2\right] \tag{7}$$

which equals 1 if the point $(\cos\phi, z + \sin\phi)$, unit distance apart from $(0, z)$, has the same color as $(0, z)$, and zero otherwise. Then the probability $p_s(h)$ is computed by averaging over the

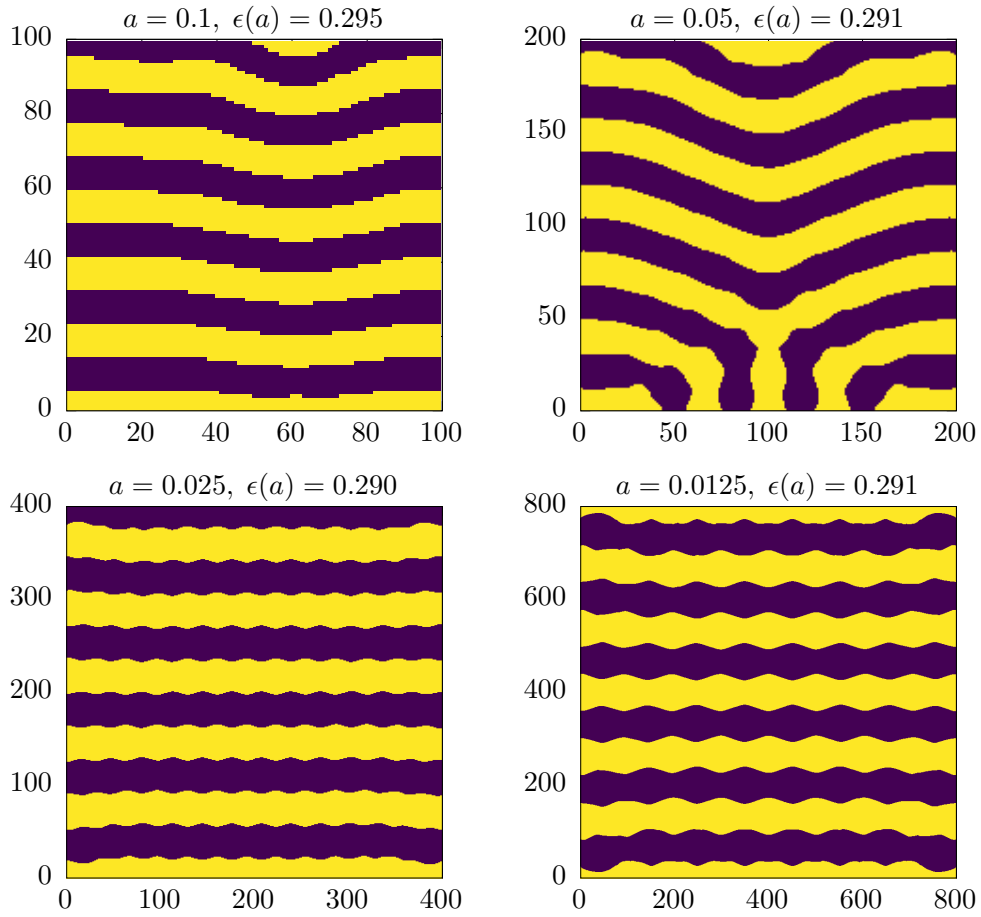

Figure 2: Evolution of the lattice coloring in the two–color case. The area of the lattice is kept at constant 100 unit areas, while the lattice constant a is reduced. The number of sites is shown in the axes. The coarser lattices show clear effects of the boundaries, which are reduced at smaller a.

reference stripe

$$p_s(h) = \frac{1}{2\pi h} \int_0^h \mathrm{d}\, z \int_0^{2\pi} \mathrm{d}\, \phi\ f(z, \phi, h). \tag{8}$$

For the checkerboard, the probability $p_c(h)$ is computed similarly. First introduce the indicator function

$$f(z_1, z_2, \phi, h) = 1 - \mathrm{mod}\left[\mathrm{floor}\left(\frac{z_1 + \cos\phi}{h}\right) + \mathrm{floor}\left(\frac{z_2 + \sin\phi}{h}\right), 2\right], \tag{9}$$

where coordinates z_1 and z_2 run over a reference square, whose lower–left corner is the origin. Averaging over the reference square gives the desired probability:

$$p_c(h) = \frac{1}{2\pi h^2} \int_0^h \mathrm{d}\, z_1 \int_0^h \mathrm{d}\, z_2 \int_0^{2\pi} \mathrm{d}\, \phi\ f(z_1, z_2, \phi, h). \tag{10}$$

Numerical estimates for $p_s(h)$ and $p_c(h)$, as a function of h, are shown in Fig. 4. The minima of the probabilities can be inferred approximately from the curves, showing that for stripes the minimum probability is realized in the region $h \geq 1/2$, whereas for the checkerboard the

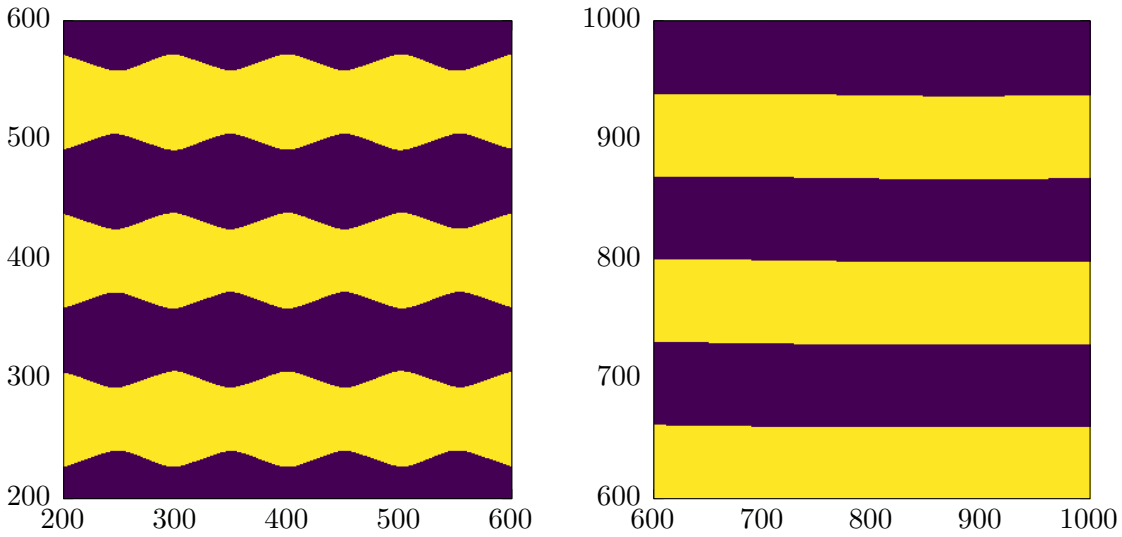

Figure 3: Computed lattice colorings for a 10×10 (left) and 20×20 (right) unit areas with lattice constant $a = 0.0125$. Note that only central portions of lattices are shown.

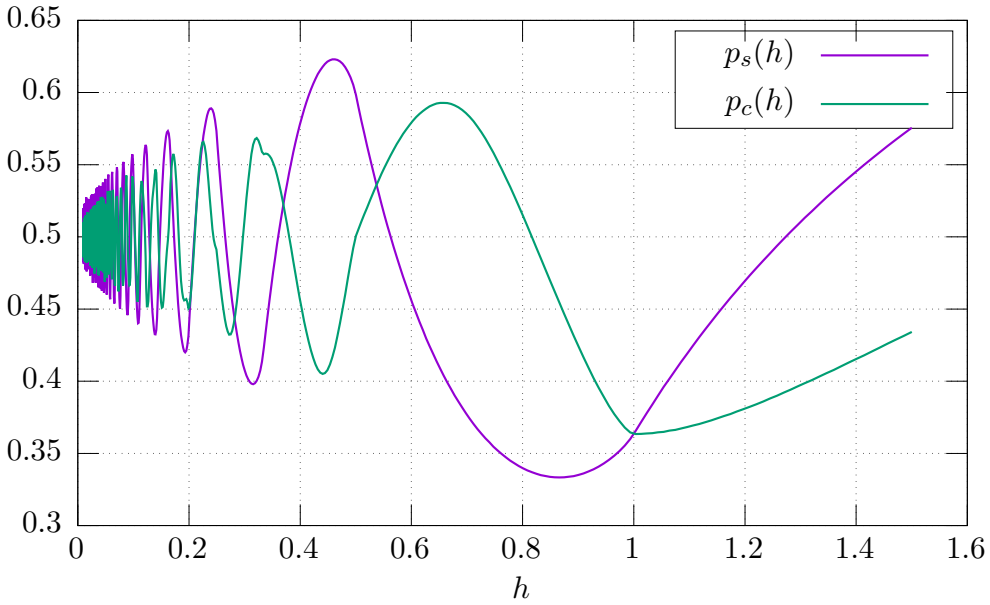

Figure 4: Probability that two randomly chosen points unit distance apart have the same color on a plane two–colored with alternating straight stripes of width h (curve $p_s(h)$) and with a checkerboard coloring with squares of side length h (curve $p_c(h)$), as a function of h.

minimum is at $h \geq 1$. In these regions the averagings according to (8) and (10) can be performed analytically, giving

$$p_s(h) = 1 - \mathrm{Re}\left[\frac{4h\arccos(h) - 4\sqrt{1-h^2} + 2}{\pi h}\right], \; h \geq 1/2 \tag{11}$$

for the striped plane, and

$$p_c(h) = 1 - \frac{4h-2}{\pi h^2}, \; h \geq 1 \tag{12}$$

for the checkerboard. The locations of the mimima and the corresponding probabilities are accordingly

$$\begin{aligned} h &= \frac{\sqrt{3}}{2}, \quad p_s = \frac{1}{3} && \text{(optimal straight stripes)} \\ h &= 1, \qquad p_c = 1 - \frac{2}{\pi} \approx 0.363 && \text{(optimal checkerboard)} \end{aligned}$$

showing that indeed the striped configuration realizes a smaller energy for the two–spin system, corresponding to smaller probability of finding equally–colored points unit distance apart. The analytical result for the optimal width of the stripes $\sqrt{3}/2 \approx 0.866$ agrees with the simulation result 0.863 for the 20×20 lattice within the resolution permitted by the lattice constant.

What is important in this comparison is that for striped configurations simulated annealing is seen to reach the correct global minimum among the landscape of solutions exhibiting an infinite amount of local minima on the left–hand side of the global one. The checkerboard has similar behavior, and while the lowest probability is rather close to the striped plane optimum, those configurations were never realized in the computations. These results, while far from conclusive, give faith that the method is able to reach optimal configurations, even when the underlying set of solutions is very complex.

There is still the question of how closely $\epsilon(a)$ reproduces the theoretical concept of probability. The computed values for the 10×10 and 20×20 lattices were 0.291 and 0.311, respectively. The latter lattice might, however, realize a lower probability. To investigate what regions of the lattice contribute to $\epsilon(a)$ in what way, we use a lattice consisting of straight stripes as a test. We artificially construct an 10×10 area lattice with $a = 0.0125$, having straight stripes of width an_s, where $n_s = \text{int}\left[\sqrt{3}/2a\right] = 69$ is the number of lattice sites covered by the stripe (int: nearest integer value). We then compare $\epsilon(a)$ of this lattice (denoted as ϵ_{gen}) to the corresponding simulated lattice (ϵ_{sim}) in Fig. 2. According to (3), $\epsilon(a)$ represents the average over the whole lattice. If we compute the average cumulatively, starting from the central part of the lattice, we can plot the quantity

$$\epsilon^{(n)}(a) = \frac{1}{n} \sum_{k=1}^{n} \epsilon_k(a) \tag{13}$$

for the central n site sublattices as $n = 1, \ldots, N$. This gives a way to estimate how the boundary regions of the lattice affects the value, and how we can get a reliable estimate for the theoretical probability from the computations. Fig. 5 shows the probabilities computed according to (3) for the artificially generated and the simulated lattices. The figure also shows, as a straight line, the theoretical probability acording to (11). The results show, that the small value of $\epsilon(a)$ for the simulated 10×10 area lattice is indeed due to the boundary region, as the value of $\epsilon(a)$ drops below the straight stripe lattice only just before the boundary is reached. The inner parts the straight stripes lattice show comparable or even smaller probability. The figure also shows that the value of $\epsilon(a)$ is indeed similar to the theoretical probability in (11) and the finite latice of 10×10 unit areas is large enough to realize this connection. This requires that we consider the portion of the lattice, where all spins have all interacting neighbours inside the lattice. There is no obvious best choice for comparing the value $\epsilon(a)$ and the theoretical concept of the probability, but in what follows we estimate the "correct" values as being approximately where the cumulative average as defined by (13) has reached a plateau, similar to what is seen

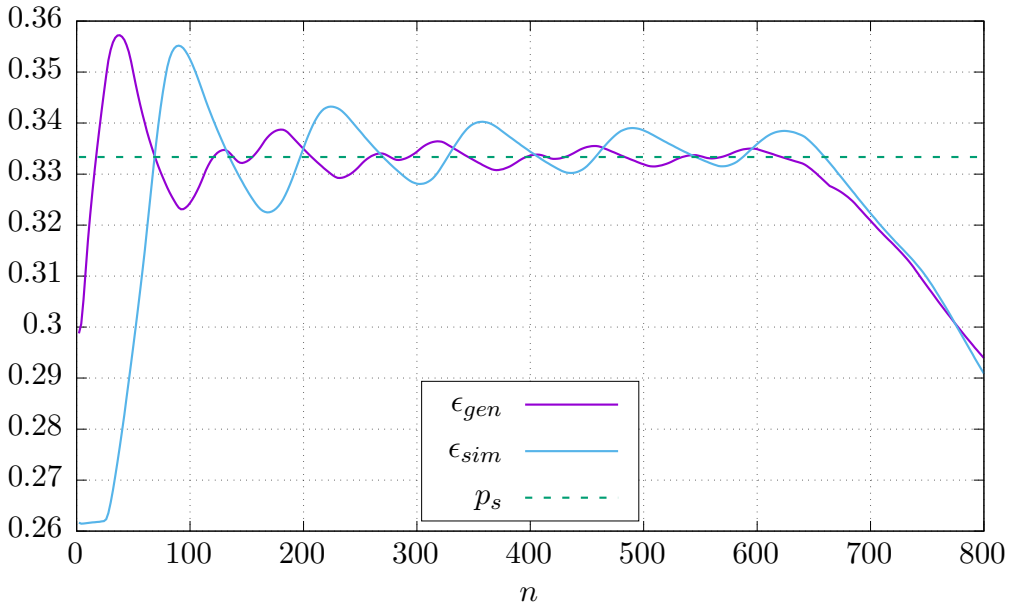

Figure 5: The quantities $\epsilon^{(n)}(a)$ in (13) for a 10×10 area (800×800 sites) lattice with $a = 0.0125$. The curve ϵ_{gen} is for the artificially generated lattice of straights stripes, ϵ_{sim} shows the probability for the simulated lattice in Fig. 2. The theoretical result p_s for the stripe lattice is shown as a dashed line.

in Fig. 5. For larger number of colors below, the behaviour of the cumulative average is more difficult to analyze though.

3.2 Three and four colors

The lattice simulation for a three–coloring is shown in Fig. 6. The right pane shows a heat map showing the values $\epsilon_k(a)$ in (4) over the lattice. It shows the locations of spins having equally–valued neighbours unit distance apart. This kind of map of "non–colorability" is useful in examining the nature of the solutions. It seems natural to take regular (periodic) frustrations to describe "true" effects; ones which are not due to numerical errors or noise, but originating from the fact that the lattice spin system has reached at least a local minimum where there is an obstacle for further lowering of energy.

The computed optimal three–coloring map is not regular, but consists of elements resembling distorted hexagons. The hexagonal structure is more visible in the heat map, showing that the frustrations (spins with maximal energy) are arranged hexagonally, albeit with wiggly boundaries. The value of $\epsilon(a)$ is best inferred in a method similar to Fig. 5, as a cumulative average. The optimum probability for the three–colored lattice is $\epsilon_{opt} \approx 0.12$.

The four color simulation results also in a hexagonal coloring, as shown in the left hand side Fig. 7. The converged value for the relative energy $\epsilon(a)$ was ~ 0.01, an order–of–magnitude improvement over the three–color case. The individual colored areas are this time regular

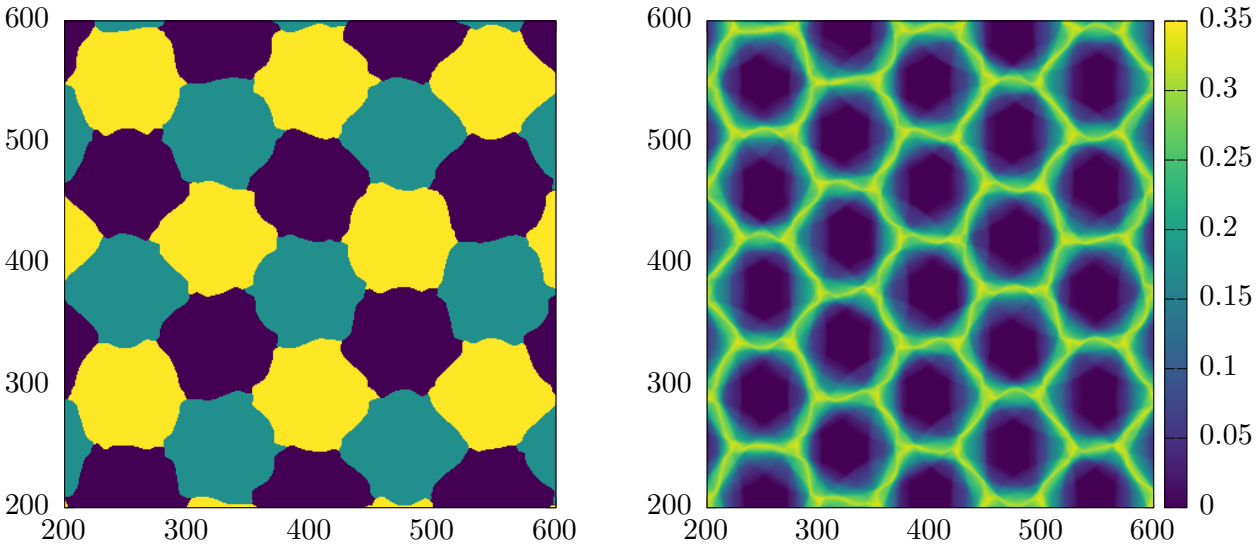

Figure 6: Computed optimal three–coloring of the lattice (left) for lattice constant $a = 0.0125$, and the associated heat map showing the distribution of the values $\epsilon_k(a)$ defined in (4), corresponding to numbers of neighbours with same color unit distance apart (right). Only the central parts of the lattices are shown for clarity. The lattice portion shown is 25 unit areas.

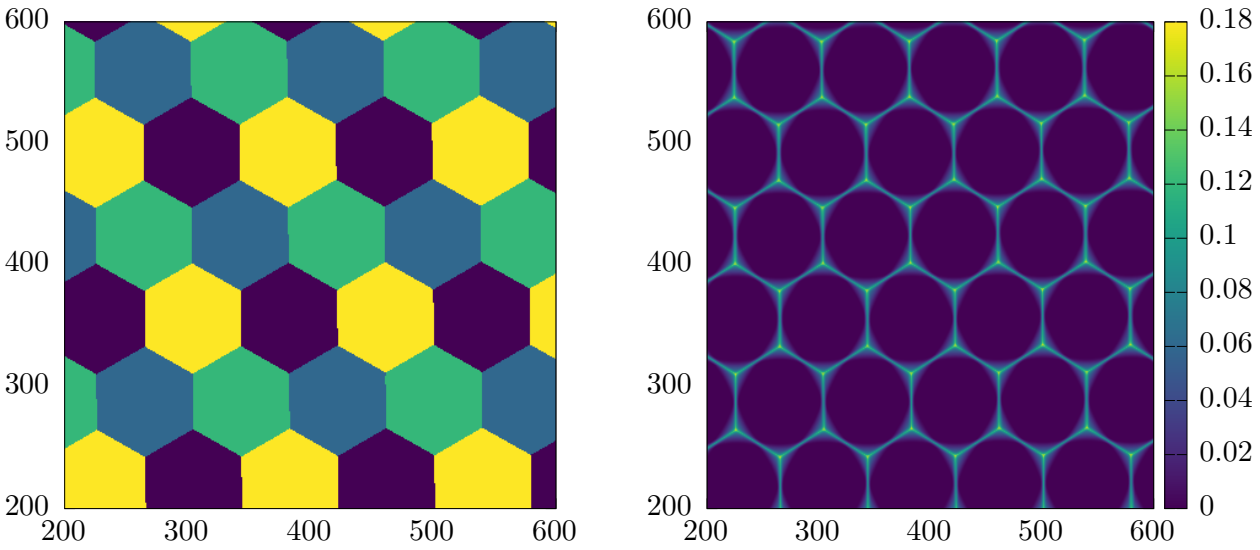

Figure 7: Computed optimal four–coloring of the lattice (left), and the associated heat map showing the numbers of neighbours with same color unit distance apart of the sites (right). The lattice had size of 10×10 unit areas and the lattice constant was $a = 0.0125$. The central 25 unit areas of the lattices are shown.

hexagons, which themselves form a hexagonal lattice with the usual basis vectors

$$\mathbf{u}_1 = \alpha\,\mathbf{i}, \quad \mathbf{u}_2 = \frac{\alpha}{2}\,\mathbf{i} + \frac{\sqrt{3}\alpha}{2}\,\mathbf{j}, \tag{14}$$

where the unit of the hexagonal lattice $\alpha = \sqrt{3}s$, in terms of the side length s of the individual

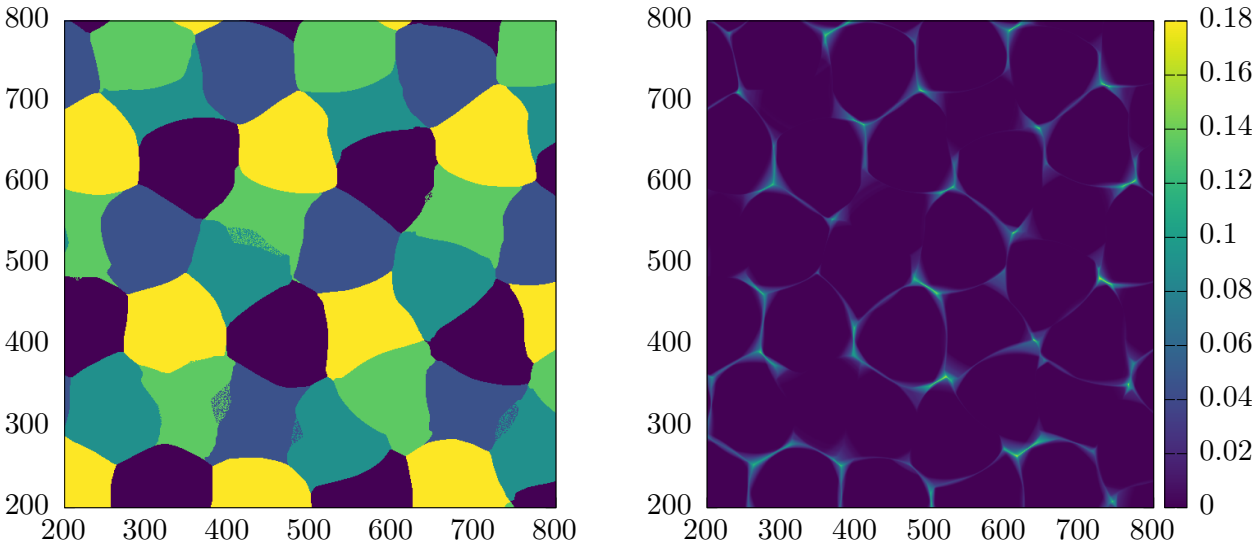

Figure 8: Computed optimal five–coloring of the lattice (left), and the associated heat map showing the numbers of neighbours with same color unit distance apart of the sites (right). The lattice used in the simulation had the size of 8×8 unit areas and lattice constant $a = 0.008$. The central 4.8×4.8 unit areas of the lattices are shown.

hexagons. Hexagons forming a monocolor sublattice are described by lattice vectors $2n\mathbf{u}_1 + 2m\mathbf{u}_2$, with integer n, m. Based on the simulation result with the $a = 0.0125$ lattice constant, an individual hexagon covers 79 lattice sites in the horizontal direction, corresponding to $\alpha \approx 0.988$, just below one (at this resolution, 80 sites would already cover a unit distance). I was not able to analytically estimate an optimal value for α, using the method that worked for the two–color case. However, some other method might well do the trick and it would be interesting to see if the numerical result is indeed optimal for the regular hexagonal configuration.

3.3 Five and six colors

Unlike the map–like colorings above, the computed optimal five–coloring of the lattice includes diffuse areas, indicating locations where spins can have either of the two values without affecting the energy. The heat map of the distribution of $\epsilon_k(a)$ values exhibits a more regular structure showing the border regions of the colored regions have frustrations preventing further lowering of energy. The value of $\epsilon(a)$, estimated as in Sec. 3.1, is roughly $\sim 4 \times 10^{-3}$.

There are reasons to believe that the case of six colors is very difficult to resolve with this method. This is due to the existence "almost six–colorings": seven colorings with a small amount of the seventh color. An example shown in [4] is one with ratio $\sim 1/300$ for the areas of the seventh color to others. It is possible that the annealing method could converge towards the correct minimum, but be unable to tell the difference between zero and finite but very small $\epsilon(a)$. In computations the value $\epsilon(a)$ was seen to decrease for decreasing a, until a plateau was reached around $a = 0.01$. The smallest values obtained for $\epsilon(a)$ were $\sim 2 \times 10^{-4}$. This is hardly

the absolute minimum, but probably shows the correct order of magnitude.

The distribution of frustrations in the lattice for the all the best converged calculations with six spins formed regular structures, as seen in the example calculation in Fig. 9, where the central portions of 700×700 lattices with lattice constant $a = 0.01$ are shown. The heat map

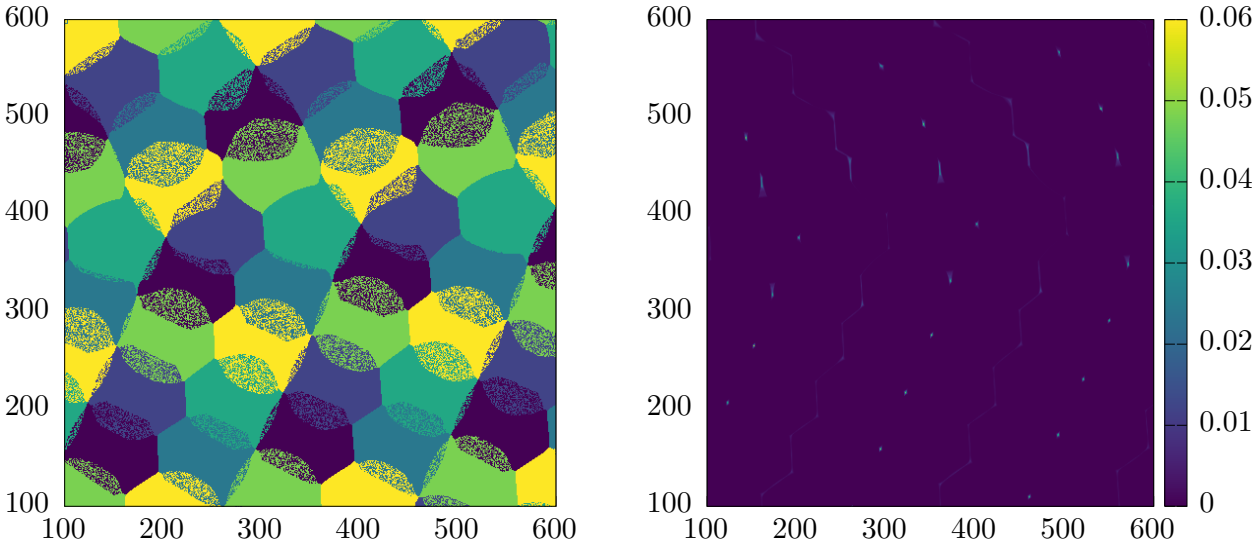

Figure 9: Computed optimal six–coloring of the lattice (left) for lattice constant $a = 0.01$, and the associated heat map showing the numbers of neighbours with same color unit distance apart of the sites (right). The central 25 unit areas of the lattices are shown.

may not be easy to inspect visually, but the distribution of values of $\epsilon_k(a)$ consists of lines of point–like structures, with wiggly lines in between. This particular computation shows a wedge in the middle of the lattice, where the direction of the structures changes. This is most likely a finite–size effect, induced by border regions. The lattice structure formed by frustrations $\epsilon_k(a)$ were similar in other converged six–color computations. This phenomenon can be contrasted to the seven color case discussed below, where even in the calculations not converging to zero the frustrations were irregular domains, giving the impression of numerical noise.

3.4 Seven colors

The case $q = 7$ is important, as it is known through explicit constructions that the plane is seven–colorable. Our method should therefore, if it works as advertised, converge towards zero energy for seven and higher colors. This indeed happened in the computations. A typical computation did not converge to zero, but a successful example is shown in Fig. 10. What is important is that the computation was done with a relatively modest lattice having 480×480 sites and lattice constant $a = 0.0125$, corresponding to lattice size of 6×6 unit areas. The simulation used 110 steps in temperature, with the scaling factor 0.83 between temperature steps, and for each temperature value the lattice was updated 300 times. The values $\epsilon_k(a)$ in (4) all converged to zero within the double precision arithmetic. This amounts to all lattice sites

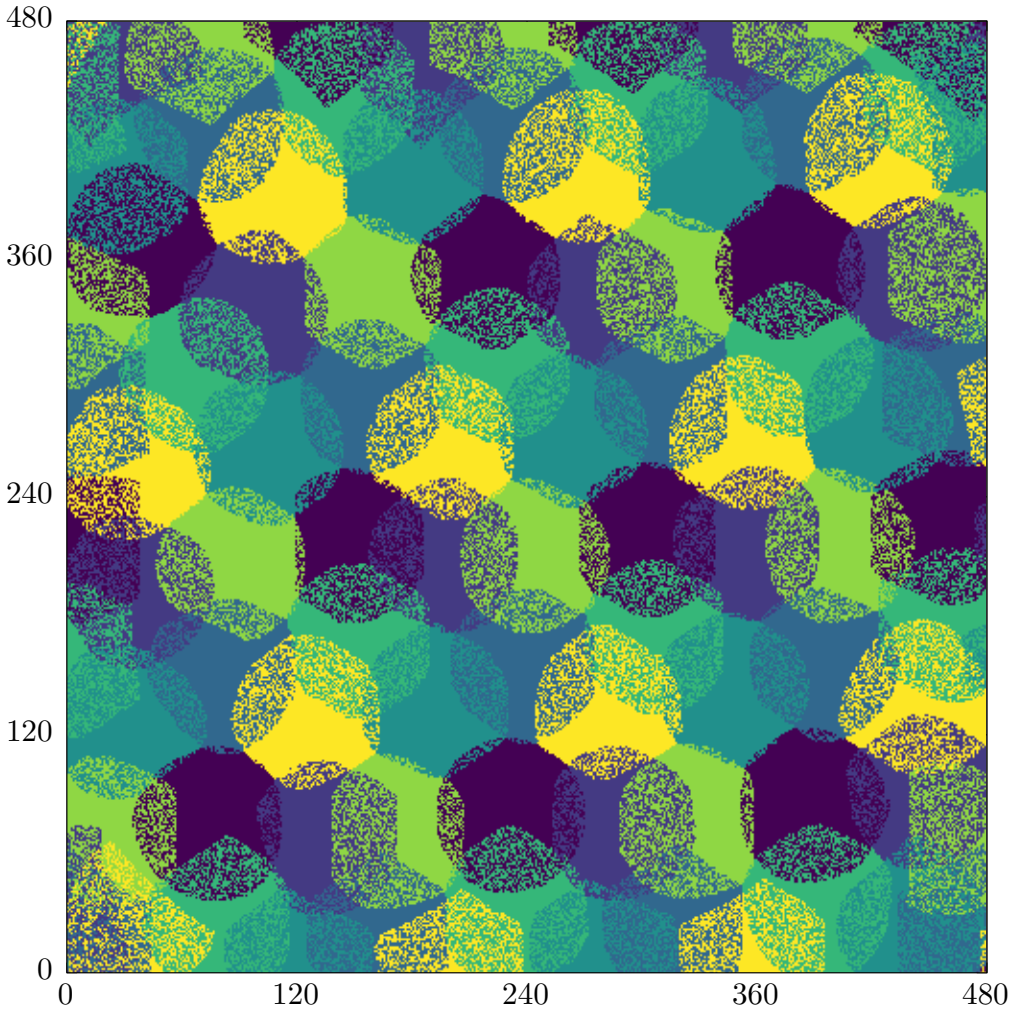

Figure 10: Computed optimal seven–coloring of the lattice for a lattice constant $a = 0.0125$. The whole lattice (6×6 unit areas) is shown. The computation converged to zero energy within the precision used in the computations.

unit distance apart having a different spin, and thus a coloring of the lattice was achieved. This is important as it shows that the lattice constant does not need to be very small for convergence to zero to happen, or that the presence of the boundaries does not spoil the computation. This is one of the reasons (besides computational cost) that the previous cases $q = 5, 6$ were not done with very much smaller lattice constants. There are lot of diffuse domains visible in the lattice, meaning that there is apparent freedom in choosing the colors in these areas. There is some resemblance in the picture to the known explicit construction (see e.g. Soifer's book [4]), which consists of hexagonal elements. The shapes of individual colored areas are not very regular, which points to lot of freedom and probably a continuum of seven–colorings. Since the individual colored areas are not strictly periodic, this particular computed lattice cannot be generalized to obtain a coloring of the infinite lattice. One is tempted to believe however that the above result is not an artefact of the finite lattice and that the there is a corresponding coloring for larger lattices also.

4 Conclusions

Simulations using the fixed–range interaction multicomponent spin model introduced here produce approximations to optimal plane colorings. These optimal colorings realize a minimum probability for points unit distance apart in the plane having the same color. The simulations confirm essential previous knowledge about plane colorings: there are no colorings with less than five colors, such that none of the colors realizes distance one. On the other hand for seven colors a convergence towards zero probability was seen. For the open cases of five and six colors definite conclusions could not be given based on simulations performed here, but the results were suggestive: for five and six colors the simulations leading to smallest probabilities were not essentially improved by lowering the lattice constant below 0.01. For all cases with less than seven colors the lattice colorings showed periodic frustrations. These were areas where the spins had the highest energies, corresponding to equal colors unit distance apart. The regularity of these areas points to systematic sources of non–colorability, which are not due to numerical errors. The other possibility is that simulated annealing could not find the true minima in the energy landscape of the spin system. This is an inherent limitation of the method, preventing absolute conclusions. The potential source of ending up in the wrong minimum is the most drastic approximation needed here: finite area of the lattice. It is possible that the border regions induce finite–size effects, which prevent finding the true optima for the ideal infinite lattice. An example of this was probably present in the two–color case discussed above. On the other hand, finite–size effects were not always fatal, as is evident from the converged seven–color computation shown in Fig. 10, where a rather small lattice of 6×6 unit areas was used. It is therefore possible to capture important properties of the infinite plane with these lattices, at least when enough periodicity is present.

For the two–color case, comparison of the simulation result and an analytical solution in the case of a striped plane showed that annealing found the true global minimum, even though an infinite amount of local minima were available. In addition, checkerboard configurations with rather similar probabilities were systematically absent in the results, further strengthening the belief that simulated annealing produced optimal results. Of course, this is not a proof that optimal two–coloring of the plane consists of straight stripes, not only due to the fact that other similar configurations were produded by the simulations, whose probability was equally low within the accuracy permitted.

There are many options not investigated here, which could affect the quality of the simulations. These include other possible forms of the distribution function used to approximate the delta function in the lattice. Also, only square lattices were used in this work, but for example hexagonal lattices might bring some advantages. There are also numerous possibilities considering the implementation of the simulated annealing algorithm, most importantly the rules employed to update the spins.

The method and results presented here give rise to several open questions:

1. What is the optimal two–coloring of the plane? Straight stripes of width $\sqrt{3}/2$ lead to probability $1/3$ for points unit distance apart having the same color, but are there configurations giving still smaller probability?

2. Are the simulation results for three and four color lattices optimal? Especially the regular hexagonal lattice in the four–color case is interesting. Simulation result gave the unit of the lattice just below one, and there is probably an analytical derivation giving the exact value for lattice unit, and especially the probability, which was numerically estimated to be roughly 0.01.

3. Are there true five or six colorings of the plane, or are the apparent convergence of energy and the regular frustrations in lattice calculations a sign of fundamental non–colorability? More extensive simulations could point to correct answers. Here convergence towards zero energy (probability) was only seen for seven colors, together with the absence of the periodic frustrations. This points to the chromatic number of the plane equaling seven.

4. Could the energy functional $\epsilon(a)$ in (3) be investigated by methods other than simulated annealing? For example, could the simulated optimal configurations be used as a starting point for further analysis?

In addition, the method presented here can be generalized or applied to different situations. Some possibilities are listed here:

1. By changing the boundary conditions of the lattice spins, other topologies can be studied. For example, periodic boundary conditions in one dimension give approximations to cylinder colorings, periodicity in two dimensions to colorings on a torus.

2. The method can in principle be applied to the chromatic number in higher dimensions, but the obvious downside is the large computational cost associated.

3. Other coloring problems can be studied by modifying the Hamiltonian. For example, multiple distances could be avoided by adding the appropriate delta function terms, or different colors can be made to avoid different distances. The second case refers to the *polychromatic number of the plane* [4], as defined by Erdős.

I hope this paper has been of interest to friends of recreational mathematics, and stimulates readers to further tests and investigations.

References

[1] L. Moser and W. Moser, Can. Math. Bull. **4**, 187 (1961).

[2] A. D. N. J. de Grey, ArXiv e-prints (2018), arXiv:1804.02385 [math.CO] .

[3] Polymath project, `http://michaelnielsen.org/polymath1/index.php?title=Hadwiger-Nelson_problem` (2018), [Online; accessed 25-November-2018].

[4] A. Soifer, *The Mathematical Coloring Book* (Springer-Verlag, 2009).

[5] O. Goulko and A. Kent, Proceedings of the Royal Society of London Series A **473**, 20170494 (2017), arXiv:1705.07621 [cond-mat.stat-mech] .

[6] C. S. Peskin, Acta Numerica **11**, 479–517 (2002).

[7] A. Soifer, Geombinatorics **1**, 13 (1992).

[8] S. Heinäsmäki, "Colorplane," `https://github.com/heinasmaki/colorplane` (2019).

[9] X. Yang, X. Zhang, Z. Li, and G.-W. He, Journal of Computational Physics **228**, 7821 (2009).