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The Many Proofs and Applications of Perron's Theorem*

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Abstract. This paper chronicles the wide dispersal of Perron's 1907 result on positive matrices into many fields of science. The many proofs given during the last 93 years are categorized and critiqued (including Perron's original two proofs), and a more natural proof is presented. This simple-to-understand result of Perron provides a unequaled vehicle for taking students on a tour of many applied areas with some depth.

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1. Perron's theorem. A workhorse of much practical mathematics is the 1907 theorem of Oskar Perron [1]:

The eigenvalue of largest absolute value of a positive (square) matrix A is both simple and positive and belongs to a positive eigenvector. All other eigenvalues are smaller in absolute value.[1]

If the entries are at the very least nonnegative, we may conclude only that

the spectral radius ρ of a nonnegative matrix $A \geq 0$ is an eigenvalue belonging to a nonnegative eigenvector.

The importance of Perron's theorem seems to flow from two sources. First, in real world situations, measured interactions are most often positive, or at least nonnegative. The time a_{ij} to drive from warehouse i to warehouse j is nonnegative. The percentage $a_{i,i+1}$ of age group i that survive into age group $i+1$ is positive. The jth economic sector consumes $a_{ij}x_j$ nonnegative units of the output x_i of sector i as it produces x_j units. And so forth. In practical matters, matrices more often than not have nonnegative entries.

Second, an astonishing number of important models are simple linear iterative processes—they begin with an initial state x^0, then evolve recursively by the rule $x^{k+1} = Ax^k$, or in short, $x^k = A^k x^0$. It is of central importance to know when such a

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[1]A matrix A or vector v is *positive* if each entry is positive, in which case we write $A > 0$ or $v > 0$.

process will converge. The answer is quite simple (in principle) and follows easily by going over to the Jordan canonical form:

An iteration scheme $x^k = A^k x^0$ *converges for every initial state* x^0 *if and only if the eigenvalues* λ *of* A *distinct from* 1 *have modulus* $|\lambda| < 1$, *and, if* $\lambda = 1$ *is an eigenvalue, its eigenspace is of full rank; i.e., its rank equals the multiplicity of the root* $\lambda = 1$ *in the characteristic equation* $\phi(\lambda) = \det(\lambda I - A)$.

When $A > 0$, Perron tells us that the iterative process will converge exactly when the (positive) dominant eigenvalue $\lambda = \rho$ is at most 1; the process converges to the value 0 if $\rho < 1$, and when $\rho = 1$, to the component of the initial condition in the (reducing) eigenspace spanned by the associated eigenvector $v > 0$.

When A is merely nonnegative, it possesses a nonnegative dominant eigenvalue ρ belonging to a nonnegative eigenvector $v \geq 0$, but there may other eigenvalues of the same modulus [2]. Frobenius [3] extended Perron's theorem to certain *unzerlegbar* nonnegative matrices (see below), spawning an even wider scope of applications. But having multiple eigenvalues of modulus ρ complicates convergence: When $\rho = 1$ such eigenvalues may yield periodic or nearly periodic orbits or even orbits that may diverge to infinity. All these convergence results follow from a careful examination of the Jordan canonical form of A.

My delightful task here will be to survey the startlingly wide range of applications of positive and nonnegative matrices, to sketch the many varied and interesting proofs of Perron's theorem given in the past 93 years, and to present my own proof, which I recently found was first discovered 40 years earlier by Samuel Karlin [4]. Although the focus is on Perron's result and its applications and proofs, it is but one result in a large body of results known collectively as *Perron–Frobenius theory.* The extended theory is essential in such applications as the iterative solution of the linear systems arising in numerical treatment of partial differential equations, in the theory of finite Markov chains, and in queueing theory. For an entry into this area see the book *Nonnegative Matrices in the Mathematical Sciences* by Berman and Plemmons [5].

2. Nonnegative and Positive Matrices in Application.

The Three-Point Numerical Method. Consider the first numerical approach to solving the heat conduction problem

$$u_t = \alpha u_{xx}, \quad a < x < b,$$

subject to $u(x,0) = f(x), u(a,t) = 0 = u(b,t)$ (see [6]). Dividing the spatial interval $[a,b]$ into n equal subintervals in the usual way with $x_i = a + i\Delta x$, where $\Delta x = (b-a)/n$, yields the famous numerical model

$$u_i^{j+1} = u_i^j + h[u_{i+1}^j - 2u_i^j + u_{i-1}^j] \tag{2.1}$$

with $u_0^j = 0 = u_n^j$ for all j. Thus the evolution in time of the temperature u_i^j at node i at time j is modeled by

$$\begin{vmatrix} u_1^{j+1} \\ . \\ . \\ . \\ u_{n-1}^{j+1} \end{vmatrix} = \begin{pmatrix} 1-2h & h & 0 & . & . & 0 \\ h & 1-2h & h & 0 & . & 0 \\ & & . & & & \\ & & & & . & h \\ 0 & . & . & 0 & h & 1-2h \end{pmatrix} \begin{vmatrix} u_1^j \\ . \\ . \\ . \\ u_{n-1}^j \end{vmatrix}, \tag{2.2}$$

where $h = \alpha \Delta t / \Delta x^2$ with initial state $u^0 = [f(x_1), \ldots, f(x_{n-1})]^T$; i.e., future temperatures u^j at time slice j are given by $u^j = A^j u^0$.

Assuming $h < 0.5$, the matrix $A = (a_{ij})$ becomes nonnegative and by Perron possesses a nonnegative dominant eigenvalue $\lambda = \rho \geq 0$ belonging to a nonnegative eigenvector $v \geq 0$. Normalize v so that its largest entry is 1. The relation $\rho v = Av$ is coordinate-by-coordinate the relation

$$\rho v_i = hv_{i-1} + (1-2h)v_i + hv_{i+1} \leq h + 1 - 2h + h = 1$$

(with $v_0 = 0 = v_n$). In particular, if i is the least subscript for which $v_i = 1$, then

$$\rho = hv_{i-1} + (1-2h) + hv_{i+1} < 1.$$

Thus the numerical model (2.1) converges, relaxing to the eventual steady state 0.

This coordinate-by-coordinate argument with small modification also yields *Gerschgorin's theorem:*

Each eigenvalue λ of the real or complex matrix $A = (a_{ij})$ lies in at least one of the closed disks

$$|\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|.$$

A Population Model. Divide a population into n bins of increasing age, say, by decades. Let x_i^j be the number of individuals in the ith age group recorded at the jth snapshot in time. The $(j+1)$th snapshot will reveal that some fraction of each age group has advanced into the next:

$$\text{(2.3)} \qquad \begin{vmatrix} x_1^{j+1} \\ . \\ . \\ . \\ x_n^{j+1} \end{vmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1N} \\ a_{21} & 0 & 0 & 0 & . & . & 0 \\ 0 & a_{32} & 0 & . & . & . & . \\ & & & & . & . & 0 \\ 0 & . & . & 0 & . & a_{n,n-1} & 0 \end{pmatrix} \begin{vmatrix} x_1^j \\ . \\ . \\ . \\ x_n^j \end{vmatrix},$$

where the first row of $A = (a_{ij})$ displays the contribution of each age group to reproduction.

If the dominant nonnegative eigenvalue ρ guaranteed by Perron exceeds 1, there are initial populations that grow without bound. If $\rho < 1$, all populations become extinct. If $\rho = 1$, a population distribution may oscillate (when there are complex eigenvalues $|\lambda| = 1$), may extinguish or explode, or may converge to a steady state distribution in the eigenspace of $\rho = 1$, all depending on the Jordan form of A (see [2] and [7]).

Markov Chains. After each of a series of experiments, a system falls into one of its n distinct states $s_1, s_2, \ldots, s_n$. Suppose we know a priori that the probability of the experimental outcome s_j is always p_{ij} when starting from state s_i. This *transition* matrix $P = (p_{ij}) \geq 0$ has each row sum equal to 1 since the system must, after each experiment, reside in one of the n states. Thus if $\lambda = \rho \geq 0$ is the Perron dominant real eigenvalue of P belonging to $v \geq 0$, then as above, when the maximum entry in v has been normalized to 1 and $v_i = 1$,

$$\rho = \sum_{j=1}^{n} p_{ij} v_j \leq \sum_{j=1}^{n} p_{ij} = 1.$$

But in fact $\rho = 1$, since $Pv^0 = v^0$, where $v^0 = [1, 1, 1, \ldots, 1]^T$.

Assume $P > 0$. Perron guarantees that $\rho = 1$ is simple and hence must belong to v^0. Moreover

$$P^\infty = \lim_{n\to\infty} P^n$$

must be of rank 1 and act as the identity on v^0. Thus for some vector $p = [p_1, p_2, \dots, p_n]^T \geq 0$,

$$P^\infty = v^0 p^T = \begin{pmatrix} p_1 & p_2 & \cdots & p_n \\ p_1 & p_2 & \cdots & p_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}, \tag{2.4}$$

where $p_1 + p_2 + \cdots + p_n = 1$. Each p_i is the probability that the system will, in the limit, be in state s_i. Moreover, p_i is independent of the initial state at the initiation of the series of experiments. This vector p of "final states" is determined by $p^T P = p^T$, since

$$P^\infty = \lim_{n\to\infty} P^{n+1} = (\lim_{n\to\infty} P^n)P = P^\infty P.$$

Feller's First Example [8]. Engrave the symbol 1 on one face of a coin, the symbol 2 on the obverse. Do the same to a second coin. Suppose neither coin is fair, that the probability of tossing face j with coin i is p_{ij}. The Markov chain is defined by the following experiment: *Given coin i, toss this coin once, obtaining side j. The outcome for the experiment is then coin j.*

It is an invigorating exercise to obtain the final states p of this chain.

Eherenfest's Example. There are n molecules (n even) trapped in a box. Partition the box into two congruent halves A and B by an imaginary partition. Let the state of this system be the number of molecules in A, a number between 0 and n. Experiment as follows: *A molecule is chosen at random and moved into the other half-box.* At each iteration the transition probabilities are therefore

$$p_{01} = 1, \qquad p_{i,i-1} = \frac{i}{n}, \qquad p_{i,i+1} = 1 - \frac{i}{n}, \qquad p_{n,n-1} = 1, \tag{2.5}$$

with all other entries 0.

Mark Kac [9] called this "one of the most instructive models in the whole of Physics" because it confronts the irreversibility of time in the second law of thermodynamics. It seems clear that if all molecules start in one half or the other, eventually the system should equilibrate, with the molecules distributed equally between A and B. Numerical experiments will verify this. On the other hand, since the laws of motion are reversible in time, is it possible that the molecules might momentarily rearrange themselves disproportionately in one half or the other?

Let us see what the model (2.5) reveals. Although the transition matrix $P = (p_{ij})$ given in (2.5) is sparse, it is *irreducible* (*indecomposable, unzerlegbar*); i.e., each state can be reached from any other with repeated iterations, albeit with small probability. In other language, if the p_{ij} represent the connection numbers, the directed graph of states is completely connected (see [2] or [10]). Even more is true. It is a rewarding exercise to show that all high powers of P are positive; in fact, $P^{n+2} > 0$. But as I prove below, nonnegative matrices whose powers are eventually positive enjoy Perron's

Table 1

	Ag.	Indust.	Serv.	Consumer	Total prod.
Ag.	$0.3x_1$	$0.2x_2$	$0.3x_3$	4	x_1
Indust.	$0.2x_1$	$0.4x_2$	$0.3x_3$	5	x_2
Serv.	$0.2x_1$	$0.5x_2$	$0.1x_3$	12 (bill of goods)	x_3

result for positive matrices. Thus repetitions of Ehrenfest's experiment lead to a final state p as in (2.4). But since $p^T P = p^T$, we see from (2.5) that

$$p_i = \left(1 - \frac{i-1}{n}\right) \cdot p_{i-1} \; + \; \frac{i+1}{n} \cdot p_{i+1} \quad \text{for } \; 1 \le i \le n-1, \tag{2.6}$$

with $p_0 = p_1/n$ and $p_n = p_{n-1}/n$. It is another charming exercise to see that recursion (2.6) has solution

$$p_i = \binom{n}{i} 2^{-n} = \binom{n}{i} \left(\frac{1}{2}\right)^i \left(1 - \frac{1}{2}\right)^{n-i}, \tag{2.7}$$

which is the binomial distribution with mean $\mu = n/2$. Thus eventually, on average half the molecules are found in A.

This is not to say that it is impossible for the molecules to momentarily rearrange themselves unevenly, for this would violate Poincaré's recurrence theorem. See Kac's wonderful expositions [9], [11] of these delicate matters arising from Boltzmann's work.

Leontiev Input/Output Economic Model. Imagine a simplified national economy with only three producing sectors—agriculture, industry, and service—together with the nonproducing consumer sector, as tabulated in Table 1. The first row of the table depicts how agriculture's total production of x_1 units is consumed: 30% is consumed by agriculture itself, $0.2x_2$ is used by industry as it produces x_2 units, $0.3x_3$ is used by the service industry as it produces x_3 units, and the remaining four units are absorbed by the consumer sector. In short, the basic assumption is that *the jth sector input of the ith sector's output x_i is proportional to the jth sector's output x_j*. As one sector increases production, its consumption of other sectors' production increases proportionally.

When Table 1 is rewritten as the matrix relation

$$Ax + b = x, \tag{2.8}$$

the question becomes, *Does the vector equation* (2.8) *have a nonnegative solution $x \ge 0$ for $b \ge 0$*? A nonnegative solution x is certainly obtainable when $I - A$ has a nonnegative inverse, for then $x = (I - A)^{-1}\, b \ge 0$. A sufficient condition to guarantee a nonnegative inverse is that the *the Perron dominant eigenvalue ρ of A is less than* 1. *Every (nonnegative) bill of goods b is then realizable by a (nonnegative) production output x.*

To see this, suppose $\rho < 1$. We may use a similarity to send A to its Jordan canonical form, whereupon it is clear that the geometric series

$$I + A + A^2 + A^3 + \cdots = (I - A)^{-1}$$

converges and hence converges in the original coordinate system to a nonnegative matrix.

Conversely, if some positive bill of goods $b > 0$ of (2.8) is realizable by some $x \geq 0$, then $(I - A)^{-1}$ exists and is nonnegative, from which it follows that $\rho < 1$ (see [12]).

Walrasian Stability of Competitive Markets. Suppose n similar commodities are competing for the consumer's dollar [12]. Excess demand f_i for commodity i (demand less supply) is approximately (at least locally) a linear function of prices p less equilibrium prices p^0, i.e.,

$$f_i \approx \sum_{j=1}^{n} a_{ij}(p_j - p_j^0). \tag{2.9}$$

Since higher prices for one commodity will increase excess demand for the others, $a_{ij} \geq 0$ for $i \neq j$. Of course $a_{ii} < 0$.

The rate of price readjustment, once disturbed from equilibrium, must be proportional to excess demand, i.e.,

$$\dot{p} = KA(p - p^0), \tag{2.10}$$

where $K = \operatorname{diag}(k_1, k_2, \ldots, k_n)$, a diagonal matrix of positive *adjustment speeds.* Thus future prices are given by

$$p = p^0 + e^{KAt}c, \tag{2.11}$$

where $c = p(0) - p^0$.

As we well know, the qualitative dynamic behavior of system (2.10) depends upon the eigenvalues of $M = KA$: If all eigenvalues lie in the open left plane $\operatorname{Re}(s) < 0$, prices return to equilibrium. If some eigenvalues lie on the imaginary axis, there can be exploding or oscillating prices. If one or more eigenvalues lie in the open right half-plane, some prices may skyrocket.

The matrix $M = KA$ is of *Metzler* type, i.e., its off-diagonal entries are nonnegative. Metzler matrices M enjoy a surprising property: *the rightmost eigenvalue of M is real; all others are strictly to its left.* The proof is beautiful: Translate the spectrum of M to the right by adding a multiple of the identity so that $M + \epsilon I \geq 0$. By Perron, the translate has real dominant eigenvalue $\rho \geq 0$. Translating back, M must have real eigenvalue $\beta = \rho - \epsilon$, and all other eigenvalues must lie within a disk (of radius ρ) to the left and tangent to the vertical line $\operatorname{Im}(s) = \beta$ at β.

Price stability is therefore determined by β.

Low-Dimensional Topology (as communicated by J. D. McCarthy). The Perron theorem is a crucial step in Thurston's classification of surface diffeomorphisms, originally announced in his preprint "On the Geometry and Dynamics of Diffeomorphisms of Surfaces, I" [13] and described in detail in "Travaux de Thurston sur les surfaces" [14]. In this context, as Penner states in his article [15], the Perron theorem is "one of the fundamental tools in the study of PA maps."

Epidemiology. The Perron dominant eigenvalue ρ is behind the so called "Kermack–McKendrick" threshold in certain deterministic models of epidemics; see [16].

Statistical Mechanics. The partition function [17] of a thermodynamic system at the nth transition can be written in the form $Q = \operatorname{tr}(A^n)$, where A is the transition matrix [18]. Thus the limiting free energy per particle is $-kT \lim_{n\to\infty} \ln Q^{1/n} = -kT \ln \rho$, provided ρ is a simple, positive eigenvalue of A that exceeds all other eigenvalues in modulus; its existence is guaranteed by Perron.

Matrix Iterative Analysis. The *Stein–Rosenberg* theorem [19] employs the Frobenius extensions of Perron's result to compare the convergence rates of the Gauss–Seidel versus the Jacobi iterative methods for solving linear equations. Also see the encyclopedic [5].

3. The Many Proofs of Perron's Theorem. Perron's goal in his *Habilitationsschrift,* published in 1907 as the paper *Jacobischer Kettenbruchalgorithmus* [1], was a convergence theory for Jacobi multidimensional continued fractions. At one point (in section 14) he considers the special case of real, nonnegative partial quotients, states and proves what he apparently thought to be a mere technical lemma, then deduces a convergence result. Later he realized the importance of this technical *Hilfsatz* and placed it at the centerpiece of his *Zur Theorie der Matrices* [20].

Perron's first proof [1] goes by induction on the size of $A = (a_{ij}) > 0$, $i, j = 0, 1, \ldots, n$. The objective is to find a positive $\rho > 0$ and a positive vector $x > 0$ (with 0th coordinate 1) so that $Ax = \rho x$, i.e., for the 0th row

$$\rho - a_{00} - a_{01}x_1 - a_{02}x_2 - \cdots - a_{0n}x_n = 0, \tag{3.1}$$

and for the n remaining rows

$$\rho x_i = a_{i0} + a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \qquad (i = 1, 2, \ldots, n). \tag{3.2}$$

By induction, the remaining n equations (3.2) have (by Cramer's rule) a unique solution $x = x(\lambda)$ for all $\lambda > \rho_0$, where ρ_0 is the maximal positive eigenvalue of the minor $A_0 = (a_{ij})$ with $i, j = 1, \ldots, n$. Each entry $x_i(\lambda)$ of $x(\lambda)$ is analytic at infinity with an expansion

$$x_i(\lambda) = \frac{b_{i1}}{\lambda} + \frac{b_{i2}}{\lambda^2} + \cdots, \qquad b_{i1} = a_{i0} > 0,$$

which is arbitrarily small for all large λ and seen to be positive for all $\lambda > \rho_0$. A quiet moment of thought about Cramer's rule reveals that the left-hand side of (3.1) is actually $\phi(\lambda)/\phi_0(\lambda) = \det(\lambda - A)/\det(\lambda - A_0)$. Because at least one $x_i(\lambda)$ has a pole at ρ_0, the left-hand side of (3.1) must be negative for λ close to ρ_0, and since $\phi_0(\lambda)$ is constant in sign, there is some $\rho > \rho_0$ where $\phi(\rho) = 0$. Perron deduces the remaining facts using observations about the derivatives of the $x_i(\lambda)$.

Perron's second proof in [20] had at its core the *resolvent,* the mechanism employed in the modern and most powerful proof of Karlin. Perron derived (what is more easily seen as a consequence of the adjoint formula) that

$$\phi(\lambda)(\lambda - A)^{-1} = G(\lambda), \tag{3.3}$$

where $\phi(\lambda) = \det(\lambda - A)$ is the characteristic equation for the positive $n \times n$ matrix A, and where $G(\lambda) = (g_{ij}(\lambda))$ is a matrix of polynomials $g_{ij}(\lambda)$ of degree at most $n - 1$ in λ. He writes out the partial fraction expansion

$$\frac{\lambda^m}{\phi(\lambda)} = E_m(\lambda) + \sum_{k=1}^{n} \frac{b_k}{\lambda - \lambda_k},$$

where $E_m(\lambda)$ is the polynomial dividend and where $\lambda_1, \ldots, \lambda_n$ are the distinct eigenvalues.[2] Of course, $b_k = \lambda_k^m/\phi'(\lambda_k)$. Perron now multiplies through by $\phi(\lambda)$ and specializes λ at A to obtain via (3.3) that

$$A^m = 0 + \sum_{k=1}^{n} \frac{\lambda_k^m}{\phi'(\lambda_k)} \cdot \frac{\phi(\lambda_k)}{\lambda_k - A} = \sum_{k=1}^{n} \frac{\lambda_k^m G(\lambda_k)}{\phi'(\lambda_k)}. \tag{3.4}$$

[2]Here I present the simpler case of no repeated roots.

After introducing the notation $A^m = (a_{ij}^{(m)})$, entry by entry (3.4) reads

$$a_{ij}^{(m)} = \sum_{k=1}^{n} \frac{\lambda_k^m g_{ij}(\lambda_k)}{\phi'(\lambda_k)}.$$

Dividing through by the eigenvalue of largest modulus, (say) λ_1, gives

$$\frac{a_{ij}^{(m)}}{\lambda_1^m} = \frac{g_{ij}(\lambda_1)}{\phi'(\lambda_1)} + o(1)$$

and hence

$$\lim_{m\to\infty} \frac{a_{ij}^{(m+1)}}{a_{ij}^{(m)}} = \lambda_1,$$

giving that λ_1 is both real and nonnegative. The remainder of the proof is the cleaning up of loose ends: handling the case of repeated roots, multiple eigenvalues of maximum modulus, the positivity and simplicity of $\rho = \lambda_1$.

Most textbooks (e.g., [10], [12], [19], [21], [22], [23], [24]) present some variation of Wielandt's 1950 proof [25]. Wielandt's approach is in turn built on Frobenius's maximum-minimum idea developed during his 1908–1909 and 1912 extensions [3], [26] of Perron's result. Wielandt defines for $0 < A = (a_{ij})$ the nonnegative real

$$r = \sup_{x\geq 0, x\neq 0} \inf_i \frac{\sum_{j=1}^n a_{ij}x_j}{x_i}, \tag{3.5}$$

then shows via inequalities that the value $r > 0$ is actually achieved at some $x > 0$, that $Ax = rx$, that the eigenvalue is simple, and so on. I do not find these inequality arguments esthetically satisfying even though the approach provides effective computational bounds for the spectral radius ρ. Frobenius generalized the Perron result to nonnegative irreducible matrices, showing that there may be other (simple) complex eigenvalues on the spectral radius circle. Wielandt extended and clarified Frobenius's work. All these extensions are best explained by Varga [19].

Often one encounters proofs of Perron's result that employ the Brouwer fixed point theorem; see [27], [28], [29], [30], [31]. The common thread is this: the hyper octant $x \geq 0$, $x \neq 0$, in $\mathbf{R}^n$ is projected radially to the (compact convex) portion of the unit sphere in the hyper octant, upon which $A > 0$ induces a continuous map. Since all compact convex sets of $\mathbf{R}^{n-1}$ with interior are homeomorphic, Brouwer guarantees a fixed point, i.e., a nonnegative eigenvector of A. The remaining details are obtained with various ad hoc methods.

Birkhoff obtained a generalization of Perron's result for closed cones via a clever manipulation of the Jordan canonical form [32]. Ninio found a proof of only several lines for positive, symmetric matrices [33]. Bellman [24] cites several other proofs.

4. Karlin's Proof. Karlin was after bigger game—a Perron result for "positive" operators on Banach spaces [4]. I hold that Karlin's argument cut back to $\mathbf{R}^n$ is the most natural demonstration of Perron's theorem because it flows naturally from the beautiful circle of ideas surrounding the theory of the resolvent. (The approach also enjoys the added charm of a link with Perron's own proof.) Although Karlin's proof lives in the wider context of Banach-valued analytic functions of a complex variable, one may think of the proof below as involving only matrices with power series entries arising from the partial fraction expansion of $R(\lambda) = G(s)/\phi(\lambda)$ of (3.3).

LEMMA. *A nonnegative matrix A has its spectral radius ρ as an eigenvalue. Moreover, the eigenspace of $\lambda = \rho$ contains nonnegative eigenvectors.*

Proof. The resolvent

$$R(\lambda) = (\lambda - A)^{-1} \tag{4.1}$$

is analytic except at its spectrum (the eigenvalues). It must be analytic on the annulus $|\lambda| > \rho$ and must have at least one singularity somewhere on the spectral radius circle $|\lambda| = \rho$.

The Neumann series for the resolvent

$$R(\lambda) = \frac{1}{\lambda} + \frac{A}{\lambda^2} + \frac{A^2}{\lambda^3} + \cdots \tag{4.2}$$

will converge down to the first singularity of $R(\lambda)$, i.e., on the annulus $|\lambda| > \rho$. If $\lambda = \rho$ is not a singularity, then

$$\lim_{\lambda \to \rho^+} R(\lambda) < \infty$$

and thus

$$R(\rho) = \frac{1}{\rho} + \frac{A}{\rho^2} + \frac{A^2}{\rho^3} + \cdots < \infty$$

since each coefficient $A^n \geq 0$. But then the series (4.2) converges (entry by entry) absolutely and uniformly on the closed annulus $|\lambda| \geq \rho$, contradicting the necessary presence of a pole on the spectral radius circle.

To see that the eigenspace of $\lambda = \rho$ contains nonnegative vectors, expand the resolvent in its Laurent series about $\lambda = \rho$:

$$R(\lambda) = \sum_{k=-h}^{\infty} A_k (\lambda - \rho)^k, \tag{4.3}$$

where $A_{-h} \neq 0$. Note that $A_{-h} \geq 0$ since

$$A_{-h} = \lim_{\lambda \to \rho^+} (\lambda - \rho)^h R(\lambda) \geq 0. \tag{4.4}$$

By Cauchy's integral formula (see the appendix)

$$A_k \cdot A_m = \begin{cases} -A_{k+m+1} & \text{if} \quad k, m \geq 0, \\ 0 & \text{if} \quad k < 0, m \geq 0, \\ A_{k+m+1} & \text{if} \quad k, m < 0, \end{cases} \tag{4.5}$$

and so

$$R(\lambda) A_{-h} = \sum_{k=-h}^{\infty} (\lambda - \rho)^k A_k A_{-h} = (\lambda - \rho)^{-1} A_{-1} A_{-h} = \frac{A_{-h}}{\lambda - \rho}, \tag{4.6}$$

i.e.,

$$AA_{-h} = \rho A_{-h}. \tag{4.7}$$

Thus each nonzero column of A_{-h} is a (nonnegative) eigenvector of A belonging to $\lambda = \rho$.

Remark A. Assume that A is positive and v a nonnegative eigenvector belonging to the spectral radius $\lambda = \rho$. Then $0 < Av = \rho v$, giving that both ρ and v are positive. Thus each nonzero (nonnegative) column of A_{-h} is in fact a positive eigenvector of $A > 0$ belonging to $\lambda = \rho$. But if $h > 1$, by (4.5), the matrix A_{-h} is nilpotent, i.e.,

$$A_{-h}^2 = 0, \tag{4.8}$$

yet every nonzero column of A_{-h} is positive—an impossibility. Hence $h = 1$; the pole at $\lambda = \rho$ has order 1.

Because of (4.6), the nonzero elements of the range of A_{-1} are eigenvectors of A belonging to $\lambda = \rho$. Conversely, for any eigenvector v of A belonging to $\lambda = \rho$, $Av = \rho v$ and hence

$$R(\lambda)v = \frac{v}{\lambda - \rho}$$

for all λ near ρ giving by uniqueness of expansion (4.3) at v that

$$A_{-1}v = v. \tag{4.9}$$

Thus the range of A_{-1} (its column space) is the eigenspace of $\lambda = \rho$.

The rank of A_{-1} is in fact 1, for if two of its positive columns c_1, c_2 are not multiples of one another, then as in the "theta trick" of the simplex method [34], for some first positive t, the eigenvector $v = c_1 - tc_2$ is nonnegative and at least one of its coordinates is 0. But as noted above, nonnegative eigenvectors v belonging to $\rho > 0$ are in fact positive when A is positive.

The operator A cannot be cyclic at $\lambda = \rho$, for if $v = (\rho - A)x \neq 0$ is in the eigenspace belonging to $\lambda = \rho$, then by (4.9) and (4.7), $A_{-1}v = v = A_{-1}(\rho - A)x = (\rho - A)A_{-1}x = 0$. But $v \neq 0$. Thus $\lambda = \rho$ is a simple eigenvalue of A.

Finally, there are no other eigenvalues on the spectral radius circle. For consider $A - \epsilon I > 0$ for small $\epsilon > 0$. By the spectral mapping theorem its largest positive eigenvalue is $\rho - \epsilon$, which by the lemma above is its spectral radius. Translating this smaller circle back to the right by ϵ we see that all remaining eigenvalues of A lie within the open disk $|\lambda| < \rho$. (Note that this argument requires only that $A \geq 0$ with positive diagonal elements.)

In summary, we give the following theorem, from Perron.

THEOREM. *The dominant eigenvalue of a positive matrix is positive and simple and belongs to a positive eigenvector. All other eigenvalues are smaller in modulus.*

Remark B. Perron's theorem also holds under the weaker assumption that $A \geq 0$ yet $A^m > 0$ for some m, since by the theorem, $B = A^m$ has a simple positive eigenvalue but $n - 1$ eigenvalues, properly counted, of smaller moduli, where n is the rank of the underlying space. Thus the results of Perron's theorem must also hold for A by the spectral mapping theorem.

Appendix.

LEMMA. *The coefficients of the Laurent expansion of the resolvent*

$$R(\lambda) = (\lambda - A)^{-1} = \sum_{k=-h}^{\infty} A_k(\lambda - \lambda_0)^k \tag{A.1}$$

about a pole at $\lambda = \lambda_0$ *satisfy*

$$A_k \cdot A_m = \begin{cases} -A_{k+m+1} & \text{if } k, m \geq 0, \\ 0 & \text{if } k < 0, m \geq 0, \\ A_{k+m+1} & \text{if } k, m < 0. \end{cases} \tag{A.2}$$

Proof. We may assume $\lambda_0 = 0$. Integrating both sides of (A.1) counterclockwise about a circular contour C centered at the origin that includes only the single singularity 0 yields the Cauchy formula

$$A_k = \frac{1}{2\pi i}\int_C \frac{R(\lambda)}{\lambda^{k+1}}\,d\lambda. \tag{A.3}$$

Using two such contours C_1 and C_2 of radii $r_1 < r_2$,[3]

$$\begin{aligned}
A_k \cdot A_m &= \frac{1}{2\pi i}\int_{C_1} \frac{R(\lambda)}{\lambda^{k+1}}\,d\lambda \cdot \frac{1}{2\pi i}\int_{C_2} \frac{R(\mu)}{\mu^{m+1}}\,d\mu \\
&= \frac{1}{4\pi^2 i^2}\int_{C_1}\int_{C_2} \frac{R(\lambda)R(\mu)}{\lambda^{k+1}\mu^{m+1}}\,d\lambda\,d\mu \\
&= -\frac{1}{4\pi^2 i^2}\int_{C_1}\int_{C_2} \frac{R(\lambda)-R(\mu)}{\lambda^{k+1}\mu^{m+1}(\lambda-\mu)}\,d\lambda\,d\mu && \text{(A.4)} \\
&= \frac{1}{2\pi i}\int_{C_1} \frac{R(\lambda)}{\lambda^{k+1}}\left(\frac{1}{2\pi i}\int_{C_2}\frac{d\mu}{\mu^{m+1}(\mu-\lambda)}\right)d\lambda && \text{(A.5a)} \\
&\quad + \frac{1}{2\pi i}\int_{C_2} \frac{R(\mu)}{\mu^{m+1}}\left(\frac{1}{2\pi i}\int_{C_1}\frac{d\lambda}{\lambda^{k+1}(\lambda-\mu)}\right)d\mu. && \text{(A.5b)}
\end{aligned}$$

But counterclockwise around the contour $C : |z| = r$,

$$\frac{1}{2\pi i}\int_C \frac{dz}{z^n(z-w)} = \begin{cases} 0 & \text{if} \quad n \le 0 \text{ and } |w| > r, \\ -1/w^n & \text{if} \quad n > 0 \text{ and } |w| > r, \\ 1/w^n & \text{if} \quad n \le 0 \text{ and } |w| < r, \\ 0 & \text{if} \quad n > 0 \text{ and } |w| < r, \end{cases} \tag{A.6}$$

as can be seen by integrating either the expansion

$$\frac{1}{z^n(z-w)} = -\frac{1}{wz^n}\left[1 + \frac{z}{w} + \left(\frac{z}{w}\right)^2 + \cdots\right]$$

or

$$\frac{1}{z^n(z-w)} = \frac{1}{z^{n+1}}\left[1 + \frac{w}{z} + \left(\frac{w}{z}\right)^2 + \cdots\right].$$

The relations of (A.2) now follow. For example, when $k < 0$ and $m \ge 0$, both inner integrals of (A.5) are 0, giving that $A_k A_m = 0$. Or if both k and m are negative, the inner integral of (A.5a) is

$$\frac{1}{2\pi i}\int_{C_2} \frac{d\mu}{\mu^{m+1}(\mu-\lambda)}\,d\lambda = \frac{1}{\lambda^{m+1}},$$

while the inner integral of (A.5b) is 0, giving by (A.5a) that

$$A_k \cdot A_m = \frac{1}{2\pi i}\int_{C_1} \frac{R(\lambda)}{\lambda^{k+m+2}}\,d\lambda = A_{k+m+1}.$$

[3]The third line of the equation, (A.4), is via the easy *first resolvent equation:* $R(\lambda) - R(\mu) = -(\lambda-\mu)R(\lambda)R(\mu)$.

REFERENCES

[1] O. Perron, *Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus*, Math. Ann., 64 (1907), pp. 11–76.
[2] V. G. Rumchev and D. J. G. James, *The role of nonnegative matrices in discrete-time mathematical modelling*, Internat. J. Math. Ed. Sci. Tech., 21 (1990), pp. 169–182.
[3] G. Frobenius, *Über Matrizen aus nicht negativen Elementen*, Sitzungsberichte Preussische Akademie der Wissenschaft, Berlin, 1912, pp. 456–477.
[4] S. Karlin, *Positive operators*, J. Math. Mech., 8 (1959), pp. 907–937.
[5] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences,* SIAM, Philadelphia, 1994.
[6] C. R. MacCluer, *Boundary Value Problems and Orthogonal Expansions,* IEEE Press, Piscataway, NJ, 1994.
[7] N. Keyfitz, *Applied Mathematical Demography,* 2nd ed., Springer-Verlag, New York, 1977.
[8] W. Feller, *An Introduction to Probability Theory and Its Applications,* Vol. 1, 1st ed., Wiley, New York, 1950.
[9] M. Kac, *Probability and Related Topics in Physical Sciences,* Interscience, New York, 1959.
[10] A. Graham, *Nonnegative Matrices and Applicable Topics in Linear Algebra,* Halsted Press, New York, 1987.
[11] M. Kac, *Random walk and the theory of Brownian motion*, Amer. Math. Monthly, 54 (1947), pp. 369–391.
[12] A. Takayama, *Mathematical Economics,* Dryden Press, Hinsdale, IL, 1974.
[13] W. P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.), 19 (1988), pp. 417–431.
[14] Séminaire Orsay, *Travaux de Thurston sur les surfaces*, Soc. Math. France, Paris, 1979, and Astérisque, 66–67 (1991), pp. 1–286.
[15] R. C. Penner, *A construction of pseudo-Anosov homeomorphisms*, Trans. Amer. Math. Soc., 310 (1988), pp. 179–197.
[16] D. J. Daley and J. Gani, *A deterministic general epidemic model in a stratified population*, in Probability, Statistics and Optimization, Wiley, Chichester, UK, 1994, pp. 117–132.
[17] R. P. Feynman, *Statistical Mechanics,* Benjamin, Reading, MA, 1972.
[18] G. F. Newell and E. W. Montroll, *On the theory of the Ising model of ferromagnetism*, Rev. Modern Phys., 25 (1953), pp. 353–389.
[19] R. S. Varga, *Matrix Iterative Analysis,* Prentice-Hall, Englewood Cliffs, NJ, 1962, pp. 68–73.
[20] O. Perron, *Zur Theorie der Matrices*, Math. Ann., 64 (1907), pp. 248–263.
[21] R. A. Horn and C. A. Johnson, *Matrix Analysis,* Cambridge University Press, Cambridge, UK, 1985.
[22] N. J. Pullman, *Matrix Theory and Its Applications,* Marcel Dekker, New York, 1976.
[23] H. Minc, *Nonnegative Matrices,* Wiley-Interscience, New York, 1988.
[24] R. Bellman, *Introduction to Matrix Analysis,* 2nd. ed., McGraw-Hill, New York, 1970.
[25] H. Wielandt, *Unzerlegbare, nicht negative Matrizen*, Math. Z., 52 (1950), pp. 642–648.
[26] G. Frobenius, *Über Matrizen aus positiven Elementen*, Sitzungsberichte Preussische Akademie der Wissenschaft, Berlin, 1908, 1909, pp. 471–476, 514–518.
[27] G. Malteses, *A simple proof of the fundamental theorem of finite Markov chains*, Amer. Math. Monthly, 93 (1986), pp. 629–630.
[28] A. Borobia and U. R. Trías, *A geometric proof of the Perron-Frobenius theorem*, Rev. Mat. Univ. Complut. Madrid, 5 (1992), pp. 57–63.
[29] N. J. Pullman, *A geometric approach to the theory of nonnegative matrices*, Linear Algebra Appl., 4 (1971), pp. 297–312.
[30] G. Debreu and I. N. Herstein, *Nonnegative square matrices*, Econometrica, 21 (1953), pp. 597–607.
[31] H. Samelson, *On the Perron-Frobenius theorem*, Michigan Math. J., 4 (1957), pp. 57–59.
[32] G. Birkhoff, *Linear transformations with invariant cones*, Amer. Math. Monthly, 74 (1967), pp. 274–276.
[33] F. Ninio, *A simple proof of the Perron-Frobenius theorem for positive symmetric matrices*, J. Phys. A, 9 (1976), pp. 1281–1282.
[34] C. R. MacCluer, *Industrial Mathematics,* Prentice–Hall, Upper Saddle River, NJ, 1999.