LOOP-NEARRINGS

Silvia Pianta
Dipartimento di Matematica e Fisica, Università Cattolica, Via Trieste 17, I-25121 Brescia
pianta@dmf.unicatt.it

Introduction

The notion of nearring, due to H.Zassenhaus (1936, see [23]), can be considered as a generalization of the notion of nearfield.

In fact, according to the more recent definitions (see e.g. [4], [5], [21]), a *nearring* is a triple $(N, +, \cdot)$ where N is a non empty set, "+" and " $\cdot$ " are binary operations on N such that $(N, +)$ is a group with identity 0 (not necessarily abelian), $(N, \cdot)$ is a semigroup and one of the two distributive laws of multiplication with respect to addition is satisfied.

A *nearfield* is a nearring $(N, +, \cdot)$ such that $(N \setminus \{0\}, \cdot)$ is a group. Nearfields attained particular relevance in two main areas of investigation.

The former area is group theory, dealing with sharply 2-transitive permutation groups: it is well known (see, e.g., [22]) that, if $(N, +, \cdot)$ is a nearfield and $T_2(N) := \{\tau_{a,b} : N \to N,\ x \mapsto a + bx \mid a \in N, b \in N^*\}$ is the *affine group* of N, then $T_2(N)$ acts sharply 2-transitively on N; conversely, if G is a sharply 2-transitive permutation group acting on a finite set N, then N can be provided with the structure of a nearfield $(N, +, \cdot)$ such that $G = T_2(N)$. But if the set N is infinite, then the additive structure of $(N, +, \cdot)$, constructed with the same procedure as in the finite case, fails to be a group in general (it loses the associative law), and what one gets is a weaker structure, called *neardomain* (introduced by H.Karzel in 1968, [14]). It is still an open problem whether it is possible to find any example of a proper neardomain which is not actually a nearfield.

Neardomains result to be special instances of *left neofields*, introduced and studied from both algebraic and geometric point of view by D.F.Hsu and A.D. Keedwell in 1984 (see [13]) in order to generalize the older structure of *neofield* $(N, +, \cdot)$, due to L.J.Paige (1949, see [20]), where $(N, +)$ is a loop, $(N \setminus \{0\}, \cdot)$ is a group and both distributive laws hold

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true; a left neofield is required to fulfil only the left distributivity. A left neofield is a generalization of a nearfield: one relaxes the associativity for the additive structure. Also a nearring is a generalization of a nearfield: one relaxes the group property for the multiplicative structure.

Well, why not consider both ways of generalizing a nearfield?

We will just do that, introducing the notion of *loop-nearring*.

The latter area of interest for nearfields is geometry, precisely the study of coordinate structures of non desarguesian projective planes: for this purpose nearfields require an additional "key-property", that is *planarity*.

In this work we want to go through with this concept, that was introduced by J.Clay in 1967 also in the more general nearring structure (see [1], [2]) and intensively studied by G.Ferrero (e.g. [9], [10]), who gave a standard method to construct planar nearrings from Frobenius groups and showed how to obtain different classes of designs from finite planar nearrings.

Starting from here, in §2 we extend the definition of planarity to the loop-nearring case and moreover we show how this notion can be weakened in order to obtain a sort of "graduation" of planarity into different steps (*0-planarity*, *semiplanarity*, *full planarity*), each of them being realized in some concrete examples (see §3).

Finally, just as planar nearrings can be constructed starting from *Ferrero pairs*, i.e. pairs of groups (N, Φ), where Φ is a non trivial group of *uniform*[1] and *fixed-point-free* automorphisms of the group N, so by a similar construction 0-planar, semiplanar and planar loop-nearrings correspond respectively to *weak-*, *semi-* and *full l-Ferrero pairs*, i.e. pairs (N, Φ), where N is, more generally, a loop, and Φ is a non trivial group of automorphisms of N admitting at least one regular orbit on N in the weak case; the automorphisms of Φ are, in addition, fixed-point-free (or a bit more) in the semi-l-Ferrero pairs and also uniform in the full l-Ferrero pairs.

1. Preliminary definitions

Let N be a non empty set, furnished with two binary operations "$+$" and "$\cdot$". We call $(N, +, \cdot)$ a *loop-nearring* if:

[1] A loop (or group) automorphism $\varphi : N \to N$ is said to be *uniform* if for all $c \in N$ the mapping $c^{+}\varphi : N \to N;\ x \mapsto c + \varphi(x)$ admits a fixed point. If $(N, +)$ is a group, this condition is equivalent to saying that the map $\varphi - 1 : N \to N;\ x \mapsto \varphi(x) - x$ is surjective (see [6]).

L1. $(N,+)$ is a *loop*, i.e. $\forall a,b \in N \ \exists_1 (x,y) \in N \times N$ such that $a+x=b=y+a$ and there is a neutral element $0 \in N$ such that $\forall a \in N$: $a+0=0+a=a$,

L2. $(N,\cdot)$ is a *semigroup*, i.e. " $\cdot$ " is associative,

L3. One of the two distributive laws holds, assume $\forall a,b,c \in N$: $a(b+c)=ab+ac$.

In any loop $(N,+)$ consider the maps $a^+ : N \to N;\ x \mapsto a+x$, called *left translations*, or *left additions*, for all $a \in N$ (in the literature also denoted by λ_a or $\mathrm{L}(a)$).

All these maps are bijective, hence $N^+ := \{a^+ \mid a \in N\}$ generates a group, the *left translation group* $T_l :=< a^+ \mid a \in N >$, which is a subgroup of the whole permutation group $SymN$.

Moreover, for all $a,b \in N$, the *precession maps*

$$\delta_{a,b} := \left((a+b)^+\right)^{-1} a^+ b^+ \quad (=\lambda_{a+b}^{-1}\lambda_a\lambda_b)$$

are permutations fixing the element 0, and their algebraic meaning is cleared up by the identity $a+(b+x)=(a+b)+\delta_{a,b}(x)$, which follows directly from the definition of $\delta_{a,b}$.

We shall denote by $\Delta :=< \delta_{a,b} \mid a,b \in N >$ the subgroup of T_l generated by the precession maps, called *left inner mapping group* (or *structure group*) of N.

Now, in a loop-nearring $(N,+,\cdot)$ another significant mapping set can be regarded: denoting by $a^{\cdot} : N \to N;\ x \mapsto a \cdot x$ the *left multiplication by* a, for all $a \in N$, the left distributive law (L3) entails that $N^{\cdot} := \{a^{\cdot} \mid a \in N\}$ is a subsemigroup of

$$End(N,+) := \{\varphi : N \to N \mid \varphi(x+y) = \varphi(x) + \varphi(y),\ \forall x,y \in N\},$$

the *endomorphism* semigroup of $(N,+)$.

In general, we can say that whenever in a loop $(N,+)$ there is defined a map $^{\cdot} : N \to End(N,+);\ a \mapsto a^{\cdot}$ such that $(a^{\cdot}(b))^{\cdot} = a^{\cdot}b^{\cdot}$, then one gets a loop-nearring $(N,+,\cdot)$, where $a \cdot b := a^{\cdot}(b)$ for all $a,b \in N$.

Here we want to remark that, if in a loop-nearring $(N,+,\cdot)$ the loop $(N,+)$ admits a unique inverse to each $a \in N$, i.e. $a+b=0$ implies $b+a=0$ for all $a,b \in N$, and the precession maps are left multiplications, i.e. $\{\delta_{a,b} \mid a,b \in N\} \subseteq N^{\cdot}$, then $(N,+,\cdot)$ becomes a *complete K-nearring* in the sense of Karzel (cf. [15]), which, in turn, becomes a *(complete) neardomain* exactly when $(N \setminus \{0\},\cdot)$ is a group; finally, a neardomain is a *nearfield* if and only if $(N,+)$ is a group.

2. Planarity and its graduation

In a loop-nearring $(N,+,\cdot)$ the left multiplications are endomorphisms of $(N,+)$ hence $a\cdot 0 = a^{\cdot}(0) = 0$ holds true for all $a \in N$.

On the other hand one cannot say anything, in general, about the products $0\cdot a$, so just as in nearring theory, the loop-nearring is called *0-symmetric* if $0^{\cdot}$ is the 0-endomorphism, i.e. if $0\cdot a = 0$ for all $a \in N$.

Now, in order to copy the definition of planar nearring there are two possibilities: a loop-nearring $(N,+,\cdot)$ such that $|N^{\cdot}| \geq 3$ is called *planar on the right*, resp. *left*, side if the following property (Pr), resp. (Pl), is satisfied:

(Pr) $\forall a,b,c \in N$ with $a^{\cdot} \neq b^{\cdot}$ $\exists_1 x \in N : a\cdot x = b\cdot x + c$

(Pl) $\forall a,b,c \in N$ with $a^{\cdot} \neq b^{\cdot}$ $\exists_1 x \in N : a\cdot x = c + b\cdot x$.

In the sequel we shall regard to the left side planarity and simply call *planar* a loop-nearring which satisfies (P_l).

2.1 0-planarity

It is an easy exercise to check that any loop-nearring $(N,+,\cdot)$ which is planar on the right or on the left side has all properties stated e.g. by J.Clay in the "main structure theorem" of planar nearrings (cf. [4], (4.9)), in particular it is 0-symmetric. Since the proofs of such properties make use of the planarity equation not in its complete form, but just in the form $a\cdot x = c + 0\cdot x$, the idea is to start from such simpler equation in order to introduce a notion which is weaker than planarity, but yields the same structure as described in the cited theorem.

We proceed as follows. Given a loop-nearring $(N,+,\cdot)$, for $a,b,c \in N$ with $a^{\cdot} \neq b^{\cdot}$ let us define $S_{a,b,c} := \{x \in N \mid a\cdot x = c + b\cdot x\}$.

(2.1) *The loop-nearring $(N,+,\cdot)$ is 0-symmetric if and only if for all $a \in N$ with $a^{\cdot} \neq 0^{\cdot}$: $|S_{a,0,0}| = 1$ (i.e. the equation $a\cdot x = 0\cdot x$ admits a unique solution).*

Proof. From $a\cdot 0 = 0 = 0\cdot 0$ and $a\cdot(0\cdot x) = (a\cdot 0)\cdot x = 0\cdot x = (0\cdot 0)\cdot x = 0\cdot(0\cdot x)$ it follows that $|S_{a,0,0}| = 1$ if and only if $0\cdot x = 0$ for all $x \in N$. □

Denote by 1 the identity of N, by 0 the 0-endomorphism of $(N,+)$ and by $A := \{a \in N \mid a^{\cdot} = 0\}$ the set of *0-multipliers* of $(N,+)$, and set $N^* := N \setminus A$.

(2.2) *In a loop-nearring $(N,+,\cdot)$, if $(N^*)^{\cdot} \subseteq SymN$ then either $0^{\cdot} = 0$ or $0^{\cdot} = 1$.*

Proof. If $0^{\cdot} \neq 0$ then $0 \in N^*$ hence $0^{\cdot} \in SymN$ and from $0 = 0 \cdot 0 = 0^{\cdot}(0)$ it follows $0 = (0^{\cdot})^{-1}(0)$, that entails $0^{\cdot} = \left((0^{\cdot})^{-1}(0)\right)^{\cdot}$. But $0^{\cdot}\left((0^{\cdot})^{-1}(0)\right)^{\cdot} = \left(0^{\cdot}(0^{\cdot})^{-1}(0)\right)^{\cdot}$ hence $1 = \left((0^{\cdot})^{-1}(0)\right)^{\cdot} = 0^{\cdot}$. □

(2.3) *Let $(N, +, \cdot)$ be a loop-nearring with $|N^{\cdot}| \geq 2$. Then $(N^*)^{\cdot} \subseteq SymN$ if and only if $|S_{a,0,c}| = 1$ for all $a, c \in N$ with $a^{\cdot} \neq 0^{\cdot}$.*

Proof. ($\Leftarrow$) By (2.1) $0^{\cdot} = 0$ and this implies that $\forall a \in N^*$, $\forall c \in N$ the equation $a^{\cdot}(x) = c$ is equivalent to $a \cdot x = c + 0 = c + 0 \cdot x$, so for $|S_{a,0,c}| = 1$, $\exists_1 x \in N : a^{\cdot}(x) = c$ hence $a^{\cdot} \in SymN$.

($\Rightarrow$) Suppose $0^{\cdot} = 1$. Then, for any $a \in N$, $a^{\cdot} = a^{\cdot}0^{\cdot} = (a^{\cdot}(0))^{\cdot} = (a \cdot 0)^{\cdot} = 0^{\cdot} = 1$, a contradiction to $|N^{\cdot}| \geq 2$. Hence, by (2.2), $0^{\cdot} = 0$, so $(N, +, \cdot)$ is 0-symmetric and $S_{a,0,c}$ is the set of solutions of the equation $a \cdot x = c$, i.e. $|S_{a,0,c}| = 1$ for all $a, c \in N$ with $a^{\cdot} \neq 0^{\cdot}$. □

Let us say that a loop-nearring $(N, +, \cdot)$ is *0-planar* if $|N^{\cdot}| \geq 2$ and for all $a, c \in N$ with $a^{\cdot} \neq 0^{\cdot}$ the condition $|S_{a,0,c}| = 1$ holds.

In particular, any 0-planar loop-nearring is 0-symmetric by (2.1). Moreover:

(2.4) *Let $(N, +, \cdot)$ be a 0-planar loop-nearring and $\Phi := (N^*)^{\cdot}$, then*

1. $\Phi \leq Aut(N, +)$;
2. $\forall \varphi \in \Phi \setminus \{1\} : Fix\,\varphi \cap N^* = \emptyset$;[2]
3. $\Phi(A) = A$ *and* $\Phi(N^*) = N^*$.

Proof. *1.* By (2.3) we already know $(N^*)^{\cdot} \subseteq Aut(N, +)$. Moreover $S_{a,0,a} =: \{e\}$ implies $a \cdot e = a$ and $a^{\cdot}e^{\cdot} = (a^{\cdot}(e))^{\cdot} = a^{\cdot}$, that entails $e^{\cdot} = 1 \in (N^*)^{\cdot}$. Finally, let $\{b\} := S_{a,0,e}$, then $a \cdot b = e$ and $e^{\cdot} = (a^{\cdot}(b))^{\cdot} = a^{\cdot}b^{\cdot}$, that means $(a^{\cdot})^{-1} = b^{\cdot} \in (N^*)^{\cdot}$.
2. For all $a, b \in N^*$, $b = \varphi(b) := a^{\cdot}(b)$ implies $a^{\cdot}b^{\cdot} = (a^{\cdot}(b))^{\cdot} = b^{\cdot}$, hence, $\varphi = a^{\cdot} = 1$.
3. For $a \in N^*$, $b \in A$, $(a^{\cdot}(b))^{\cdot} = a^{\cdot}b^{\cdot} = a^{\cdot}0 = 0$, hence for all $\varphi \in \Phi$, $\varphi(A) = A$, and this implies also $\varphi(N^*) = N^*$. □

Now we are ready to show the structure of 0-planar loop-nearrings:

Structure Theorem. *Let $(N, +, \cdot)$ be a 0-planar loop-nearring, $E := \{e \in N \mid e^{\cdot} = 1\}$ and $\Phi := (N^*)^{\cdot} \leq Aut(N, +)$ (by (2.4)), then $\emptyset \neq E$ and E is a complete set of representatives for the orbits of Φ on N^* (i.e.*

[2]For a permutation $\alpha \in SymN$ we denote by $Fix\,\alpha := \{x \in N \mid \alpha(x) = x\}$.

$N^* = \dot{\bigcup}_{e \in E} \Phi(e))$ *and* Φ *acts regularly on each of these orbits* $\Phi(e)$ *(i.e.* Φ *is semiregular on* N^**).*

Moreover, for each $e \in E$, $(\Phi(e), \cdot)$ *is a subgroup of the semigroup* $(N, \cdot)$*, isomorphic to* Φ.

Proof. Let $x \in N^*$, hence $x^{\cdot} \in \Phi$ and if $e := (x^{\cdot})^{-1}(x)$ then $x^{\cdot}(e) = x$, so that $x^{\cdot} = x^{\cdot} e^{\cdot}$, i.e. $e^{\cdot} = 1$, hence $e \in E$ and $N^* \subseteq \Phi(E)$. Conversely, let, $x \in \Phi(E)$, hence $x = \varphi(e)$ for some $e \in E$ and $\varphi \in \Phi = (N^*)^{\cdot}$. Then there is an $n \in N^*$ with $n^{\cdot}(e) = \varphi(e) = x$, i.e. $x^{\cdot} = (n^{\cdot}(e))^{\cdot} = n^{\cdot} e^{\cdot} = n^{\cdot} \neq 0$, hence $x \in N^*$ and we get $\Phi(E) = N^*$. Thus $N^* = \Phi(E) = \bigcup_{e \in E} \Phi(e)$.

Let us prove that the latter union is disjoint: if $f \in \Phi(e) \cap E$ then there is a $b \in N^*$ with $b \cdot e = f \in E$, hence $1 = f^{\cdot} = b^{\cdot} e^{\cdot} = b^{\cdot}$ implying $f = e$. By (2.4) Φ acts semiregularly on N^*, which means that the action of Φ on $\Phi(e)$ is regular for each $e \in E$, thus Φ induces on $\Phi(e)$ a group structure with identity e and, for each $x \in \Phi(e)$, x^{-1} is defined by $\left(x^{-1}\right)^{\cdot}(x) = e$. □

Now, let $(N, +)$ be a loop and $\{1\} \neq \Phi \leq Aut(N, +)$. For any complete set T of orbit representatives of Φ on N (i.e. T is such that $N = \Phi(T)$ and $\Phi(t) \cap T = \{t\}$ for all $t \in T$), denote by $T^* := \{t \in T \mid \forall \varphi \in \Phi : \varphi(t) = t \Rightarrow \varphi = 1\}$. Since Φ is non trivial $0 \notin T^*$.

Then we call (N, Φ) a *weak l-Ferrero pair* if for a (hence for any) set of orbit representatives T of Φ on N the condition $T^* \neq \emptyset$ holds; in other words, if (N, Φ) admits at least one regular orbit. By the previous theorem, if $(N, +, \cdot)$ is a planar loop-nearring and $\Phi := (N^*)^{\cdot}$ then the set E can be extended to a complete set T of orbit representatives on the whole N such that $T^* \supseteq E \neq \emptyset$, hence $\left((N, +), (N^*)^{\cdot}\right)$ is a weak l-Ferrero pair.

Conversely,

Theorem 1. *Let* (N, Φ) *be a weak l-Ferrero pair,* T *a complete set of orbit representatives of* Φ *on* N *with* $T^* \neq \emptyset$ *and* $\emptyset \neq E \subseteq T^*$*. Define, for each* $a \in N$*:*

$$a^{\cdot} := \begin{cases} 0 & if \quad a \in N \setminus \Phi(E) \\ \varphi & if \quad a = \varphi(e) \ \ for\ e \in E,\ \varphi \in \Phi, \end{cases}$$

and $a \cdot b := a^{\cdot}(b)$ *for all* $a, b \in N$*. Then* $(N, +, \cdot)$ *is a 0-planar loop-nearring.*

Proof. Since $\Phi \leq Aut(N, +) \subseteq End(N, +)$ and $0 \in End(N, +)$, the left distributive law holds in $(N, +, \cdot)$. The associative law $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ is equivalent to $a^{\cdot} b^{\cdot} = (a^{\cdot}(b))^{\cdot}$. In order to prove this identity, first note that $0 \in N \setminus \Phi(E)$, hence $0^{\cdot} = 0$.

Now if $a^{\cdot} = 0$ then $a^{\cdot}b^{\cdot} = 0b^{\cdot} = 0$ and $(a^{\cdot}(b))^{\cdot} = (0^{\cdot}(b))^{\cdot} = 0^{\cdot} = 0$ for all $b \in N$.

If $b^{\cdot} = 0$ then $b \in N \setminus \Phi(E)$ and so $a^{\cdot}(b) \in N \setminus \Phi(e)$ which implies $(a^{\cdot}(b))^{\cdot} = 0 = a^{\cdot}0 = a^{\cdot}b^{\cdot}$.

If $a^{\cdot}, b^{\cdot} \neq 0$ then $a^{\cdot}, b^{\cdot} \in \Phi$ and if $e \in E$ is such that $b = b^{\cdot}(e)$ then $a^{\cdot}(b) = a^{\cdot}b^{\cdot}(e) \in \Phi(e)$, thus $(a^{\cdot}(b))^{\cdot} = a^{\cdot}b^{\cdot}$.

This shows that $(N, +, \cdot)$ is a loop-nearring with $A := N \setminus \Phi(E)$, $N^* := \Phi(E)$ and $(N^*)^{\cdot} = \Phi \leq Aut(N, +)$. By definition $|N^{\cdot}| \geq 2$, so by (2.3) $(N, +, \cdot)$ is 0-planar. □

Let us call "*Ferrero derivation*" the procedure, described in Theorem 1, to construct a (not uniquely determined) loop-nearring from a weak l-Ferrero pair (N, Φ). One sees immediately that this is the same procedure used in [2], [9], [10] to construct planar nearrings from Ferrero pairs: the main fact here is that the Ferrero derivation keeps on working under the weaker assumptions of Theorem 1, in which $(N, +)$ is just a loop and $\Phi \leq Aut(N, +)$ is only required to admit a regular orbit.

2.2 Planarity and semiplanarity

So far we have used planarity only under the restriction $b = 0$ or equivalently $a = 0$ (0-planarity) in the equation $a \cdot x = c + b \cdot x$.

Therefore, we may assume now that, in the full planarity condition, $a^{\cdot}$ is different from $0^{\cdot}$, i.e. by (2.3) that $a^{\cdot} \in Aut(N, +)$, thus condition (P_l) becomes equivalent to:

$(P_l)'$ $\forall b, c \in N$ with $b^{\cdot} \neq 1$, $|S_{1,b,c}| = 1$, i.e. $|Fix(c^{+}b^{\cdot})| = 1$.

We call a loop N together with an automorphism group $\Phi \leq AutN$ an *l-Ferrero pair* (N, Φ) if $|\Phi| \geq 2$ and $|Fix(c^{+}\varphi)| = 1$ for all $c \in N$ and $\varphi \in \Phi \setminus \{1\}$.

In a loop $(N, +)$ with the *inverse property*[3] (e.g. in a *Moufang loop*) it is straightforward to verify that, given a non trivial automorphism $\varphi \in AutN \setminus \{1\}$, the mapping $\varphi - 1 : N \to N; x \mapsto \varphi(x) - x$ is injective if and only if $|Fix(c^{+}\varphi)| \leq 1$, and surjective if and only if $|Fix(c^{+}\varphi)| \geq 1$ hence bijective exactly when $|Fix(c^{+}\varphi)| = 1$, for all $c \in N$.

[3]In a loop $(N, +)$, for each $a \in N$, let $-a$ be the unique solution of the equation $a + x = 0$. The *left inverse property* (L.i.p.) is expressed by the identity $-a + (a + x) = x$ (for $a, x \in N$). When this property holds, the right inverse $-a$ is also the left inverse of a, i.e. $-a + a = 0$, so the loop has unique inverses, and one can express also the *right inverse property* (R.i.p.) by the identity $(x + a) - a = x$ (for $a, x \in N$). The loop is said to have the *inverse property* (I.p.) if both L.i.p. and R.i.p. hold. For further algebraic properties of loops and the definition of Moufang loops, see e.g. [3].

Now if (N, Φ) is an l-Ferrero pair then all $\varphi \in \Phi \setminus \{1\}$ act *fixed-point-free* (i.e. $Fix\ \varphi = \{0\}$) on N, hence in this case, chosen any complete set T of orbit representatives of Φ on N, $T^* = T \setminus \{0\} \subseteq N \setminus \{0\}$ and $T^* \neq \emptyset$ since $|N| \geq 2$.

Therefore any l-Ferrero pair (N, Φ) is a weak l-Ferrero pair as well, so by Theorem 1 the loop-nearring $(N, +, \cdot)$ obtained by the Ferrero derivation is 0-planar and, indeed, planar because $(P_l)'$ is fulfilled.

Thus we have proved the following theorem, that provides a direct generalization to the loop case of the classical procedure to construct planar nearrings starting from Ferrero pairs of groups:

Theorem 2. *If (N, Φ) is an l-Ferrero pair, then the "Ferrero derivation" of Theorem 1 gives rise to a planar loop-nearring.*

Let us consider the class of *complete K-nearrings* introduced in [15] by H.Karzel, i.e. loop-nearrings $(N, +, \cdot)$ characterized by the properties:
K1. $(N, +)$ is a loop with unique inverses;
K2. For all $a, b \in N$ the precession map $\delta_{a,b}$ is a left multiplication, i.e. $\delta_{a,b} \in N^{\cdot} \cap SymN$. So the loop $(N, +)$ is a so called *weak K-loop* (cf. e.g. [15], [18]).

Here planarity becomes so strong that it forces the additive loop to be a group. In fact we can show the following

Theorem 3. *Any planar complete K-nearring is a nearring.*

Proof. First observe that $\forall a \in N \setminus \{0\}$, $a = a + (-a + a) = (a + (-a)) + \delta_{a,-a}(a) = \delta_{a,-a}(a)$, hence $\delta_{a,-a} = 1$ since any non trivial left multiplication is fixed-point-free by hypothesis.

Now assume $\delta_{a,b} \neq 1$ for some $a, b \in N$. Since $\delta_{a,b} \in (N^*)^{\cdot}$, by $(P_l)'$ there exists an x such that $x = (a + b) + \delta_{a,b}(x) = a + (b + x)$; but we have also $x = (a - a) + x = (a + (-a)) + \delta_{a,-a}(x) = a + (-a + x)$, hence $b + x = -a + x$, that implies $b = -a$, so $\delta_{a,b} = \delta_{a,-a} = 1$, in contradiction with the original assumption. □

Owing to the previous observations, it is quite of interest to consider also a condition which, at least in the infinite case, is a bit weaker than planarity.

A loop-nearring $(N, +, \cdot)$ is called *semiplanar* if it is 0-planar and $|S_{1,b,c}| \leq 1$, i.e. $|Fix\,(c^+ b^{\cdot})| \leq 1$, for all $b, c \in N$ with $b^{\cdot} \neq 1$.

One can construct also semiplanar loop-nearrings via the "Ferrero derivation" as before, starting now from *semi-l-Ferrero pairs* (N, Φ), where N is a loop and $\Phi \leq AutN$ an automorphism group such that for all $c \in N$ and $\varphi \in \Phi$, $\varphi \neq 1$, $|Fix\,(c^+\varphi)| \leq 1$.

We remark that, if (N,Φ) is a semi-l-Ferrero pair then there is a Ferrero derivation (cf. Th. 1) such that for the obtained semiplanar loop-nearring $(N,+,\cdot)$ we have $A=\{0\}$, hence $N^*=N\setminus\{0\}$.

Note that $|Fix\,(c^+\varphi)\,|\leq 1$ for all $c\in N$ and $\varphi\neq 1$ always implies (set $c=0$) that φ is fixed-point-free, while the converse, that can be easily proved if N is a group, does not hold in general for a loop, even when the inverse property holds.

In the following section we will show an example of a proper semiplanar loop-nearring, which is a complete K-nearring.

2.3 Examples

Many examples of 0-planar loop-nearrings can be constructed starting from weak l-Ferrero pairs.

In particular we get properly 0-planar complete K-nearrings starting from pairs (K,Φ), where K is a K-loop (for the definition, see e.g. [18]) Φ a non trivial subgroup of the left inner mapping group Δ and the *right nucleus* $N_r := \{a\in K \mid \forall x,y\in K : x+(y+a)=(x+y)+a\}$ is non trivial and coincides with $Fix\ \delta_{x,y}$ for all $x,y\in K$ (examples can be found e.g. in [11], [16], [19]).

If we restrict to 0-planar nearrings, then a rich class of examples is furnished by those pairs (N,Φ) where N is a group and $\Phi\leq AutN$ is such that for all $\varphi,\vartheta\in\Phi$, $Fix\varphi = Fix\vartheta\leq N$ (expressed by condition (K)' in [12]) as for instance when N is the additive group of a vector space and Φ is a subgroup of elations with a fixed hyperplane as axis, or a subgroup of rotations with fixed axis of a euclidean space N with $\dim N\geq 3$.

Semiplanar loop-nearrings include infinite neofields, left neofields, neardomains and nearfields. An interesting semiplanar complete K-nearring can be constructed from the semi-l-Ferrero pair (K,Δ) where K is the K-loop of hyperbolic translations of the real hyperbolic plane (cf. e.g. [17]) and Δ is the whole structure group of K, consisting of all hyperbolic rotations with centre the point 0 of K: it is straightforward to verify that for each $a\in K$ and $\delta\in\Delta$ the isometry $a^+\delta$ has exactly one fixed point except the case of "limit rotations" for which $Fix\,(a^+\delta)=\emptyset$, hence the general condition $|Fix\,(a^+\delta)\,|\leq 1$ holds for all $a\in K,\ \delta\in\Delta$.

Finally, any planar nearring or nearfield, any finite (left) neofield can be regarded to as an instance of a planar loop-nearring.

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