Iteration Semirings

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Abstract. A Conway semiring is a semiring S equipped with a unary operation $^* : S \to S$, called star, satisfying the sum star and product star equations. An iteration semiring is a Conway semiring satisfying Conway's group equations. In this extended abstract, we review the role of iteration semirings in the axiomatization of regular languages and rational power series, and in the axiomatization of the equational theory of continuous and complete semirings.

1 Introduction

One of the most fundamental algebraic structures introduced in Computer Science are the algebras of regular languages over an alphabet. These algebras are idempotent semirings equipped with a star operation, denoted * and called Kleene iteration. Other structures with similar operations include binary relations equipped with the operations of union, relation composition and reflexive-transitive closure which provide a theoretical basis for nondeterministic imperative programming. The reflexive-transitive closure operation on binary relations and the operation of Kleene iteration on languages can both be defined as least upper bounds of a chain of approximations, or by an infinite geometric sum. More generally, each ω-continuous or ω-complete semiring gives rise to a star operation. It is known that regular languages satisfy the very same equations as binary relations, or as idempotent ω-continuous semirings, or ω-complete semirings with an infinitary idempotency condition: the sum of every nonempty countable sum of any element a with itself is a. Moreover, the *-semirings of regular languages can be characterized as the free algebras in the class of all models of these equations.

The equational theory of regular languages has received a lot of attention since the late 1950's. Early results by Redko [40,41] and Conway [13] show that there is no finite basis of equations. On the other had, there are several finitely axiomatized first-order theories capturing the equational theory of regular languages. The first such first-order theory was described by Salomaa [42]. He proved that a finite set of equations, together with a guarded unique fixed point rule is sound and complete for the equational theory of regular languages. This result has been the model of several other complete axiomatizations involving power series, trees and tree languages, traces, bisimulation semantics, etc., see Morisaki and Sakai [38], Elgot [17],

* Supported in part by grant no. MTM 2007-63422 from the Ministry of Science and Education of Spain. On leave from University of Szeged, Hungary.

M. Ito and M. Toyama (Eds.): DLT 2008, LNCS 5257, pp. 1–20, 2008.

Elgot, Bloom, Tindell [18], Ito and Ando [28], Milner [36,37], Rabinovitch [39], Corradini, De Nicola, Labella [14], to mention a few examples. Salomaa's axiomatization has been further refined by Archangelsky and Gorshkov [1], Krob [34], Kozen [30,31] and Boffa [10,11]. See also [7]. In particular, the system found by Kozen relies on the least pre-fixed point rule for left and right linear fixed point equations. This result is important since the (guarded) unique fixed point rule is not sound in several important models such as binary relations over a nontrivial set.

Another direction of research has been trying to describe infinite nontrivial equational basis of regular languages, cf. [34,5]. A Conway semiring [6] is a *-semiring (i.e., a semiring equipped with a star operation) satisfying the well-known sum star and product star equations. In [13], Conway associated an equation with each finite group in any *-semiring. An iteration semiring is a Conway semiring which is a model of these group equations. Conway conjectured that the iteration semiring equations, together with a strengthened form of idempotency (the equation $1^* = 1$) are complete for regular languages. His conjecture was confirmed by Krob. (See also [19] for a proof from a more general perspective.) The completeness of this system readily implies the completeness of all known first-order axiomatizations, including the completeness of the systems given by Krob, Kozen, Boffa and others mentioned above. The equations associated with the finite groups are rather complex. For commutative (or even solvable) groups, they can be reduced to the classical "power equations", see Conway [13] and Krob [34], but the simplification of the general group equations is still open.

In this paper our aim is to review the role of Conway and iteration semirings in some recent axiomatization results extending those mentioned above. These recent results show that the iteration semiring equations, together with three simple additional equations, provide a complete account of the equations that hold in ω-continuous, or ω-complete semirings without any idempotency conditions. Moreover, the same equations provide a complete account of the valid equations of rational power series over the semiring $\mathbb{N}_\infty$, obtained by adding a point of infinity to the usual semiring $\mathbb{N}$ of nonnegative integers. We also provide finite first-order axiomatizations extending the results of Krob and Kozen. We will also include results regarding the equational theory of rational series over $\mathbb{N}$, or the semirings $\mathbf{k}$ on the set $\{0, 1, \cdots, k-1\}$ obtained from $\mathbb{N}_\infty$ by collapsing ∞ with the integers $\geq k-1$. For proofs we refer to [8] and [9].

2 Preliminaries

A *semiring* [24] is an algebra $S = (S, +, \cdot, 0, 1)$ such that $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, moreover, the following hold for all $a, b, c \in S$:

$$0 \cdot a = 0$$
$$a \cdot 0 = 0$$
$$a(b + c) = ab + ac$$
$$(b + c)a = ba + ca$$

The operation $+$ is called *sum* or *addition*, and the operation $\cdot$ is called *product* or *multiplication*. A semiring S is called *idempotent* if

$$1 + 1 = 1, \quad \text{or equivalently,} \quad a + a = a$$

for all $a \in S$. A morphism of semirings preserves the sum and product operations and the constants 0 and 1. Since semirings are defined by equations, the class of all semirings is a variety (see e.g., [25]) as is the class of all idempotent semirings.

For any integer $k \geq 1$, we will also write k for the *term* which is the k-fold sum of 1 with itself: $1 + \cdots + 1$, where 1 appears k-times. Thus, a semiring S is idempotent if the equation $1 = 2$ holds in S. More generally, we will find occasion to deal with semirings satisfying the equation $k - 1 = k$, for some $k \geq 2$.

An important example of a semiring is the semiring $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ of nonnegative integers equipped with the usual sum and product operations, and an important example of an idempotent semiring is the *boolean semiring* $\mathbb{B}$ whose underlying set is $\{0, 1\}$ and whose sum and product operations are disjunction and conjunction. Actually $\mathbb{N}$ and $\mathbb{B}$ are respectively the initial semiring and the initial idempotent semiring. More generally, for every $k \geq 2$, we let $\mathbf{k}$ denote the semiring obtained from $\mathbb{N}$ by identifying all integers $\geq k - 1$. We may represent $\mathbf{k}$ as the semiring on the set $\{0, \cdots, k-1\}$ where sum and product are the usual operations if the result of the operation is in this set, otherwise the operation returns $k - 1$. For each k, the semiring $\mathbf{k}$ is initial in the class of semirings satisfying $k - 1 = k$.

We end this section by describing two constructions on semirings. For more information on semirings, the reader is referred to Golan's book [24].

2.1 Polynomial Semirings and Power Series Semirings

Suppose that S is a semiring and A is a set. Let A^* denote the free monoid of all words over A including the empty word ϵ. A *formal power series*, or just *power series* over S in the (noncommuting) letters in A is a function $s : A^* \to S$. It is a common practice to represent a power series s as a formal sum $\sum_{w \in A^*}(s, w)w$, where the *coefficient* (s, w) is ws, the value of s on the word w. The *support* of a series s is the set $\mathrm{supp}(s) = \{w : (s, w) \neq 0\}$. When $\mathrm{supp}(s)$ is finite, s is called a *polynomial*. We let $S\langle\langle A^*\rangle\rangle$ and $S\langle A^*\rangle$ respectively denote the collection of all power series and polynomials over S in the letters A.

We define the sum $s + s'$ and product ss' of two series $s, s' \in S\langle\langle A^*\rangle\rangle$ as follows. For all $w \in A^*$,

$$\begin{aligned}(s + s', w) &= (s, w) + (s', w)\\ (ss', w) &= \sum_{uu'=w} (s, u)(s', u').\end{aligned}$$

We may identify any element $s \in S$ with the series, in fact polynomial that maps ϵ to s and all other elements of A^* to 0. In particular, 0 and 1 may be viewed as polynomials. It is well-known that equipped with the above operations and constants, $S\langle\langle A^*\rangle\rangle$ is a semiring which contains $S\langle A^*\rangle$ as a subsemiring.

The semiring $S\langle A^*\rangle$ can be characterized by a universal property. Consider the natural embedding of A into $S\langle A^*\rangle$ such that each letter $a \in A$ is mapped to the polynomial whose support is $\{a\}$ which maps a to 1. By this embedding, we may view A as a subset of $S\langle A^*\rangle$. Recall also that each $s \in S$ is identified with a polynomial. The following fact is well-known.

Theorem 1. *Given any semiring S', any semiring morphism $h_S : S \to S'$ and any function $h : A \to S'$ such that*

$$(sh_S)(ah) = (ah)(sh_S) \tag{1}$$

for all $a \in A$ and $s \in S$, there is a unique semiring morphism $h^\sharp : S\langle A^*\rangle \to S'$ which extends both h_S and h.

The condition (1) means that for any $s \in S$ and letter $a \in A$, sh_S *commutes with* ah. In particular, since $\mathbb{N}$ is initial, and since when $S = \mathbb{N}$ the condition (1) holds automatically, we obtain that any map $A \to S'$ into a semiring S' extends to a unique semiring morphism $\mathbb{N}\langle A^*\rangle \to S'$, i.e., the polynomial semiring $\mathbb{N}\langle A^*\rangle$ is *freely generated by A in the class of semirings.* In the same way, for each $k \geq 2$, $\mathbf{k}\langle A^*\rangle$ is freely generated by A in the class of semirings satisfying the equation $k - 1 = k$.

Note that a series in $\mathbb{B}\langle\langle A^*\rangle\rangle = \mathbf{2}\langle\langle A^*\rangle\rangle$ may be identified with its support. Thus a series in $\mathbb{B}\langle\langle A^*\rangle\rangle$ corresponds to a language over A and a polynomial in $\mathbb{B}\langle A^*\rangle$ to a finite language. The sum operation corresponds to set union and the product operation to concatenation. The constants 0 and 1 are the empty set and the singleton set $\{\epsilon\}$.

The power series semirings $S\langle\langle A^*\rangle\rangle$ can be generalized in a straightforward way to semirings of series $S\langle\langle M\rangle\rangle$, where $M = (M, \cdot, 1)$ is a *locally finite partial monoid.*[1] Here, a partial monoid is a set M equipped with a partially defined product $(m, m') \mapsto mm'$ and a constant 1 such that for any m, m', m'' in M, $(mm')m''$ is defined iff $m(m'm'')$ is defined in which case they are equal. Moreover, the products $1m$ and $m1$ are always defined and equal to m. We say that a partial monoid is locally finite if each $m \in M$ can be written only a finite number of different ways as a product $m_1 \cdots m_k$ with $m_i \neq 1$ for all i. Clearly, every free monoid is locally finite. Another example is given by *data word* monoids. Suppose that A is an alphabet and D is set of *data values.* A *data word* over (A, D) is either the symbol 1 or a word in $D(AD)^+$. Let $(A, D)^*$ denote this set. We define a partial product operation on $(A, D)^*$ called *fusion* and denoted $\bullet$: For any $u = d_0 a_1 \cdots a_n d_n$ and $u' = d'_0 a'_1 \cdots a'_m d'_m$ in $D(AD)^+$ we define $u \bullet u' = d_0 a_1 \cdots a_n d'_n a'_1 \cdots a'_m d'_m$ if $d_n = d'_0$, and leave $u \bullet u'$ undefined otherwise. Moreover, we define $1 \bullet u = u \bullet 1 = u$ for all $u \in (A, D)^*$. Let M be a locally finite partial monoid. We call an element $m \in M$ *irreducible* if $m \neq 1$ and m cannot be written as a nontrivial product of elements different from 1. It is clear that each $m \in M$ is a product of irreducibles (where the empty product is 1).

[1] This notion generalizes the locally finite monoids of Eilenberg [16].

2.2 Matrix Semirings

When S is a semiring, then for each $n \geq 0$ the set $S^{n \times n}$ of all $n \times n$ matrices over S is also a semiring denoted $S^{n \times n}$. The sum operation is defined pointwise and product is the usual matrix product. The constants are the matrix 0_{nn} all of whose entries are 0 (often denoted just 0), and the diagonal matrix E_n whose diagonal entries are all 1.

We can associate an $n \times n$ zero-one matrix with each function or relation ρ from the set $\{1, \cdots, n\}$ to the set $\{1, \cdots, m\}$, whose (i, j)th entry is 1 when $i \rho j$, and 0 otherwise. We usually identify ρ with the associated matrix, called a *functional* or *relational matrix*. A matrix associated with a bijective function is called a *permutation matrix*.

3 Conway Semirings

The definition of Conway semirings involves two important equations of regular languages. Conway semirings appear implicitly in Conway [13] and were defined explicitly in [6]. See also [35]. On the other hand, the applicability of Conway semirings is limited due to the fact that the star operation is total, whereas many important semirings only have a partially defined star operation. Moreover, it is not true that all such semirings can be embedded into a Conway semiring with a totally defined star operation. The following definition is taken from [9].

Definition 1. *A* partial *-semiring *is a semiring S equipped with a partially defined star operation $^* : S \to S$ whose domain is an* ideal *of S. A* *-semiring *is a partial *-semiring S such that * is defined on the whole semiring S. A morphism $S \to S'$ of (partial) *-semirings is a semiring morphism $h : S \to S'$ such that for all $s \in S$, if s^* is defined then so is $(sh)^*$ and $s^* h = (sh)^*$.*

Thus, in a partial *-semiring S, 0^* is defined, and if a^* *and* b^* are defined then so is $(a + b)^*$, finally, if a^* *or* b^* is defined, then so is $(ab)^*$. When S is a partial *-semiring, we let $D(S)$ denote the domain of definition of the star operation.

Definition 2. *A* partial Conway semiring *is a partial *-semiring S satisfying the following two axioms:*

1. Sum star equation*:*

$$(a + b)^* = a^*(ba^*)^*$$

for all $a, b \in D(S)$.

2. Product star equation*:*

$$(ab)^* = 1 + a(ba)^* b,$$

for all $a, b \in S$ such that $a \in D(S)$ or $b \in D(S)$.

A Conway semiring *is a partial Conway semiring S which is a *-semiring (i.e., $D(S) = S$). A morphisms of (partial) Conway semirings is a (partial) *-semiring morphism.*

Note that in any partial Conway semiring S,

$$\begin{aligned} aa^* + 1 &= a^* \\ a^*a + 1 &= a^* \\ 0^* &= 1 \end{aligned}$$

for all $a \in D(S)$. Moreover, if $a \in D(S)$ or $b \in D(S)$, then

$$(ab)^*a = a(ba)^*.$$

It follows that also

$$\begin{aligned} aa^* &= a^*a \\ (a+b)^* &= (a^*b)^*a^* \end{aligned}$$

for all $a, b \in D(S)$. When $a \in D(S)$ we will denote $aa^* = a^*a$ by a^+ and call $^+$ the *plus* operation.

When S is a (partial) Conway semiring, each semiring $S^{n\times n}$ may be turned into a (partial) Conway semiring.

Definition 3. *Suppose that S is a partial Conway semiring with $D(S) = I$. We define a partial star operation on the semirings $S^{k\times k}$, $k \geq 0$, whose domain of definition is $I^{k\times k}$, the ideal of those $k \times k$ matrices all of whose entries are in I. When $k = 0$, $S^{k\times k}$ is trivial as is the definition of star. When $k = 1$, we use the star operation on S. Assuming that $k > 1$ we write $k = n + 1$. For a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $I^{k\times k}$, define*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{2}$$

where $a \in S^{n\times n}$, $b \in S^{n\times 1}$, $c \in S^{1\times n}$ and $d \in S^{1\times 1}$, and where

$$\begin{aligned} \alpha &= (a + bd^*c)^* & \beta &= \alpha bd^* \\ \gamma &= \delta ca^* & \delta &= (d + ca^*b)^*. \end{aligned}$$

Theorem 2. (Conway[13], Krob [34,35], Bloom, Ésik [6], Bloom, Ésik, Kuich [9]) *Suppose that S is a partial Conway semiring with $D(S) = I$. Then, equipped with the above star operation, each semiring $S^{k\times k}$ is a partial Conway semiring with $D(S^{k\times k}) = I^{k\times k}$.*

Theorem 3. (Conway[13], Krob [34], Bloom, Ésik [6], Bloom, Ésik, Kuich [9]) *Suppose that S is a partial Conway semiring with $D(S) = I$. Then the following equations hold in $S^{k\times k}$:*

1. *The* matrix star equation *(2) for all possible decompositions of a square matrix over I into four blocks as above such that a and d are square matrices.*

2. The permutation equation

$$(\pi A \pi^T)^* = \pi A^* \pi^T,$$

for all A in $I^{k\times k}$ and any $k \times k$ permutation matrix π, where π^T denotes the transpose of π.

We note the following variant of the matrix star equation. Let S be a partial Conway semiring, $I = D(S)$. Then if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix with entries in I, partitioned as above, then

$$A^* = \begin{pmatrix} (a + bd^*c)^* & a^*b(d + ca^*b)^* \\ d^*c(a + bd^*c)^* & (d + ca^*b)^* \end{pmatrix}$$

4 Iteration Semirings

Many important (partial) Conway semirings satisfy the group equations associated with the finite groups, introduced by Conway [13]. When a (partial) Conway semiring satisfies the group equations, it will be called a (partial) iteration semiring. Below we will consider groups of order n defined on the set $\{1, \cdots, n\}$ of positive integers with multiplication $(i, j) \mapsto ij$ and inverse $i \mapsto i^{-1}$.

Definition 4. *We say that the* group equation associated with a finite group G *of order n holds in a partial Conway semiring S if*

$$e_1 M_G^* u_n = (a_1 + \cdots + a_n)^* \tag{3}$$

holds, where $a_1, \cdots, a_n$ are arbitrary elements of $D(S)$, and where M_G is the $n \times n$ matrix whose (i, j)th entry is $a_{i^{-1}j}$, for all $1 \le i, j \le n$, and e_1 is the $1 \times n$ functional matrix whose first entry is 1 and whose other entries are 0, finally u_n is the $n \times 1$ matrix all of whose entries are 1.

Equation (3) asserts that the sum of the entries of the first row of M_G^* is $(a_1 + \cdots + a_n)^*$. For example, the group equation associated with the group of order 2 is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}^* \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (a_1 + a_2)^*$$

which by the matrix star equation can be written as

$$(a_1 + a_2 a_1^* a_2)^* (1 + a_2 a_1^*) = (a_1 + a_2)^*.$$

(It is known that in Conway semirings, this equation is further equivalent to the *power identity* $(a^2)^*(1 + a) = a^*$.)

Definition 5. *We say that a Conway semiring S is an* iteration semiring *if it satisfies all group equations. We say that a partial Conway semiring S is a* partial iteration semiring *if it satisfies all group equations (3) where $a_1, \cdots, a_n$ range over $D(S)$. A morphism of (partial) iteration semirings is a (partial) Conway semiring morphism.*

Proposition 1. *Suppose that the partial Conway semiring S satisfies the group equation (3) for all $a_1, \cdots, a_n \in D(S)$. Then S also satisfies*

$$u_n^T M_G^* e_1^T = (a_1 + \cdots + a_n)^*, \tag{4}$$

for all $a_1, \cdots, a_n \in D(S)$, where e_1, M_G and u_n are defined as above. Thus, if S is an iteration semiring, then (4) holds for all finite groups G.

Remark 1. In Conway semirings, the group equation (3) is equivalent to (4).

Remark 2. Let $\mathcal{G}$ denote a class of finite groups. It is known, cf. [13,34] that the defining equations of Conway semirings, in conjunction with the group equations associated with the groups in $\mathcal{G}$ are complete for iteration semirings iff every finite simple group is a quotient of a subgroup of a group in $\mathcal{G}$.

The group equations seem to be extremely difficult to verify in practice. However, they are implied by the simpler functorial star conditions defined below.

Definition 6. *Suppose that S is a partial Conway semiring so that each matrix semiring $S^{n \times n}$ is also a Conway semiring. Let $I = D(S)$, and let $\mathcal{C}$ be a class of rectangular matrices over S. We say that S has a* functorial star *with respect to $\mathcal{C}$ if for all matrices $A \in I^{m \times m}$ and $B \in I^{n \times n}$, and for all $m \times n$ matrices C in $\mathcal{C}$, if $AC = CB$ then $A^*C = CB^*$. Finally, we say that S has a functorial star if it has a functorial star with respect to the class of all rectangular matrices.*

Proposition 2. (Bloom, Ésik [6], Bloom, Ésik, Kuich [9]) *Suppose that S is a (partial) Conway semiring.*

1. *S has a functorial star with respect to the class of all injective functional matrices and their transposes.*
2. *If S has a functorial star with respect to the class of functional matrices $m \to 1$, $m \geq 2$, then S has a functorial star with respect to the class of all functional matrices.*
3. *If S has a functorial star with respect to the class of transposes of functional matrices $m \to 1$, $m \geq 2$, then S has a functorial star with respect to the class of transposes of all functional matrices.*
4. *If S has a functorial star with respect to the class of all functional matrices $m \to 1$, $m \geq 2$, or with respect to the class of transposes of functional matrices $m \to 1$, $m \geq 2$, then S is a (partial) iteration semiring.*

Theorem 4. (Conway [13], Krob [34], Ésik [19], Bloom, Ésik, Kuich [9]) *If S is a (partial) iteration semiring, then so is $S^{k \times k}$ for each $k \geq 0$.*

Later we will see that the class of (partial) iteration semirings is also closed under taking power series semirings. We now give two classes of (partial) iteration semirings.

4.1 Partial Iterative Semirings

In this section we exhibit a class of partial iteration semirings. The definition of these semirings is motivated by Salomaa's axiomatization [42] of regular languages.

Definition 7. *A* partial iterative semiring *is a partial* **-semiring* S *such that for every* $a \in D(S)$ *and* $b \in S$, a^*b *is the* unique solution *of the equation* $x = ax + b$. *A morphism of partial iterative semirings is a* **-semiring morphism.*

We note that any semiring S with a distinguished ideal I such that for all $a \in I$ and $b \in S$, the equation $x = ax + b$ has a unique solution can be turned into a partial iterative semiring, where star is defined on I. Indeed, when $a \in I$, define a^* as the unique solution of the equation $x = ax+1$. It follows that $aa^*b+b = a^*b$ for all b, so that a^*b is the unique solution of $x = ax + b$. We also note that when S, S' are partial iterative semirings, then any semiring morphism $h : S \to S'$ with $D(S)h \subseteq D(S')$ automatically preserves star.

Theorem 5. (Bloom, Ésik, Kuich [9]) *Every partial iterative semiring is a partial iteration semiring with a functorial star.*

Thus, when S is a partial iterative semiring with $D(S) = I$, then by Definition 3, each matrix semiring $S^{n\times n}$ is a partial iteration semiring with $D(S) = I^{n\times n}$.

Theorem 6. (Bloom, Ésik, Kuich [9]) *Suppose that* S *is a partial iterative semiring with* $D(S) = I$. *Then for any* $A \in I^{n\times n}$ *and* $B \in S^{n\times p}$, A^*B *is the unique solution of the matrix equation* $X = AX + B$. *In particular,* $S^{n\times n}$ *is a partial iterative semiring where the star operation is defined on* $I^{n\times n}$.

Theorem 7. (Bloom, Ésik, Kuich [9]) *Suppose that* S *is a partial iterative semiring with* $D(S) = I$ *and* $A \in S^{n\times n}$ *and* $B \in S^{n\times p}$ *such that* $A^k \in I^{n\times n}$ *for some* $k \geq 1$. *Then the equation* $X = AX + B$ *in the variable* X *ranging over* $S^{n\times p}$ *has* $(A^k)^*(A^{k-1}B + \cdots + B)$ *as its unique solution.*

We give an example of a partial iterative semiring. Let S be a semiring and M a locally finite partial monoid, and consider the semiring $S\langle\langle M\rangle\rangle$. We call a series $s \in S\langle\langle M\rangle\rangle$ *proper* if $(s, 1) = 0$. Clearly, the proper series form an ideal. For any series s, r, if s is proper, then the equation $x = sx + r$ has a unique solution. (For the case when M is a locally finite monoid, see [16].) Moreover, this unique solution is s^*r, where s^* is the unique solution of the equation $y = sy + 1$.

Proposition 3. *For any semiring* S *and locally finite partial monoid* M, $S\langle\langle M\rangle\rangle$, *equipped with the above star operation defined on the proper series, is a partial iterative semiring and thus a partial iteration semiring.*

Theorem 8. (Bloom, Ésik [6]) *Suppose that* S *is partial iterative semiring with star operation defined on* $D(S) = I$, *and suppose that* S_0 *is a subsemiring of* S *which is equipped with a unary operation* $^\otimes$. *Moreover, suppose that* S *is the* direct sum *of* S_0 *and* I, *so that each* $s \in S$ *has a unique representation as a sum*

$x + a$ with $x \in S_0$ and $a \in I$. If S_0, equipped with the operation ${}^{\otimes}$, is a Conway semiring, then there is a unique way to turn S into a Conway semiring whose star operation extends ${}^{\otimes}$. This operation also extends the star operation defined on I. Moreover, when S_0 is an iteration semiring, then S is also an iteration semiring. In particular, if S is a Conway or an iteration semiring, then so is $S\langle\langle A^* \rangle\rangle$, for any set A.

The last sentence of the previous result can be generalized.

Theorem 9. *For every locally finite partial monoid M, if S is a Conway or iteration semiring, then so is $S\langle\langle M \rangle\rangle$ in an essentially unique way.*

4.2 ω-Complete *-Semirings

Power series semirings only have a partial star operation defined on proper series. In order to make star a totally defined operation, Eilenberg introduced complete semirings in [16]. A variant of this notion is defined below, see also Krob [32], Hebisch [27], Karner [29] and Bloom, Ésik [6].

Definition 8. *We call a semiring S* ω-complete *if it is equipped with a* summation operation $\sum_{i \in I} s_i$, *defined on countable families s_i, $i \in I$ over S such that $\sum_{i \in \emptyset} s_i = 0$, $\sum_{i \in \{1,2\}} s_i = s_1 + s_2$, moreover,*

$$a(\sum_{i \in I} b_i) = \sum_{i \in I} ab_i \quad (\sum_{i \in I} b_i)a = \sum_{i \in I} b_i a \quad \sum_{j \in J} \sum_{i \in I_j} a_i = \sum_{i \in I} a_i$$

where in the last equation the countable set I is the disjoint union of the sets I_j, $j \in J$. A morphism of ω-complete semirings also preserves summation.

We note that ω-complete semirings are sometimes called *countably complete semirings*. An example of an ω-complete semiring is the semiring $\mathbb{N}_\infty$ obtained by adding a point of infinity ∞ to the semiring $\mathbb{N}$, where a sum $\sum_{i \in I} s_i$ is defined to be ∞ if there is some i with $s_i = \infty$ or the number of i with $s_i \neq 0$ is infinite. In all other cases each s_i is in $\mathbb{N}$ and the number of i with $s_i \neq 0$ is finite, so that we define $\sum_{i \in I} s_i$ as the usual sum of those s_i with $s_i \neq 0$. Suppose that S is an ω-complete semiring. Then we define a star operation by $a^* = \sum_{n \geq 0} a^n$, for all $a \in S$.

Definition 9. *An ω-complete *-semiring is a *-semiring S which is an ω-complete semiring whose star operation is derived from the ω-complete structure as above. A morphism of ω-complete *-semirings is a semiring morphism which preserves all countable sums and thus the star operation.*

Theorem 10. (Bloom, Ésik [6]) *Any ω-complete *-semiring is an iteration semiring with a functorial star.*

The fact that any ω-complete *-semiring is a Conway semiring was shown in [27].

Proposition 4. *If S is an ω-complete semiring, then equipped with the pointwise summation, so is each matrix semiring $S^{n\times n}$ as is each power series semiring $S\langle\langle M\rangle\rangle$, where M is any locally finite partial monoid.*

(Actually for the last fact it suffices that each element of M has an at most countable number of nontrivial decompositions into a product.) Thus, when S is an ω-complete *-semiring, where the star operation is derived from an ω-complete structure, then we obtain two star operations on each matrix semiring $S^{n\times n}$: the star operation defined by the matrix star equation, and the star operation derived from the ω-complete structure on $S^{n\times n}$. However, these two star operations are the same. A similar fact holds for each power series semiring $S\langle\langle M\rangle\rangle$, where M is a locally finite partial monoid.

Theorem 11. (Bloom, Ésik [6]) *Let S be an ω-complete *-semiring, so that S is an iteration semiring.*

1. *For each $n \geq 1$, the star operation determined on $S^{n\times n}$ by Definition 3 is the same as the star operation derived from the ω-complete structure on S.*
2. *For each locally finite partial monoid M, the star operation determined on $S\langle\langle M\rangle\rangle$ by Theorem 8 is the same as that derived from the ω-complete structure on S.*

Actually, the last fact is stated in [6] for free monoids, but the extension is clear. It is easy to show that all free ω-complete *-semirings exist.

Theorem 12. *For each set A, $\mathbb{N}_\infty\langle\langle A^*\rangle\rangle$ is freely generated by A in the class of all ω-complete *-semirings. Moreover, for each $k \geq 2$, the *-semiring $\mathbf{k}\langle\langle A^*\rangle\rangle$ is freely generated by A in the class of all ω-complete *-semirings satisfying $1^* = k-1$.*

Note that if an ω-complete semiring S satisfies $1^* = k-1$, then any countable sum of at least $k-1$ copies of an element a with itself is $(k-1)a$, the sum of exactly $k-1$ copies.

Complete semirings [16] have a summation operation defined on *all* families of elements over the semiring. There is a weaker notion of *rationally additive* semirings, see [21]. These semirings also give rise to iteration semirings with a functorial star.

5 Kleene Theorem

The classical Kleene theorem equates languages recognizable by finite automata with the regular languages, and its generalization by Schützenberger equates power series recognizable by weighted finite automata with rational power series. Since Kleene's theorem can be formulated in equational logic, by the completeness results presented in Section 6, it can be proved by equational reasoning using the equational axioms. Actually the Conway semiring equations suffice for that purpose. In this section we show a Kleene theorem for partial Conway semirings. To this end, we define a general notion of (finite) automaton in partial Conway semirings. Our presentation follows [9].

Definition 10. *Suppose that S is a partial Conway semiring, S_0 is a subsemiring of S and A is a subset of $D(S)$. An* automaton *in S over (S_0, A) is a triplet $\mathbf{A} = (\alpha, M, \beta)$ consisting of an* initial vector *$\alpha \in S_0^{1\times n}$, a* transition matrix *$M \in (S_0A)^{n\times n}$, where S_0A is the set of all linear combinations over A with coefficients in S_0, and a* final vector *$\beta \in S_0^{n\times 1}$. The* behavior *of $\mathbf{A}$ is $|\mathbf{A}| = \alpha M^*\beta$.*

(Since $M \in D(S)^{n\times n}$, M^* exists.)

Definition 11. *We say that $s \in S$ is* recognizable *over (S_0, A) if s is the behavior of some automaton over (S_0, A). We let $\mathbf{Rec}_S(S_0, A)$ denote the set of all elements of S which are recognizable over (S_0, A).*

Next we define rational elements.

Definition 12. *Let S, S_0 and A be as above. We say that $s \in S$ is* rational *over (S_0, A) if $s = x + a$ for some $x \in S_0$ and some $a \in S$ which is contained in the least set $\mathbf{Rat}'_S(S_0, A)$ containing $A \cup \{0\}$ and closed under the* rational operations *$+$, $\cdot$, $^+$ and left and right multiplication with elements of S_0. We let $\mathbf{Rat}_S(S_0, A)$ denote the set of rational elements over (S_0, A).*

Note that $\mathbf{Rat}'_S(S_0, A) \subseteq D(S)$.

Proposition 5. *Suppose that S is a partial Conway semiring, S_0 is a subsemiring of S and A is a subset of $D(S)$. Then $\mathbf{Rat}_S(S_0, A)$ contains S_0 and is closed under sum and product. Moreover, it is closed under star iff it is closed under the plus operation.*

Proposition 6. *Suppose that S is a partial Conway semiring, S_0 is a subsemiring of S and A is a subset of $D(S)$. Then $\mathbf{Rat}_S(S_0, A)$ is contained in the least subsemiring of S containing S_0 and A which is closed under star.*

We give two sufficient conditions under which $\mathbf{Rat}_S(S_0, A)$ is closed under star.

Proposition 7. *Let S, S_0 and A be as above. Assume that either $S_0 \subseteq D(S)$ and S_0 is closed under star, or the following condition holds:*

$$\forall x \in S_0 \forall a \in D(S) \quad (x + a \in D(S) \Rightarrow x = 0). \tag{5}$$

Then $\mathbf{Rat}_S(S_0, A)$ is closed under star. Moreover, in either case, $\mathbf{Rat}_S(S_0, A)$ is the least subsemiring of S containing S_0 and A which is closed under star.

Remark 3. Note that the second condition in the above proposition holds whenever each $s \in S$ has at most one representation $s = x + a$ with $x \in S_0$ and $a \in D(S)$. This happens when S is the *direct sum* of S_0 and $D(S)$.

Theorem 13. (Bloom, Ésik, Kuich [9]) *Suppose that S is a partial Conway semiring, S_0 is a subsemiring of S, $A \subseteq D(S)$. Then $\mathbf{Rec}_S(S_0, A)$=$\mathbf{Rat}_S(S_0, A)$.*

Corollary 1. *Suppose that S is a Conway semiring, S_0 is a Conway subsemiring of S and $A \subseteq S$. Then $\mathbf{Rec}_S(S_0, A) = \mathbf{Rat}_S(S_0, A)$ is the least Conway subsemiring of S which contains $S_0 \cup A$.*

Corollary 2. *Suppose that S is a partial Conway semiring, S_0 is a subsemiring of S and $A \subseteq D(S)$. Suppose that condition (5) holds. Then $\mathbf{Rec}_S(S_0, A) = \mathbf{Rat}_S(S_0, A)$ is the least partial Conway subsemiring of S which contains $S_0 \cup A$.*

The case when the partial Conway semiring is a power series semiring over a free monoid deserves special attention. Let A be set and S be a semiring, and consider the partial iteration semiring $S\langle\langle A^*\rangle\rangle$ whose star operation is defined on proper series. Alternatively, let S be a Conway semiring, and consider the Conway semiring $S\langle\langle A^*\rangle\rangle$ (cf. Theorem 8). We denote $\mathbf{Rat}_{S\langle\langle A^*\rangle\rangle}(S, A)$ by $S^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$ and $\mathbf{Rec}_{S\langle\langle A^*\rangle\rangle}(S, A)$ by $S^{\mathrm{rec}}\langle\langle A^*\rangle\rangle$.

Remark 4. Note that when S is a Conway semiring, then it is also a semiring. So we may view $S\langle\langle A^*\rangle\rangle$ in two different ways: as a Conway semiring (see Theorem 8), or as a partial Conway semiring where star is defined on the proper series. Thus, we obtain two different notion of rationality: two different definitions of $S^{\mathrm{rat}}\langle A^*\rangle\rangle$, one being a Conway semiring (in fact an iteration semiring) and the other being a partial Conway semiring. However, by Proposition 7 the elements of these two semirings are the same, and the star operation agrees on proper series in the two semirings.

Corollary 3. *Suppose that S is a semiring and A is a set. Then $S^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$ is the least partial iteration subsemiring of $S\langle\langle A^*\rangle\rangle$ containing $S \cup A$. Moreover, $S^{\mathrm{rat}}\langle\langle A^*\rangle\rangle = S^{\mathrm{rec}}\langle\langle A^*\rangle\rangle$.*

Corollary 4. *Suppose that S is a Conway semiring. Then $S^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$ is the least Conway subsemiring of $S\langle\langle A^*\rangle\rangle$ containing $S \cup A$. Moreover, $S^{\mathrm{rat}}\langle\langle A^*\rangle\rangle = S^{\mathrm{rec}}\langle\langle A^*\rangle\rangle$.*

The previous facts can be generalized. Suppose that M is a locally finite partial monoid and let S be a semiring, or a Conway semiring. In the first case, $S\langle\langle M\rangle\rangle$ is a partial iteration semiring whose star operation is defined on the proper series, and in the second case, $S\langle\langle M\rangle\rangle$ is a Conway semiring. Moreover, in the first case, each series associated with an irreducible of M is proper. Now let A denote the set of all irreducibles. We let $S^{\mathrm{rat}}\langle\langle M\rangle\rangle = \mathbf{Rat}_{S\langle\langle M\rangle\rangle}(S, A)$ and $S^{\mathrm{rec}}\langle\langle M\rangle\rangle = \mathbf{Rec}_{S\langle\langle M\rangle\rangle}(S, A)$.

Corollary 5. *Suppose that S is a semiring and M is a locally finite monoid. Then $S^{\mathrm{rat}}\langle\langle M\rangle\rangle$ is the least partial iteration subsemiring of $S\langle\langle M\rangle\rangle$ containing $S \cup A$. Moreover, $S^{\mathrm{rat}}\langle\langle M\rangle\rangle = S^{\mathrm{rec}}\langle\langle M\rangle\rangle$.*

Corollary 6. *Suppose that S is a Conway semiring and M is locally finite partial monoid. Then $S^{\mathrm{rat}}\langle\langle M\rangle\rangle$ is the least Conway subsemiring of $S\langle\langle M\rangle\rangle$ containing $S \cup A$. Moreover, $S^{\mathrm{rat}}\langle\langle M\rangle\rangle = S^{\mathrm{rec}}\langle\langle M\rangle\rangle$.*

The above corollaries essentially cover the Kleene theorems for timed automata in [12,15]. To obtain these results, one only needs to specialize M to certain data word monoids.

6 Completeness

Theorem 14. (Bloom, Ésik [8]) *For each set A, the semiring $\mathbb{N}^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$ is freely generated by A in the class of all partial iteration semirings. In more detail, given any partial iteration semiring S and function $h : A \to S$ with $Ah \subseteq D(S)$, there is a unique morphism of partial iteration semirings $h^\sharp : \mathbb{N}^{\mathrm{rat}}\langle\langle A^*\rangle\rangle \to S$ extending h.*

The proof of Theorem 14 uses the results of Section 5 and some recent results from Béal, Lombardy, Sakarovitch [2,3].

Corollary 7. *For each set A, the semiring $\mathbb{N}^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$ is freely generated by the set A in the class of all partial iterative semirings.*

The notion of a *(*-semiring) term* (or *rational expression*) over a set A is defined as usual: Each letter $a \in A$ is a term as are the symbols 0 and 1, and if t and t' are terms then so are $t + t'$, tt' and t^*. When S is a *-semiring and t is term, then for any valuation $A \to S$, t evaluates to an element of S. We say that an equation $t = t'$ between terms t and t' over A *holds in* S, or *is satisfied by* S, if t and t' evaluate to the same element for each valuation $A \to S$. We may assume that terms are over a fixed countably infinite set of letters.

The above definition does not make sense for partial *-semirings. Therefore we define *ideal terms* (over A) as follows. Each letter $a \in A$ is an ideal term as is the symbol 0. When t, t' are ideal terms and s is any term over A, then $t + t'$, ts, st are ideal terms. A *guarded term* is a term t such that whenever t has a "subterm" of the form s^*, then s is an ideal term. When S is a partial *-semiring and t is a guarded term over A, then t evaluates to an element of S under each *ideal valuation* $A \to D(S)$. Let t and t' be guarded. We say that $t = t'$ holds in S if t and t' evaluate to the same element of S under each guarded evaluation. The following result follows from Theorem 14.

Theorem 15. *The following conditions are equivalent for guarded terms t and t'.*

1. *The equation $t = t'$ holds in all partial iteration semirings.*
2. *The equation $t = t'$ holds in all partial iterative semirings.*
3. *The equation $t = t'$ holds in all semirings $\mathbb{N}^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$.*
4. *The equation $t = t'$ holds in $\mathbb{N}^{\mathrm{rat}}\langle\langle \{a, b\}^*\rangle\rangle$.*

We now turn to the semirings $\mathbb{N}_\infty^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$.

Theorem 16. (Bloom, Ésik [8]) *For any set A, the *-semiring $\mathbb{N}_\infty^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$ is freely generated by A in the class of all iteration semirings satisfying (6), (7) and (8):*

$$1^*1^* = 1^* \tag{6}$$

$$1^*a = a1^* \tag{7}$$

$$(1^*a)^*1^* = 1^*a^*. \tag{8}$$

Moreover, for any $k \geq 2$, the *-semiring $\mathbf{k}^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$ is freely generated by A in the class of all iteration semirings satisfying $1^* = k - 1$.

Using this fact, we obtain:

Theorem 17. (Bloom, Ésik [8]) *The following conditions are equivalent for an equation.*

1. *The equation holds in all ω-complete *-semirings.*
2. *The equation holds in all iteration semirings satisfying (6), (7), (8).*
3. *The equation holds in all *-semirings $\mathbb{N}_\infty^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$.*
4. *The equation holds in $\mathbb{N}_\infty^{\mathrm{rat}}\langle\langle \{a,b\}^*\rangle\rangle$.*

We note that in conjunction with the iteration semiring equations, (6) is equivalent to $1^* = 1^{**}$ and (8) is equivalent to $(1+a)^* = 1^*a^*$ or $a^{**} = 1^*a^*$.

Theorem 18. (Bloom, Ésik [8]) *For any $k \geq 2$, following conditions are equivalent for an equation.*

1. *The equation holds in all ω-complete *-semirings satisfying* $1^* = k-1$.
2. *The equation holds in all iteration semirings satisfying* $1^* = k-1$.
3. *The equation holds in all *-semirings $\mathbf{k}^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$.*
4. *The equation holds in $\mathbf{k}^{\mathrm{rat}}\langle\langle \{a,b\}^*\rangle\rangle$.*

In Theorems 16 and 18, the case when $k = 2$ is due to Krob [34].

7 Ordered Iteration Semirings

A semiring S is called *ordered* if it is equipped with a partial order which is preserved by the sum and product operations. An ordered semiring S is *positive* if 0 is the least element of S. Morphisms of (positive) ordered semirings are also monotone.

Note that when S is a positive ordered semiring, then $a \leq a+b$ for all $a, b \in S$. Moreover, the relation $\preceq$ defined by $a \preceq b$ iff there is some c with $a + c = b$ is also a partial order on S which is preserved by the operations. Since $0 \preceq a$ for all $a \in S$, it follows that equipped with the relation $\preceq$, S is a positive ordered semiring, called a *sum ordered semiring*. Every idempotent semiring S is sum ordered, moreover, the sum order agrees with the *semilattice order*: $a \preceq b$ iff $a + b = b$, for all $a, b \in S$. For later use observe that if S and S' are positive ordered semirings such that S is sum ordered, then any semiring morphism $S \to S'$ is monotone and thus a morphism of ordered semirings.

Definition 13. *An* ordered iteration semiring *is an iteration semiring which is a positive ordered semiring. A morphism of ordered iteration semirings is an iteration semiring morphism which is also an ordered semiring morphism.*

It is clear that if S is an ordered positive semiring, then so is any matrix semiring $S^{n\times n}$ and any power series semiring $S\langle\langle A^*\rangle\rangle$, or more generally, any semiring $S\langle\langle M\rangle\rangle$ where M is a locally finite partial monoid. It follows easily that if S is an ordered iteration semiring, then $S^{n\times n}$ and $S\langle\langle M\rangle\rangle$ are also ordered iteration semirings.

Below we give two classes of ordered iteration semirings.

7.1 ω-Continuous *-Semirings

There are several definitions of continuous, or ω-continuous semirings in the literature. Our definition is taken from [6]. See also [29], where the same semirings are termed *finitary.*[2] Recall that a poset is an *ω-cpo* if it has a least element and suprema of ω-chains. Moreover, a function between ω-cpo's is *ω-continuous* if it preserves suprema of ω-chains. It is clear that if P is an ω-cpo, then so is each P^n equipped with the pointwise order.

Definition 14. *An* ω-continuous semiring *is a positive ordered semiring S such that S is an ω-cpo and the sum and product operations are ω-continuous. An idempotent* ω-continuous *semiring is an ω-continuous semiring which is idempotent. A morphism of (idempotent) ω-continuous semirings is a semiring morphism which is a continuous function.*

Note that each finite positive ordered semiring is ω-continuous. Thus, for each integer $k \geq 1$, the semiring $\mathbf{k}$, equipped with the usual order is ω-continuous. Moreover, $\mathbb{N}_\infty$, ordered as usual, is also ω-continuous.

Theorem 19. (Karner [29]) *Every ω-continuous semiring can be turned into an ω-complete semiring by defining*

$$\sum_{i \in I} a_i = \sup_{F \subseteq I} \sum_{i \in F} a_i$$

where F ranges over the finite *subsets of F and a_i, $i \in I$ is a countable family of elements of S. Morphisms of ω-continuous semirings preserve all countable sums.*

Since each ω-continuous semiring is ω-complete and esch ω-complete semiring is a *-semiring, we deduce that each ω-continuous semiring S is a *-semiring where $a^* = \sum_n a^n = \sup_n \sum_{i=0}^n a^i$ for all $a \in S$.

Definition 15. *An* ω-continuous *-semiring *is an ω-continuous semiring which is a *-semiring where star is determined by the ordered structure as above. A morphism of ω-continuous *-semirings is a morphism of ω-continuous semirings.*

Note that any morphism automatically preserves star. As an example, consider the 3-element semiring $S_0 = \{0, 1, \infty\}$ obtained by adding the element ∞ to the boolean semiring $\mathbb{B}$ such that $a + \infty = \infty + a = \infty$ for all $a \in \{0, 1, \infty\}$ and $b\infty = \infty b = \infty$ if $b \neq 0$. Equipped with the order $0 < 1 < \infty$, it is an

[2] In [33,29], a continuous semiring is a semiring equipped with the *sum order* which is a continuous semiring as defined here. Our terminology stems from the commonly accepted definition of an *ω-continuous algebra* [23,26] as an algebra equipped with a partial order which is an ω-cpo such that the operations are ω-continuous in each argument.

ω-continuous semiring and thus an ω-complete semiring. Let us denote by S this ω-complete semiring. Note that in S, a countably infinite sum of 1 with itself is 1 and thus $1^* = 1$ holds. However, there is another way of turning S_0 into an ω-complete semiring. We define infinite summation so that an infinite sum is ∞ iff either one of the summands is ∞, or the number of nonzero summands is infinite. In the resulting ω-complete semiring S', it holds that a countably infinite sum of 1 with itself is ∞ and thus $1^* = \infty$.

Theorem 20. *The class of ω-continuous semirings is closed under taking matrix semirings and power series semirings over all locally finite partial monoids.*

This fact is well-known (the second claim at least for free monoids). In each case, the order is the pointwise order. Since each ω-complete *-semiring is an iteration semiring, the same holds for ω-continuous *-semirings.

Corollary 8. *Every ω-continuous *-semiring is an ordered iteration semiring.*

On the semirings $\mathbf{k}$ and $\mathbb{N}_\infty$, the order is the sum order. It follows that the pointwise order is the sum order on each semiring $\mathbf{k}\langle\langle M\rangle\rangle$ or $\mathbb{N}_\infty\langle\langle M\rangle\rangle$. The following fact is easy to establish.

Theorem 21. *For each set A, the ω-continuous *-semiring $\mathbb{N}_\infty\langle\langle A^*\rangle\rangle$, equipped with the pointwise order, is freely generated by A in the class of all ω-continuous *-semirings. Moreover, for each set A, and for each $k \geq 2$, the ω-continuous *-semiring $\mathbf{k}\langle\langle A^*\rangle\rangle$, equipped with the pointwise order, is freely generated by A in the class of all ω-continuous semirings satisfying $k - 1 = k$.*

7.2 Inductive *-Semirings

Inductive *-semirings, introduced in [22], are a generalization of ω-continuous *-semirings. As opposed to ω-continuous *-semirings, inductive *-semirings are defined by first-order conditions.

Definition 16. *Suppose that S is a *-semiring which is an ordered semiring. We call S an* inductive *-semiring *if the following hold for all $a, b, x \in S$:*

$$\begin{aligned} aa^* + 1 &\leq a^* \\ ax + b \leq x \quad &\Rightarrow \quad a^*b \leq x. \end{aligned}$$

A morphism of inductive *-semirings is a *-semiring morphism which is an ordered semiring morphism.

It then follows that for any a, b in an inductive *-semiring S, a^*b is the *least pre-fixed point* of the map $x \mapsto ax + b$, which is actually a fixed point. In particular, every inductive *-semiring is a positive ordered semiring. Moreover, it follows that the star operation is also monotone.

Definition 17. *A* symmetric inductive *-semiring *is an inductive *-semiring S which also satisfies*

$$xa + b \leq x \Rightarrow ba^* \leq x$$

for all $a, b, x \in S$. A morphism of symmetric inductive *-semirings is an inductive *-semiring morphism.

In [31], Kozen defines a *Kleene algebra* as an *idempotent* symmetric inductive *-semiring. A morphism of Kleene algebras is a *-semiring morphism (which is necessarily an ordered semiring morphism).

Theorem 22. (Ésik, Kuich [22]) *Every inductive *-semiring is an ordered iteration semiring.*

Thus if S is an inductive *-semiring, then, equipped with the star operation of Definition 3, each matrix semiring $S^{n\times n}$ is an iteration semiring. Moreover, for any locally finite partial monoid M, $S\langle\langle M\rangle\rangle$ is a also an iteration semiring. Actually we have:

Theorem 23. (Ésik, Kuich [22]) *If S is a (symmetric) inductive *-semiring, then each matrix semiring over S is a (symmetric) inductive *-semiring. Moreover, for any locally finite partial monoid M, $S\langle\langle M\rangle\rangle$ is a (symmetric) inductive *-semiring.*

(The second part of this theorem is proved in [22] only for free monoids.)

8 Completeness, again

Although the sum order and the pointwise order coincide on $\mathbb{N}_\infty\langle\langle A^*\rangle\rangle$ (and in fact on any semiring $S\langle\langle A^*\rangle\rangle$, where S is sum ordered), these two orders are in general different on the semiring $\mathbb{N}_\infty^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$. In this section, we will consider the sum order on the semiring $\mathbb{N}_\infty^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$. On the other hand, the sum order and the pointwise order coincide on the semirings $\mathbf{k}^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$. The following fact can be derived from Theorem 16.

Theorem 24. *The semiring $\mathbb{N}_\infty^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$, equipped with the sum order, is freely generated by the set A in the class of all ordered iteration semirings satisfying the equations (6),(7) and (8). Moreover, for each $k \geq 2$ and any set A, $\mathbf{k}\langle\langle A^*\rangle\rangle$, equipped with the sum order is freely generated by A in the class of all ordered iteration semirings satisfying $1^* = k - 1$.*

Theorem 25. (Bloom, Ésik [8]) *For each set A, $\mathbb{N}_\infty^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$, equipped with the sum order, is freely generated by A both in the class of all inductive *-semirings satisfying (6) and in the class of all symmetric inductive *-semirings. Similarly, for each set A and integer $k > 1$, $\mathbf{k}\langle\langle A^*\rangle\rangle$ equipped with the sum order is freely generated by A in the class of all inductive (symmetric inductive) *-semirings satisfying $k - 1 = k$.*

The case when $k = 2$ is due to Krob [34] and Kozen [30,31].

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