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Lyapunov Functions, Stability and Input-to-State Stability Subtleties for Discrete-Time Discontinuous Systems

Mircea Lazar, W. P. Maurice H. Heemels, and Andy R. Teel

Abstract—In this note we consider stability analysis of discrete-time discontinuous systems using Lyapunov functions. We demonstrate via simple examples that the classical second method of Lyapunov is precarious for discrete-time discontinuous dynamics. Also, we indicate that a particular type of Lyapunov condition, slightly stronger than the classical one, is required to establish stability of discrete-time discontinuous systems. Furthermore, we examine the robustness of the stability property when it was attained via a discontinuous Lyapunov function, which is often the case for discrete-time hybrid systems. In contrast to existing results based on smooth Lyapunov functions, we develop several input-to-state stability tests that explicitly employ an available discontinuous Lyapunov function.

Index Terms—Discontinuous systems, discrete-time, input-to-state stability, Lyapunov methods, stability.

I. INTRODUCTION

Discrete-time discontinuous systems, such as piecewise affine (PWA) systems, form a powerful modeling class for the approximation of hybrid and non-smooth nonlinear dynamics [1], [2]. Many numerically efficient tools for stability analysis and stabilizing controller synthesis for discrete-time PWA systems have already been developed, see, for example, [3]–[7] for static feedback methods and [8]–[11] for model predictive control (MPC) techniques. Most of these methods make use of classical Lyapunov methods [12]. The first contribution of this note is to illustrate the precariousness of the second method of Lyapunov, as presented in [12], for discontinuous system dynamics. We illustrate via a simple example that existence of a Lyapunov function in the sense of Corollary 1.2 of [12] (and hence, a continuous function) does not necessarily guarantee global asymptotic stability (GAS) for discrete-time discontinuous systems. In the presence of discontinuity of the dynamics one needs stronger properties, e.g., the one-step difference of the Lyapunov function should be upper bounded by a class $\mathcal{K}_\infty$ function with a minus sign in front, to attain GAS.

The second contribution of this note concerns robustness of stability in terms of input-to-state stability (ISS) [13]. First, we present a simple example inspired from [14] (see also [15] for a similar example in MPC) to illustrate that even the global exponential stability (GES) property is precarious for discrete-time discontinuous systems affected by arbitrary small perturbations. The severe lack of inherent robustness is related to the absence of a continuous Lyapunov function. This example establishes that there exist GES discrete-time systems

that admit a discontinuous Lyapunov function, but not a continuous one. Notice that previous results on stability of discrete-time PWA systems [3]–[7] only indicated that continuous Lyapunov functions may be more difficult to find than discontinuous ones, while in fact a continuous Lyapunov function might not even exist. As such, a valid warning regarding nominally stabilizing state-feedback synthesis methods for discrete-time discontinuous systems, including both static feedback approaches [3]–[7] and MPC techniques [8]–[11] arises. These synthesis methods lead to a stable, possibly discontinuous closed-loop system and often rely on discontinuous Lyapunov functions. For example, in MPC the most natural candidate Lyapunov function is the value function corresponding to the MPC cost, which is generally discontinuous when PWA systems are used as prediction models [10]. Hence, these controllers may result in closed-loop systems that are GAS, or even GES, but may not be ISS to arbitrarily small perturbations, which are always present in practice.

This brings us to the second contribution of this note: for discrete-time systems for which only a discontinuous Lyapunov function is known, we propose several robustness tests that can establish ISS solely based on the available discontinuous Lyapunov function.

II. PRELIMINARIES

A. Nomenclature and Basic Definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For every subset Π of $\mathbb{R}$ we define $\mathbb{R}_\Pi := \{k \in \mathbb{R} | k \in \Pi\}$ and $\mathbb{Z}_\Pi := \{k \in \mathbb{Z} | k \in \Pi\}$. Let $\|\cdot\|$ denote an arbitrary norm on $\mathbb{R}^n$ and let $|\cdot|$ denote the absolute value of a real number. For a sequence $\mathbf{w} := \{w(l)\}_{l \in \mathbb{Z}_+}$ with $w(l) \in \mathbb{R}^n$, $l \in \mathbb{Z}_+$, let $\|\mathbf{w}\| := \sup\{\|w(l)\| \mid l \in \mathbb{Z}_+\}$ and let $\mathbf{w}_{[k]} := \{w(l)\}_{l \in \mathbb{Z}_{[0,k]}}$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, we denote by $\text{int}(\mathcal{S})$ the interior, by $\partial \mathcal{S}$ the boundary and by $\text{cl}(\mathcal{S})$ the closure of $\mathcal{S}$. For two arbitrary sets $\mathcal{S} \subseteq \mathbb{R}^n$ and $\mathcal{P} \subseteq \mathbb{R}^n$, let $\mathcal{S} \oplus \mathcal{P} := \{x + y | x \in \mathcal{S}, y \in \mathcal{P}\}$ denote their Minkowski sum. The distance of a point $x \in \mathbb{R}^n$ from a set $\mathcal{P}$ is denoted by $d(x, \mathcal{P}) := \inf_{y \in \mathcal{P}} \|x - y\|$. For any $\mu \in \mathbb{R}_{(0,\infty)}$ we define $\mathcal{B}_\mu := \{x \in \mathbb{R}^n | \|x\| \leq \mu\}$. A polyhedron (or a polyhedral set) in $\mathbb{R}^n$ is a set obtained as the intersection of a finite number of open and/or closed half-spaces. The p-norm of a vector $x \in \mathbb{R}^n$ is defined as $\|x\|_p := (|x_1|^p + \cdots + |x_n|^p)^{1/p}$ for $p \in \mathbb{Z}_{[1,\infty)}$ and $\|x\|_\infty := \max_{i=1,\ldots,n} |x_i|$, where x_i, $i = 1, \ldots, n$ is the i-th component of x. For a matrix $Z \in \mathbb{R}^{m \times n}$ let $\|Z\|_p := \sup_{x \neq 0} \|Zx\|_p / \|x\|_p$, $p \in \mathbb{Z}_{[1,\infty)}$ or $p = \infty$ denote its induced matrix norm. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ ($\varphi \in \mathcal{K}$) if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}_\infty$ ($\varphi \in \mathcal{K}_\infty$) if $\varphi \in \mathcal{K}$ and $\lim_{s \to \infty} \phi(s) = \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is decreasing and $\lim_{k \to \infty} \beta(s, k) = 0$.

B. Stability and Input-to-State Stability

To study robustness, we will employ the ISS framework [13], [16]. Consider the discrete-time perturbed nonlinear system

$$\xi(k+1) = g(\xi(k), w(k)), \quad k \in \mathbb{Z}_+ \tag{1}$$

where $\xi : \mathbb{Z}_+ \to \mathbb{R}^n$ is the state trajectory, $w : \mathbb{Z}_+ \to \mathbb{R}^{d_v}$ is an unknown disturbance input trajectory and $g : \mathbb{R}^n \times \mathbb{R}^{d_v} \to \mathbb{R}^n$ is a nonlinear, possibly discontinuous function. For simplicity, we assume that the origin is an equilibrium for (1) with zero disturbance, i.e., $g(0, 0) = 0$.

Definition II.1: A set $\mathcal{P} \subseteq \mathbb{R}^n$ with $0 \in \text{int}(\mathcal{P})$ is called a *robustly positively invariant (RPI) set with respect to* $\vee \subseteq \mathbb{R}^{d_v}$ for system (1)

if for all $x \in \mathcal{P}$ it holds that $g(x, v) \in \mathcal{P}$ for all $v \in \mathbb{V}$. A set $\mathcal{P} \subseteq \mathbb{R}^n$ with $0 \in \text{int}(\mathcal{P})$ is called a *positively invariant (PI) set* for system (1) with zero input if for all $x \in \mathcal{P}$ it holds that $g(x, 0) \in \mathcal{P}$. ☐

Definition II.2: Let $\mathbb{X}$ with $0 \in \text{int}(\mathbb{X})$ be a subset of $\mathbb{R}^n$. We call system (1) with zero input (i.e., $w(k) = 0$ for all $k \in \mathbb{Z}_+$) asymptotically stable in $\mathbb{X}$, or shortly $AS(\mathbb{X})$, if there exists a $\mathcal{KL}$-function β such that, for each $\xi(0) \in \mathbb{X}$ it holds that $\|\xi(k)\| \leq \beta(\|\xi(0)\|, k)$, $\forall k \in \mathbb{Z}_+$, where $\xi(k)$ is the state trajectory corresponding to $\xi(0)$ and zero disturbance input. If the property holds with $\beta(s, k) := \theta \rho^k s$ for some $\theta \in \mathbb{R}_{[0,\infty)}$ and $\rho \in \mathbb{R}_{[0,1)}$ we call system (1) with zero input exponentially stable in $\mathbb{X}(ES(\mathbb{X}))$. We call system (1) with zero input globally asymptotically (exponentially) stable if it is $AS(\mathbb{R}^n)(ES(\mathbb{R}^n))$. ☐

Definition II.3: Let $\mathbb{X}$ and $\mathbb{V}$ be subsets of $\mathbb{R}^n$ and $\mathbb{R}^{d_v}$, respectively, with $0 \in \text{int}(\mathbb{X})$. We call system (1) *input-to-state stable in $\mathbb{X}$ for inputs in $\mathbb{V}$*, or shortly $ISS(\mathbb{X}, \mathbb{V})$, if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that, for each initial condition $\xi(0) \in \mathbb{X}$ and all $\mathbf{w} = \{w(l)\}_{l \in \mathbb{Z}_+}$ with $w(l) \in \mathbb{V}$ for all $l \in \mathbb{Z}_+$, it holds that the corresponding state trajectory of (1) with initial state $\xi(0)$ and input trajectory $\mathbf{w}$ satisfies $\|\xi(k)\| \leq \beta(\|\xi(0)\|, k) + \gamma(\|\mathbf{w}_{[k-1]}\|)$ for all $k \in \mathbb{Z}_{[1,\infty)}$. The system (1) is *globally ISS* if it is $ISS(\mathbb{R}^n, \mathbb{R}^{d_v})$. ☐

Throughout this article we will employ the following sufficient conditions for analyzing ISS.

Theorem II.4: [13], [17] Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$ and let $\mathbb{V}$ be a subset of $\mathbb{R}^{d_v}$. Let $\mathbb{X}$ with $0 \in \text{int}(\mathbb{X})$ be a RPI set with respect to $\mathbb{V}$ for system (1) and let $V : \mathbb{X} \to \mathbb{R}_+$ be a function with $V(0) = 0$. Consider the following inequalities:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{2a}$$

$$V(g(x, v)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|v\|). \tag{2b}$$

If inequalities (2) hold for all $x \in \mathbb{X}$ and all $v \in \mathbb{V}$, then system (1) is $ISS(\mathbb{X}, \mathbb{V})$. If inequalities (2) hold for all $x \in \mathbb{R}^n$ and all $v \in \mathbb{R}^{d_v}$, then system (1) is *globally ISS*. If $\mathbb{X}$ with $0 \in \text{int}(\mathbb{X})$ is a PI set for system (1) with zero input and inequalities (2) hold for all $x \in \mathbb{X}$ ($x \in \mathbb{R}^n$) and $v \in \mathbb{V} = \{0\}$, then system (1) with zero input is $AS(\mathbb{X})$ (GAS). ☐

A function V that satisfies the hypothesis of Theorem II.4 is called an *ISS Lyapunov function*. Note the following aspects regarding Theorem II.4. (i) The hypothesis of Theorem II.4 allows that both g and V are discontinuous. The hypothesis *only* requires continuity at the point $x = 0$, and *not* necessarily on a neighborhood of $x = 0$. (ii) If the inequalities (2) are satisfied for $\alpha_1(s) = as^\lambda$, $\alpha_2(s) = bs^\lambda$, $\alpha_3(s) = cs^\lambda$, for some $a, b, c, \lambda \in \mathbb{R}_{(0,\infty)}$, then the hypothesis of Theorem II.4 implies exponential stability of system (1) with zero input [18]; (iv) A counter part of these results for continuous-time discontinuous dynamical systems and non-differentiable ISS Lyapunov functions can be found in [19].

C. Lyapunov Functions

As an extension of classical Lyapunov functions (see Corollary 1.2 and Corollary 1.3 of [12]), which are assumed to be continuous and only required to have a negative one step forward difference, we will introduce the following known types of Lyapunov functions for the zero input system corresponding to (1), i.e., $\xi(k+1) = g(\xi(k), 0), k \in \mathbb{Z}_+$. Let $\mathbb{X} \subseteq \mathbb{R}^n$ be a positively invariant set for $\xi(k+1) = g(\xi(k), 0)$ with $0 \in \text{int}(\mathbb{X})$, let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, let $V : \mathbb{X} \to \mathbb{R}_+$ denote a possibly discontinuous function with $V(0) = 0$, and consider the inequalities

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathbb{X}, \tag{3a}$$

$$V(g(x, 0)) - V(x) \leq 0, \quad \forall x \in \mathbb{X}, \tag{3b}$$

$$V(g(x, 0)) - V(x) < 0, \quad \forall x \in \mathbb{X} \setminus \{0\}, \tag{3c}$$

$$V(g(x, 0)) - V(x) \leq -\alpha_3(\|x\|), \quad \forall x \in \mathbb{X}. \tag{3d}$$

Definition II.5: A function V that satisfies (3a) and (3b) is called a *Lyapunov function*. A function V that satisfies (3a) and (3c) is called a *strict Lyapunov (SL) function*. A function V that satisfies (3a) and (3d) is called a *uniformly strict Lyapunov (USL) function*. ☐

For continuous V and *discrete-time* continuous system dynamics it is known that SL functions and USL functions can be related and both imply asymptotic stability and inherent robustness (ISS, under certain conditions); see, for example, [14], [18], [20]. In the following section we will investigate whether these properties still hold when either the system dynamics or the Lyapunov function is discontinuous, or both.

III. ILLUMINATING EXAMPLES

Consider the following discrete-time PWA systems, which form one of the simplest classes of discontinuous systems and will serve as a support for setting up the examples:

$$\begin{aligned} \xi(k+1) &= G(\xi(k)) \\ &:= A_j \xi(k) + f_j \quad \text{if } \xi(k) \in \Omega_j \end{aligned} \tag{4a}$$

$$\begin{aligned} \tilde{\xi}(k+1) &= g(\tilde{\xi}(k), w(k)) \\ &:= A_j \tilde{\xi}(k) + f_j + w(k) \quad \text{if } \tilde{\xi}(k) \in \Omega_j \end{aligned} \tag{4b}$$

with $w(k) \in \mathcal{B}_\mu$ for some small $\mu \in \mathbb{R}_{(0,\infty)}$, $k \in \mathbb{Z}_+$, and where $A_j \in \mathbb{R}^{n \times n}$, $f_j \in \mathbb{R}^n$ for all $j \in \mathcal{S}$ (a finite set of indexes) and $\{\Omega_j \subseteq \mathbb{R}^n | j \in \mathcal{S}\}$ defines a partition of $\mathbb{X}$, meaning that $\cup_{j \in \mathcal{S}} \Omega_j = \mathbb{X}$ and $\Omega_i \cap \Omega_j = \emptyset$, with the sets Ω_j not necessarily closed. First, we present a simple one-dimensional example of a discontinuous system that admits a continuous SL function but it is not GAS.

Example 1: Consider the discrete-time system (4a) with $j \in \mathcal{S} := \{1, 2\}$, $A_1 = f_1 = 0$, $A_2 = 0.5$, $f_2 = 0.5$ and the partition given by $\Omega_1 = \{x \in \mathbb{R} | x \leq 1\}$, $\Omega_2 = \{x \in \mathbb{R} | x > 1\}$. One can easily check that $\lim_{k \to \infty} \xi(k) = 1$ for any $\xi(0) \in \mathbb{R}_{(1,\infty)} = \Omega_2$ and thus, this system is not GAS. Consider the function $V(x) := |x|$. Clearly, for $x \in \Omega_1 \setminus \{0\}$ we have $V(G(x)) - V(x) = -V(x) < 0$ and, for $x \in \Omega_2$ we have $V(G(x)) - V(x) = 0.5|x+1| - |x| < |x| - |x| = 0$. Hence, V is a continuous SL function. However, V is not a USL function, as for any $\alpha_3 \in \mathcal{K}_\infty$ it holds that $\lim_{x \downarrow 1} (V(G(x)) - V(x)) = \lim_{x \downarrow 1} (0.5|x + 1| - x) = 0 > -\alpha_3(1)$. ☐

As illustrated above, the system of Example 1 admits a continuous SL function but the trajectories do not converge to the origin globally. *This indicates that SL functions (even continuous ones), which are not USL functions, do not necessarily guarantee GAS for discrete-time discontinuous systems.* Hence, one must strive for a USL function to guarantee GAS of a discrete-time discontinuous system. For a proof that (discontinuous) USL functions imply GAS see, for example, [18]. The interested reader is also referred to [20] for a proof that a GAS discrete-time system always admits a (possibly discontinuous) USL function.

Example 2: Consider now the discrete-time system (4a) with $j \in \mathcal{S} := \{1, 2\}$, $A_1 = A_2 = 0$, $f_1 = 0$, $f_2 = 1$ and the partition given by $\Omega_1 = \{x \in \mathbb{R} | x \leq 1\}$, $\Omega_2 = \{x \in \mathbb{R} | x > 1\}$. Fig. 1 shows the values of the function G. One can easily observe that any trajectory ξ of system (4a) starting from an initial condition $\xi(0) \in \mathbb{R}$ satisfies $|\xi(k)| \leq |\xi(0)|$ (even $|\xi(k)| < |\xi(0)|$ when $\xi(0) \neq 0$) and converges exponentially to the origin. Actually, any trajectory ξ reaches the origin in 2 discrete-time steps or less. Furthermore, it can be proven that $V(x) := \sum_{i=0}^\infty \xi(i)^2$ is a USL function, where ξ denotes the trajectory of system (4a) obtained from initial condition $\xi(0) = x \in \mathbb{R}$. Indeed, since $V(x) = \sum_{i=0}^\infty \xi(i)^2 = \xi(0)^2 + \xi(1)^2$ for any $\xi(0) = x \in \mathbb{R}$, it holds that $V(G(x)) - V(x) \leq -\alpha_3(|x|)$ for all $x \in \mathbb{R}$, where $\alpha_3(s) := s^2$. An explicit expression for V is

$$V(x) = \sum_{i=0}^\infty \xi(i)^2 = \xi(0)^2 + \xi(1)^2 = \begin{cases} x^2 + 1, & \text{if } x > 1, \\ x^2, & \text{if } x \leq 1 \end{cases}$$

which shows that V is discontinuous at $x = 1$.
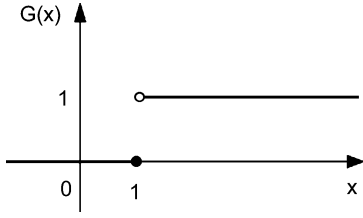
Fig. 1. Function G for the system of Example 2.

Next consider the case when $w(k) = \mu \in \mathbb{R}_{(0,\infty)}$ for all $k \in \mathbb{Z}_+$ in (4b). Then, the origin of the perturbed system (4b) corresponding to the nominal system (4a) is not ISS, as $1 + \mu$ is an equilibrium of (4b) to which all trajectories with initial conditions $\xi(0) \in \mathbb{R}_{(1,\infty)} = \Omega_2$ converge. Hence, no matter how small $\mu \in \mathbb{R}_{(0,\infty)}$ is taken, the system (4b) is not ISS($\mathbb{R}, \mathcal{B}_\mu$). □

The following conclusions can be drawn from Example 2: (i) *GES discrete-time discontinuous systems are not necessarily ISS, even to arbitrarily small inputs*; (ii) *existence of a discontinuous USL function does not guarantee ISS, even to arbitrarily small inputs*. This indicates that additional conditions must be imposed on USL functions to attain ISS. For example, continuity of the USL function is known to guarantee inherent ISS [18], but this condition is too restrictive for discrete-time discontinuous systems such as PWA systems. Thus, in the next section we will propose ISS tests that can deal with discontinuous USL functions.

Remark III.1: The GES discrete-time system of Example 2 also admits a *continuous* SL function, namely $V(x) := |x|$, which satisfies $V(G(x)) - V(x) < 0$ for all $x \neq 0$. However, as it was the case in Example 1, $V(x) = |x|$ is not a USL function, as for any $\alpha_3 \in \mathcal{K}_\infty$ it holds that $\lim_{x \downarrow 1}(V(G(x)) - V(x)) = \lim_{x \downarrow 1}(1 - x) = 0 > -\alpha_3(1)$. Hence, the existence of a continuous SL function does not necessarily guarantee any robustness for discontinuous systems. □

Remark III.2: By Theorem 14 of [14], Example 2 implies that there exist GES discrete-time systems that do not admit a continuous USL function. However, as shown above, the PWA system of Example 2 does admit a discontinuous USL function, which is conform with the converse stability result for discrete-time discontinuous systems presented in [20]. □

IV. ISS TESTS BASED ON DISCONTINUOUS USL FUNCTIONS

In this section we consider piecewise continuous (PWC) nonlinear systems of the form

$$\xi(k+1) = G(\xi(k)) := G_j(\xi(k)) \text{ if } \xi(k) \in \Omega_j, \quad k \in \mathbb{Z}_+ \quad (5)$$

where each $G_j : \Omega_j \to \mathbb{R}^n$, $j \in \mathcal{S}$, is assumed to be a continuous function. PWA systems are obtained as a particular case by setting $G_j(x) = A_j x + f_j$. Consider also a perturbed version of the above system, by including additive disturbances, i.e.

$$\tilde{\xi}(k+1) = g(\tilde{\xi}(k), w(k))$$
$$:= G_j(\tilde{\xi}(k)) + w(k) \text{ if } \tilde{\xi}(k) \in \Omega_j, \ k \in \mathbb{Z}_+. \quad (6)$$

Furthermore, we consider discontinuous USL functions $V : \mathbb{R}^n \to \mathbb{R}_+$, with $V(0) = 0$

$$V(x) := V_i(x) \text{ if } x \in \Gamma_i, \quad i \in \mathcal{J} \quad (7)$$

where *for each* $i \in \mathcal{J}$, $V_i : \mathbb{R}^n \to \mathbb{R}_+$ is a continuous function that satisfies

$$|V_i(x) - V_i(y)| \leq \sigma_i(\|x - y\|), \quad \forall x, y \in \mathrm{cl}(\Gamma_i) \quad (8)$$

for some $\sigma_i \in \mathcal{K}$. Examples of functions that satisfy this property include uniformly continuous functions on compact sets and Lipschitz continuous functions. This captures a wide range of frequently used Lyapunov functions for PWA systems, such as piecewise quadratic (PWQ), PWA or piecewise polyhedral functions (i.e., functions defined using the infinity norm or the 1-norm), including the value functions that arise in model predictive control of PWA systems.

In (5) and (7), $\{\Omega_j | j \in \mathcal{S}\}$ and $\{\Gamma_i | i \in \mathcal{J}\}$ with $\mathcal{S} := \{1, \ldots, s\}$ and $\mathcal{J} := \{1, \ldots, \overline{M}\}$ finite sets of indices, denote partitions of $\mathbb{R}^n$. More precisely, we assume that $\cup_{j \in \mathcal{S}} \Omega_j = \mathbb{R}^n$, $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j, (i,j) \in \mathcal{S} \times \mathcal{S}$ and $\mathrm{int}(\Omega_i) \neq \emptyset$ for all $i \in \mathcal{S}$ and likewise for the regions $\Gamma_i, i \in \mathcal{J}$. Suppose that a discontinuous USL function of the form (7) is available for system (5). We have seen from Example 2 in the previous section that this does not guarantee anything in terms of ISS. However, the goal is now to develop tests for ISS of system (6) based on the discontinuous USL function (7).

The first result is based on examining the trajectory of the PWC system (5) with respect to the set of states at which V may be discontinuous. Let $\mu \in \mathbb{R}_{(0,\infty)}$ and let $\mathcal{P} \subseteq \mathbb{R}^n$ with $0 \in \mathrm{int}(\mathcal{P})$ be a RPI set for system (6) with respect to $\mathcal{B}_\mu$, i.e., $\mathcal{R}_1(\mathcal{P}) \oplus \mathcal{B}_\mu \subseteq \mathcal{P}$, where $\mathcal{R}_1(\mathcal{P}) := \{G(x) | x \in \mathcal{P}\}$ is the one-step reachable set for system (5) from states in $\mathcal{P}$. Let $\mathbb{X}_D \subset \mathcal{P}$ denote the set of all states in $\mathcal{P}$ at which V is not continuous. If one can verify that any state trajectory $\{\xi(k)\}_{k \in \mathbb{Z}_+}$ of (5) is a distance $\mu \in \mathbb{R}_{(0,\infty)}$ away from the set $\mathbb{X}_D$ for all $\xi(0) \in \mathcal{P}$ and all $k \in \mathbb{Z}_{[1,\infty)}$, then it can be proven that ISS($\mathcal{P}, \mathcal{B}_\mu$) is achieved, as formulated in the following result. Its proof is given in Appendix A.

Theorem IV.1: Suppose that the PWC system (5) admits a discontinuous [1] USL function of the form (7) and consequently, (5) is GAS. Furthermore, suppose that there exist a $\mu \in \mathbb{R}_{(0,\infty)}$ and a set $\mathcal{P} \subseteq \mathbb{R}^n$ with $0 \in \mathrm{int}(\mathcal{P})$ such that

$$d(x, \mathbb{X}_D) > \mu \quad \text{for all} \quad x \in \mathcal{R}_1(\mathcal{P}) \quad (9)$$

and $\mathcal{P}$ is a RPI set [2] for system (6) with respect to $\mathcal{B}_\mu$. Then, the PWC system (6) is ISS($\mathcal{P}, \mathcal{B}_\mu$).

The constant μ can be chosen as follows:

$$0 < \mu \leq \mu^* := \min_{j \in \mathcal{S}} \left\{ \inf_{y \in \Omega_j \cap \mathcal{P}, \bar{y} \in \mathbb{X}_D} \|G_j(y) - \bar{y}\| \right\}. \quad (10)$$

If the set $\mathbb{X}_D$ is the union of a finite number of polyhedra, the sets Ω_j, $j \in \mathcal{S}$ and $\mathcal{P}$ are polyhedra, each $G_j, j \in \mathcal{S}$ is an affine function and the infinity norm (or the 1-norm) is used in (10), a solution to the optimization problem in (10) can be obtained by solving a finite number of linear programming problems (quadratic programming problems if the 2-norm is used). If the optimization problem in (10) yields a strictly positive μ^*, then $\mu^* \in \mathbb{R}_{(0,\infty)}$ can be considered as a measure of the (worst case) inherent robustness of system (5). The sufficient condition (9) can be relaxed, as shown by the next result, in the sense that the trajectory $\{\xi(k)\}_{k \in \mathbb{Z}_+}$ of system (5) is now allowed to intersect the set $\mathbb{X}_D$.

Proposition IV.2: Let $\mathcal{P} \subseteq \mathbb{R}^n$ with $0 \in \mathrm{int}(\mathcal{P})$ be a RPI set for system (6) with respect to $\mathcal{B}_\mu$ for some $\mu \in \mathbb{R}_{(0,\infty)}$. Suppose that the PWC system (5) admits a function of the form (7) that satisfies (3a) for all $x \in \mathcal{P}$. Furthermore, suppose that there exists $\tilde{\alpha}_3 \in \mathcal{K}_\infty$ such that

$$\max_{i \in \mathcal{I}} V_i(G(x)) - V(x) \leq -\tilde{\alpha}_3(\|x\|), \quad \forall x \in \mathcal{P}. \quad (11)$$

Then, the PWC system (6) is ISS ($\mathcal{P}, \mathcal{B}_\mu$).

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[1] Note that the result also holds for continuous USL functions, as then $\mathbb{X}_D = \emptyset$.

[2] Observe that $\mathcal{P} = \mathbb{R}^n$ is a possible choice of a RPI set with respect to $\mathcal{B}_\mu$ for any $\mu \in \mathbb{R}_{(0,\infty)}$.

The above result is based on a stronger, more conservative extension of the stabilization conditions from [3]–[7], as it requires that the Lyapunov function is decreasing irrespective of which dynamics might be active at the next step. The proof of Proposition IV.2 follows from the proof of the less conservative result formulated next in Theorem IV.3.

The sufficient condition (11) can be significantly relaxed, as follows. Consider the set $\mathcal{Z} := \{x \in \mathcal{P} \,|\, (G(x) \oplus \mathcal{B}_\mu) \cap \mathbb{X}_D \neq \emptyset\}$ and define for $x \in \mathcal{Z}$

$$\mathcal{M}(x) := \{i \in \mathcal{J} \,|\, G(x) \notin \Gamma_i, (G(x) \oplus \mathcal{B}_\mu) \cap \Gamma_i \neq \emptyset\}.$$

Theorem IV.3: Suppose that the PWC system (5) admits a (discontinuous) USL function of the form (7). Furthermore, suppose that there exist a $\mu \in \mathbb{R}_{(0,\infty)}$, a $\mathcal{K}_\infty$-function $\tilde{\alpha}_3$ and a set $\mathcal{P} \subseteq \mathbb{R}^n$ with $0 \in \text{int}(\mathcal{P})$ such that

$$\max_{i \in \mathcal{M}(x)} V_i(G(x)) - V(x) \leq -\tilde{\alpha}_3(\|x\|), \quad \forall x \in \mathcal{Z} \quad (12)$$

and $\mathcal{P}$ is a RPI set for system (6) with respect to $\mathcal{B}_\mu$. Then, the PWC system (6) is ISS($\mathcal{P}, \mathcal{B}_\mu$).

The proof of Theorem IV.3 is presented in Appendix B. Observe that (9) amounts to an a posteriori check that must be performed on a given USL function of the form (7). In contrast, condition (11) can be a priori specified when computing a USL function of the form (7), and it can be casted as a semidefinite programming problem for piecewise quadratic (PWQ) functions and PWA systems, provided that the regions Γ_i are chosen (see [18], Chapter 4, for an example). On the same issue, condition (12) involves the set $\mathbb{X}_D$ and hence, amounts to an a posteriori check that must be performed on a given USL function of the form (7). Under certain reasonable assumptions (e.g., $\mathbb{X}_D$ is the union of a finite number of polyhedra, the regions Ω_j, $j \in \mathcal{S}$ and Γ_i, $i \in \mathcal{I}$ are polyhedra, the system is PWA, the USL function is convex) checking (12) amounts to solving a finite number of convex optimization problems.

Remark IV.4: The result of Theorem IV.3 also holds when condition (12) is replaced by

$$\max_{i \in \mathcal{M}(x)} V_i(G(x)) - V(G(x)) \leq c\alpha_3(\|x\|), \quad \forall x \in \mathcal{Z} \quad (13)$$

for some $c \in \mathbb{R}_{[0,1)}$, which might be easier to check than (12). □

Remark IV.5: The tests developed in this section require that for each $i \in \mathcal{J}$, V_i is a continuous function that satisfies (8), and as such must be defined on $\text{cl}(\Gamma_i)$ and, furthermore, it is defined on: (i) $\mathcal{P} \subseteq \mathbb{R}^n$ for Proposition IV.2 and (ii) $\text{cl}(\Gamma_i) \oplus \mathcal{B}_\mu$ for some $\mu \in \mathbb{R}_{(0,\infty)}$ for Theorem IV.3. These are additional requirements with respect to USL functions, which in principle, only require that each V_i is defined on Γ_i. An alternative to the tests presented in this section is to directly check condition (2b), which for PWA dynamics and PWQ candidate ISS Lyapunov functions can lead to tractable optimization problems, as shown recently in [21]. □

V. CONCLUSION

In this note we analyzed two types of Lyapunov functions in terms of their suitability for establishing stability and input-to-state stability of discrete-time discontinuous systems. Via examples we exposed certain subtleties that arise in the classical Lyapunov methods when they are applied to discrete-time discontinuous systems, as follows:

- The existence of a continuous SL function does not necessarily imply GAS—Example 1;
- The existence of a continuous SL function or discontinuous USL function does not necessarily imply ISS, even to arbitrarily small inputs—Example 2;
- GES does not necessarily imply the existence of a continuous USL function—Example 2 (see also [14]).

These results, together with the fact that existence of a possibly discontinuous USL function is equivalent to GAS [18], [20], issue a strong warning regarding existing nominally stabilizing state-feedback synthesis methods for discrete-time discontinuous systems, including both static feedback approaches [3]–[7] and MPC techniques [8]–[11]. This warning motivates the recent results on global input-to-state stabilization of discrete-time PWA systems [21] and input-to-state stabilizing (sub-optimal) MPC of discontinuous systems [22].

To render the many available procedures for obtaining Lyapunov functions, which typically yield discontinuous Lyapunov functions (e.g., value functions in MPC or PWQ Lyapunov functions), applicable to discontinuous systems, we presented several ISS tests based on discontinuous Lyapunov functions. These tests can be employed to establish ISS of nominally asymptotically stable discrete-time PWC systems in the case when a discontinuous USL function is available.

APPENDIX

Proof of Theorem IV.1: First, we will prove that there exists a $\mathcal{K}$-function σ (independent of x) such that for all x and for any two points $y, \bar{y} \in G(x) \oplus \mathcal{B}_\mu$ it holds that $|V(y) - V(\bar{y})| \leq \sigma(\|y - \bar{y}\|)$.

By (8), for each $i \in \mathcal{J}$ and any two points $y, \bar{y} \in \text{cl}(\Gamma_i)$ there exists a $\mathcal{K}$-function σ_i such that $|V_i(y) - V_i(\bar{y})| \leq \sigma_i(\|y - \bar{y}\|)$. The inequality (9) implies that V is continuous on the set $G(x) \oplus \mathcal{B}_\mu$ for any $x \in \mathcal{P}$. For any two points $y, \bar{y} \in G(x) \oplus \mathcal{B}_\mu$ consider the line segment $L(y, \bar{y}) := \{y + \alpha(\bar{y} - y) | 0 \leq \alpha \leq 1\}$ between y and $\bar{y}$. We will construct a set of points $\{z_0, \ldots, z_M\} \subset L(y, \bar{y})$ with $M \leq \overline{M}$ on this line segment such that: (i) $z_0 = y$; $z_M = \bar{y}$ and (ii) $(z_{p-1}, z_p) \in \text{cl}(\Gamma_{i_{p-1}}) \times \text{cl}(\Gamma_{i_{p-1}})$ for some $i_{p-1} \in \mathcal{J}$, for all $p = 1, \ldots, M$. To construct this set, take $i_0 \in \mathcal{J}$ such that $z_0 = y \in \text{cl}(\Gamma_{i_0})$, $\alpha_0 = 0$ and $\alpha_1 := \max\{\alpha \in [0, 1] | y + \alpha(\bar{y} - y) \in \text{cl}(\Gamma_{i_0})\}$. Note that due to closedness of $\text{cl}(\Gamma_{i_0})$ the maximum is attained and $z_1 := y + \alpha_1(\bar{y} - y) \in \text{cl}(\Gamma_{i_0})$. In addition, for all $\alpha \in (\alpha_1, 1]$ it holds that $y + \alpha(\bar{y} - y) \notin \text{cl}(\Gamma_{i_0})$. If $\alpha_1 = 1$ (and thus $\bar{y} \in \text{cl}(\Gamma_{i_0})$) the construction is complete. If $\alpha_1 \neq 1$, then there is an $i_1 \in \mathcal{J} \setminus \{i_0\}$ with $z_1 \in \text{cl}(\Gamma_{i_1})$. Take $\alpha_2 := \max\{\alpha \in [\alpha_1, 1] | y + \alpha(\bar{y} - y) \in \text{cl}(\Gamma_{i_1})\}$ and observe that $z_2 := y + \alpha_2(\bar{y} - y) \in \text{cl}(\Gamma_{i_1})$ and for all $\alpha \in (\alpha_2, 1]$ we have that $y + \alpha(\bar{y} - y) \notin \text{cl}(\Gamma_{i_0}) \cup \text{cl}(\Gamma_{i_1})$. If $\alpha_2 = 1$ the construction is complete. Otherwise, continue the construction. This construction will terminate in at most $\overline{M}$ steps as the number of regions $\text{cl}(\Gamma_i)$, $i = 1, \ldots, \overline{M}$, is finite and $\bar{y}$ lies in at least one of them. At termination, we arrived at the set of points $\{z_0, \ldots, z_M\}$ with the mentioned properties. Due to continuity of V in the region $G(x) \oplus \mathcal{B}_\mu$, continuity of V_i, $i = 1, \ldots, \overline{M}$ in $\mathcal{P}$ and $z_p \in \text{cl}(\Gamma_{i_{p-1}}) \cap \text{cl}(\Gamma_{i_p})$, $p = 1, \ldots, M$, we have that $V(z_p) = V_{i_{p-1}}(z_p) = V_{i_p}(z_p)$. Then, for any $y, \bar{y} \in G(x) \oplus \mathcal{B}_\mu$, it follows that:

$$|V(y) - V(\bar{y})| = \left| \sum_{p=1}^{M} (V(z_{p-1}) - V(z_p)) \right|$$

$$\leq \sum_{p=1}^{M} |V(z_{p-1}) - V(z_p)|$$

$$= \sum_{p=1}^{M} |V_{i_p}(z_{p-1}) - V_{i_p}(z_p)|$$

$$\leq \sum_{p=1}^{M} \sigma_{i_p}(\|z_{p-1} - z_p\|)$$

$$\leq \sum_{p=1}^{M} \sigma_{i_p}(\|y - \bar{y}\|).$$

Letting $\sigma(s) := \overline{M} \max_{i \in \mathcal{J}} \sigma_i(s) \in \mathcal{K}$, one obtains $|V(y) - V(\bar{y})| \leq \sigma(\|y - \bar{y}\|)$ for any $y, \bar{y} \in G(x) \oplus \mathcal{B}_\mu$.

Since for any $v \in \mathcal{B}_\mu$ it holds that $g(x,v) = G(x) + v \in G(x) \oplus \mathcal{B}_\mu$, it follows that:

$$V(g(x,v)) - V(G(x)) \leq \sigma(\|v\|), \forall x \in \mathcal{P}, \ \forall v \in \mathcal{B}_\mu. \tag{14}$$

As by the hypothesis V is a USL function for the PWC system (5), we have that $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$ for all $x \in \mathbb{R}^n$ and

$$V(G(x)) - V(x) \leq -\alpha_3(\|x\|), \quad \forall x \in \mathcal{P} \tag{15}$$

for some $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$. Adding (14) and (15) yields

$$V(g(x,v)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|v\|), \forall x \in \mathcal{P}, \forall v \in \mathcal{B}_\mu.$$

Hence, V is an ISS Lyapunov function for the PWC system (6). The statement then follows from Theorem II.4. □

Proof of Theorem IV.3: As done in the proof of Theorem IV.1, we will show that V satisfies the ISS inequalities (2). For any $x \in \mathcal{P}$ only the following situations can occur: (A) $(G(x) \oplus \mathcal{B}_\mu) \cap \mathbb{X}_D = \emptyset$ or (B) $x \in \mathcal{Z}$. In case (A), as shown in the proof of Theorem IV.1, by continuity of V on $G(x) \oplus \mathcal{B}_\mu$ and (8), there exists a $\sigma \in \mathcal{K}$ (independent of x) as constructed in the proof of Theorem IV.1 such that

$$V(g(x,v)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|v\|), \quad \forall v \in \mathcal{B}_\mu. \tag{16}$$

In case (B), suppose that $v \in \mathcal{B}_\mu$ is such that $G(x) \in \Gamma_p$ and $G(x) + v \in \Gamma_p$ for some $p \in \mathcal{J}$. In this case $p \notin \mathcal{M}(x)$. Then, since $V(G(x)) = V_p(G(x))$ and $V(G(x) + v) = V_p(G(x) + v)$, by continuity of V_i and (8), inequality (16) holds with the same $\mathcal{K}$-function σ constructed in the proof of Theorem IV.1.

Otherwise, if $v \in \mathcal{B}_\mu$ is such that $G(x) \in \Gamma_p$ and $G(x) + v \in \Gamma_i$ for some $p, i \in \mathcal{J}, p \neq i$, we have that $V(G(x)) = V_p(G(x))$, $V(G(x) + v) = V_i(G(x) + v)$ and $i \in \mathcal{M}(x)$. Then, by continuity of V_i, (8) and inequality (12) we obtain

$$\begin{aligned}
&V(G(x) + v) - V(x) \\
&= V_i(G(x) + v) - V(x) \\
&= V_i(G(x)) - V(x) + V_i(G(x) + v) - V_i(G(x)) \\
&\leq \max_{i \in \mathcal{M}(x)} V_i(G(x)) - V(x) + \sigma_i(\|v\|) \\
&\leq -\tilde{\alpha}_3(\|x\|) + \sigma(\|v\|)
\end{aligned}$$

with σ_i and σ as defined in the proof of Theorem IV.1. Letting $\hat{\alpha}_3(s) := \min(\alpha_3(s), \tilde{\alpha}_3(s))$ gives $\hat{\alpha}_3 \in \mathcal{K}_\infty$ and

$$V(g(x,v)) - V(x) = V(G(x) + v) - V(x) \leq -\hat{\alpha}_3(x) + \sigma(\|v\|)$$

for all $x \in \mathcal{P}$ and $v \in \mathcal{B}_\mu$. Therefore, V is an ISS Lyapunov function for system (6). The statement then follows from Theorem II.4. □

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