Higher rank tropical geometry and the variation of the demand

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Joint work with Jaime Tobar

The University of Texas at Austin

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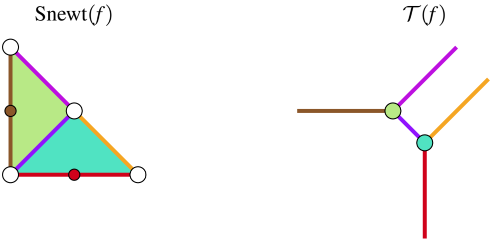

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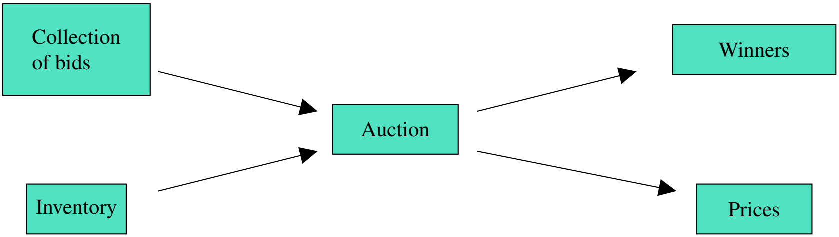

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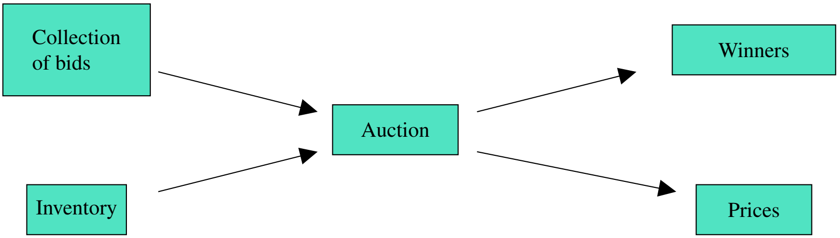

Slogan: The input data in the diagram above is tropical geometry data in a natural way. Therefore, the auction process should depend on tropical geometry as well.

Elizabeth Baldwin y Paul Klemperer (2019). *Understanding preferences:"demand types", and the existence of equilibrium with indivisibilities.* Econometrica, 87(3), 867-932.

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The function f_u is an instance of an n-th variable *tropical polynomial*.

Tropical Geometry

The tropical semiring is

$$\mathbb{T} := (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$$

with the *tropical sum* $\oplus$ and *tropical multiplication* $\odot$ given by

$$a \oplus b := \max(a, b) \quad , \quad a \odot b := a + b. \quad \forall a, b \in \mathbb{T}.$$

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A tropical Laurent polynomial is an expression of the form

$$f_u = \bigoplus_{a \in A} u_a \odot x^a$$

for some $A \subseteq \mathbb{Z}^n$. Written in a different form

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A tropical zero of f_u is an element $x \in \mathbb{R}^n$ for which f_u achieves the maximum at least twice. The set of all tropical zeros is called a tropical hypersurface.

Tropical Hypersurfaces

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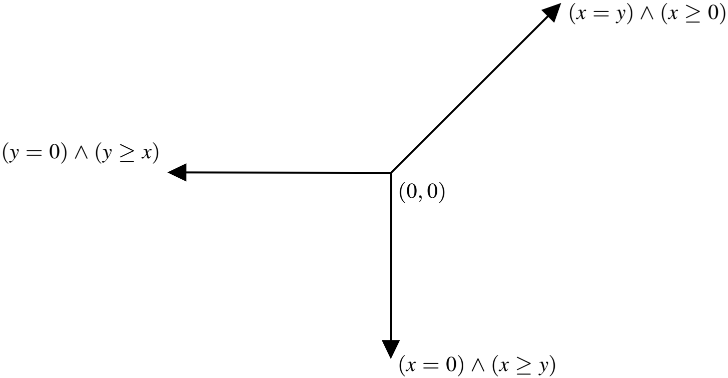
$$\{(x = y) \wedge (x \geq 0)\} \cup \{(x = 0) \wedge (x \geq y)\} \cup \{(y = 0) \wedge (y \geq x)\}.$$

which gives the tropical hypersurface $\mathcal{T}(p_1)$



Using **polymake**:

$$\begin{aligned} p_2(x,y) = 2 \oplus 4x \oplus 3x^2 \oplus 5x^3 \oplus 7x^4 \oplus 6x^5 \oplus 8x^6 \oplus 10x^7 \oplus 8x^8 \oplus 9y \\ \oplus 10y^2 \oplus 8y^3 \oplus 7y^4 \oplus 5y^5 \oplus 6y^6 \oplus 3y^7 \oplus 4y^8 \oplus 15x^4y^4 \end{aligned}$$

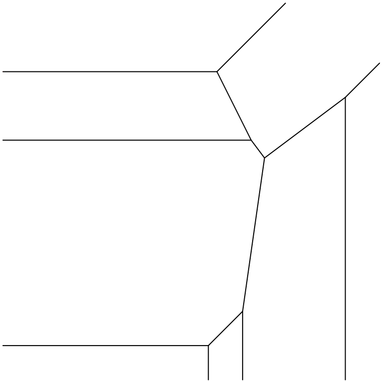
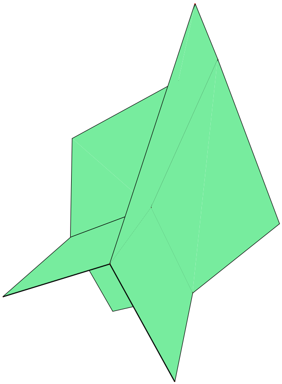


Figure 1: $\mathcal{T}(p_2)$

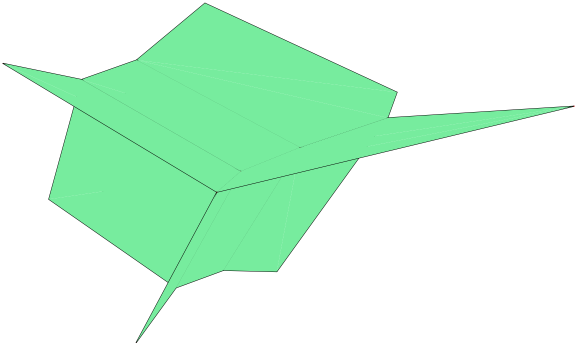
Tropical Hypersurfaces

$$p_3(x, y, z) = 1 \oplus 1x \oplus 1y \oplus 1z$$

$$p_4(x, y, z) = 6 \oplus 5x \oplus 4y \oplus 3z \oplus 3x^2 \oplus 2y^2 \oplus 1z^2$$



(a) $\mathcal{T}(p_3)$



(b) $\mathcal{T}(p_4)$

Back to Auctions

Given a market with n different types of goods and a set $A \subseteq \mathbb{Z}^n$ of bundles of goods. An agent has a valuation $u : A \to \mathbb{R}$ which defines a utility function

$$f_u(p) = \max_{x \in A} \left(u_j(x) - \langle x, p \rangle \right).$$

- The *demand* of the agent for a given price p is the set $D_u(p) = \operatorname{argmax}_{x \in A}(f_u(p))$.

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2. Surprisingly, the demands of the agent also fit into a polyhedral complex: The *regular subdivision* of the Newton polytope of f_u.
3. Moreover, the demand complex is dual to the tropical hypersurface.

Regular subdivision

$$p(x,y) = 1 \oplus 1x \oplus 1x^2 \oplus 1x^3 \oplus 1y \oplus 2yx \oplus 2yx^2 \oplus 1yx^3 \oplus 1y^2 \oplus$$
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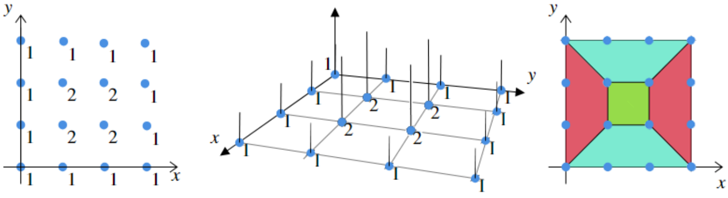

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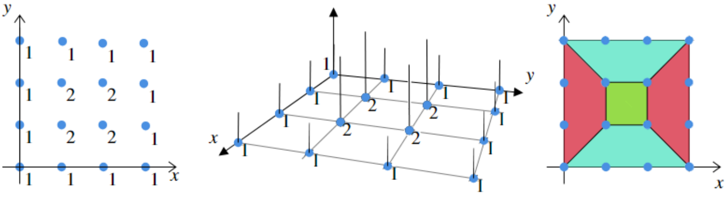

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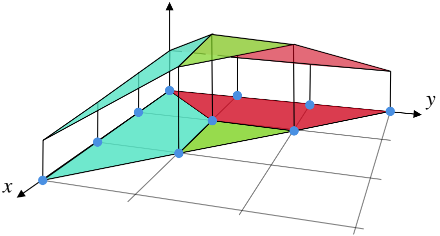
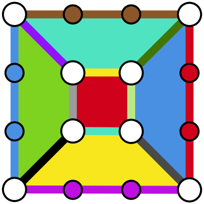
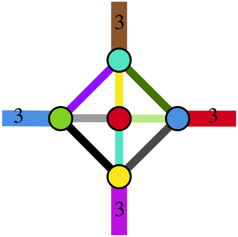


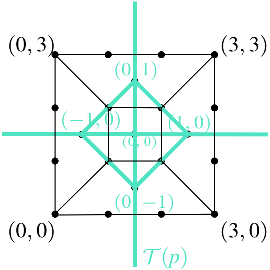
Hypersurface Duality



Newt(p)



$\mathcal{T}(p)$



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Other big advantage of the tropical geometry setting is that it handles easily multiple agents.

Definition

Given a family of agents J, their **aggregate demand** at a price $p \in \mathbb{R}^n$ is the Mikowski sum

$$D_{u_J}(p) := \sum_{j \in J} D_{u_j}(p).$$

The aggregate demand coincides to the demand of a fictional *aggregate agent*. The utility function of this aggregate agent will be obtained as a product of the polynomials

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In other words, $x = x_1 + \cdots + x_{\#J}$ where $x_i \in D_{u_i}(p)$ for each i, or equivalently, $x \in D_{u_J}(p)$.

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Remark: Notice that an auction will be successful exactly if a competitive equilibrium exists.

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To study this question, given a set $A \subseteq \mathbb{Z}^n$ we introduce the valuation space $\mathrm{Val}(A) = \{u : A \to \mathbb{R}\}$. This is the space of all possible valuations of an agent over a set of bundle of goods A.

Theorem (Hemicontinuity Theorem)

Given $A \subseteq Z^n$, the map

$$\begin{aligned} D : \mathrm{Val}(A) \times \mathbb{R}^n &\to \mathcal{P}(\mathbb{Z}^n) \\ (u, p) &\mapsto D_u(p) \end{aligned}$$

satisfy that for each $u \in \mathrm{Val}(A)$ and $p \in \mathbb{R}^n$ there exists an open neighborhood of $V \subseteq \mathrm{Val}(A) \times \mathbb{R}^n$ of (u, p) such that $\forall (u', p') \in V$

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Do we have a way to understand how is the change exactly?

Tropical Geometry of Higher Rank

The *tropical semiring of rank k* $\mathbb{T}_k = (\mathbb{R}^k \cup \{-\infty\}, \oplus, \odot)$ is the semiring over $\mathbb{R}^k$ in which $\odot$ is the addition and $\oplus$ the lexicographic order

$$(a^{(1)}, \dots, a^{(k)}) \prec (b^{(1)}, \dots, b^{(k)}) \iff a_i < b_i \text{ for the minimum } i \text{ such that } a_i \neq b_i$$

Elements of $a \in \mathbb{T}_k$ should be thought as

$$a^{(1)} + \varepsilon a^{(2)} + \dots + \varepsilon^{(k-1)} a^{(k)}$$

where ε is a very small but positive element.

We can introduce tropical polynomials $f_u = \bigoplus_{a \in A} u_a \odot x^a$ and tropical hypersurfaces $\mathcal{T}(f_u) \subseteq \mathbb{T}_k^n$ in the same way as we did before.

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How do we visualize $\mathcal{T}(f_u)$?

Iterated Fibrations

There are natural projections maps

$$\begin{aligned} \pi_r \colon \mathbb{T}_k &\longrightarrow \mathbb{T}_r \\ a &\longmapsto a^{[r]} := (a^{(1)}, \ldots, a^{(r)}) \end{aligned}$$

This projection maps extend to maps elements in $\mathbb{T}_k^n$ and to polynomials.
Then, for any Laurent polynomial f we have

$$\mathcal{T}(f^{[r]}) = \mathcal{T}(f)^{[r]}.$$

Which gives us a sequence of projections

$$\mathcal{T}(f^{[r]}) \xrightarrow{\pi_{k-1}} \mathcal{T}(f^{[r-1]}) \xrightarrow{\pi_{k-2}} \ldots \xrightarrow{\pi_1} \mathcal{T}(f^{[1]})$$

The base of this fibration is a tropical hypersurface of rank 1, and all the fibers of points are tropical hypersurfaces of rank 1. Moreover, the hypersurface duality generalize to this context.

Layered regular subdivisions

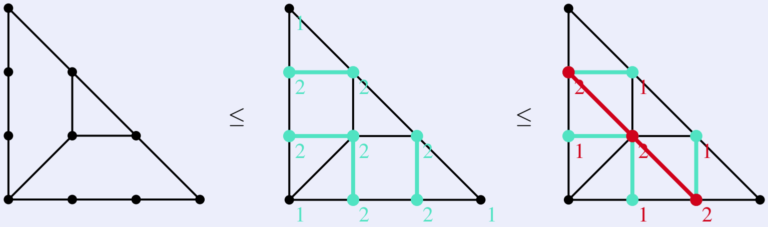
Consider the set

$$A = \{(0,0),\ (1,0),\ (2,0),\ (3,0),\ (0,1),\ (0,2),(0,3),(1,2),(2,1),(1,1)\}$$

In this case, the layered regular subdivision induced by the map $u : A \to \mathbb{T}_3$

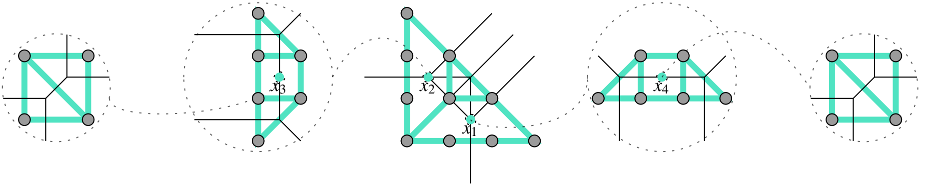
$$\begin{aligned} u(0,0) &= 1 + 1\varepsilon + 1\varepsilon^2 \\ u(1,0) &= 1 + 2\varepsilon + 1\varepsilon^2 \\ u(2,0) &= 1 + 2\varepsilon + 2\varepsilon^2 \\ u(3,0) &= 1 + 1\varepsilon + 1\varepsilon^2 \end{aligned} \qquad \begin{aligned} u(0,1) &= 1 + 2\varepsilon + 1\varepsilon^2 \\ u(0,2) &= 1 + 2\varepsilon + 2\varepsilon^2 \\ u(0,3) &= 1 + 1\varepsilon + 1\varepsilon^2 \\ u(1,2) &= 2 + 2\varepsilon + 1\varepsilon^2 \end{aligned} \qquad \begin{aligned} u(2,1) &= 2 + 2\varepsilon + 1\varepsilon^2 \\ u(1,1) &= 2 + 2\varepsilon + 2\varepsilon^2. \end{aligned}$$

Will be the following:



Theorem (Hypersurface Duality)

Given a higher rank tropical polynomial and its corresponding layered fibration of its Newton polytope, there is a way to read from this subdivisions the combinatorial structure of the iterated fibration in its tropical hypersurface.



$\mathcal{T}_{x_2,x_3}\left(f_u^{[3]}\right)$ $\mathcal{T}_{x_2}\left(f_u^{[2]}\right)$ $\mathcal{T}\left(f_u^{[1]}\right)$ $\mathcal{T}_{x_1}\left(f_u^{[2]}\right)$ $\mathcal{T}_{x_1,x_4}\left(f_u^{[3]}\right)$

Part of the usefulness of this framework is that it mixes two perspectives.

1. On one hand, the elements of $\mathbb{T}_k$ are rigid, and this allow us to draw the diagrams that generalize the ideas from $\mathbb{T}_1$.
2. On the other hand, given an element

$$x^{(1)} + \varepsilon x^{(2)} + \ldots + \varepsilon^{(k-1)} x^{(k)} = x \in \mathbb{T}_k$$

   we can replace ε by a concrete small real number, giving rise to a perturbation of the element $x^{(1)}$. More generally, "finitelly generated" objects $X/\mathbb{T}_k$ should give rise to perturbations X_ε in this way.

As working with perturbations is generally a difficult thing (What is the perturbation of the demand $D_u(p)$ as u changes?), the formal point of view of working directly in $\mathbb{T}_k$ simplify the study.

Theorem (Demand)

Consider a map $u : A \subseteq \mathbb{Z}^n \to \mathbb{R}_k$. Then, for $\delta > 0$ a small real number, the demand

$$D_{u^{(1)}+\delta u^{(2)}+\cdots+\delta^{k-1}u^{(k)}}(p^{(1)} + \delta p^{(2)} + \cdots + \delta^{k-1}p^{(k)}).$$

coincides with the corresponding cell in the layered subdivision dual to the cell in the tropical hypersurface containing $p = p^{(1)} + \varepsilon p^{(2)} + \cdots + \varepsilon^{k-1}p^{(k)}$

Theorem (Perturbation of Competitive Equilibria)

Consider a family of agents, each with a valuations which has been perturbed by functions $\{u^j : A \subseteq \mathbb{Z}^n \to \mathbb{D}_k\}_{j\in J}$. This family posses a competitive equilibrium for $x \in A$ for each $\delta > 0$ small iff the corresponding valuations have *formally* a competitive equilibrium over $\mathbb{T}_k$.