2 Exponents

2.1 Mon., Mar. 5: Expansion of b^x

We often want to consider powers of a number where:

$$b^x = \underbrace{b \cdot b \cdot b}_{x \text{ times}} = y$$

From this idea we can derive "rules" for exponents.

$$x^0 = 1 \tag{12}$$

$$x^a \cdot x^b = x^{a+b} \tag{13}$$

$$(x^a)x^b = x^{a \cdot b} \tag{14}$$

$$\frac{x^a}{x^b} = x^{a-b} \tag{15}$$

$$\frac{x^a}{x^a} = (\frac{x}{x})^a \tag{16}$$

$$x^{a/b} = \sqrt[b]{x^a} \tag{17}$$

Exercise 23. *Rewrite* 81^2 *with a base of 2.*

Exercise 24. *Find* $\sqrt[4]{81}$.

Exercise 25. *Simplify* $\frac{\frac{1}{2}}{2^{-1}}$.

Exercise 26. *Find* $\sqrt[a]{x^a}$.

Exercise 27. *Find* $(\frac{x^2}{x^3})^2$.

Exercise 28. *Find* $(\frac{1}{2})^{-1}$.

2.2 Wednes., Mar. 7: Logarithms

The logarithm language rewrites this in the format:

$$\log_b y = x \tag{18}$$

Where:

$$b^x = y \tag{19}$$

I've used the variables x and y in a consistent manner here, so you can copy between the formulas. But usually logs will be written such that the dependent variable, y, is the exponent... making them the inverse of b^x. The change of base format is very useful and helpful to consider as we understand logs:

$$\log_a x = \frac{\log_b x}{\log_b a} \tag{20}$$

Which we can prove by observing that:

$$y \log_b x = \log_b x^y \tag{21}$$

This is the power property of logs, and there are also properties for products

$$\log_a uv = \log_a u + \log_a v \tag{22}$$

...and quotients:

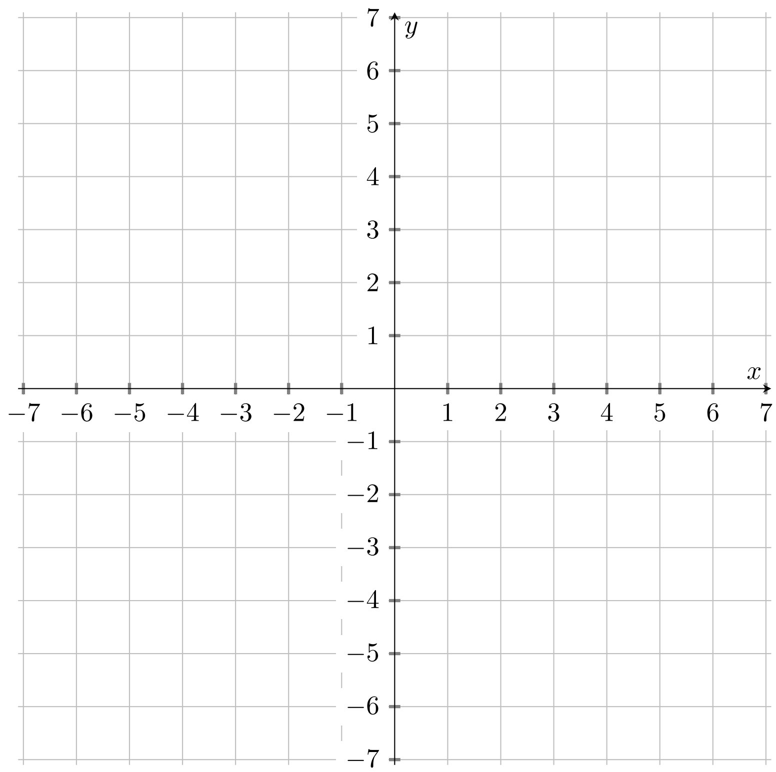
$$\log_a \frac{u}{v} = \log_a u - \log_a v \tag{23}$$

Another important identity that we should think about is:

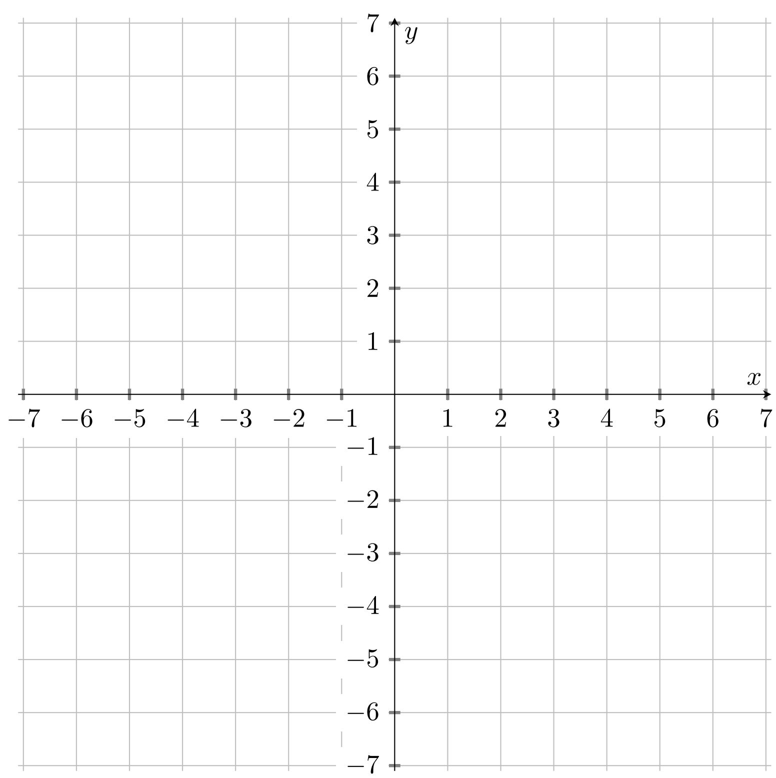
$$b^{\log_b x} = x \tag{24}$$

2.3 Fri, Mar. 8: Putting exponents and logs together with graphs

Exercise 29. *Graph* x^2. *Graph* 2^x.



Exercise 30. *Graph* $\log_2 x$. *Graph* ln. *Graph* $\log_{10} x$.



Exercise 31. *Find* $\log_{10} 1$

Exercise 32. *Find* $\log_{10} 10^7$

Exercise 33. *Find the approximate value of* $\log_2 10^3$ *rounded to the nearest whole number. No calculator! (Think about powers of 2.)*

Exercise 34. *Find* $\log_{10} 1,000,000,000$

Exercise 35. *Simplify* $\frac{a^3 b^7}{a^{-4} b^4}$

Exercise 36. *Simplify* $\frac{x^{2(z+8)}}{x^{-z}}$

Exercise 37. *Simplify* $\frac{8^3}{2^3 2^7}$

Exercise 38. *Simplify* $(\frac{a^{1/3}}{b^{1/6}})^3$

Exercise 39. *Find* $\log_5 \frac{1}{125}$

2.4 Fri, Mar. 8: Deriving e

There are two ways to derive e. The first is to expand our idea of the slope of a line:

$$y = mx + b \tag{25}$$

We know slope is a ratio of $\frac{\Delta x}{\Delta y}$ and for large x and y this can be expressed:

$$\frac{y_2 - y_1}{x_2 - x_1} \tag{26}$$

And we know y is really just $f(x)$ so this becomes:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{27}$$

But often we want to look at non-constant slopes... *i.e.* where the size of slope depends on where you are in the function. Here it's useful to think about what happens when x_2 and x_1 are very close together, *i.e.* when $x_2 \to x_1$. This is the beginning of the calculus idea. But for now, we'll just use this idea to compute e^x to get some experience with it, and to learn about e.

Exercise 40. *Let's say we want to consider something that grows at 2^t where t is time. What is the relationship between the growth rate of the function at a place and the value of the function at that place. Is there a function where this ratio is $1:1$?*

Exercise 16 offers us one way to prove e. We can also think about e as doing something, then doing it more times but less each time. Let's consider a thing that's continuously growing... and measuring it at smaller units of time but more frequently, perhaps at a half unit of time. We would find that we have to multiply two terms:

$$(1 + \frac{1}{2}) \cdot (1 + \frac{1}{2}) \tag{28}$$

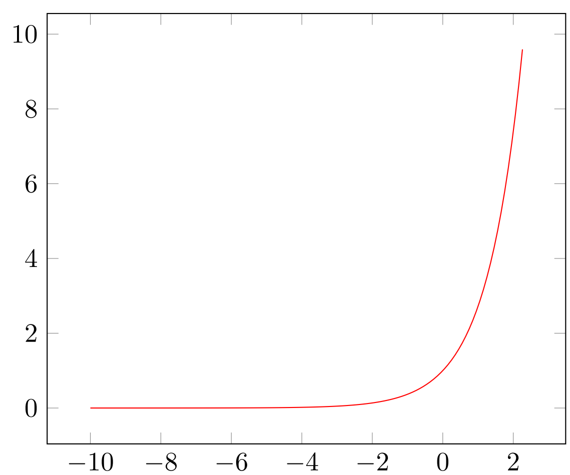
As we do this more and more, we'd get better approximations for e like:

$$(1 + \frac{1}{100}) \cdot (1 + \frac{1}{100}) \cdot \ldots \cdot (1 + \frac{1}{100}) = (1 + \frac{1}{100})^{100} \tag{29}$$

Throw this into the calculator and you'll find it's e. So in general:

$$e^x = \lim_{x \to 2} f(x) = (1 + \frac{1}{n})^n \tag{30}$$

The graph of e^x is:



More on e