

4 Exponents

4.1 Tues., Feb. 27: Expansion of b^x

We often want to consider powers of a number where:

$$b^x = \underbrace{b \cdot b \cdot b}_{x \text{ times}} = y$$

From this idea we can derive “rules” for exponents.

$$x^0 = 1 \tag{18}$$

$$x^a \cdot x^b = x^{a+b} \tag{19}$$

$$(x^a)x^b = x^{a \cdot b} \tag{20}$$

$$\frac{x^a}{x^b} = x^{a-b} \tag{21}$$

$$\frac{x^a}{x^a} = \left(\frac{x}{x}\right)^a \tag{22}$$

$$x^{a/b} = \sqrt[b]{x^a} \tag{23}$$

Exercise 93. Rewrite 81^2 with a base of 3.

Exercise 94. Find $\sqrt[4]{81}$.

Exercise 95. Simplify $\frac{1}{2^{-1}}$.

Exercise 96. Find $\sqrt[a]{x^a}$.

Exercise 97. Find $\left(\frac{x^2}{x^3}\right)^2$.

Exercise 98. Find $\left(\frac{1}{2}\right)^{-1}$.

4.2 Thurs., Feb. 29: Logarithms

The logarithm language rewrites this in the format:

$$\log_b y = x \quad (24)$$

Where:

$$b^x = y \quad (25)$$

I've used the variables x and y in a consistent manner here, so you can copy between the formulas. But usually logs will be written such that the dependent variable, y , is the exponent... making them the inverse of b^x . The change of base format is very useful and helpful to consider as we understand logs:

$$\log_a x = \frac{\log_b x}{\log_b a} \quad (26)$$

Which we can prove by observing that:

$$y \log_b x = \log_b x^y \quad (27)$$

This is the power property of logs, and there are also properties for products

$$\log_a uv = \log_a u + \log_a v \quad (28)$$

...and quotients:

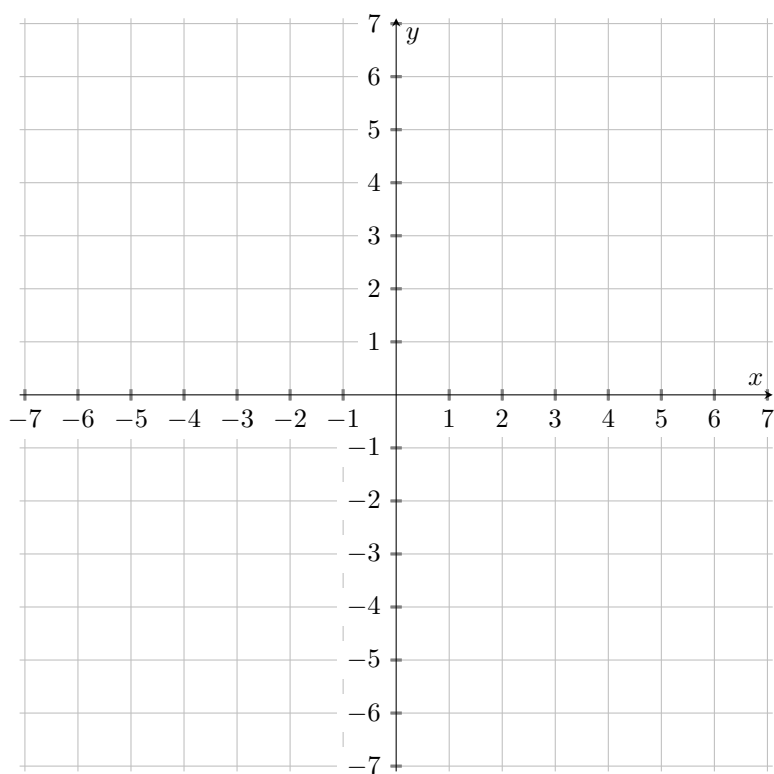
$$\log_a \frac{u}{v} = \log_a u - \log_a v \quad (29)$$

Another important identity that we should think about is:

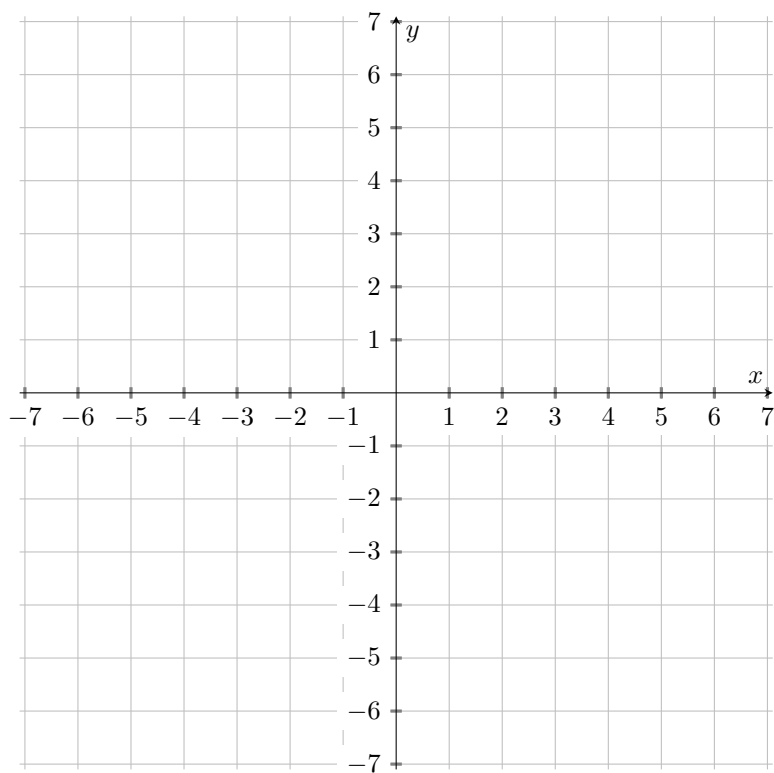
$$b^{\log_b x} = x \quad (30)$$

4.2.1 Putting exponents and logs together with graphs

Exercise 99. Graph x^2 . Graph 2^x .



Exercise 100. Graph $\log_2 x$. Graph \ln . Graph $\log_{10} x$.



Exercise 101. Find $\log_{10} 1$

Exercise 102. Find $\log_{10} 10^7$

Exercise 103. Find the approximate value of $\log_2 10^3$ rounded to the nearest whole number. No calculator! (Think about powers of 2.)

Exercise 104. Find $\log_{10} 1,000,000,000$

Exercise 105. Simplify $\frac{a^3 b^7}{a^{-4} b^4}$

Exercise 106. Simplify $\frac{x^{2(z+8)}}{x^{-z}}$

Exercise 107. Simplify $\frac{8^3}{2^3 2^7}$

Exercise 108. Simplify $(\frac{a^{1/3}}{b^{1/6}})^3$

Exercise 109. Find $\log_5 \frac{1}{125}$

4.3 Fri, Mar. 1: Deriving e

There are two ways to derive e . The first is to expand our idea of the slope of a line:

$$y = mx + b \quad (31)$$

We know slope is a ratio of $\frac{\Delta x}{\Delta y}$ and for large x and y this can be expressed:

$$\frac{y_2 - y_1}{x_2 - x_1} \quad (32)$$

And we know y is really just $f(x)$ so this becomes:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad (33)$$

But often we want to look at non-constant slopes... *i.e.* where the size of slope depends on where you are in the function. Here it's useful to think about what happens when x_2 and x_1 are very close together, *i.e.* when $x_2 \rightarrow x_1$. This is the beginning of the calculus idea. But for now, we'll just use this idea to compute e^x to get some experience with it, and to learn about e .

Exercise 110. *Let's say we want to consider something that grows at 2^t where t is time. What is the relationship between the growth rate of the function at a place and the value of the function at that place. Is there a function where this ratio is 1 : 1?*

Exercise 16 offers us one way to prove e . We can also think about e as doing something, then doing it more times but less each time. Let's consider a thing that's continuously growing... and measuring it at smaller units of time but more frequently, perhaps at a half unit of time. We would find that we have to multiply two terms:

$$\left(1 + \frac{1}{2}\right) \cdot \left(1 + \frac{1}{2}\right) \quad (34)$$

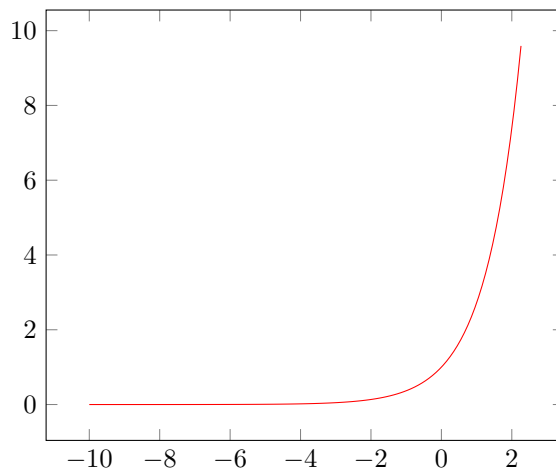
As we do this more and more, we'd get better approximations for e like:

$$\left(1 + \frac{1}{100}\right) \cdot \left(1 + \frac{1}{100}\right) \cdot \dots \cdot \left(1 + \frac{1}{100}\right) = \left(1 + \frac{1}{100}\right)^{100} \quad (35)$$

Throw this into the calculator and you'll find it's e . So in general:

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n \quad (36)$$

The graph of e^x is:



4.3.1 Interest

Motivating question: how often should interest compound?

If we think about a bank, we might imagine someone offering us to hold our money in exchange for paying us a small fee to use the money while we're not. Let's say this fee is 8% a year.

What would the bank owe us after a year? If we loaned 100\$, this would be:

$$100\$ \cdot 1.08 = 108\$ \quad (37)$$

But there's nothing special about a year, instead of giving us 8\$ at the end of the year, the bank could give us the $\frac{8}{4}$ \$ after $\frac{1}{4}$ of the year, and do so 4 times throughout the year.

Exercise 111. Which is better? A bank that compounds interest annually or one that compounds $\frac{1}{4}$ of interest quarterly?

In general, the formula for the current amount of money for money earning interest at an interest rate of r that is compounded n times a year for some years t from an initial deposit of P is:

$$M = P\left(1 + \frac{r}{n}\right)^{(nt)} \quad (38)$$

Exercise 112. If you deposit 3,000\$ in a bank that pays 9% interest and compounds twice a year, how much money would you have after 14 years?

Exercise 113. Would a bank that compounds every day, or even every second, be far better than a bank that compounds annually or only a little better? Compute some examples to prove this.

Now recall our formula for e :

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n \quad (39)$$

By visual inspection it's clear this is just the continuously compounding interest case!

Since $e = \left(1 + \frac{1}{n}\right)^n$ for large n , we can replace $\left(1 + \frac{1}{n}\right)^n$ with e so (32) becomes and r factors out by log rules:

$$M = Pe^{rt} \quad (40)$$