3 Quadratics

3.1 Tues, Feb. 27: Practice with distributive property of multiplication

We note that multiplication is distributive. This means multiplying each part of a sum is the same as taking the sum and multiplying that... so:

$$A(B+C) = A \cdot B + A \cdot C \tag{35}$$

Verify that this is true with real numbers! The same is true with unknowns.

Exercise 44. *Multiply* $x(2+x)$

Exercise 45. *Multiply* $(x-1)(x+1)$

Exercise 46. *Find a solution for* $7(2+x) = 63$

Exercise 47. *Multiply* $(x+y+z)(a+b+c)$

Exercise 48. *Find some ways to factor* $x^3 + x^2 + x$

We often want to know all the *factors* of a number, these are the primes that can be multiplied to make that number. For instance, the prime factorization of 144 is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$.

Exercise 49. *Draw the tree for the prime factorization of* 81

Exercise 50. *Factor* 124

Exercise 51. *Factor* 1000

Exercise 52. *Factor* 3402

Exercise 53. *Factor* 512

Exercise 54. *Factor* 221

Exercise 55. *Factor* 3599. *Hint: any* $(x^2 - 1) = (x+1)(x-1)$.

What is the geometric nature of multiplication and distribution. Let's explore this with an example?

Exercise 56. *Solve* $3 \cdot (3+7)$ *by finding the area of the rectangle with sides* 3 *and* 3*... and the rectangle with sides* 3 *and* 7*. And then find the area of the rectangle with sides* 3 *and* 10*.*

In fact, any number can be expressed as a difference of squares... so for any numbers r, s you can think about a midpoint number m where r, s are $m+d, m-d$.

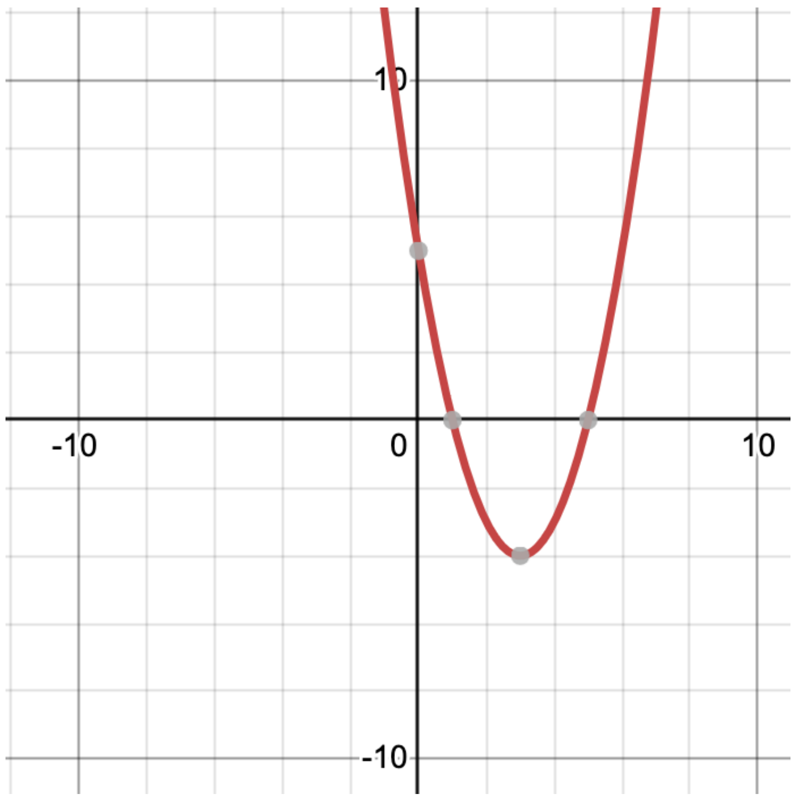
3.1.1 Guessing roots of some quadratics

We want to consider something like $ax^2 + bx + c$ and find when it equals zero. It's much easier to think about where that leading coefficient is 1 because we're left with just the simplification:

$$(x - r)(x - s) = x^2 - (r + s)x + rs \tag{36}$$

Prove that changing the leading coefficient of x doesn't change the roots of a quadratic by doing the following exercise...

Exercise 57. *I've given the plot of $x^2 - 6x + 5$ below. Now use a graphing calculator to plot $S(x^2 - 6x + 5)$ where $S = \{4, 2, -5, 10, ...\}$*



We can do a pretty good job of finding these r, s just by guessing and checking.

Exercise 58. *What is x when $x^2 + 13x + 35 = 0$*

Exercise 59. *What is x when $x^2 + x - 2 = 0$*

Exercise 60. *What is x when $x^2 - x - 12 = 0$*

Exercise 61. *What is x when $x^2 - x + 12 = 0$*

Exercise 62. *What is x when $x^2 - 100 = 0$*

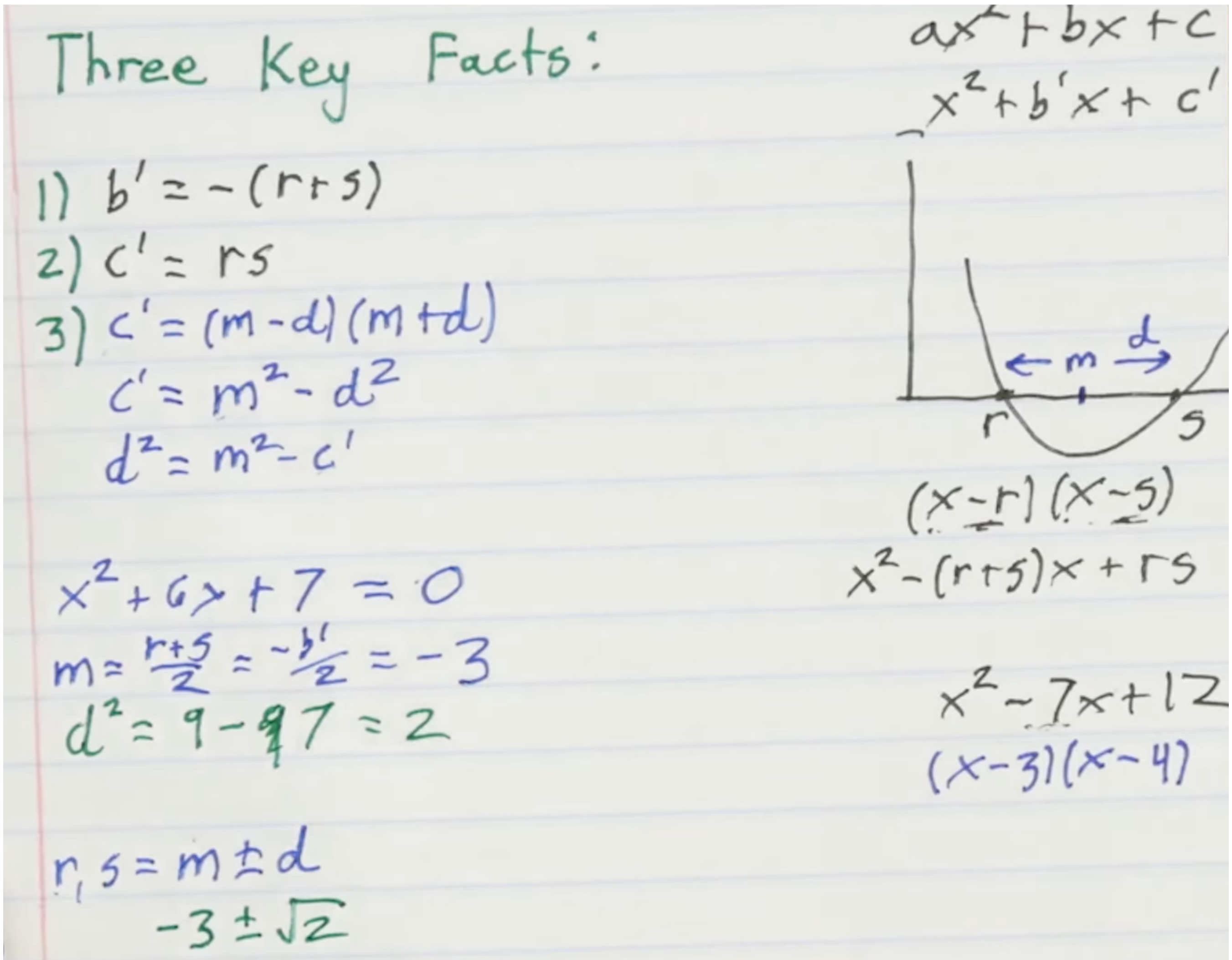
Exercise 63. *What is x when $x^2 - 6x - 55 = 0$*

3.2 Thurs., Feb. 29: The Quadratic formula

But there is a systematic way to do this work... we can think about the midpoint of our two numbers and the distance between that midpoint and the numbers! If we consider our equations:

$$ax^2 + bx + c \tag{37}$$

$$(x-r)(x-s) = x^2 - (r+s)x + rs \tag{38}$$



So practically, to find the roots of a quadratic, we take the middle coefficient (of x), divide by -2 and write it down... the roots will be that plus and minus the square root of the difference of that and the last term, c. In even simpler terms, we find $m = -\frac{b}{2}$ then take $\sqrt{m^2 - c}$.

Exercise 64. *Solve* $4a^2 + 6 = 0$

Exercise 65. *Solve* $2x^2 - 8x - 2 = 0$

Exercise 66. *Solve* $2m^2 - 3 = 0$

Exercise 67. *Solve* $3r^2 - 2r - 1 = 0$

Exercise 68. *Solve* $4n^2 - 36 = 0$

Exercise 69. *Solve* $v^2 - 4v - 5 = -8$

Exercise 70. *Solve* $2x^2 + 3x + 14 = 6$

Exercise 71. *Solve* $3x^2 + 3x - 4 = 7$

Exercise 72. *Solve* $7x^2 + 3x - 16 = -2$

Exercise 73. *Solve* $2x^2 + 6x - 16 = 4$

Exercise 74. *Solve* $3x^2 + 3x = -3$

Exercise 75. *Solve* $2x^2 = -7x + 49$

Exercise 76. *Solve* $5x^2 = 7x + 7$

Exercise 77. *Solve* $8x^2 = -3x - 8$

Exercise 78. *Solve* $2x^2 + 5x = -3$

Exercise 79. *Solve* $4x^2 - 64 = 0$

Exercise 80. *Solve* $4x^2 + 5x - 36 = 3x^2$

3.3 Fri, Mar. 1: Polynomials and the general form of quadratics

Polynomials are important in mathematics and life. It's worth thinking about why polynomials appear so often... I don't expect that you fully understand, but I do want us to spend some time thinking about how they relate important ideas about space and distance. Does this connection seem familiar? It is related to the two interpretations (graphic, algebraic) of linear equations in the form $y = mx + b$.

> Why are quadratic polynomials useful? There seem to be two different but interacting reasons. The first is that quadratic functions of a real variable are always either convex or concave and therefore have a unique maximum or minimum. The second is that quadratic functions are intimately related to bilinear forms and therefore can be accessed using linear algebra. —Paul Siegel, *Mathoverflow*

Polynomials have the form:

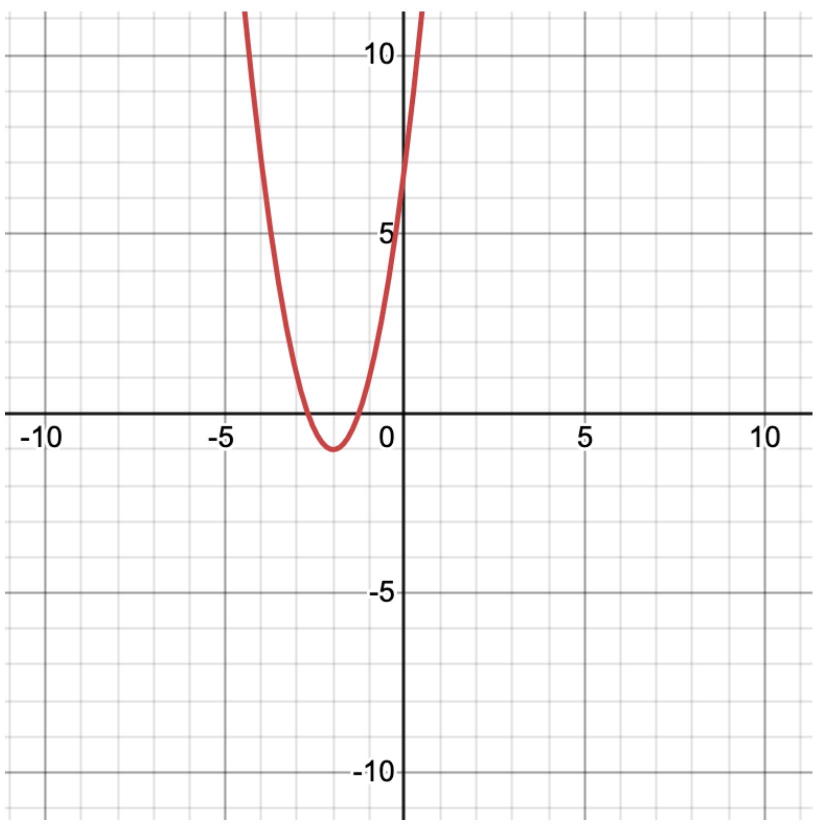
$$a_n x^n + \ldots + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{39}$$

Where the "poly" means many and the "nomial" refers to terms... or things added to each other. A polynomial with only one term is called a mononomial, one with two is called a binomial... and the highest order (or power) term designates whether it is quadratic, cubic, etc. The behavior of a polynomial is largely determined by its first term. Why is this? Think about properties of exponents. Is there a way we could say that x^5 is more "powerful" than x? This gives us the "leading coefficient test". And tells us how to graph polynomials. (Weird behavior often happens between the zeros, not outside of them). We begin investigating polynomials by thinking about quadratics. The standard form of a quadratic is:

$$f(x) = a(x-h)^2 + k \tag{40}$$

How could we derive this form from others? In this form, (h, k) will be the vertex. What does f look like if $a > 0$? If $a < 0$? Will the vertex be a min or max? For $ax^2 + bx + c = 0$, where a,b,c $\in \mathbb{R}$ and $a \neq 0$, $h = \frac{-b}{2a}$ and $k = f(h) = f(\frac{-b}{2a})$.

Exercise 81. *Write $f(x) = 2x^2 + 8x + 7$ in standard form by factoring around x, then working on the $x^2 + x$ term and inventing a constant value to make the factorization work... then adding that value to k. I've included the graph of this function below.*



3.3.1 Graphing quadratics

Without solving, graph:

Exercise 82. $3x^2 + 2 = 0$

Exercise 83. $6x^2 - 1 = 0$

Exercise 84. $5x^2 + 2x + 6 = 0$

Exercise 85. $2x^2 - 2x - 15 = 0$

Exercise 86. $3x^2 + 6 = 0$

Exercise 87. $2x^2 + 4x + 12 = 8$

Exercise 88. $6x^2 - 3x + 3 = -4$

Exercise 89. $4x^2 - 14 = -2$

Exercise 90. $4x^2 + 5x = 7$

Exercise 91. $x^2 + 4x - 48 = -3$

Exercise 92. $3x^2 - 3 = 8x$

Exercise 93. $3x^2 + 4 = -6x$

Exercise 94. $6x^2 = -5x + 13$

Exercise 95. $6x^2 = 4 + 6x$

Exercise 96. $x^2 = 8$

Exercise 97. $2x^2 + 6x - 16 = 2x$

Exercise 98. $12x^2 + x + 7 = 5x^2 + 5x$