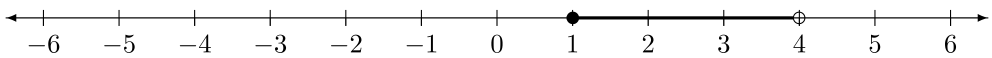
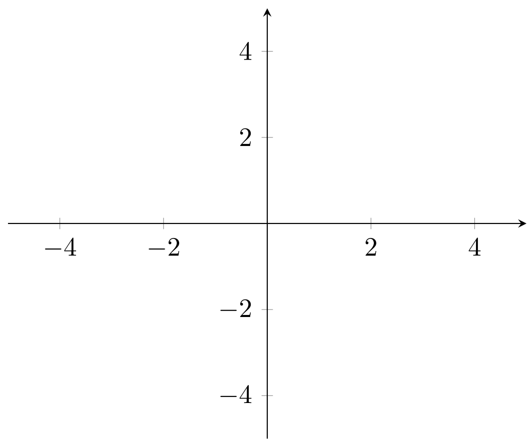
1 Graphing and solving linear equations through rearrangement, substitution and subtraction

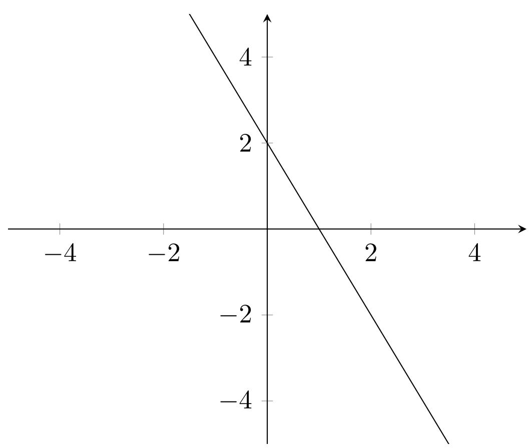
We begin our study of algebra by remembering the number line. If you haven't seen one in a while, it looks like this:



Here I used the number line to show adding $1 + 3$. But often we want to draw on two number lines at the same time, to show the *relationship* of two numbers, or what happens to the second when we increase or decrease the size of the first number:



This is good because we can now plot not just one number, but two at the same time... and these relations can be expressed in line equations where each point on the line is a relationship of x and y that is true in the line equation:



These lines are expressed with equations like $y = mx + b$ where:

$$y = \underbrace{m}_{\text{the ratio of x and y}} x + \underbrace{b}_{\text{a constant value}}$$

It can be helpful to think about where m comes from:

$$\frac{y}{x} = \frac{y}{x} \to y = (\underbrace{\frac{y}{x}}_{\text{m}})x \tag{1}$$

We know slope is a ratio of $\frac{\Delta x}{\Delta y}$ and for any x and y this can be expressed:

$$\frac{y_2 - y_1}{x_2 - x_1} \tag{2}$$

1.1 Graphing a line and finding the equation of a line with any two points that lie on it

We can draw the graph for a line equation $y = mx + b$ by:

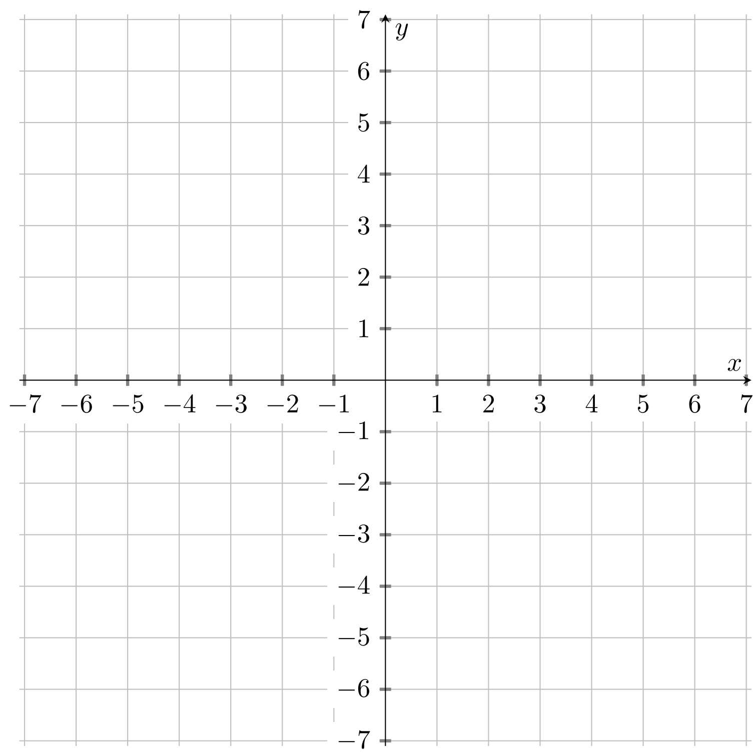
- identifying the b-value on the y-axis... (hint: this is found by imagining x to be 0)
- drawing from that point the line slope given by m

We can also find a line equation from any two points on the graph $(x_2, y_2)$and (x_1, y_1) by reversing the process:

- finding m, or the slope with $\frac{y_2-y_2}{x_2-x_1}$
- substituting one of our points for x and y and the value we just found for m to solve for b

We can also find the distance between any two points in a graph with the help of our old friend, Pythagoras.

Exercise 1. *Find the distance between the points $(-2, -3)$ and $(3, 5)$ with the Pythagorean theorem. Write this down for your notes as the "distance formula". Find their midpoint. Verify this with the graph by carefully estimating the midpoint of these two points. Write down the midpoint formula for your notes. (Hint: it is like taking an average.)*



Note that distance in the y and x are independent.

1.2 Shapes of common functions

Consider the functions:

$$f(x) = c \tag{3}$$

$$f(x) = x^2 \tag{4}$$

$$f(x) = x^3 \tag{5}$$

$$x^2 + y^2 = 1 \tag{6}$$

(Note: You may have heard of the vertical line test. Does this pass the vertical line test? If not a function, what is it? Are there other ways we could describe a circle that would pass the test?)

$$f(x) = |x| \tag{7}$$

$$f(x) = \sqrt{x} \tag{8}$$

$$f(x) = \frac{1}{x} \tag{9}$$

$$f(x) = \frac{1}{x^2} \tag{10}$$

What is the basic shape of each of these functions? How do their compositions with other functions affect their shapes?

Exercise 2. *Let's invent some points and functions and work on translating between the two. Use the method of* 2^n *steps to find points on the graph with the brute force method.*

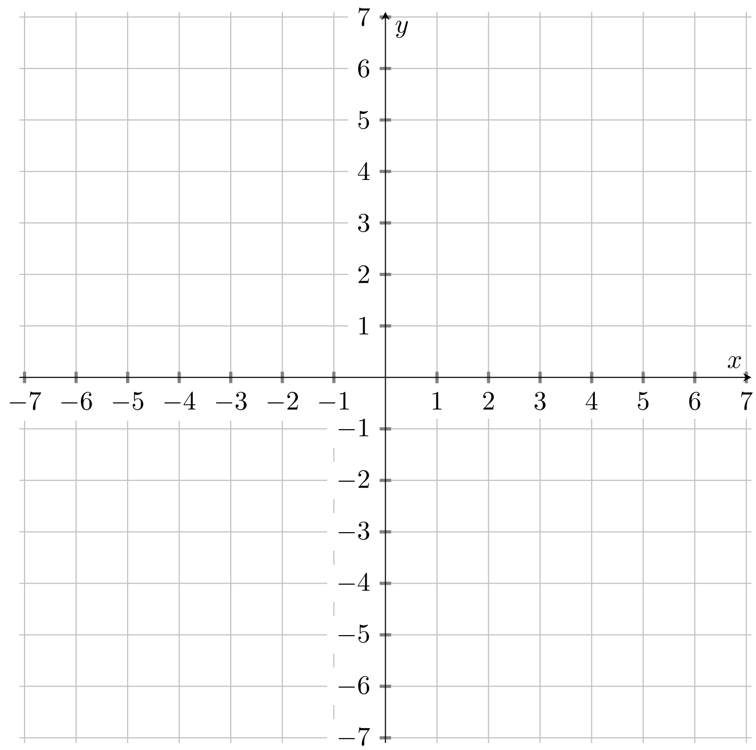
1.2.1 Inverse functions

It's often useful to think of the "opposite" of a function. If we have a function that takes some A and maps them to B, we might want to think about the "undo" function that takes B and turns them into A again. We call this the inverse function. We can define it this way:

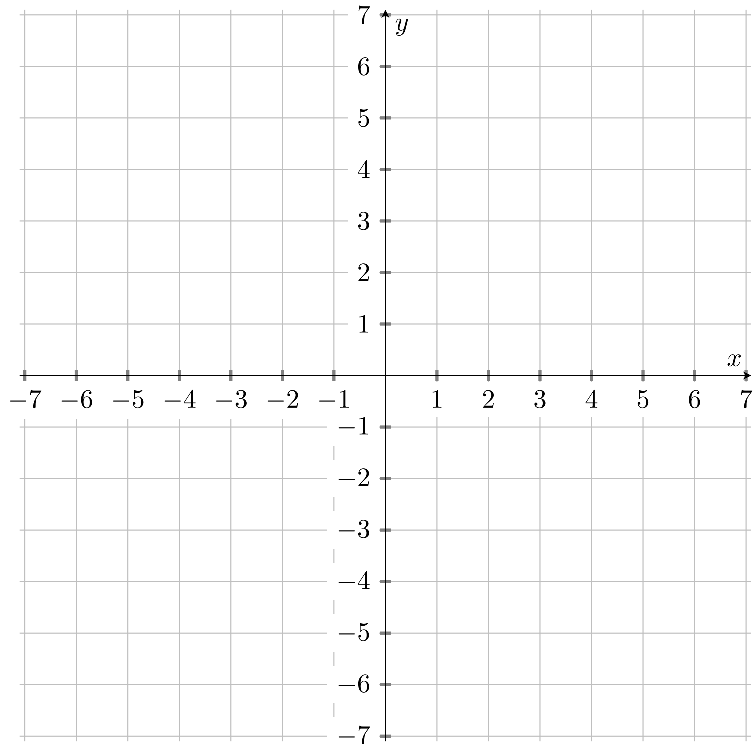
$$f^{-1}(f(x)) = x \tag{11}$$

And remember that $f(x) = y$ so this is the same as saying that $f(x) = x$ and $f(y) = y$. What are the properties of these kinds of functions? If a function f has some domain and range, or some domain and range. Then for the inverse function f^{-1} the domain and range will be switched. In other words, the output values will become the input values and vice versa. A helpfu trick is to remember that inverse functions are reflections about the line $y = x$.

Exercise 3. *Graph $f(x) = 2x + 1$ and find its inverse.*



Exercise 4. *Graph $f(x) = \sqrt{x+1}$ and find its inverse. Note that while the domain of the inverse function will be larger than the range of the initial function, it must be restricted according to what the original gives to match that function in a perfect reflection about $x = y$.*



1.2.2 Algebraic simplification practice

We often need to rearrange algebraic expressions to solve for x.

Exercise 5. $2x = 8$

Exercise 6. $\frac{x}{3} = 3$

Exercise 7. $3x + 1 = 4$

Exercise 8. $\frac{3}{2}x - 1 = x + 6$

Exercise 9. $x - 3 = \frac{1}{2}$

Exercise 10. $\frac{1}{2}x = (6 + 1)\frac{1}{7}$

Exercise 11. $x - 18 = (3 \cdot 3)2 - 1$

Exercise 12. $x + x = \frac{28}{4}$

Exercise 13. $2a = (a - 4)$

Exercise 14. $\frac{7}{x} = 14$

Exercise 15. $\frac{x-3}{2x} = 2$

Exercise 16. $\frac{118}{2} = \frac{x}{6}$

1.3 Shapes... derivation of π

Exercise 17. *What is the perimeter of a circle with radius 4?*

Exercise 18. *What is the volume of a sphere with diameter 6?*

Exercise 19. *What is the volume of a regular pyramid with side length 3?*

Exercise 20. *Imagine a sphere inside a cube. The sphere touches the cube once on all six sides. What is the volume between the shapes?*

Exercise 21. *Why is* π 3.14*? Can you come up with a proof for the value for* π*? Can you come up with a proof for the area of a circle?*

Exercise 22. *Write down all the shape equations for surface and volume for future reference... (You can look these up on the internet. Try to have about 10.)*