

## 2 Exponents

### 2.1 Tues., Feb. 27: Expansion of $b^x$

We often want to consider powers of a number where:

$$b^x = \underbrace{b \cdot b \cdot b}_{x \text{ times}} = y$$

From this idea we can derive “rules” for exponents.

$$x^0 = 1 \tag{12}$$

$$x^a \cdot x^b = x^{a+b} \tag{13}$$

$$(x^a)x^b = x^{a \cdot b} \tag{14}$$

$$\frac{x^a}{x^b} = x^{a-b} \tag{15}$$

$$\frac{x^a}{x^a} = \left(\frac{x}{x}\right)^a \tag{16}$$

$$x^{a/b} = \sqrt[b]{x^a} \tag{17}$$

**Exercise 23.** Rewrite  $81^2$  with a base of 3.

**Exercise 24.** Find  $\sqrt[4]{81}$ .

**Exercise 25.** Simplify  $\frac{1}{2^{-1}}$ .

**Exercise 26.** Find  $\sqrt[a]{x^a}$ .

**Exercise 27.** Find  $\left(\frac{x^2}{x^3}\right)^2$ .

**Exercise 28.** Find  $\left(\frac{1}{2}\right)^{-1}$ .

## 2.2 Thurs., Feb. 29: Logarithms

The logarithm language rewrites this in the format:

$$\log_b y = x \tag{18}$$

Where:

$$b^x = y \tag{19}$$

I've used the variables  $x$  and  $y$  in a consistent manner here, so you can copy between the formulas. But usually logs will be written such that the dependent variable,  $y$ , is the exponent... making them the inverse of  $b^x$ . The change of base format is very useful and helpful to consider as we understand logs:

$$\log_a x = \frac{\log_b x}{\log_b a} \tag{20}$$

Which we can prove by observing that:

$$y \log_b x = \log_b x^y \tag{21}$$

This is the power property of logs, and there are also properties for products

$$\log_a uv = \log_a u + \log_a v \tag{22}$$

...and quotients:

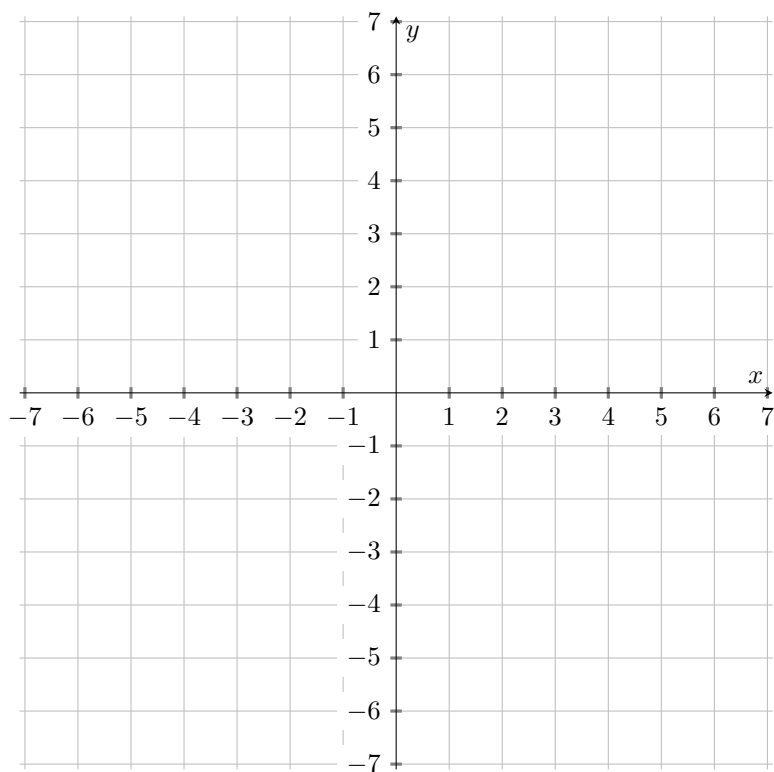
$$\log_a \frac{u}{v} = \log_a u - \log_a v \tag{23}$$

Another important identity that we should think about is:

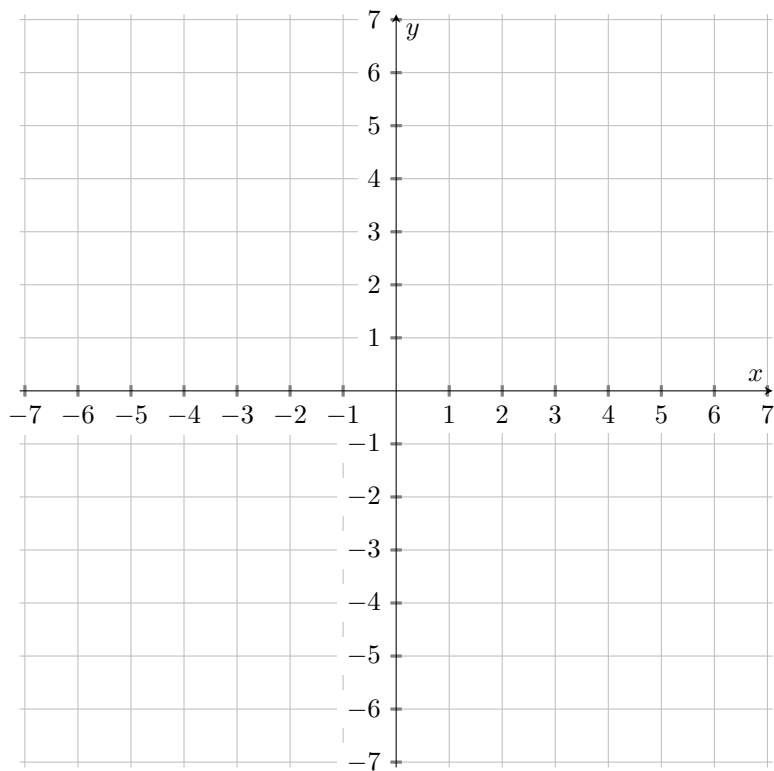
$$b^{\log_b x} = x \tag{24}$$

### 2.2.1 Putting exponents and logs together with graphs

**Exercise 29.** Graph  $x^2$ . Graph  $2^x$ .



**Exercise 30.** Graph  $\log_2 x$ . Graph  $\ln$ . Graph  $\log_{10} x$ .



**Exercise 31.** Find  $\log_{10} 1$

**Exercise 32.** Find  $\log_{10} 10^7$

**Exercise 33.** Find the approximate value of  $\log_2 10^3$  rounded to the nearest whole number. No calculator! (Think about powers of 2.)

**Exercise 34.** Find  $\log_{10} 1,000,000,000$

**Exercise 35.** Simplify  $\frac{a^3 b^7}{a^{-4} b^4}$

**Exercise 36.** Simplify  $\frac{x^{2(z+8)}}{x^{-z}}$

**Exercise 37.** Simplify  $\frac{8^3}{2^3 2^7}$

**Exercise 38.** Simplify  $(\frac{a^{1/3}}{b^{1/6}})^3$

**Exercise 39.** Find  $\log_5 \frac{1}{125}$

### 2.3 Fri, Mar. 1: Deriving $e$

There are two ways to derive  $e$ . The first is to expand our idea of the slope of a line:

$$y = mx + b \quad (25)$$

We know slope is a ratio of  $\frac{\Delta x}{\Delta y}$  and for large  $x$  and  $y$  this can be expressed:

$$\frac{y_2 - y_1}{x_2 - x_1} \quad (26)$$

And we know  $y$  is really just  $f(x)$  so this becomes:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad (27)$$

But often we want to look at non-constant slopes... *i.e.* where the size of slope depends on where you are in the function. Here it's useful to think about what happens when  $x_2$  and  $x_1$  are very close together, *i.e.* when  $x_2 \rightarrow x_1$ . This is the beginning of the calculus idea. But for now, we'll just use this idea to compute  $e^x$  to get some experience with it, and to learn about  $e$ .

**Exercise 40.** *Let's say we want to consider something that grows at  $2^t$  where  $t$  is time. What is the relationship between the growth rate of the function at a place and the value of the function at that place. Is there a function where this ratio is 1 : 1?*

Exercise 16 offers us one way to prove  $e$ . We can also think about  $e$  as doing something, then doing it more times but less each time. Let's consider a thing that's continuously growing... and measuring it at smaller units of time but more frequently, perhaps at a half unit of time. We would find that we have to multiply two terms:

$$\left(1 + \frac{1}{2}\right) \cdot \left(1 + \frac{1}{2}\right) \quad (28)$$

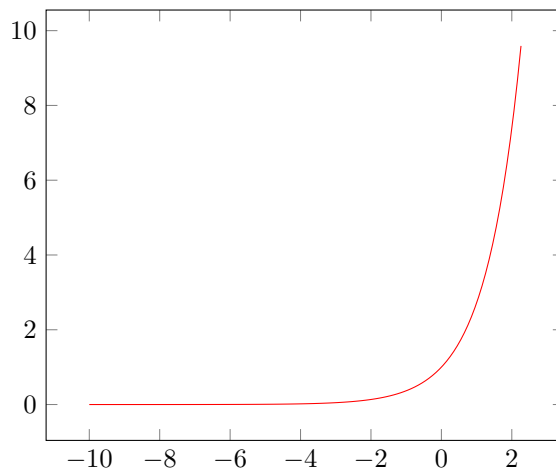
As we do this more and more, we'd get better approximations for  $e$  like:

$$\left(1 + \frac{1}{100}\right) \cdot \left(1 + \frac{1}{100}\right) \cdot \dots \cdot \left(1 + \frac{1}{100}\right) = \left(1 + \frac{1}{100}\right)^{100} \quad (29)$$

Throw this into the calculator and you'll find it's  $e$ . So in general:

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n \quad (30)$$

The graph of  $e^x$  is:



### 2.3.1 Interest

Motivating question: how often should interest compound?

If we think about a bank, we might imagine someone offering us to hold our money in exchange for paying us a small fee to use the money while we're not. Let's say this fee is 8% a year.

What would the bank owe us after a year? If we loaned 100\$, this would be:

$$100\$ \cdot 1.08 = 108\$ \quad (31)$$

But there's nothing special about a year, instead of giving us 8\$ at the end of the year, the bank could give us the  $\frac{8}{4}$ \$ after  $\frac{1}{4}$  of the year, and do so 4 times throughout the year.

**Exercise 41.** Which is better? A bank that compounds interest annually or one that compounds  $\frac{1}{4}$  of interest quarterly?

In general, the formula for the current amount of money for money earning interest at an interest rate of  $r$  that is compounded  $n$  times a year for some years  $t$  from an initial deposit of  $P$  is:

$$M = P\left(1 + \frac{r}{n}\right)^{(nt)} \quad (32)$$

**Exercise 42.** If you deposit 3,000\$ in a bank that pays 9% interest and compounds twice a year, how much money would you have after 14 years?

**Exercise 43.** Would a bank that compounds every day, or even every second, be far better than a bank that compounds annually or only a little better? Compute some examples to prove this.

Now recall our formula for  $e$ :

$$e^x = \lim_{x \rightarrow \infty} f(x) = \left(1 + \frac{1}{n}\right)^n \quad (33)$$

By visual inspection it's clear this is just the continuously compounding interest case!

Since  $e = \left(1 + \frac{1}{n}\right)^n$  for large  $n$ , we can replace  $\left(1 + \frac{1}{n}\right)^n$  with  $e$  so (32) becomes and  $r$  factors out by log rules:

$$M = Pe^{rt} \quad (34)$$