4 Systems of equations

4.1 Tues., Mar. 19: Considering what we mean by algebraic substitution

We've worked with many types of algebraic expressions. Now we can start putting them together, or thinking about where several expressions are simultaneously true. Let's work a few to warm up:

$$\begin{cases} 8 = 2x + 3y \\ -2 = x \end{cases}$$

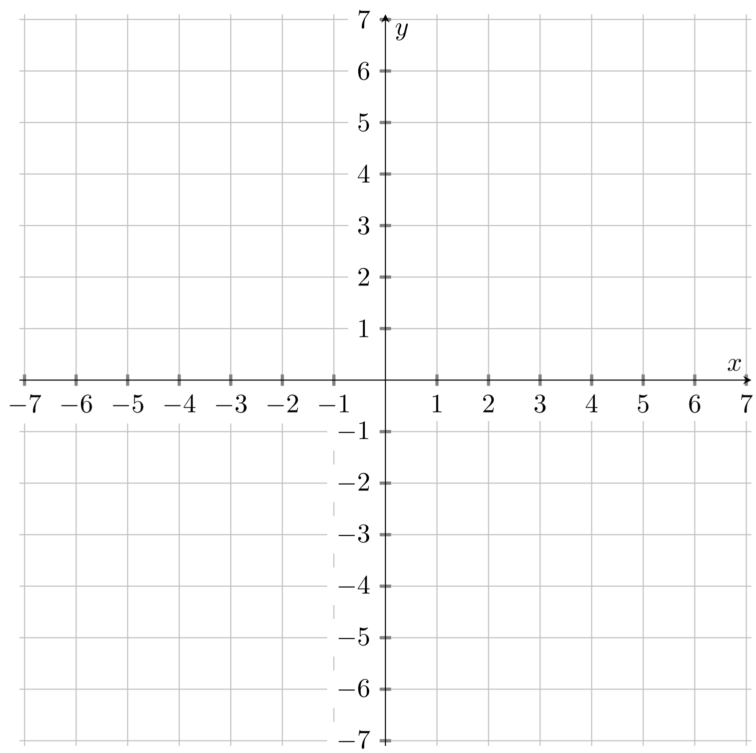
$$\begin{cases} -6x + \frac{1}{2}y = 4 \\ y = 4 \end{cases}$$

$$\begin{cases} 5x - y = 17 \\ x = y + 1 \end{cases}$$

We can solve these with substitution. But will that continue to work as the equations get more complicated? Let's consider another system of equations and solve it graphically.

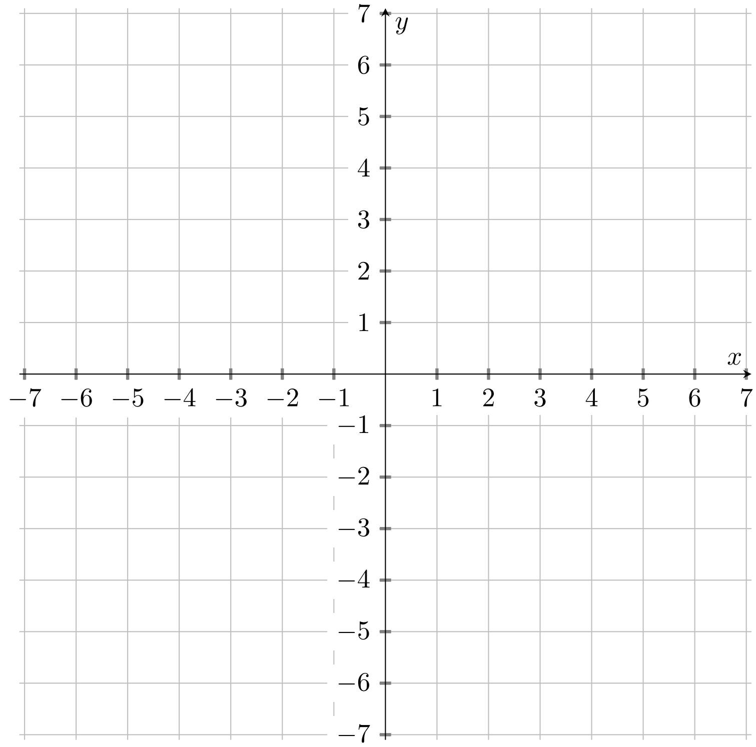
Exercise 47.

$$\begin{cases} y = -x^2 + 2x + 8 \\ y = 3x + 2 \end{cases}$$



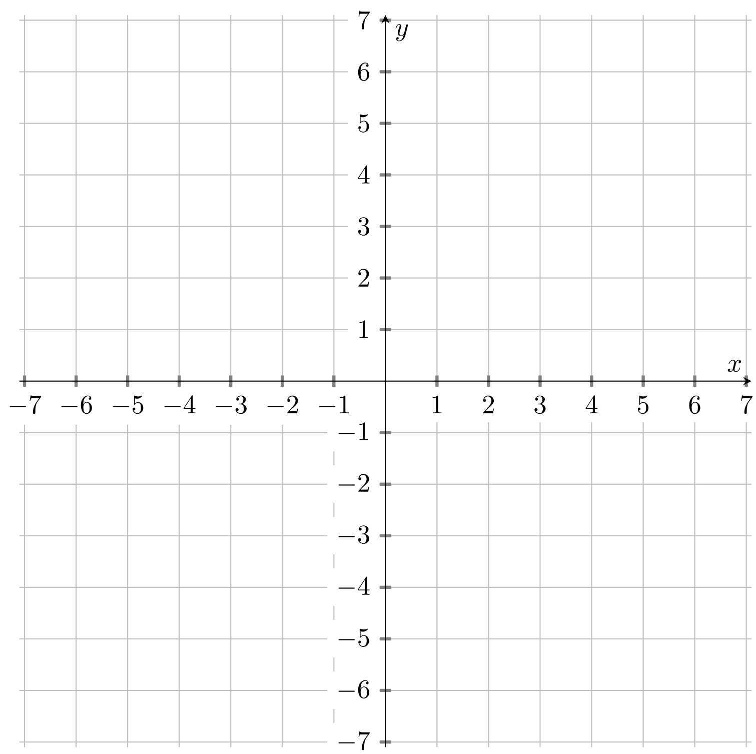
Exercise 48.

$$\begin{cases} y = x^2 - 3x - 4 \\ y = x - 8 \end{cases}$$



Exercise 49.

$$\begin{cases} y = 2x^2 + 4x + 3 \\ y = 4x - 1 \end{cases}$$



4.2 Thurs., Mar. 21: Using elimination to solve systems of equations in three unknowns

We now consider a more challenging case, where there may be three equations in three unknowns!

Exercise 50. *Let's try to solve the equation below with substitution:*

$$\begin{cases} x + y + z = -2 \\ 2x - 3y + 2z = -14 \\ 4x + 3y - z = 5 \end{cases}$$

Exercise 51. *Let's try to solve the equation below with substitution:*

$$\begin{cases} 2x - y - 2z = 3 \\ 3x + y - 2z = 11 \\ -2x - y + z = -8 \end{cases}$$

Exercise 52. *Let's try to solve the equation below with elimination:*

$$\begin{cases} x + y + z = -2 \\ 2x - 3y + 2z = -14 \\ 4x + 3y - z = 5 \end{cases}$$

Exercise 53. *Let's try to solve the equation below with elimination:*

$$\begin{cases} 2x - 3y = 7 \\ y + z = -5 \\ x + 2y + 4z = -17 \end{cases}$$

Exercise 54. *Let's try to solve the equation below with elimination:*

$$\begin{cases} 5x + y3z = 9 \\ -x - 2y - z = -16 \\ 2x + 4y + 2z = -30 \end{cases}$$

Exercise 55. *Let's try to solve the equation below with elimination:*

$$\begin{cases} x - 4y + 3z = -7 \\ 2x + 3y - 5z = 19 \\ 4x + y - z = 17 \end{cases}$$

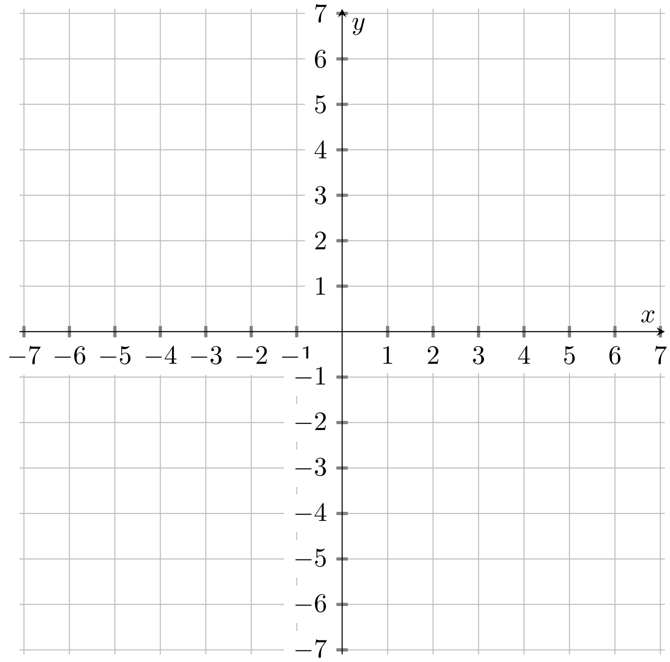
Exercise 56. *Let's try to solve the equation below with elimination:*

$$\begin{cases} 2x - 3z = 4 \\ 2x + y - 5z = -1 \\ 3y - 4z = 2 \end{cases}$$

4.2.1 Applying this to inequalities

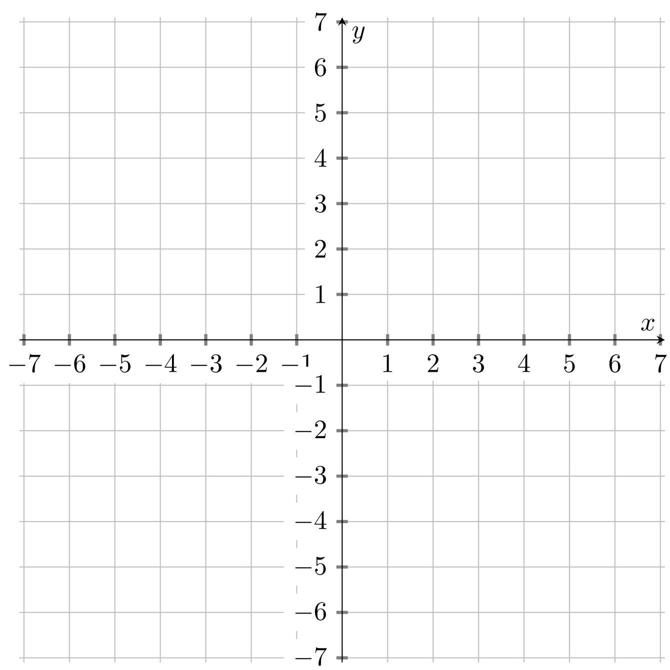
Exercise 57. *We can also think about graphing systems of inequalities. Graph the lines and shade the region defined in this system of inequalities:*

$$\begin{cases} y > x^2 - 3x + 4 \\ y < x + 1 \end{cases}$$



Exercise 58. *We can also think about graphing systems of inequalities. Graph the lines and shade the region defined in this system of inequalities:*

$$\begin{cases} y < -2x^2 - x - \frac{1}{2} \\ y > \frac{1}{4}x - 1 \end{cases}$$



4.3 Tues., Mar. 22: Motivation for a better method... the matrix

Is there a better way to do this? What if we had an equation with four unknowns... five?

Exercise 59. *Let's try to solve the equation below with substitution:*

$$\begin{cases} a+b+c+d=-1 \\ 2a-3b+2c+2d=-12 \\ 4a+3b-c-d=4 \\ 3a-4b-4c+5d=6 \end{cases}$$

So a better method is needed. We note that it's not really necessary to keep writing the variables, and that all the work we need to do can be done just with the coefficients. By grouping these into a box, called a "matrix", we can add/subtract/multiply/divide rows by each other and integers to isolate variables. The goal is to produce a diagonal matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array}\right)$$

Here are some worked examples:

Use Gaussian elimination to solve the following system of equations:

$$\begin{cases} z-3y=-6 \\ x-2y-2z=-14 \\ 4y-x-3z=5 \end{cases}$$

Solution: First write it in an augmented matrix.

$$\left(\begin{array}{ccc|c} 0 & -3 & 1 & -6 \\ 1 & -2 & -2 & -14 \\ -1 & 4 & -3 & 5 \end{array}\right) \xrightarrow{I\leftrightarrow II} \left(\begin{array}{ccc|c} 1 & -2 & -2 & -14 \\ 0 & -3 & 1 & -6 \\ -1 & 4 & -3 & 5 \end{array}\right) \xrightarrow{III+I} \left(\begin{array}{ccc|c} 1 & -2 & -2 & -14 \\ 0 & -3 & 1 & -6 \\ 0 & 2 & -5 & -9 \end{array}\right)$$

$$\xrightarrow{I-2/3II,III+2/3II} \left(\begin{array}{ccc|c} 1 & 0 & -8/3 & -10 \\ 0 & -3 & 1 & -6 \\ 0 & 0 & -13/3 & -13 \end{array}\right) \xrightarrow{III*3/-13} \left(\begin{array}{ccc|c} 1 & 0 & -8/3 & -10 \\ 0 & -3 & 1 & -6 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{I+8/3III,II-III} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & -3 & 0 & -9 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{II/-3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Thus the solution is $(-2,3,3)$.

Use Gaussian elimination to solve the following system of equations:

$$\begin{cases} x_1 - 2x_2 - 6x_3 = 5 \\ 2x_1 + 4x_2 + 12x_3 = -6 \\ x_1 - 4x_2 - 12x_3 = 9 \end{cases}$$

Solution: Using Gaussian elimination gives

$$\left(\begin{array}{ccc|c} 1 & -2 & -6 & 5 \\ 2 & 4 & 12 & -6 \\ 1 & -4 & -12 & 9 \end{array}\right) \xrightarrow{II-2I, III-I} \left(\begin{array}{ccc|c} 1 & -2 & -6 & 5 \\ 0 & 8 & 24 & -16 \\ 0 & -2 & -6 & 4 \end{array}\right)$$

$$\xrightarrow{II/8, III/-2} \left(\begin{array}{ccc|c} 1 & -2 & -6 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & -2 \end{array}\right) \xrightarrow{I+2II, III-II} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solving, we get $x_1 = 1, x_2 = -2 - 3x_3$ and x_3 can be whatever it wants. This means there are an infinite number of solutions, that the determinant of the original matrix is 0.

Exercise 60. *Let's try to solve the equation below with Gaussian elimination:*

$$\begin{cases} x + y + z = -2 \\ 2x - 3y + 2z = -14 \\ 4x + 3y - z = 5 \end{cases}$$

We make the matrix... called the "augmented" matrix when we include the values c, for $x + y + z = c$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 2 & -3 & 2 & -14 \\ 4 & 3 & -1 & 5 \end{array}\right)$$

Exercise 61. *Find conditions on a, b such that the following system has no solutions, infinitely many, and a unique solution.*

$$\begin{cases} x + ay = 2 \\ 4x + 8y = b \end{cases}$$

We want to solve the matrix:

$$\begin{pmatrix} 1 & a \\ 4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ b \end{pmatrix}$$

We know that this has a unique solution if the determinant is nonzero so we need $8 - 4a \neq 0$ or $a \neq 0$. For all $a \neq 2$ and any b this has a unique solution. But for $a = 2$ this has zero or infinite solutions... let's use the augmented matrix.

$$\left(\begin{array}{cc|c} 1 & a & 2 \\ 4 & 8 & b \end{array}\right)$$

Which when we subtract lines $II - 4I$ becomes:

$$\left(\begin{array}{cc|c} 1 & a & 2 \\ 0 & 0 & b - 8 \end{array}\right)$$

So if $b \neq 8$ the system has no solutions and if $b = 8$ there are infinitely many solutions.