3 Exponentials

3.1 Tues, Mar. 5: Expansion of b^x

We often want to consider powers of a number where:

$$b^x = \underbrace{b \cdot b \cdot b}_{x \text{ times}} = y$$

From this idea we can derive "rules" for exponents.

$$x^0 = 1 \tag{22}$$

$$x^a \cdot x^b = x^{a+b} \tag{23}$$

$$(x^a)^b = x^{a \cdot b} \tag{24}$$

$$\frac{x^a}{x^b} = x^{a-b} \tag{25}$$

And we can think about when the numerator is just 1 so that:

$$\frac{1}{x^b} = x^{-b} \tag{26}$$

$$(\frac{x}{y})^a = \frac{x^a}{y^a} \tag{27}$$

$$x^{a/b} = \sqrt[b]{x^a} \tag{28}$$

Often students (including myself when I was learning these formulas) will get stuck on the idea of a fractional power $x^{1/2}$... how could you multiply a number by itself *less* than one time? But try not to think about this and instead think how division undoes multiplication...

Exercise 72. *Consider how fractional exponents work by comparing* $2^{3/3} = \sqrt[3]{2^3}$.

3.2 Thurs, Mar. 7: Logarithms

The logarithm language rewrites this in the format:

$$\log_b y = x \tag{29}$$

Where:

$$b^x = y \tag{30}$$

I've used the variables x and y in a consistent manner here, so you can copy between the formulas. But usually logs will be written such that the dependent variable, y, is the exponent... making them the inverse of b^x. The change of base format is very useful and helpful to consider as we understand logs:

$$\log_a x = \frac{\log_b x}{\log_b a} \tag{31}$$

Which we can prove by observing that:

$$y \log_b x = \log_b x^y \tag{32}$$

This is the power property of logs, and there are also properties for products

$$\log_a uv = \log_a u + \log_a v \tag{33}$$

...and quotients:

$$\log_a \frac{u}{v} = \log_a u - \log_a v \tag{34}$$

Another important identity that we should think about is:

$$b^{\log_b x} = x \tag{35}$$

3.3 Fri, Mar. 8: Deriving e

There are two ways to derive e. The first is to expand our idea of the slope of a line:

$$y = mx + b \tag{36}$$

We know slope is a ratio of $\frac{\Delta x}{\Delta y}$ and for large x and y this can be expressed:

$$\frac{y_2 - y_1}{x_2 - x_1} \tag{37}$$

And we know y is really just $f(x)$ so this becomes:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{38}$$

But often we want to look at non-constant slopes... *i.e.* where the size of slope depends on where you are in the function. Here it's useful to think about what happens when x_2 and x_1 are very close together, *i.e.* when $x_2 \to x_1$. This is the beginning of the calculus idea. But for now, we'll just use this idea to compute e^x to get some experience with it, and to learn about e.

Exercise 73. *Let's say we want to consider something that grows at 2^t where t is time. What is the relationship between the growth rate of the function at a place and the value of the function at that place. Is there a function where this ratio is $1:1$?*

Exercise 16 offers us one way to prove e. We can also think about e as doing something, then doing it more times but less each time. Let's consider a thing that's continuously growing... and measuring it at smaller units of time but more frequently, perhaps at a half unit of time. We would find that we have to multiply two terms:

$$(1 + \frac{1}{2}) \cdot (1 + \frac{1}{2}) \tag{39}$$

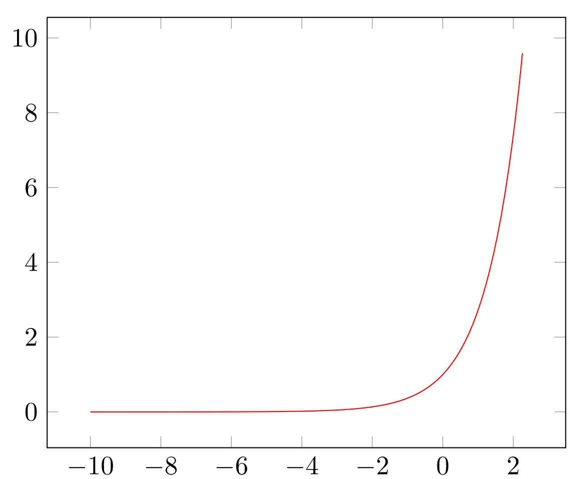
As we do this more and more, we'd get better approximations for e like:

$$(1 + \frac{1}{100}) \cdot (1 + \frac{1}{100}) \cdot ... \cdot (1 + \frac{1}{100}) = (1 + \frac{1}{100})^{100} \tag{40}$$

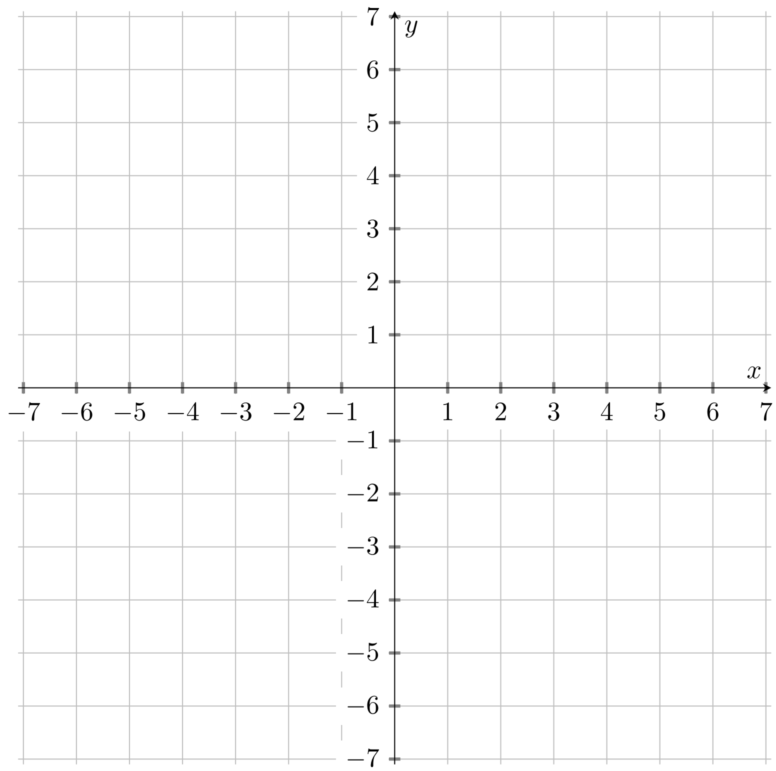
Throw this into the calculator and you'll find it's e. So in general:

$$e^x = \lim_{x \to \infty} f(x) = (1 + \frac{1}{n})^n \tag{41}$$

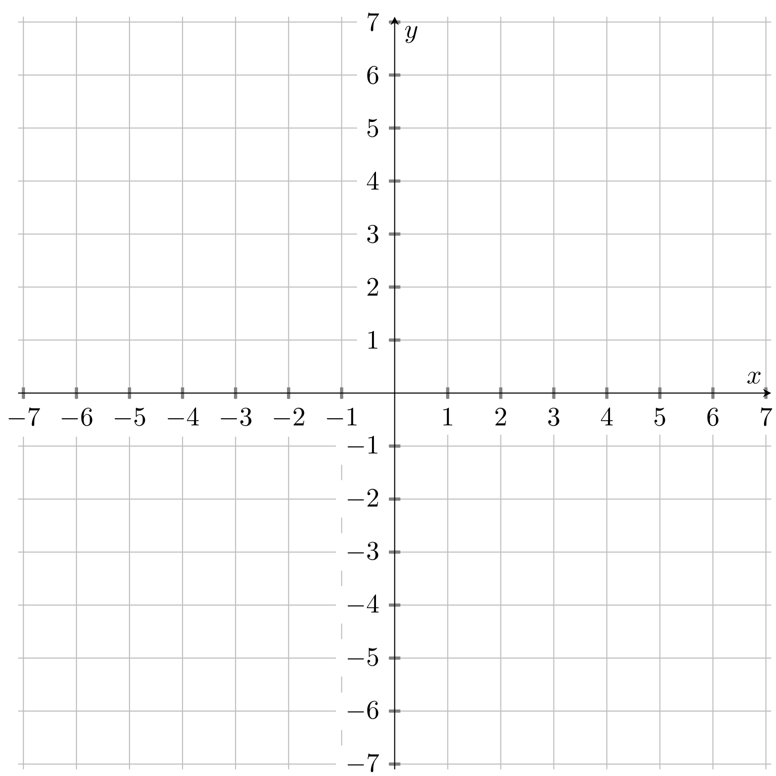
The graph of e^x is:



Exercise 74. *Graph* x^2. *Graph* 2^x.



Exercise 75. *Graph* $\log_2 x$. *Graph* ln. *Graph* $\log_{10} x$.



Exercise 76. *Find* $\log_{10} 1$

Exercise 77. *Find* $\log_{10} 10^7$

Exercise 78. *Find the approximate value of* $\log_2 10^3$ *rounded to the nearest whole number. No calculator! (Think about powers of 2.)*

Exercise 79. *Find* $e^{\ln \pi}$

Exercise 80. *Find* $\log_{10} 1{,}000{,}000{,}000$

Exercise 81. *Simplify* $\frac{a^3 b^7}{a^{-4} b^4}$

Exercise 82. *Simplify* $\frac{x^{2(z+8)}}{x^{-z}}$

Exercise 83. *Simplify* $\frac{8^3}{2^3 2^7}$

Exercise 84. *Simplify* $(\frac{a^{1/3}}{b^{1/6}})^3$

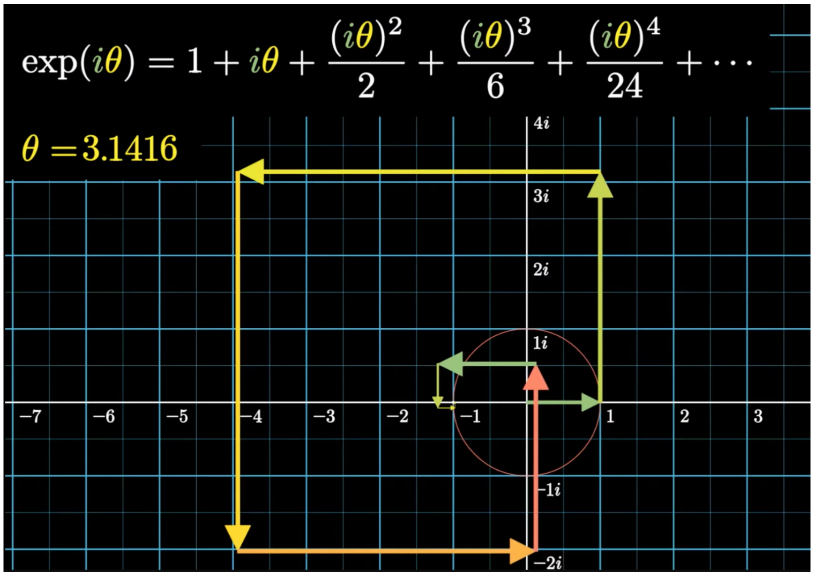
Exercise 85. *Rewrite with quotient rule for logs:* $\ln \frac{\sqrt{3x-5}}{7}$

Exercise 86. *Simplify* $\ln \frac{6}{e^2}$

Exercise 87. *Find* $\log_5 \frac{1}{125}$

Exercise 88. *Use change-of-base to rewrite this logarithm as a ratio of logarithms:* $\log_{1/2} x$*. Then graph the ratio and the original to verify equivalence.*

Exercise 89. *Assume* $\log \frac{a}{b} = m \log \frac{b}{a}$*. Find m.*



3.3.1 More about e

We learned one way to think about exponents...

$$e^x = \underbrace{e \cdot e \cdot e}_{x \text{ times}}$$

but in finding a series formula for e we also learned another way to think about exponents:

$$exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} ... \tag{42}$$

Exercise 90. *Show that when you fully expand $exp(x) \cdot exp(y)$, each term has the form $\frac{x^k y^m}{k!m!}$*

Exercise 91. *Show that when you expand $exp(x + y)$, each term has the form $\frac{1}{n!}\binom{n}{k} x^k y^{n-k}$.*

Exercise 92. *Compare results above to explain why $exp(x + y) = exp(x) \cdot exp(y)$*

3.3.2 Interest

Motivating question: how often should interest compound?

If we think about a bank, we might imagine someone offering us to hold our money in exchange for paying us a small fee to use the money while we're not. Let's say this fee is 8% a year.

What would the bank owe us after a year? If we loaned 100$, this would be:

$$100\$ \cdot 1.08 = 108\$ \tag{43}$$

But there's nothing special about a year, instead of giving us 8$ at the end of the year, the bank could give us the $\frac{8}{4}$\$ after $\frac{1}{4}$ of the year, and do so 4 times throughout the year.

Exercise 93. *Which is better? A bank that compounds interest annually or one that compounds $\frac{1}{4}$ of interest quarterly?*

In general, the formula for the current amount of money for money earning interest at an interest rate of r that is compounded n times a year for some years t from an initial deposit of P is:

$$M = P(1 + \frac{r}{n})^{(nt)} \tag{44}$$

Exercise 94. *If you deposit 3,000$ in a bank that pays 9% interest and compounds twice a year, how much money would you have after 14 years?*

Exercise 95. *Would a bank that compounds every day, or even every second, be far better than a bank that compounds annually or only a little better? Compute some examples to prove this.*

Now recall our formula for e:

$$e^x = \lim_{x\to\infty} f(x) = (1 + \frac{1}{n})^n \tag{45}$$

By visual inspection it's clear this is just the continuously compounding interest case!

Since $e = (1 + \frac{1}{n})^n$ for large n, we can replace $(1 + \frac{1}{n})^n$ with e so (32) becomes and r factors out by log rules:

$$M = Pe^{rt} \tag{46}$$