3 Things of special interest: Exponentials, Parametric Equations

3.1 Tues, Mar. 5: Expansion of b^x

We often want to consider powers of a number where:

$$b^x = \underbrace{b \cdot b \cdot b}_{x \text{ times}} = y$$

From this idea we can derive "rules" for exponents.

$$x^0 = 1 \tag{14}$$

$$x^a \cdot x^b = x^{a+b} \tag{15}$$

$$(x^a)^b = x^{a \cdot b} \tag{16}$$

$$\frac{x^a}{x^b} = x^{a-b} \tag{17}$$

And we can think about when the numerator is just 1 so that:

$$\frac{1}{x^b} = x^{-b} \tag{18}$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} \tag{19}$$

$$x^{a/b} = \sqrt[b]{x^a} \tag{20}$$

Often students (including myself when I was learning these formulas) will get stuck on the idea of a fractional power $x^{1/2}$... how could you multiply a number by itself *less* than one time? But try not to think about this and instead think how division undoes multiplication...

Exercise 15. *Consider how fractional exponents work by comparing* $2^{3/3} = \sqrt[3]{2^3}$.

3.2 Thurs, Mar. 7: Logarithms

The logarithm language rewrites this in the format:

$$\log_b y = x \tag{21}$$

Where:

$$b^x = y \tag{22}$$

I've used the variables x and y in a consistent manner here, so you can copy between the formulas. But usually logs will be written such that the dependent variable, y, is the exponent... making them the inverse of b^x. The change of base format is very useful and helpful to consider as we understand logs:

$$\log_a x = \frac{\log_b x}{\log_b a} \tag{23}$$

Which we can prove by observing that:

$$y \log_b x = \log_b x^y \tag{24}$$

This is the power property of logs, and there are also properties for products

$$\log_a uv = \log_a u + \log_a v \tag{25}$$

...and quotients:

$$\log_a \frac{u}{v} = \log_a u - \log_a v \qquad (26)$$

Another important identity that we should think about is:

$$b^{\log_b x} = x \qquad (27)$$

3.3 Fri, Mar. 8: Deriving e

There are two ways to derive e. The first is to expand our idea of the slope of a line:

$$y = mx + b \qquad (28)$$

We know slope is a ratio of $\frac{\Delta x}{\Delta y}$ and for large x and y this can be expressed:

$$\frac{y_2 - y_1}{x_2 - x_1} \qquad (29)$$

And we know y is really just $f(x)$ so this becomes:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \qquad (30)$$

But often we want to look at non-constant slopes... *i.e.* where the size of slope depends on where you are in the function. Here it's useful to think about what happens when x_2 and x_1 are very close together, *i.e.* when $x_2 \to x_1$. This is the beginning of the calculus idea. But for now, we'll just use this idea to compute e^x to get some experience with it, and to learn about e.

Exercise 16. *Let's say we want to consider something that grows at 2^t where t is time. What is the relationship between the growth rate of the function at a place and the value of the function at that place. Is there a function where this ratio is $1:1$?*

Exercise 16 offers us one way to prove e. We can also think about e as doing something, then doing it more times but less each time. Let's consider a thing that's continuously growing... and measuring it at smaller units of time but more frequently, perhaps at a half unit of time. We would find that we have to multiply two terms:

$$\left(1 + \frac{1}{2}\right) \cdot \left(1 + \frac{1}{2}\right) \qquad (31)$$
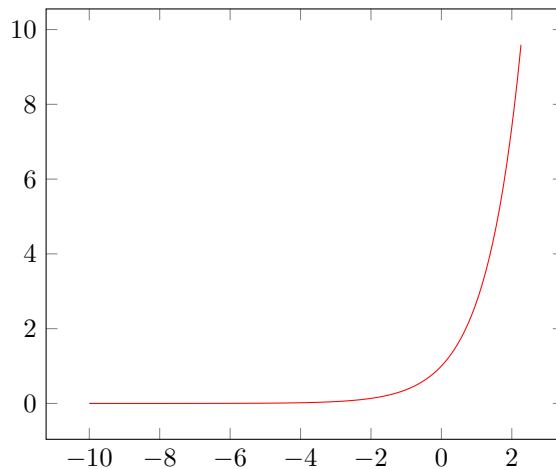
As we do this more and more, we'd get better approximations for e like:

$$\left(1 + \frac{1}{100}\right) \cdot \left(1 + \frac{1}{100}\right) \cdot ... \cdot \left(1 + \frac{1}{100}\right) = \left(1 + \frac{1}{100}\right)^{100} \qquad (32)$$
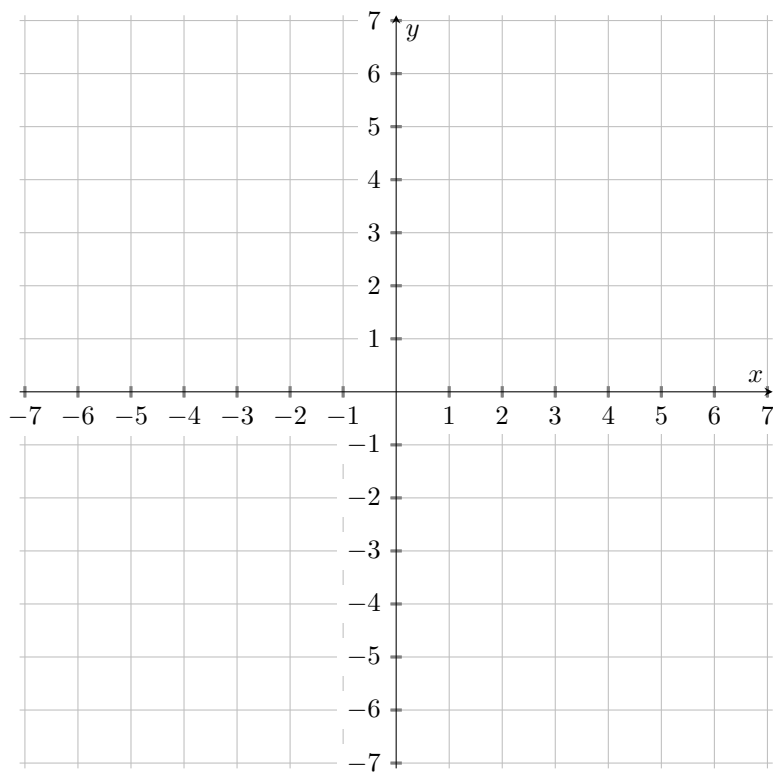
Throw this into the calculator and you'll find it's e. So in general:

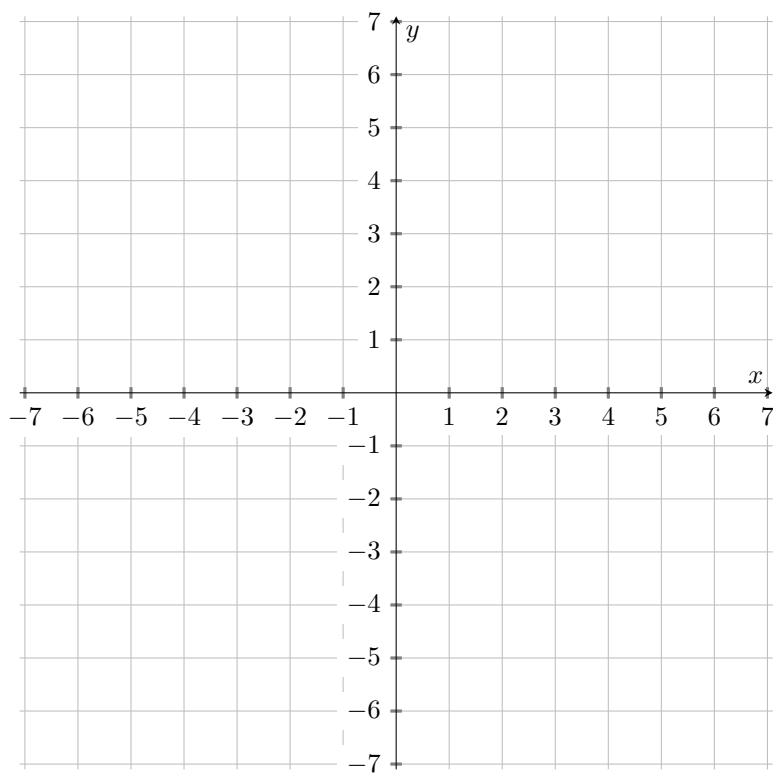$$e^x = \lim_{x \to 2} f(x) = \left(1 + \frac{1}{n}\right)^n \qquad (33)$$

The graph of e^x is:

Exercise 17. *Graph x^2. Graph 2^x.*

Exercise 18. *Graph $\log_2 x$. Graph $\ln$. Graph $\log_{10} x$.*

Exercise 19. *Find* $\log_{10} 1$

Exercise 20. *Find* $\log_{10} 10^7$

Exercise 21. *Find the approximate value of* $\log_2 10^3$ *rounded to the nearest whole number. No calculator! (Think about powers of 2.)*

Exercise 22. *Find* $e^{\ln \pi}$

Exercise 23. *Find* $\log_{10} 1,000,000,000$

Exercise 24. *Simplify* $\frac{a^3 b^7}{a^{-4} b^4}$

Exercise 25. *Simplify* $\frac{x^{2(z+8)}}{x^{-z}}$

Exercise 26. *Simplify* $\frac{8^3}{2^3 2^7}$

Exercise 27. *Simplify* $\left(\frac{a^{1/3}}{b^{1/6}}\right)^3$
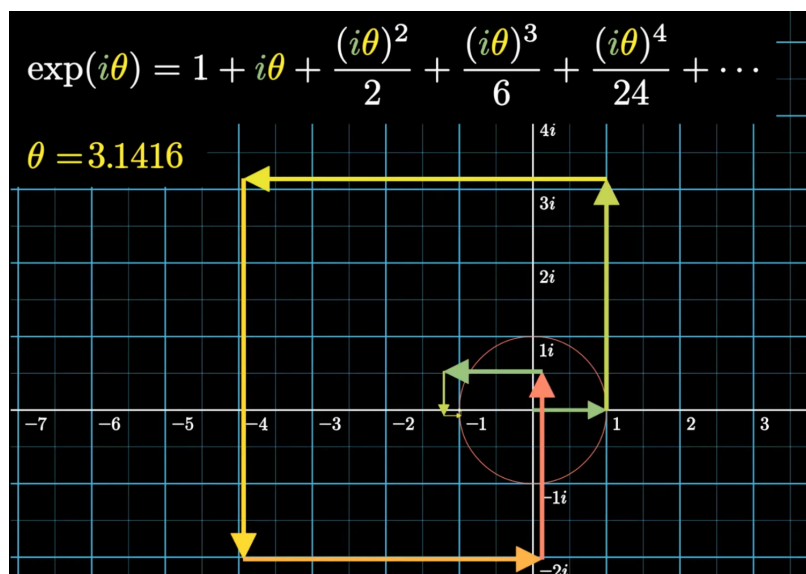
Exercise 28. *Rewrite with quotient rule for logs:* $\ln \frac{\sqrt{3x-5}}{7}$

Exercise 29. *Simplify* $\ln \frac{6}{e^2}$

Exercise 30. *Find* $\log_5 \frac{1}{125}$

Exercise 31. *Use change-of-base to rewrite this logarithm as a ratio of logarithms:* $\log_{1/2} x$. *Then graph the ratio and the original to verify equivalence.*

Exercise 32. *Assume* $\log \frac{a}{b} = m \log \frac{b}{a}$. *Find m.*

$$\exp(i\theta) = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{6} + \frac{(i\theta)^4}{24} + \cdots$$

$\theta = 3.1416$

3.4 More about e

We learned one way to think about exponents...

$$e^x = \underbrace{e \cdot e \cdot e}_{x \text{ times}}$$

but in finding a series formula for e we also learned another way to think about exponents:

$$exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}... \tag{34}$$

Exercise 33. *Show that when you fully expand $exp(x) \cdot exp(y)$, each term has the form $\frac{x^k y^m}{k! m!}$*

Exercise 34. *Show that when you expand $exp(x + y)$, each term has the form $\frac{1}{n!}\binom{n}{k} x^k y^{n-k}$.*

Exercise 35. *Compare results above to explain why $exp(x + y) = exp(x) \cdot exp(y)$*