Running Lecture Outline: Calculus of Variations

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Spring 2024

Purpose of class

Physics is local. This means that things tend to change other things around them and do so in predictable, if very complicated, ways. Calculus is a rigorous way of thinking about such changes. By giving engineers, economists, and you—if you want to know—the ability to predict and control physical systems, calculus, more than biology or the social sciences, is responsible for creating the modern world of the last 300 years. AI now accomplishes many of these goals of prediction and simulation, but it works together with calculus and understanding both is important. On a device as accessible as your laptop computer, you can simulate and predict non-trivial systems and even change those systems if you want.

Calculus also develops important ideas about functions and numbers, and as a well-constructed system of thought it is useful in developing your own ability to think about things consistently and energetically, whatever problems you choose to put those skills towards!

Grades

There will be a weekly problem set. Problem sets will come out on the first day of class in a given week and be due the first class day of the following week. We will structure our in-class time around learning to do the exercises in these problem sets together. This will be the only work that is taken for a grade. But the problems will be difficult to work without practicing in class, so attendance is strongly encouraged. There will be 12 problem sets and I will drop 2. The grading will be on a curve so that $grade = \frac{\%\ correct}{\%\ max\ correct}$. I will also grade by effort if you prefer in this manner:

A	Only attainable through > 90% correct answers on the curve
B	Strong, honest effort no matter % of problems correct
C	Very little effort
F/0	No effort/not turned in

Exceptions and other policies

Late homework can be turned in with a letter grade penalty per week and we'll find a way to work with any reasonable medical/personal issues on the honor system in a no-stress way that emphasizes learning rather than rules. I'm very open to suggestions on course content and pace. If you feel that the class is not working for you, we can try *any* combination of online/self-paced instruction or change the way class works as long as some reasonable work and learning is happening. Bring these issues to me and KAPS faculty/administration early and often. My school e-mail is *theise@kapschool.org*.

Contents

1 Functions

1.1 Tues, Feb. 20: Domain and co-domain, graphing solutions to pairs of linear equations

A function is a defined relationship between two things. These can be inputs/outputs, two numbers, a number and an idea, etc. The first thing is called the "domain" and the second thing is the "co-domain":

$$F : A \to B \tag{1}$$

The function is said to be one-to-one (injective) if every $a \in A$ maps to a unique b ∈ B, *i.e.* no two $a \in A$ map to the same $b \in B$, and "onto" if for every $b \in B$ there is an $a \in A$ that maps to it. The former function is said to be an "injection" and the latter a "surjection". A function that is both is a "bijection," or a correspondence.

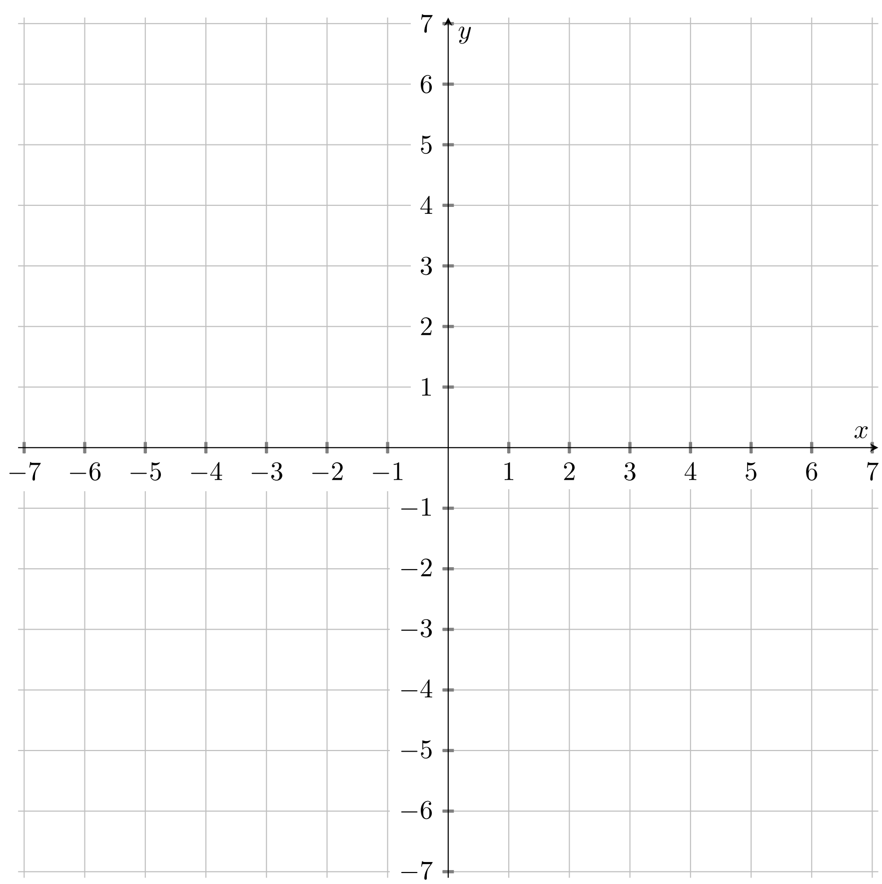
Exercise 1. *For the counting numbers* $1, 2, 3, ...,$ *not including halves or fractional numbers, name a function that is one-to-one, onto, and both one-to-one and onto.*

The "linear" or line function is fundamental for relating two numbers with a constant relationship in the Cartesian plane. You probably know about the Cartesian plane from algebra... it's just the $x - y$ coordinate graph.

Pairs of these functions can be solved visually with graphing or with algebraic rearrangement. Each pair of linear equations will have exactly one solution (x, y) that satisfies both equations, and this point will be the intersection of the two lines, whereas many (x, y) will solve a single linear equation in the form:

$$y = f(x) = mx + b \tag{2}$$

Exercise 2. *Graph the points* $(6, 7)$ *and* $(-4, -8)$*. Find the slope of the line that runs through these points and its equation. Find another point on this line. Find its intercepts. Find the equation of a line perpendicular to this line, and the equation for a line parallel to it.*



It can be helpful to think about where the equation $y = mx + b$ comes from:

$$y = \underbrace{m}_{\text{the ratio of x and y}} x + \underbrace{b}_{\text{a constant value}}$$

It can be helpful to think about where m comes from:

$$\frac{y}{x} = \frac{y}{x} \to y = (\underbrace{\frac{y}{x}}_{\text{m}})x \tag{3}$$

We know slope is a ratio of $\frac{\Delta x}{\Delta y}$ and for any x and y this can be expressed:

$$\frac{y_2 - y_1}{x_2 - x_1} \tag{4}$$

1.1.1 Graphing a line and finding the equation of a line with any two points that lie on it

We can draw the graph for a line equation $y = mx + b$ by:

- identifying the b-value on the y-axis... (hint: this is found by imagining x to be 0)
- drawing from that point the line slope given by m

We can also find a line equation from any two points on the graph $(x_2, y_2)$and (x_1, y_1) by reversing the process:

- finding m, or the slope with $\frac{y_2 - y_2}{x_2 - x_1}$
- substituting one of our points for x and y and the value we just found for m to solve for b

We can also find the distance between any two points in a graph with the help of our old friend, Pythagoras.

Exercise 3. *Find the distance between the points* $(-2, -3)$ *and* $(3, 5)$ *with the Pythagorean theorem. Write this down for your notes as the "distance formula". Find their midpoint. Write down the midpoint formula for your notes. (Hint: it is like taking an average.)*

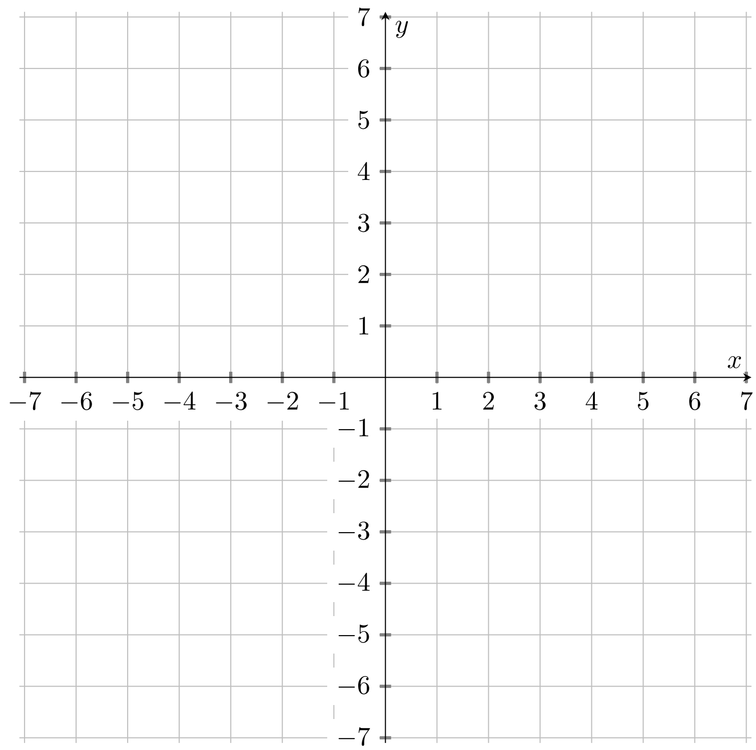
1.1.2 Inverse functions

It's often useful to think of the "opposite" of a function. If we have a function that takes some A and maps them to B, we might want to think about the "undo" function that takes B and turns them into A again. We call this the inverse function. We can define it this way:

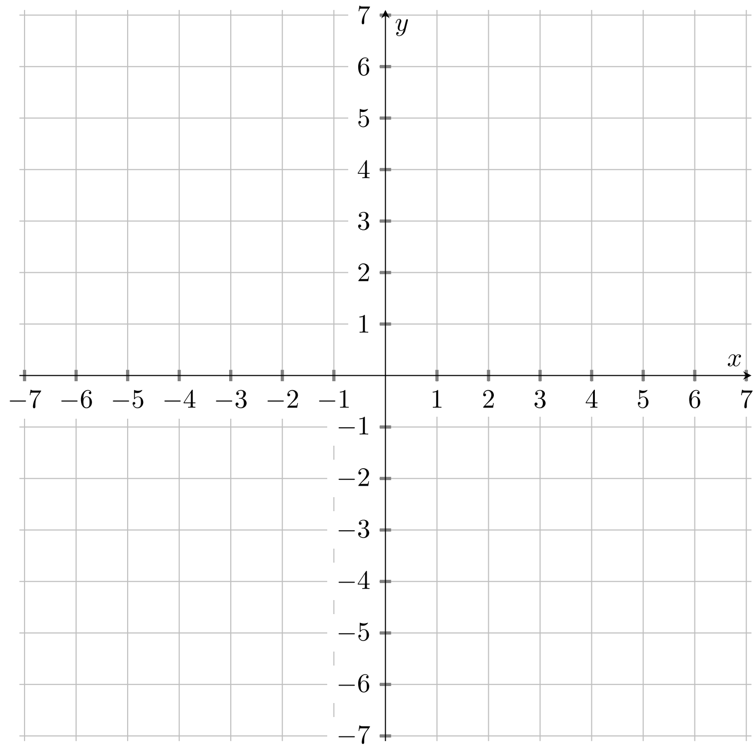
$$f^{-1}(f(x)) = x \tag{5}$$

And remember that $f(x) = y$ so this is the same as saying that $f(x) = x$ and $f(y) = y$. What are the properties of these kinds of functions? If a function f has some domain and range then for the inverse function f^{-1} the domain and range will be switched. In other words, the output values will become the input values and vice versa. A helpful trick is to remember that inverse functions are reflections about the line $y = x$.

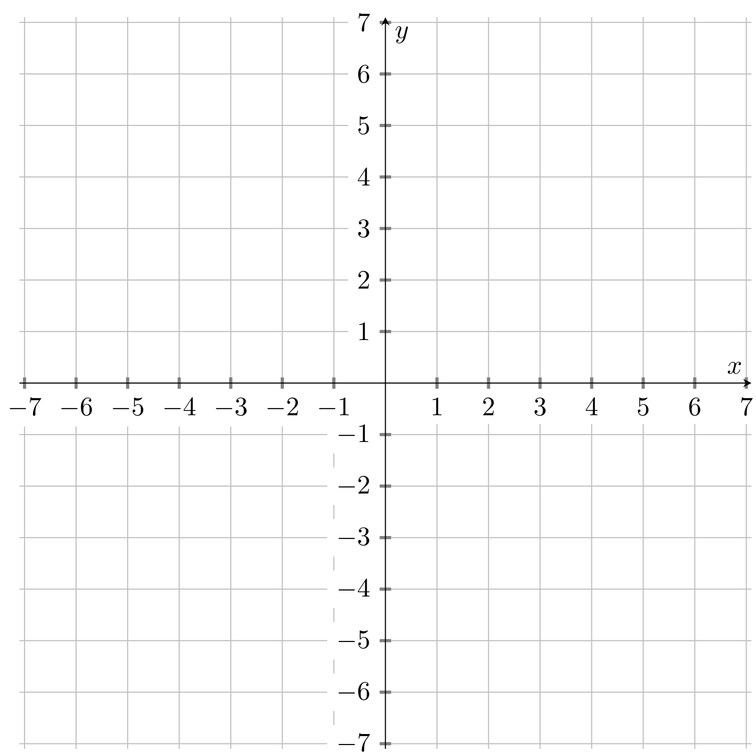
Exercise 4. *Graph $f(x) = 2x + 1$ and find its inverse.*



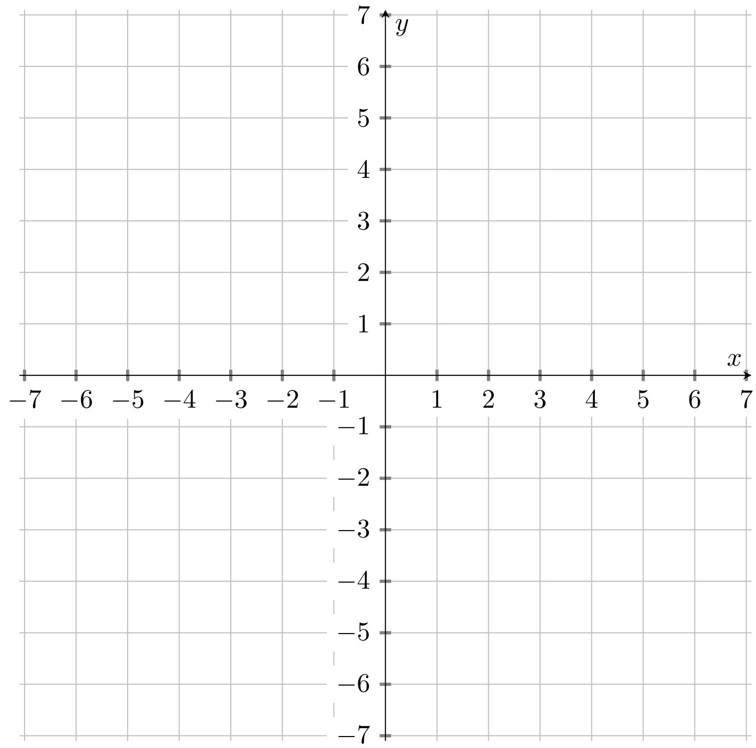
Exercise 5. *Graph* $f(x) = \sqrt{x+1}$ *and find its inverse. Note that while the domain of the inverse function will be larger than the range of the initial function, it must be restricted according to the original to match that function in a perfect reflection about* $x = y$.



Exercise 6. *Graph* $f(x) = 1/x$. *Where is the function undefined? What is the behavior of the function at its big values of* x *and* y. *Find the inverse of this function. (This might be a trick question. What is the way to find a function's inverse in general?)*



Exercise 7. *Graph* $x^2 - 5x - 6 = 0$. *Then solve. Solve first with guessing, then with completing the square, then with the quadratic formula.*



Exercise 8. *Write in standard form using complex conjuates:*

$$\frac{2+3i}{4-2i} \tag{6}$$

Exercise 9. *Write in standard form:*

$$12i + \frac{25}{5i^2} + i^7 \tag{7}$$

1.2 Thurs, Feb. 22: Shapes of common functions

Consider the functions:

$$f(x) = c \tag{8}$$

$$f(x) = x^2 \tag{9}$$

$$f(x) = x^3 \tag{10}$$

$$x^2 + y^2 = 1 \tag{11}$$

(Note: You may have heard of the vertical line test. Does this pass the vertical line test? If not a function, what is it? Are there other ways we could describe a circle that would pass the test?)

$$f(x) = |x| \tag{12}$$

$$f(x) = \sqrt{x} \tag{13}$$

$$f(x) = \frac{1}{x} \tag{14}$$

$$f(x) = \frac{1}{x^2} \tag{15}$$

What is the basic shape of each of these functions? How do their compositions with other functions affect their shapes?

Exercise 10. *Let's invent some points and functions and work on translating between the two. Use the method of* 2^n *steps to find points on the graph with the brute force method.*

1.3 Fri, Feb. 23: Shapes... derivation of π

Exercise 11. *What is the perimeter of a circle with radius 4?*

Exercise 12. *What is the volume of a sphere with diameter 6?*

Exercise 13. *What is the volume of a regular pyramid with side length 3?*

Exercise 14. *Imagine a sphere inside a cube. The sphere touches the cube once on all six sides. What is the volume between the shapes?*

Exercise 15. *Why is* π 3.14*? Can you come up with a proof for the value for* π*? Can you come up with a proof for the area of a circle?*

Exercise 16. *Write down all the shape equations for surface and volume for future reference... (You can look these up on the internet. Try to have about 10.)*

2 Exponentials

2.1 Tues, Mar. 5: Expansion of b^x

We often want to consider powers of a number where:

$$b^x = \underbrace{b \cdot b \cdot b}_{x \text{ times}} = y$$

From this idea we can derive "rules" for exponents.

$$x^0 = 1 \tag{16}$$

$$x^a \cdot x^b = x^{a+b} \tag{17}$$

$$(x^a)^b = x^{a \cdot b} \tag{18}$$

$$\frac{x^a}{x^b} = x^{a-b} \tag{19}$$

And we can think about when the numerator is just 1 so that:

$$\frac{1}{x^b} = x^{-b} \tag{20}$$

$$(\frac{x}{y})^a = \frac{x^a}{y^a} \tag{21}$$

$$x^{a/b} = \sqrt[b]{x^a} \tag{22}$$

Often students (including myself when I was learning these formulas) will get stuck on the idea of a fractional power $x^{1/2}$... how could you multiply a number by itself *less* than one time? But try not to think about this and instead think how division undoes multiplication...

Exercise 17. *Consider how fractional exponents work by comparing* $2^{3/3} = \sqrt[3]{2^3}$.

2.2 Thurs, Mar. 7: Logarithms

The logarithm language rewrites this in the format:

$$\log_b y = x \tag{23}$$

Where:

$$b^x = y \tag{24}$$

I've used the variables x and y in a consistent manner here, so you can copy between the formulas. But usually logs will be written such that the dependent variable, y, is the exponent... making them the inverse of b^x. The change of base format is very useful and helpful to consider as we understand logs:

$$\log_a x = \frac{\log_b x}{\log_b a} \tag{25}$$

Which we can prove by observing that:

$$y \log_b x = \log_b x^y \tag{26}$$

This is the power property of logs, and there are also properties for products

$$\log_a uv = \log_a u + \log_a v \tag{27}$$

...and quotients:

$$\log_a \frac{u}{v} = \log_a u - \log_a v \tag{28}$$

Another important identity that we should think about is:

$$b^{\log_b x} = x \tag{29}$$

2.3 Fri, Mar. 8: Deriving e

There are two ways to derive e. The first is to expand our idea of the slope of a line:

$$y = mx + b \tag{30}$$

We know slope is a ratio of $\frac{\Delta x}{\Delta y}$ and for large x and y this can be expressed:

$$\frac{y_2 - y_1}{x_2 - x_1} \tag{31}$$

And we know y is really just $f(x)$ so this becomes:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{32}$$

But often we want to look at non-constant slopes... *i.e.* where the size of slope depends on where you are in the function. Here it's useful to think about what happens when x_2 and x_1 are very close together, *i.e.* when $x_2 \to x_1$. This is the beginning of the calculus idea. But for now, we'll just use this idea to compute e^x to get some experience with it, and to learn about e.

Exercise 18. *Let's say we want to consider something that grows at 2^t where t is time. What is the relationship between the growth rate of the function at a place and the value of the function at that place. Is there a function where this ratio is $1:1$?*

Exercise 16 offers us one way to prove e. We can also think about e as doing something, then doing it more times but less each time. Let's consider a thing that's continuously growing... and measuring it at smaller units of time but more frequently, perhaps at a half unit of time. We would find that we have to multiply two terms:

$$(1 + \frac{1}{2}) \cdot (1 + \frac{1}{2}) \tag{33}$$

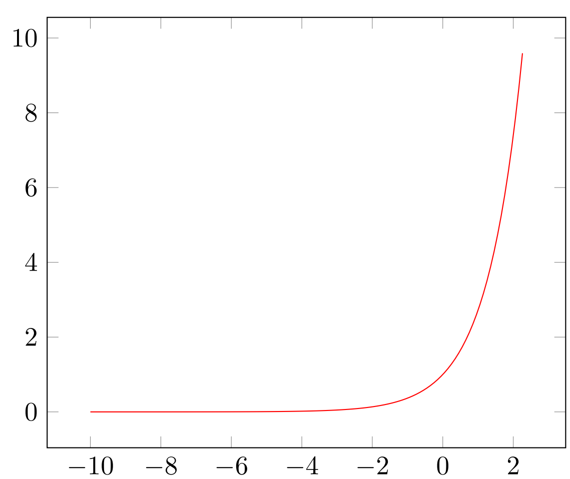
As we do this more and more, we'd get better approximations for e like:

$$(1 + \frac{1}{100}) \cdot (1 + \frac{1}{100}) \cdot ... \cdot (1 + \frac{1}{100}) = (1 + \frac{1}{100})^{100} \tag{34}$$

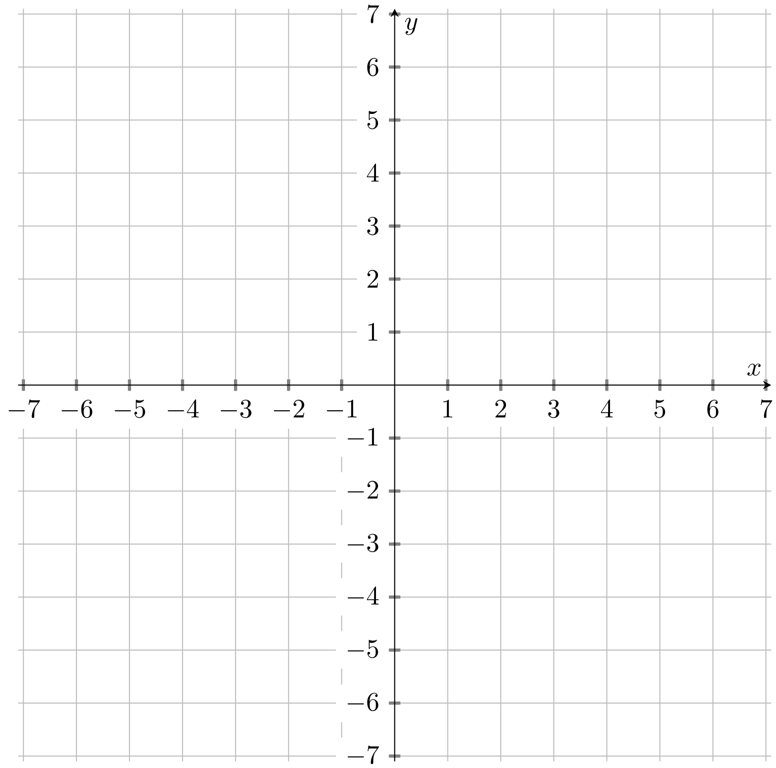
Throw this into the calculator and you'll find it's e. So in general:

$$e^x = \lim_{x\to\infty} f(x) = (1 + \frac{1}{n})^n \tag{35}$$

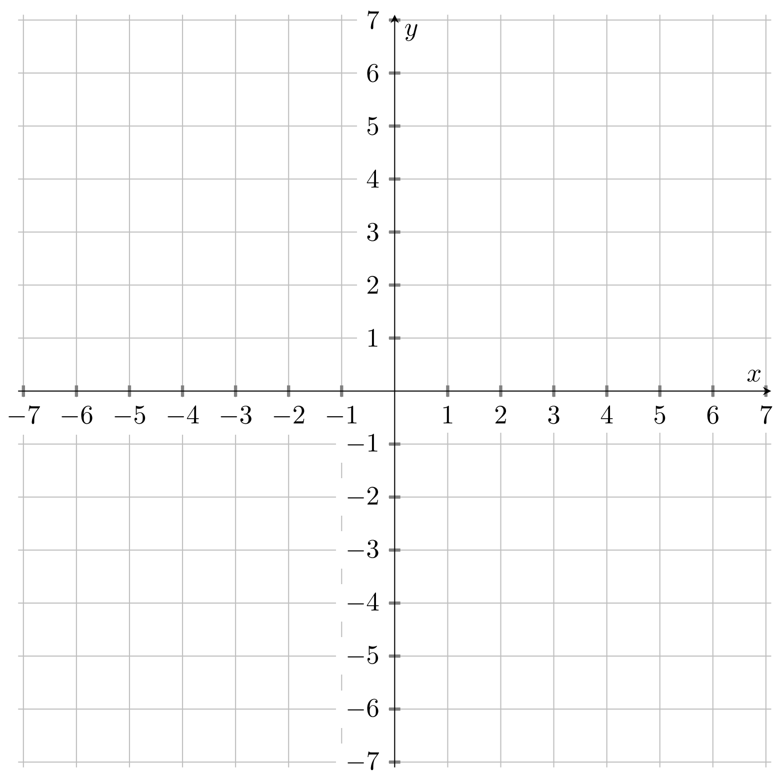
The graph of e^x is:



Exercise 19. *Graph* x^2. *Graph* 2^x.



Exercise 20. *Graph* $\log_2 x$. *Graph* ln. *Graph* $\log_{10} x$.



Exercise 21. *Find* $\log_{10} 1$

Exercise 22. *Find* $\log_{10} 10^7$

Exercise 23. *Find the approximate value of* $\log_2 10^3$ *rounded to the nearest whole number. No calculator! (Think about powers of 2.)*

Exercise 24. *Find* $e^{\ln \pi}$

Exercise 25. *Find* $\log_{10} 1,000,000,000$

Exercise 26. *Simplify* $\frac{a^3 b^7}{a^{-4} b^4}$

Exercise 27. *Simplify* $\frac{x^{2(z+8)}}{x^{-z}}$

Exercise 28. *Simplify* $\frac{8^3}{2^3 2^7}$

Exercise 29. *Simplify* $(\frac{a^{1/3}}{b^{1/6}})^3$

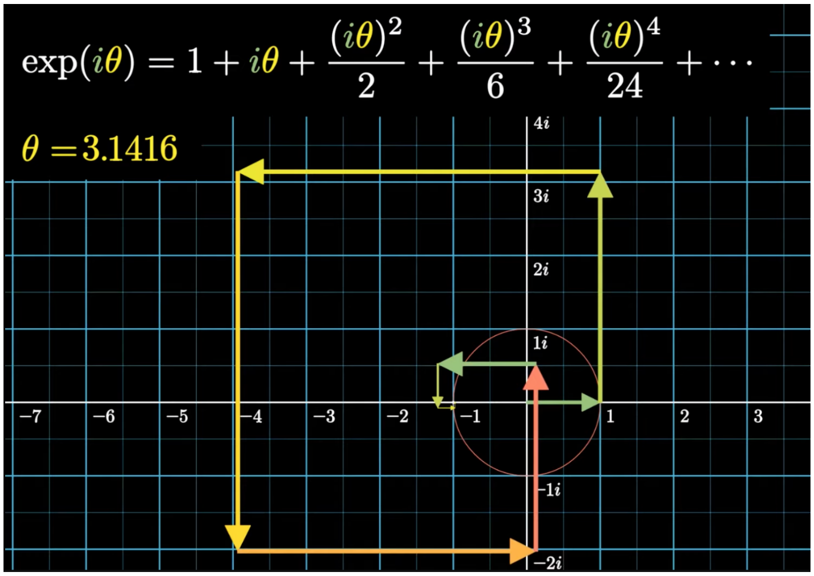
Exercise 30. *Rewrite with quotient rule for logs:* $\ln \frac{\sqrt{3x-5}}{7}$

Exercise 31. *Simplify* $\ln \frac{6}{e^2}$

Exercise 32. *Find* $\log_5 \frac{1}{125}$

Exercise 33. *Use change-of-base to rewrite this logarithm as a ratio of logarithms:* $\log_{1/2} x$. *Then graph the ratio and the original to verify equivalence.*

Exercise 34. *Assume* $\log \frac{a}{b} = m \log \frac{b}{a}$. *Find m.*



2.3.1 More about e

We learned one way to think about exponents...

$$e^x = \underbrace{e \cdot e \cdot e}_{x \text{ times}}$$

but in finding a series formula for e we also learned another way to think about exponents:

$$exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}... \tag{36}$$

Exercise 35. *Show that when you fully expand $exp(x) \cdot exp(y)$, each term has the form $\frac{x^k y^m}{k!m!}$*

Exercise 36. *Show that when you expand $exp(x+y)$, each term has the form $\frac{1}{n!}\binom{n}{k}x^k y^{n-k}$.*

Exercise 37. *Compare results above to explain why $exp(x+y) = exp(x) \cdot exp(y)$*

2.3.2 Interest

Motivating question: how often should interest compound?

If we think about a bank, we might imagine someone offering us to hold our money in exchange for paying us a small fee to use the money while we're not. Let's say this fee is 8% a year.

What would the bank owe us after a year? If we loaned 100$, this would be:

$$100\$ \cdot 1.08 = 108\$ \tag{37}$$

But there's nothing special about a year, instead of giving us 8$ at the end of the year, the bank could give us the $\frac{8}{4}$\$ after $\frac{1}{4}$ of the year, and do so 4 times throughout the year.

Exercise 38. *Which is better? A bank that compounds interest annually or one that compounds $\frac{1}{4}$ of interest quarterly?*

In general, the formula for the current amount of money for money earning interest at an interest rate of r that is compounded n times a year for some years t from an initial deposit of P is:

$$M = P(1 + \frac{r}{n})^{(nt)} \tag{38}$$

Exercise 39. *If you deposit 3,000\$ in a bank that pays 9% interest and compounds twice a year, how much money would you have after 14 years?*

Exercise 40. *Would a bank that compounds every day, or even every second, be far better than a bank that compounds annually or only a little better? Compute some examples to prove this.*

Now recall our formula for e:

$$e^x = \lim_{x \to \infty} f(x) = (1 + \frac{1}{n})^n \tag{39}$$

By visual inspection it's clear this is just the continuously compounding interest case!

Since $e = (1 + \frac{1}{n})^n$ for large n, we can replace $(1 + \frac{1}{n})^n$ with e so (32) becomes and r factors out by log rules:

$$M = Pe^{rt} \tag{40}$$

3 Trigonometry and Euclid

3.1 Tues., Mar. 19: Representative triangles

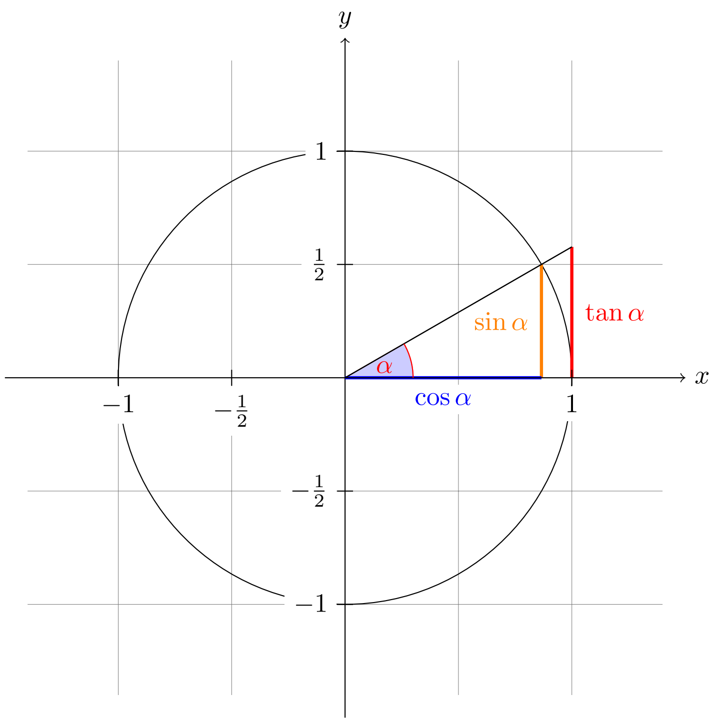
In calc we turn from degree measure to radian measure... as we learned from deriving π, there are 6.28 radius lengths in a circle and so we have the formula:

$$s = 2\pi r \tag{41}$$

So $\frac{2\pi}{2}$ is a half revolution, or 180 degrees. It's also useful to know that $\frac{\pi}{6} = 30°$, $\frac{\pi}{4} = 45°$, $\frac{\pi}{2} = 90°$. We can convert from degrees to radians easily by observing:

$$360° = 2\pi \,\text{rad} \tag{42}$$

$\frac{\pi}{180°}$ is the conversion you most often want to multiply by.



The trig functions and functions for their reciprocals are:

$$\begin{array}{ll} \sin(\theta) = \frac{opposite}{hypotenuse} & \csc(\theta) = \frac{h}{o} \\ \sin(\theta) = \frac{adjacent}{hypotenuse} & \sec(\theta) = \frac{h}{a} \\ \sin(\theta) = \frac{opposite}{adjacent} & \cot(\theta) = \frac{a}{o} \end{array}$$

Exercise 41. *Find the sides and all six trig ratios for the representative triangles $45-45-90$ and $30-60-90$ in each quadrant, i.e. when $\theta = \frac{3\pi}{4}$, $\theta = \frac{5\pi}{6}$, etc. (See the last page of this section for all the trig ratios... all you'll have to do is figure out how the signs change.)*

Exercise 42. *Compute $\sin(\theta)$ for three θ you choose... then compute $\cos(\theta - 90°)$ for the same three θ values. What is their relationship? Why?*

Exercise 43. *Find $\sin(30°)$.*

Exercise 44. *Find $\sin(45°)$.*

Exercise 45. *Find* $\cos(270°)$.

Exercise 46. *Find* $\tan(13°)$.

Exercise 47. *Find* $\sin(180°)$.

Exercise 48. *Find* $\sin(330°)$.

Exercise 49. *Find* $\cos(30°)$.

Exercise 50. *Find* $\cos(\pi)$.

Exercise 51. *Find* $\sin(\frac{\pi}{6})$.

Exercise 52. *Find* $\cos(\frac{\pi}{2})$.

Exercise 53. *Find* $\tan(\frac{5\pi}{6})$.

Exercise 54. *Find* $\cos(\frac{\pi}{3})$.

There are functions that invert the regular trig functions. Instead of taking in an angle and returning a ratio of sides, they take in a ratio of sides and return an angle:

$$\arcsin(\frac{opposite}{hypotenuse}) = \theta$$
$$\arccos(\frac{adjacent}{hypotenuse}) = \theta$$
$$\arctan(\frac{opposite}{adjacent}) = \theta$$

Exercise 55. $\arccos(\frac{1}{2}) = \theta$

Exercise 56. $\arccos(\frac{2\sqrt{2}}{2}) = \theta$

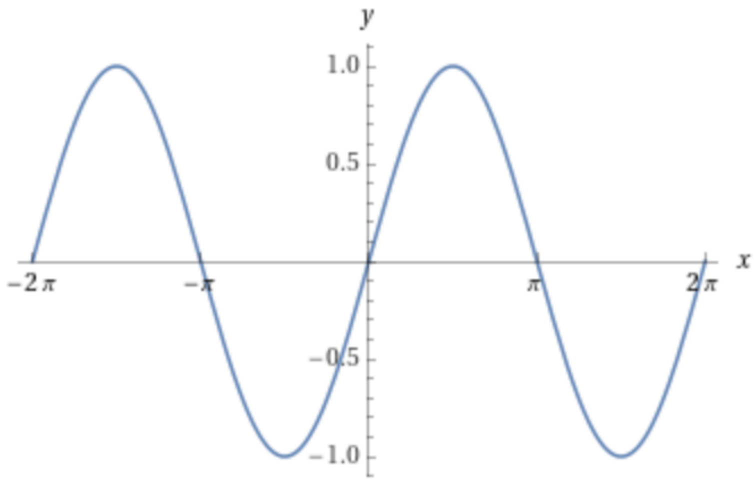
Exercise 57. $\arcsin(\frac{\sqrt{3}}{2}) = \theta$

Exercise 58. $\arctan(1) = \theta$

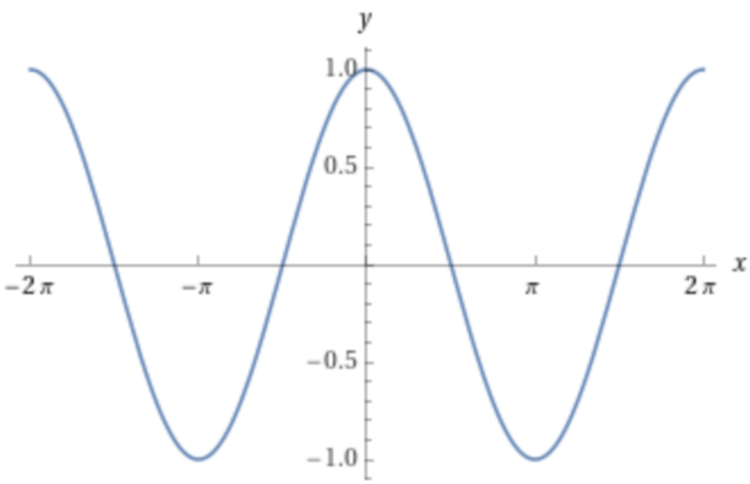
Exercise 59. $\arcsin(-1) = \theta$

3.2 Thurs., Mar. 21: Plots of trig functions

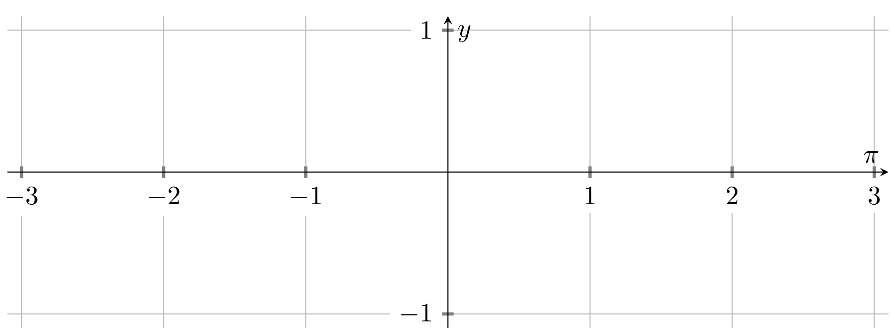
The plot of $y = sin(x)$ is:



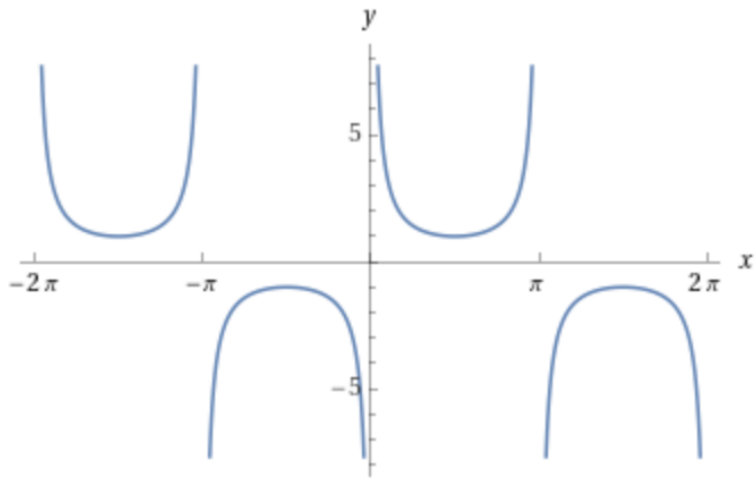
The plot of $y = cos(x)$ is:



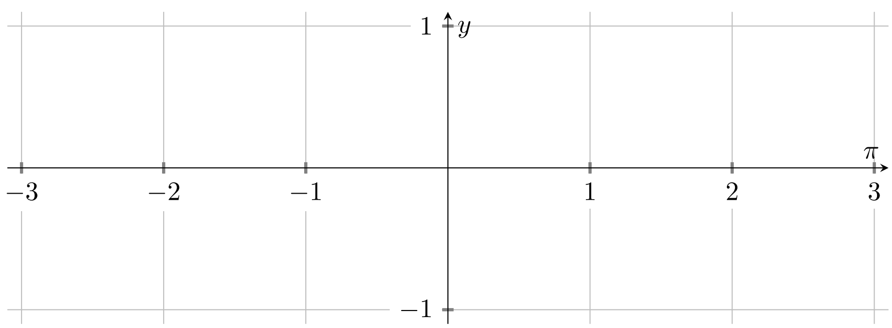
Exercise 60. *What is the plot of $y = tan(x)$? (Hint: use simple 'choose x, find y' but choose points strategically. You can use a calculator to evaluate the trig functions but don't use a graphing calculator.)*



Exercise 61. *I've given the plot of $y = csc(x)$ below. Based on what we know about these two functions, without doing any computation, what is the plot of $y = sec(x)$?*

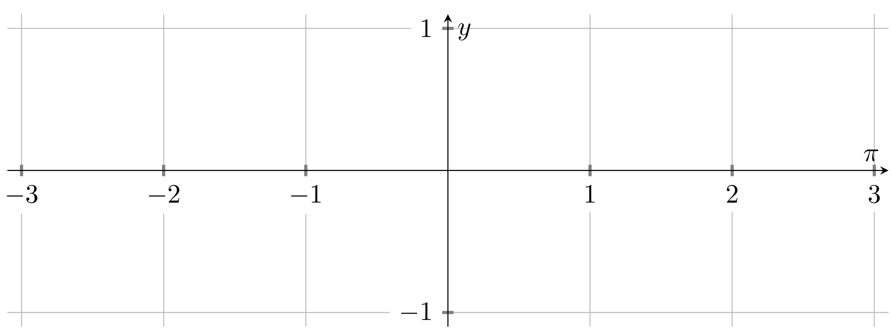


Now your turn!

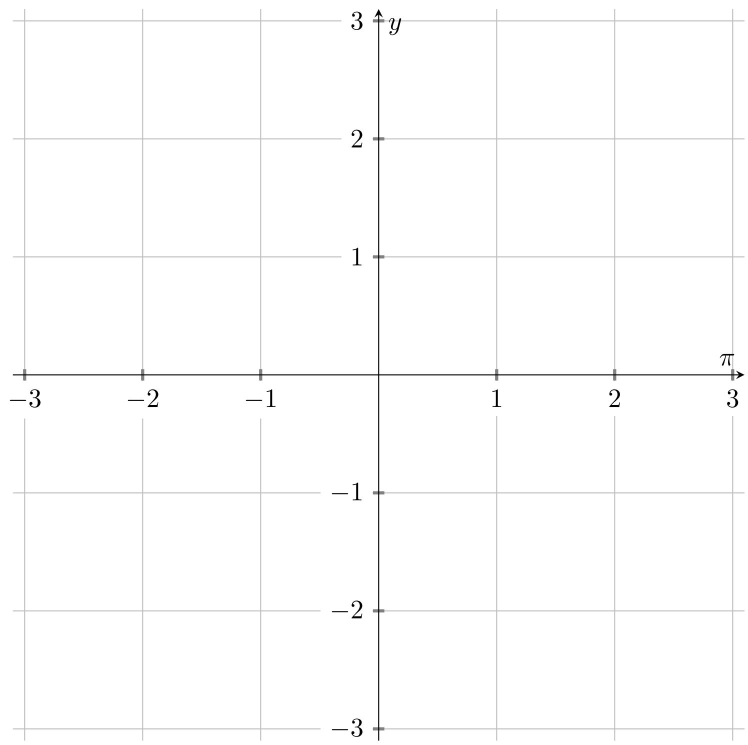


3.3 Fri., Mar. 22: Dilation and shifting of the trig functions!

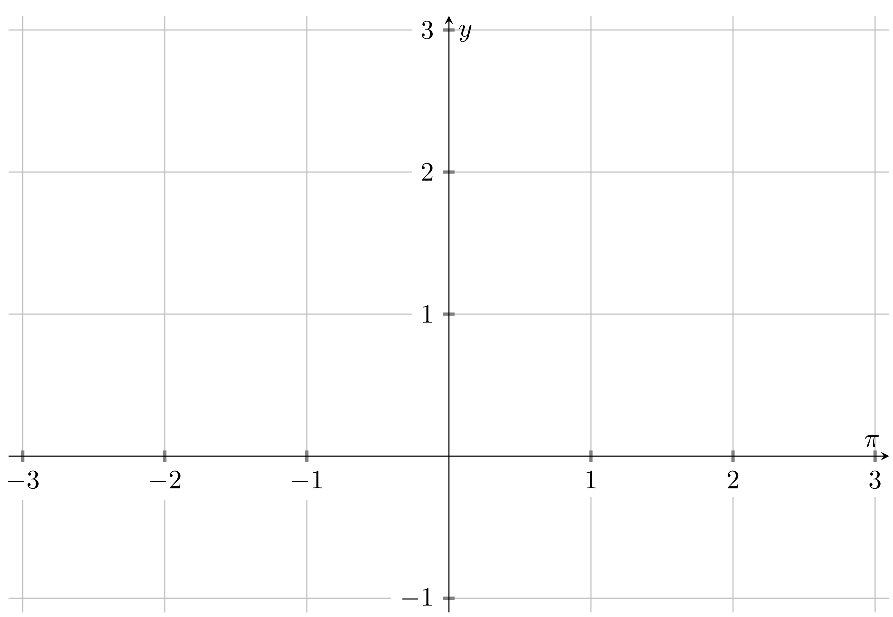
Exercise 62. *What is the plot of $y = sin(\frac{x}{3})$?*



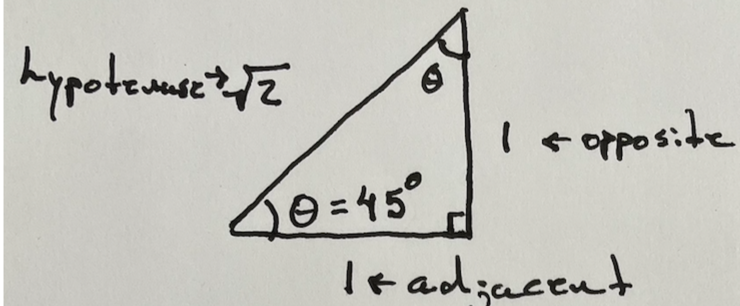
Exercise 63. *What is the plot of $y = 3sin(x)$?*

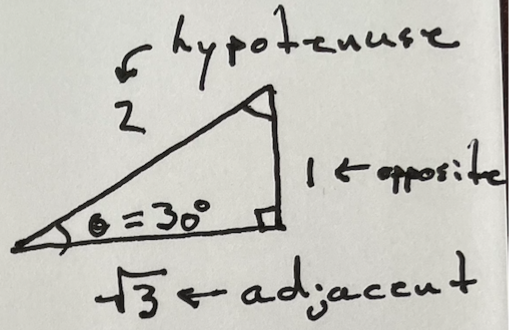


Exercise 64. *What is the plot of $y = sin(x) + 2$?*



Here are the values for several common trig functions as a reference:





So:

$$\sin(45^\circ) = \frac{o}{h} = \frac{1}{\sqrt{2}} = \frac{■\sqrt{2}}{2} = .71$$

$$\cos(45^\circ) = \frac{a}{h} = \frac{1}{\sqrt{2}} = \frac{■\sqrt{2}}{2} = .71$$

$$\tan(45^\circ) = \frac{o}{a} = \frac{1}{1} = 1$$

$$\sin(30^\circ) = \frac{o}{h} = \frac{1}{2} = .5$$

$$\cos(30^\circ) = \frac{a}{h} = \frac{\sqrt{3}}{2} = .87$$

$$\tan(30^\circ) = \frac{o}{a} = \frac{1}{\sqrt{3}} = \frac{■\sqrt{3}}{3} = .58$$

■ Note: this is a simplification to eliminate $\sqrt{\ }$ in denominator:

$$b/c \quad \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} = \frac{\sqrt{2}}{2}$$

4 Quadratics

4.1 Tues, Feb. 27: Practice with distributive property of multiplication

We note that multiplication is distributive. This means multiplying each part of a sum is the same as taking the sum and multiplying that... so:

$$A(B+C) = A \cdot B + A \cdot C \tag{43}$$

Verify that this is true with real numbers! The same is true with unknowns.

Exercise 65. *Multiply* $x(2+x)$

Exercise 66. *Multiply* $(x-1)(x+1)$

Exercise 67. *Find a solution for* $7(2+x) = 63$

Exercise 68. *Multiply* $(x+y+z)(a+b+c)$

Exercise 69. *Find some ways to factor* $x^3 + x^2 + x$

We often want to know all the *factors* of a number, these are the primes that can be multiplied to make that number. For instance, the prime factorization of 144 is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$.

Exercise 70. *Draw the tree for the prime factorization of* 81

Exercise 71. *Factor* 124

Exercise 72. *Factor* 1000

Exercise 73. *Factor* 3402

Exercise 74. *Factor* 512

Exercise 75. *Factor* 221

Exercise 76. *Factor* 3599*. Hint: any* $(x^2 - 1) = (x+1)(x-1)$*.*

What is the geometric nature of multiplication and distribution. Let's explore this with an example?

Exercise 77. *Solve* $3 \cdot (3+7)$ *by finding the area of the rectangle with sides* 3 *and* 3*... and the rectangle with sides* 3 *and* 7*. And then find the area of the rectangle with sides* 3 *and* 10*.*

In fact, any number can be expressed as a difference of squares... so for any numbers r, s you can think about a midpoint number m where r, s are $m+d, m-d$.

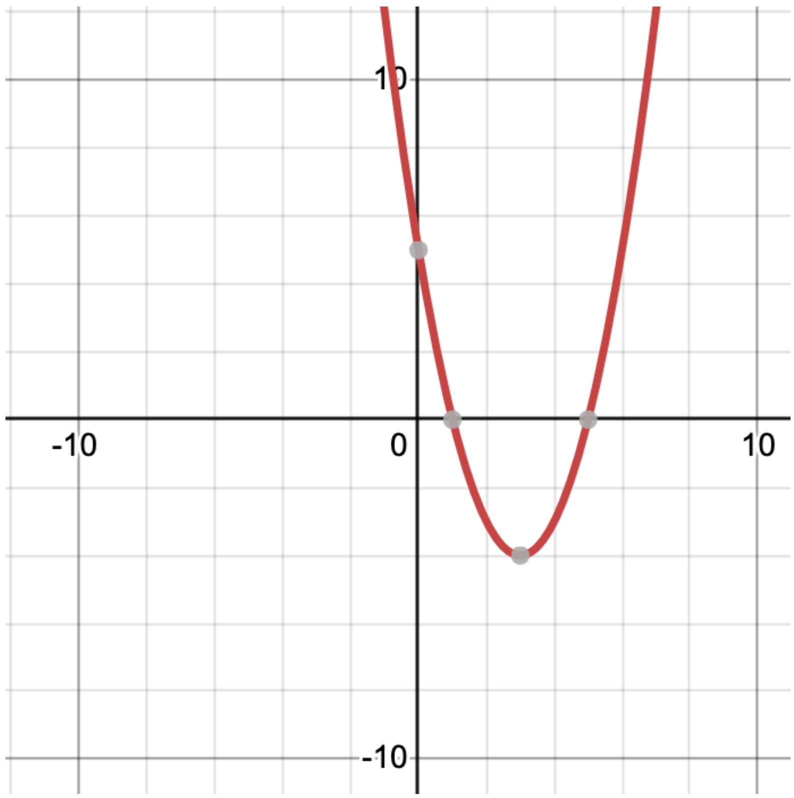
4.1.1 Guessing roots of some quadratics

We want to consider something like $ax^2 + bx + c$ and find when it equals zero. It's much easier to think about where that leading coefficient is 1 because we're left with just the simplification:

$$(x - r)(x - s) = x^2 - (r + s)x + rs \tag{44}$$

Prove that changing the leading coefficient of x doesn't change the roots of a quadratic by doing the following exercise...

Exercise 78. *I've given the plot of $x^2 - 6x + 5$ below. Now use a graphing calculator to plot $S(x^2 - 6x + 5)$ where $S = \{4, 2, -5, 10, ...\}$*



We can do a pretty good job of finding these r, s just by guessing and checking.

Exercise 79. *What is x when $x^2 + 12x + 35 = 0$*

Exercise 80. *What is x when $x^2 + x - 2 = 0$*

Exercise 81. *What is x when $x^2 - x - 12 = 0$*

Exercise 82. *What is x when $x^2 - x + 12 = 0$*

Exercise 83. *What is x when $x^2 - 100 = 0$*

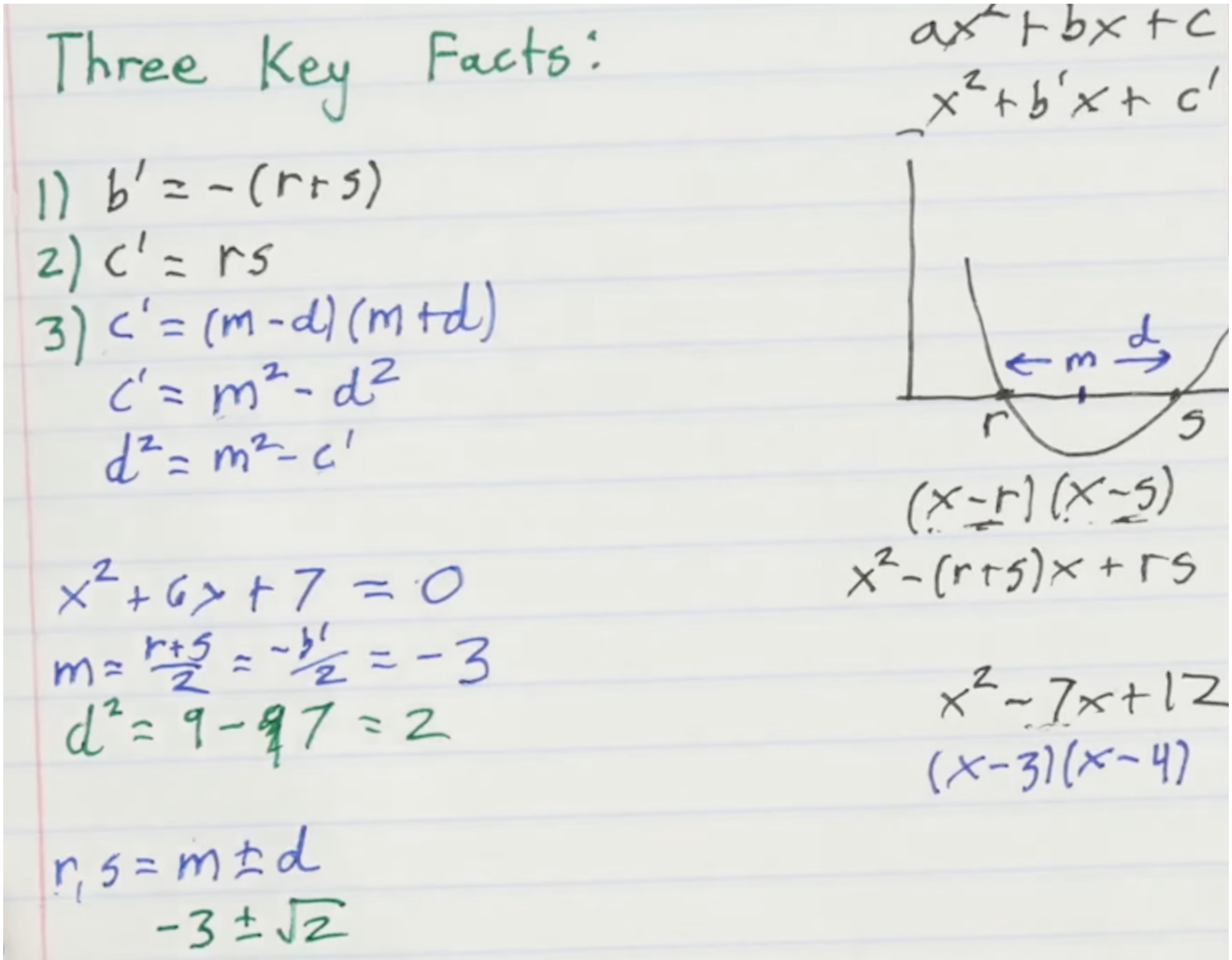
Exercise 84. *What is x when $x^2 - 6x - 55 = 0$*

4.2 Thurs., Feb. 29: The Quadratic formula

But there is a systematic way to do this work... we can think about the midpoint of our two numbers and the distance between that midpoint and the numbers! If we consider our equations:

$$ax^2 + bx + c \tag{45}$$

$$(x - r)(x - s) = x^2 - (r + s)x + rs \tag{46}$$



So practically, to find the roots of a quadratic, we take the middle coefficient (of x), divide by -2 and write it down... the roots will be that plus and minus the square root of the difference of that and the last term, c. In even simpler terms, we find $m = -\frac{b}{2}$ then take $\sqrt{m^2 - c}$.

4.2.1 Solving quadratics formally

Exercise 85. *Solve* $4a^2 + 6 = 0$

Exercise 86. *Solve* $2x^2 - 8x - 2 = 0$

Exercise 87. *Solve* $2m^2 - 3 = 0$

Exercise 88. *Solve* $3r^2 - 2r - 1 = 0$

Exercise 89. *Solve* $4n^2 - 36 = 0$

Exercise 90. *Solve* $v^2 - 4v - 5 = -8$

Exercise 91. *Solve* $2x^2 + 3x + 14 = 6$

Exercise 92. *Solve* $3x^2 + 3x - 4 = 7$

Exercise 93. *Solve* $7x^2 + 3x - 16 = -2$

Exercise 94. *Solve* $2x^2 + 6x - 16 = 4$

Exercise 95. *Solve* $3x^2 + 3x = -3$

Exercise 96. *Solve* $2x^2 = -7x + 49$

Exercise 97. *Solve* $5x^2 = 7x + 7$

Exercise 98. *Solve* $8x^2 = -3x - 8$

Exercise 99. *Solve* $2x^2 + 5x = -3$

Exercise 100. *Solve* $4x^2 - 64 = 0$

Exercise 101. *Solve* $4x^2 + 5x - 36 = 3x^2$

4.3 Fri, Mar. 1: Polynomials and the general form of quadratics

Polynomials are important in mathematics and life. It's worth thinking about why polynomials appear so often... I don't expect that you fully understand, but I do want us to spend some time thinking about how they relate important ideas about space and distance. Does this connection seem familiar? It is related to the two interpretations (graphic, algebraic) of linear equations in the form $y = mx + b$.

> Why are quadratic polynomials useful? There seem to be two different but interacting reasons. The first is that quadratic functions of a real variable are always either convex or concave and therefore have a unique maximum or minimum. The second is that quadratic functions are intimately related to bilinear forms and therefore can be accessed using linear algebra. —Paul Siegel, *Mathoverflow*

Polynomials have the form:

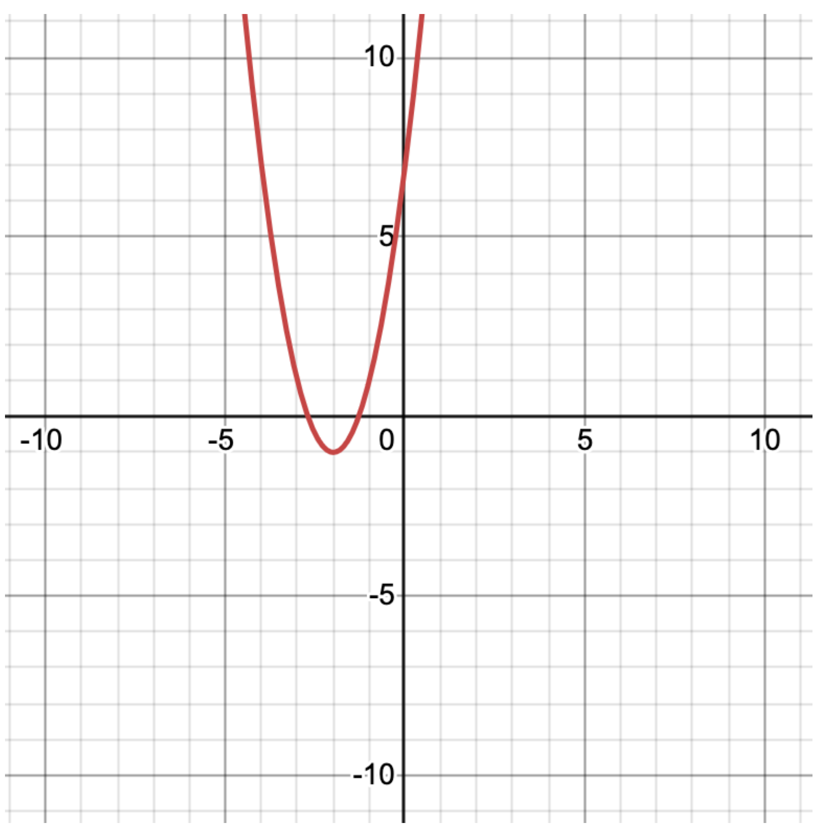
$$a_n x^n + \ldots + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{47}$$

Where the "poly" means many and the "nomial" refers to terms... or things added to each other. A polynomial with only one term is called a mononomial, one with two is called a binomial... and the highest order (or power) term designates whether it is quadratic, cubic, etc. The behavior of a polynomial is largely determined by its first term. Why is this? Think about properties of exponents. Is there a way we could say that x^5 is more "powerful" than x? This gives us the "leading coefficient test". And tells us how to graph polynomials. (Weird behavior often happens between the zeros, not outside of them). We begin investigating polynomials by thinking about quadratics. The standard form of a quadratic is:

$$f(x) = a(x-h)^2 + k \tag{48}$$

How could we derive this form from others? In this form, (h, k) will be the vertex. What does f look like if $a > 0$? If $a < 0$? Will the vertex be a min or max? For $ax^2 + bx + c = 0$, where a,b,c $\in \mathbb{R}$ and $a \neq 0$, $h = \frac{-b}{2a}$ and $k = f(h) = f(\frac{-b}{2a})$.

Exercise 102. *Write $f(x) = 2x^2 + 8x + 7$ in standard form by factoring around x, then working on the $x^2 + x$ term and inventing a constant value to make the factorization work... then adding that value to k. I've included the graph of this function below.*



4.3.1 Graphing quadratics

Without solving, graph:

Exercise 103. $3x^2 + 2 = 0$

Exercise 104. $6x^2 - 1 = 0$

Exercise 105. $5x^2 + 2x + 6 = 0$

Exercise 106. $2x^2 - 2x - 15 = 0$

Exercise 107. $3x^2 + 6 = 0$

Exercise 108. $2x^2 + 4x + 12 = 8$

Exercise 109. $6x^2 - 3x + 3 = -4$

Exercise 110. $4x^2 - 14 = -2$

Exercise 111. $4x^2 + 5x = 7$

Exercise 112. $x^2 + 4x - 48 = -3$

Exercise 113. $3x^2 - 3 = 8x$

Exercise 114. $3x^2 + 4 = -6x$

Exercise 115. $6x^2 = -5x + 13$

Exercise 116. $6x^2 = 4 + 6x$

Exercise 117. $x^2 = 8$

Exercise 118. $2x^2 + 6x - 16 = 2x$

Exercise 119. $12x^2 + x + 7 = 5x^2 + 5x$

5 Systems of equations

5.1 Tues., Apr. 9: Considering what we mean by algebraic substitution

We've worked with many types of algebraic expressions. Now we can start putting them together, or thinking about where several expressions are simultaneously true. Let's work a few to warm up:

$$\begin{cases} 8 = 2x + 3y \\ -2 = x \end{cases}$$

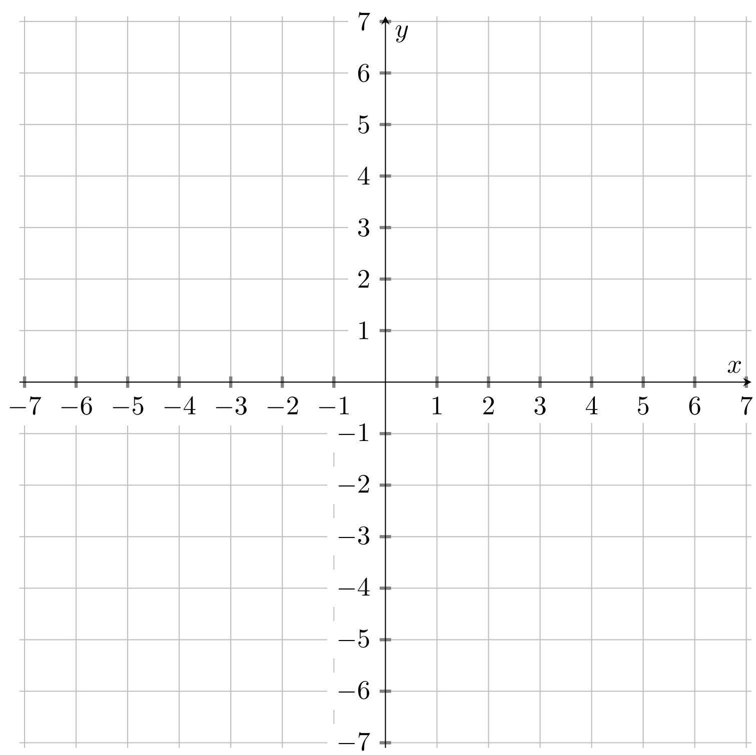
$$\begin{cases} -6x + \frac{1}{2}y = 4 \\ y = 4 \end{cases}$$

$$\begin{cases} 5x - y = 17 \\ x = y + 1 \end{cases}$$

We can solve these with substitution. But will that continue to work as the equations get more complicated? Let's consider another system of equations and solve it graphically.

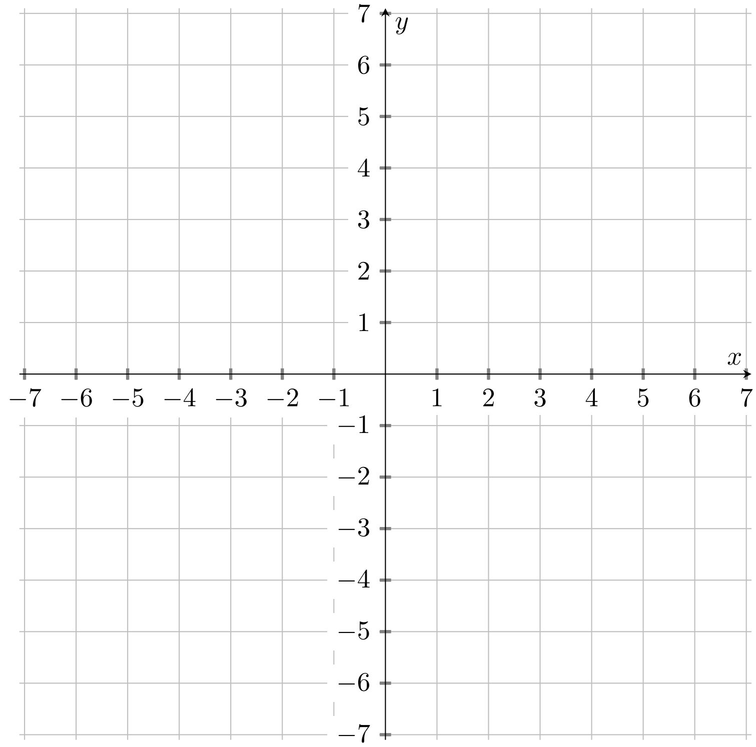
Exercise 120.

$$\begin{cases} y = -x^2 + 2x + 8 \\ y = 3x + 2 \end{cases}$$



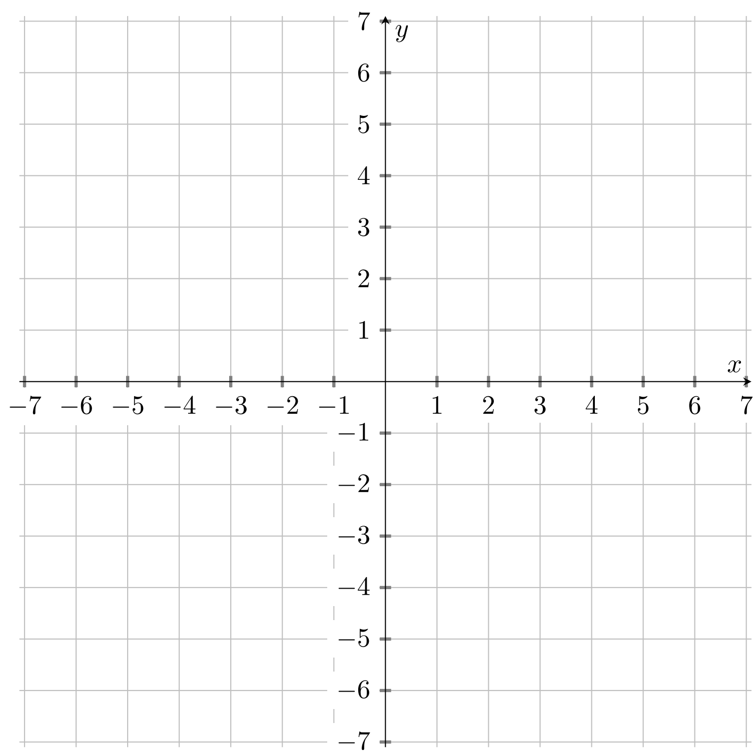
Exercise 121.

$$\begin{cases} y = x^2 - 3x - 4 \\ y = x - 8 \end{cases}$$



Exercise 122.

$$\begin{cases} y = 2x^2 + 4x + 3 \\ y = 4x - 1 \end{cases}$$



5.2 Thurs., Apr. 11: Using elimination to solve systems of equations in three unknowns

We now consider a more challenging case, where there may be three equations in three unknowns!

Exercise 123. *Let's try to solve the equation below with substitution:*

$$\begin{cases} x+y+z=-2 \\ 2x-3y+2z=-14 \\ 4x+3y-z=5 \end{cases}$$

Exercise 124. *Let's try to solve the equation below with substitution:*

$$\begin{cases} 2x-y-2z=3 \\ 3x+y-2z=11 \\ -2x-y+z=-8 \end{cases}$$

Exercise 125. *Let's try to solve the equation below with elimination:*

$$\begin{cases} x+y+z=-2 \\ 2x-3y+2z=-14 \\ 4x+3y-z=5 \end{cases}$$

Exercise 126. *Let's try to solve the equation below with elimination:*

$$\begin{cases} 2x-3y=7 \\ y+z=-5 \\ x+2y+4z=-17 \end{cases}$$

Exercise 127. *Is this system solvable,* i.e. are there values for x, y, z that jointly satisfy the equations*?*

$$\begin{cases} 5x+y3z=9 \\ -x-2y-z=-16 \\ 2x+4y+2z=-30 \end{cases}$$

Exercise 128. *Let's try to solve the equation below with elimination:*

$$\begin{cases} x-4y+3z=-7 \\ 2x+3y-5z=19 \\ 4x+y-z=17 \end{cases}$$

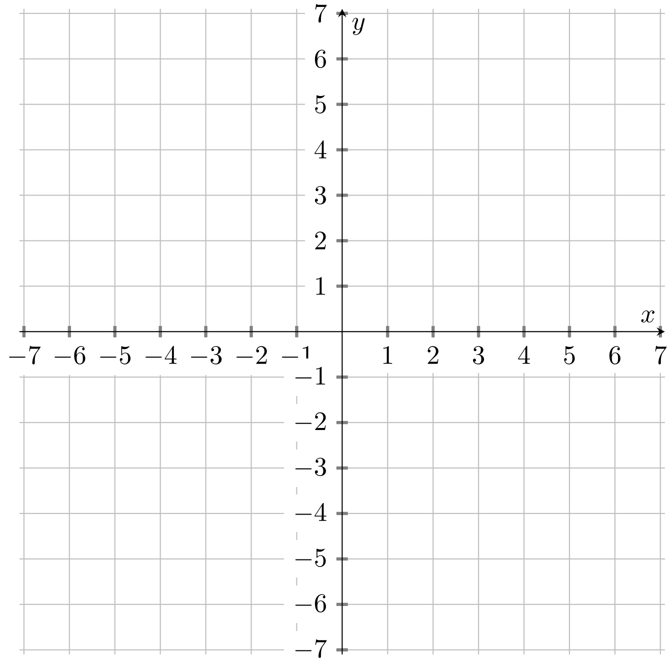
Exercise 129. *Let's try to solve the equation below with elimination:*

$$\begin{cases} 2x-3z=4 \\ 2x+y-5z=-1 \\ 3y-4z=2 \end{cases}$$

5.2.1 Applying this to inequalities

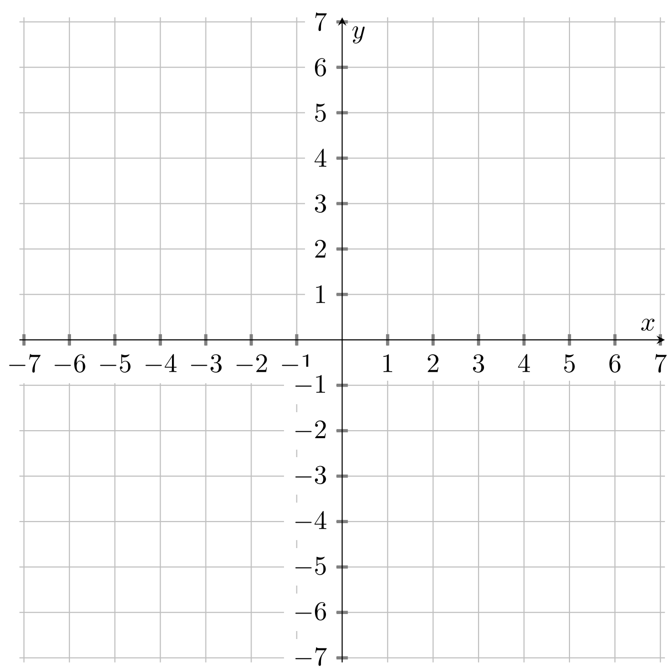
Exercise 130. *We can also think about graphing systems of inequalities. Graph the lines and shade the region defined in this system of inequalities:*

$$\begin{cases} y > x^2 - 3x + 4 \\ y < x + 1 \end{cases}$$



Exercise 131. *We can also think about graphing systems of inequalities. Graph the lines and shade the region defined in this system of inequalities:*

$$\begin{cases} y < -2x^2 - x - \frac{1}{2} \\ y > \frac{1}{4}x - 1 \end{cases}$$



5.3 Fri., Apr. 12: Motivation for a better method... the matrix

Is there a better way to do this? What if we had an equation with four unknowns... five... a million?

Exercise 132. *Let's try to solve the equation below with substitution:*

$$\begin{cases} a+b+c+d=-1 \\ 2a-3b+2c+2d=-12 \\ 4a+3b-c-d=4 \\ 3a-4b-4c+5d=6 \end{cases}$$

So a better method is needed. We note that it's not really necessary to keep writing the variables, and that all the work we need to do can be done just with the coefficients. By grouping these into a box, called a "matrix", we can add/subtract/multiply/divide rows by each other and integers to isolate variables. The goal is to produce a diagonal matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array}\right)$$

Here are some worked examples:

Use Gaussian elimination to solve the following system of equations:

$$\begin{cases} z-3y=-6 \\ x-2y-2z=-14 \\ 4y-x-3z=5 \end{cases}$$

Solution: First write it in an augmented matrix.

$$\left(\begin{array}{ccc|c} 0 & -3 & 1 & -6 \\ 1 & -2 & -2 & -14 \\ -1 & 4 & -3 & 5 \end{array}\right) \xrightarrow{I\leftrightarrow II} \left(\begin{array}{ccc|c} 1 & -2 & -2 & -14 \\ 0 & -3 & 1 & -6 \\ -1 & 4 & -3 & 5 \end{array}\right) \xrightarrow{III+I} \left(\begin{array}{ccc|c} 1 & -2 & -2 & -14 \\ 0 & -3 & 1 & -6 \\ 0 & 2 & -5 & -9 \end{array}\right)$$

$$\xrightarrow{I-2/3II,III+2/3II} \left(\begin{array}{ccc|c} 1 & 0 & -8/3 & -10 \\ 0 & -3 & 1 & -6 \\ 0 & 0 & -13/3 & -13 \end{array}\right) \xrightarrow{III*3/-13} \left(\begin{array}{ccc|c} 1 & 0 & -8/3 & -10 \\ 0 & -3 & 1 & -6 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{I+8/3III,II-III} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & -3 & 0 & -9 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{II/-3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Thus the solution is $(-2,3,3)$.

Use Gaussian elimination to solve the following system of equations:

$$\begin{cases} x_1 - 2x_2 - 6x_3 = 5 \\ 2x_1 + 4x_2 + 12x_3 = -6 \\ x_1 - 4x_2 - 12x_3 = 9 \end{cases}$$

Solution: Using Gaussian elimination gives

$$\left(\begin{array}{ccc|c} 1 & -2 & -6 & 5 \\ 2 & 4 & 12 & -6 \\ 1 & -4 & -12 & 9 \end{array}\right) \overset{II-2I, III-I}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & -2 & -6 & 5 \\ 0 & 8 & 24 & -16 \\ 0 & -2 & -6 & 4 \end{array}\right)$$

$$\overset{II/8, III/-2}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & -2 & -6 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & -2 \end{array}\right) \overset{I+2II, III-II}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solving, we get $x_1 = 1, x_2 = -2 - 3x_3$ and x_3 can be whatever it wants. This means there are an infinite number of solutions, that the determinant of the original matrix is 0.

Exercise 133. *Let's try to solve the equation below with Gaussian elimination:*

$$\begin{cases} x + y + z = -2 \\ 2x - 3y + 2z = -14 \\ 4x + 3y - z = 5 \end{cases}$$

We make the matrix... called the "augmented" matrix when we include the values c, for $x + y + z = c$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 2 & -3 & 2 & -14 \\ 4 & 3 & -1 & 5 \end{array}\right)$$

Exercise 134. *Find conditions on a, b such that the following system has no solutions, infinitely many,and a unique solution.*

$$\begin{cases} x + ay = 2 \\ 4x + 8y = b \end{cases}$$

We want to solve the matrix:

$$\begin{pmatrix} 1 & a \\ 4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ b \end{pmatrix}$$

We know that this has a unique solution if the determinant is nonzero so we need $8 - 4a \neq 0$ or $a \neq 0$. For all $a \neq 2$ and any b this has a unique solution. But for $a = 2$ this has zero or infinite solutions... let's use the augmented matrix.

$$\left(\begin{array}{cc|c} 1 & a & 2 \\ 4 & 8 & b \end{array}\right)$$

Which when we subtract lines $II - 4I$ becomes:

$$\left(\begin{array}{cc|c} 1 & a & 2 \\ 0 & 0 & b-8 \end{array}\right)$$

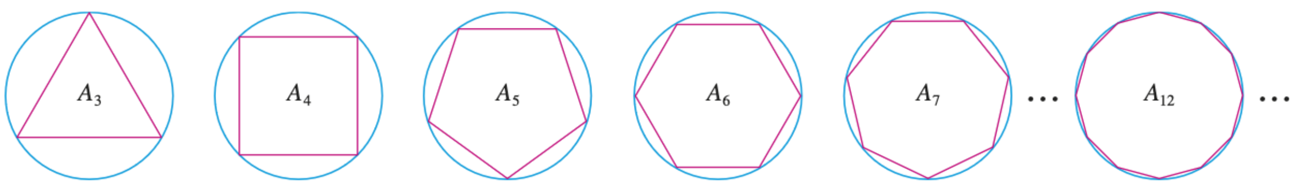
So if $b \neq 8$ the system has no solutions and if $b = 8$ there are infinitely many solutions.

6 Rethinking functions with Limits

We can turn now from algebra and geometry review towards some of the ideas we can really call "calculus". At this point I think it'd be good to take some of Stewart's diagnostic tests for calculus found at the beginning of his book *Transcendentals*, available online. This will notify you of potential weak areas and let us know where to build extra algebra and geometry knowledge as we move on.

6.1 Tues., Apr 22: Two parts of calculus: change and the effects of change

We've already been introduced to limits through our investigation of π, where 3.14 is the number approached by taking successively better approximations of a circle by a polygon with more sides:



So

$$\pi r^2 = \lim_{n\to\infty} A_n \tag{49}$$

6.1.1 Motivating question: how often should interest compound? e^x!

If we think about a bank, we might imagine someone offering us to hold our money in exchange for paying us a small fee to use the money while we're not. Let's say this fee is 8% a year.

What would the bank owe us after a year? If we loaned 100\$, this would be:

$$100\$ \cdot 1.08 = 108\$ \tag{50}$$

But there's nothing special about a year, instead of giving us 8\$ at the end of the year, the bank could give us the $\frac{8}{4}$\$ after $\frac{1}{4}$ of the year, and do so 4 times throughout the year.

Exercise 135. *Which is better? A bank that compounds interest annually or one that compounds $\frac{1}{4}$ of interest quarterly?*

In general, the formula for the current amount of money for money earning interest at an interest rate of r that is compounded n times a year for some years t from an initial deposit of P is:

$$M = P(1+\frac{r}{n})^{(nt)} \tag{51}$$

Exercise 136. *If you deposit 3,000\$ in a bank that pays 9% interest and compounds twice a year, how much money would you have after 14 years?*

Exercise 137. *Would a bank that compounds every day, or even every second, be far better than a bank that compounds annually or only a little better? Compute some examples to prove this.*

Now recall our formula for e:

$$e^x = \lim_{x\to\infty} f(x) = (1+\frac{1}{n})^n \tag{52}$$

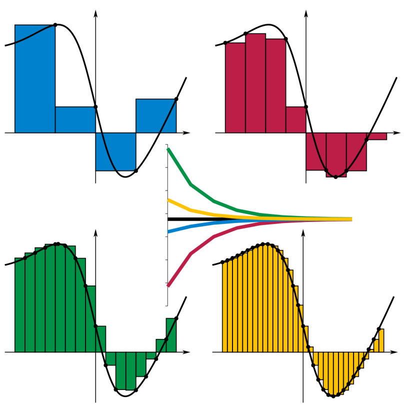
By visual inspection it's clear this is just the continuously compounding interest case!

6.1.2 Integrals with Riemann sums

In the same way a limit gets more specific with the idea of *rate*, the integral gets more specific with the idea of a *rate applied over time*. How do things accumulate?

$$\frac{Quantity}{Time} \cdot Time = Quantity \tag{53}$$

A fundamental calculus technique is to first answer a given problem with an approximation, then refine that approximation to make it better, then use limits in the refining process to find the exact answer. That is exactly what we will do here to develop a technique to find the area of more complicated regions.



When computing a Riemann sum, note that for each rectangle, length is...

$$x_1, x_2, x_3, ...x_n \tag{54}$$

And the heights are...

$$x_1^2, x_2^2, x_3^2, ...x_n^2 \tag{55}$$

Exercise 138. *Let's compute* $y = x^2$ *over the interval* $[0, 2]$*. Then find* $\int_0^2 x^2\,dx$ *and compare.*

In general a Riemann sum is:

$$A = \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx \tag{56}$$

Riemann sums are all calculated by partitioning the region into some sort of shapes (rectangles, trapezoids, parabolas, or cubics — sometimes infinitesimally small) that together form a region that is similar to the region being measured, then calculating the area for each of these shapes, and finally adding all of these small areas together. This approach can be used to find a numerical approximation for a definite integral even if the fundamental theorem of calculus does not make it easy to find a closed-form solution.

Exercise 139. *Let's explore left and right Riemann sums with this example. Find the Riemann sum of the function defined in the following table. (First draw its graph.)*

x	1	4	7	10
$f(x)$	6	8	3	5

So the choice of a starting place matters! Can we think of any better tessellations to approximate this figure?

Exercise 140. *Use a Riemann sum to integrate* $sin(x)$*. Is there an easier way to do this? Let's consider what's happening on the interval* $[0, 2\pi]$*. As always, begin by graphing or recalling the* $sin(x)$ *graph.*

$$\int_0^{2\pi} sin(x)\,dx$$

6.2 Thurs., Apr. 24: Limits

We've also already been introduced to the idea of limits through learning point-slope linear equations in the form $y = mx + b$. If we investigate m we find:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{57}$$

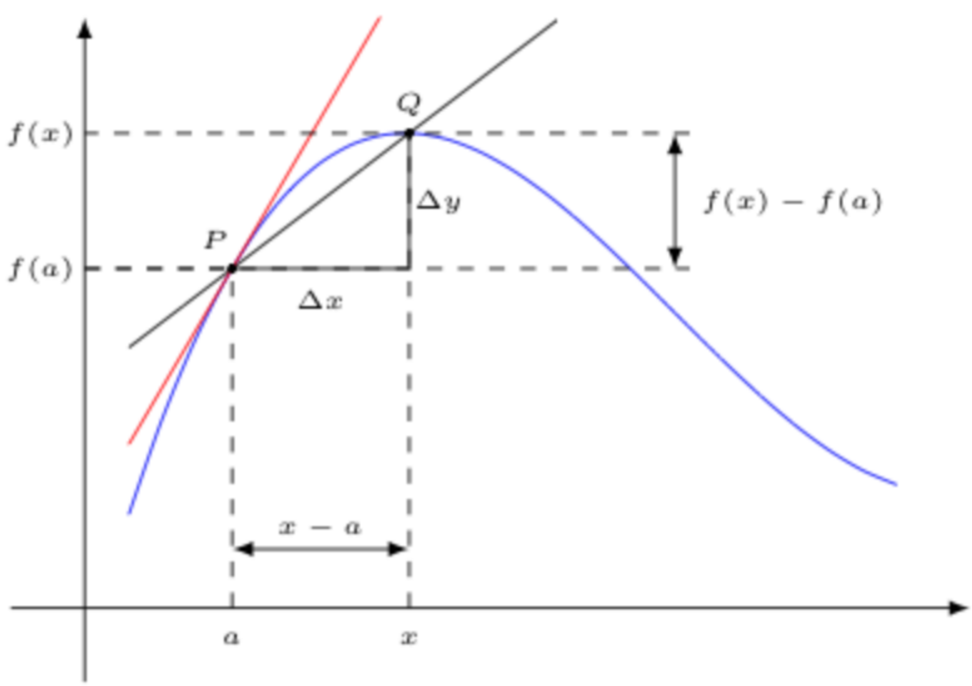
And we know y is really just $f(x)$ so this becomes:

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{58}$$

And for a smaller interval this is:

$$m = \lim_{x \to a} f(x) = \frac{f(x) - f(a)}{x - a} \tag{59}$$

What does this look like?



We can also consider a new variable, and set $x_1 = a$ and $x_2 = a + h$, considering when $h \lim 0$:
This gives:

$$\lim_{h \to 0} f(x) = \frac{f(a+h) - f(a)}{h} \tag{60}$$

Exercise 141. *Use the $h \to 0$ limit definition of derivative to find the derivative of $V(t) = 3 - 14t$.*

Exercise 142. *Use the $h \to 0$ limit definition of derivative to find the derivative of $f(x) = \frac{5}{x}$.*

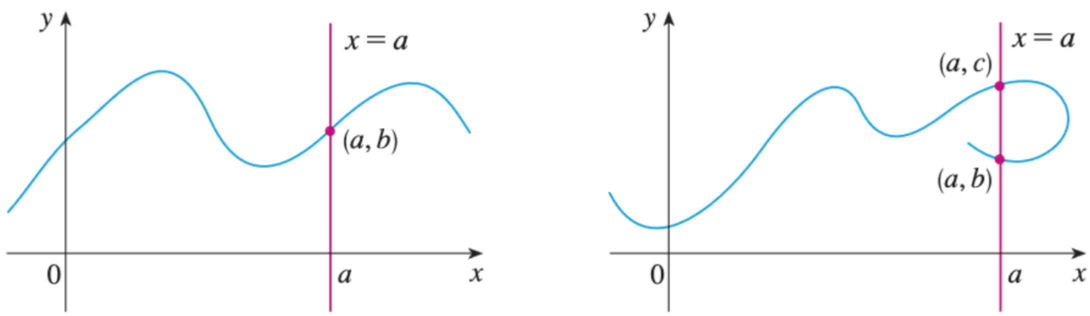
Exercise 143 (More advanced but only b/c the computation is messy.)**.** *Use the $h \to 0$ limit definition of derivative to find the derivative of $f(x) = \frac{x+1}{x+2}$.*

Exercise 144. *For $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$ evaluate $\frac{f(a+h)-f(a)}{h}$.*

6.3 Fri., Apr. 25: Continuity

6.3.1 Domain restriction

It's useful to think about general facts about functions before thinking too hard about limits. Remember a function cannot map the same x to more than one unique y:



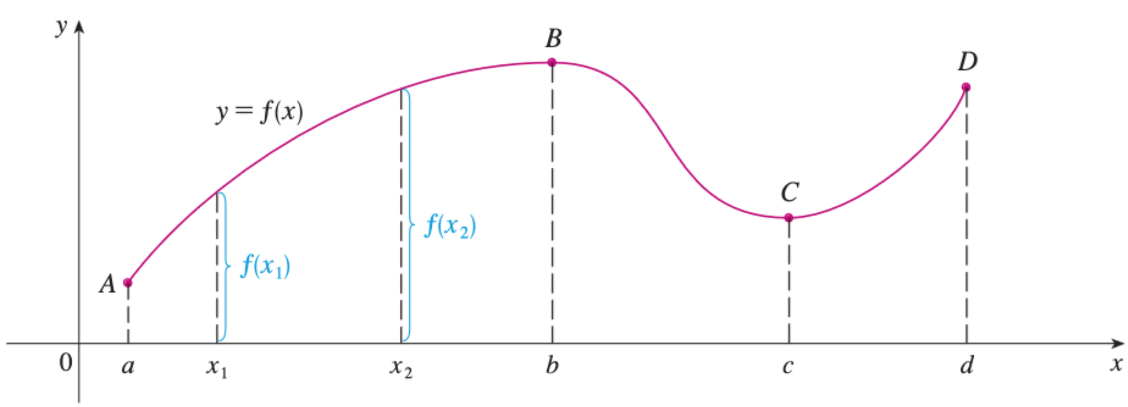
We investigate piecewise functions with the vertical line test by thinking about $x = y^2 - 2$.

Exercise 145. *Graph $x = y^2 - 2$ by factoring, and think about domain restrictions required to make it satisfy the definition of a function.*

6.3.2 Increasing or decreasing?

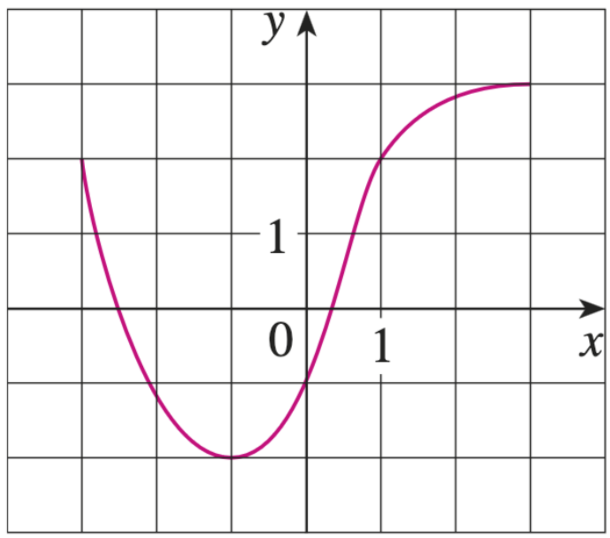
We also often want to know if a function is *increasing*, *decreasing* or neither...

Exercise 146. *For the function described in the image below, on what interval(s) is it increasing, decreasing, or neither?*



We can formalize this to say that a function is:

- If $f(x_1) < f(x_2)$ for $x_1 < X_2$
- If $f(x_1) > f(x_2)$ for $x_1 < X_2$



Exercise 147. *For the function described in the image above:*

- *What is $f(-1)$?*
- *What is $f(2)$?*
- *For what x is $f(x) = 2$?*
- *Find all x such that $f(x) = 0$*
- *What are domain and range of f?*
- *For slope, m, where is $m = 0$?*
- *On what intervals is f increasing, decreasing or neither?*

6.3.3 Limits and continuity

We say that a function f is *continuous* if

$$\lim_{x \to a} f(x) = f(a) \tag{61}$$

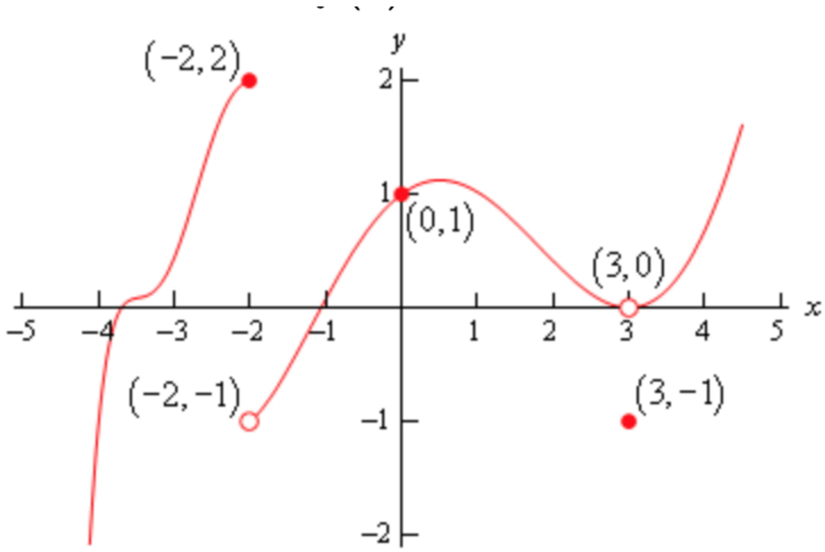
Exercise 148. *Is $y = x^3$ continuous?*

Exercise 149. *Is $\frac{4t+10}{t^2-2t-15}$ continuous? Why?*

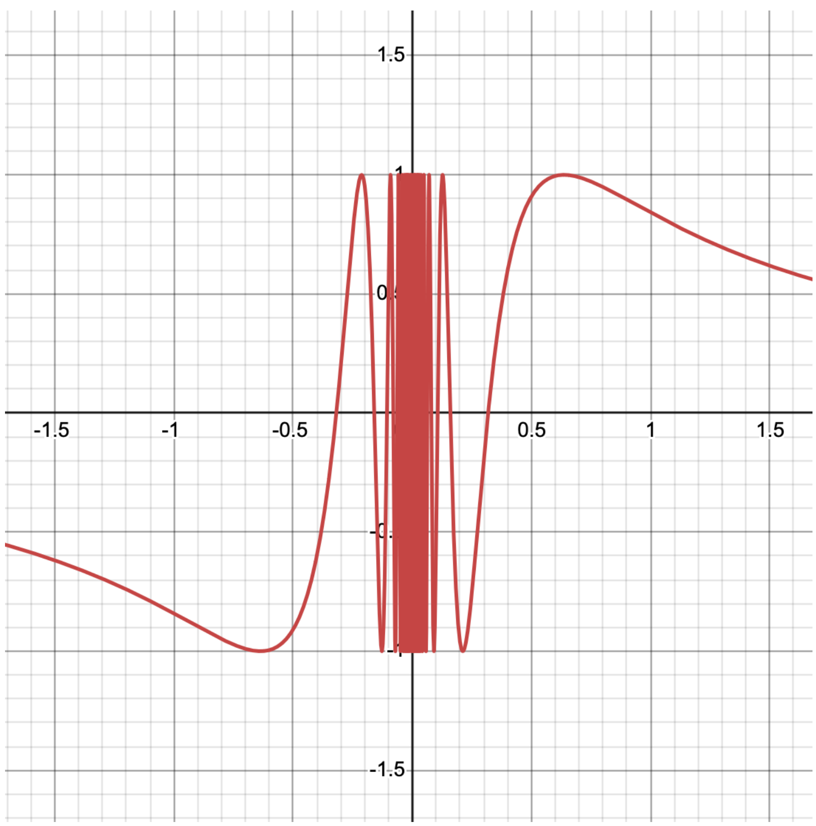
And it is continuous on an interval if it is continuous on every point in the interval. Continuity also composes... which we will find in this example.

Exercise 150. *Evaluate* $\lim_{x \to 0} e^{sin(x)}$.

Exercise 151. *For the function f defined by the graph below, determine if $f(x)$ is continuous at $x = -2$, $x = 0$, and $x = 3$.*



Exercise 152. *I've given the graph of $f(x) = sin(\frac{1}{x})$ below. Graph it yourself to zoom in! Is $f(x)$ continuous at $x = 0$?*



7 Derivatives

7.1 Product, quotient and chain rules

7.2 Work on applications of these rules

7.3 Exp and log derivatives

8 Integrals

8.1 Fundamental theorem of calc and integrals

8.2 Velocity, rotation of a solid, and an economic example

8.3 Discussion of sequences and series and analytic solutions to integral problems

9 Miscellany

9.1 L'Hôpital's Rule

9.2 Newton's method

9.3 Introduction to differential equations via the heat transfer and gravity equations and recommendations for future application/study for those continuing...