

### 3 Things of special interest: Exponentials, Parametric Equations

#### 3.1 Tues, Mar. 5: Expansion of $b^x$

We often want to consider powers of a number where:

$$b^x = \underbrace{b \cdot b \cdot b}_{x \text{ times}} = y$$

From this idea we can derive “rules” for exponents.

$$x^0 = 1 \tag{19}$$

$$x^a \cdot x^b = x^{a+b} \tag{20}$$

$$(x^a)^b = x^{a \cdot b} \tag{21}$$

$$\frac{x^a}{x^b} = x^{a-b} \tag{22}$$

And we can think about when the numerator is just 1 so that:

$$\frac{1}{x^b} = x^{-b} \tag{23}$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} \tag{24}$$

$$x^{a/b} = \sqrt[b]{x^a} \tag{25}$$

Often students (including myself when I was learning these formulas) will get stuck on the idea of a fractional power  $x^{1/2}$ ... how could you multiply a number by itself *less* than one time? But try not to think about this and instead think how division undoes multiplication...

**Exercise 23.** Consider how fractional exponents work by comparing  $2^{3/3} = \sqrt[3]{2^3}$ .

#### 3.2 Thurs, Mar. 7: Logarithms

The logarithm language rewrites this in the format:

$$\log_b y = x \tag{26}$$

Where:

$$b^x = y \tag{27}$$

I’ve used the variables  $x$  and  $y$  in a consistent manner here, so you can copy between the formulas. But usually logs will be written such that the dependent variable,  $y$ , is the exponent... making them the inverse of  $b^x$ . The change of base format is very useful and helpful to consider as we understand logs:

$$\log_a x = \frac{\log_b x}{\log_b a} \tag{28}$$

Which we can prove by observing that:

$$y \log_b x = \log_b x^y \tag{29}$$

This is the power property of logs, and there are also properties for products

$$\log_a uv = \log_a u + \log_a v \tag{30}$$

...and quotients:

$$\log_a \frac{u}{v} = \log_a u - \log_a v \quad (31)$$

Another important identity that we should think about is:

$$b^{\log_b x} = x \quad (32)$$

### 3.3 Fri, Mar. 8: Deriving $e$

There are two ways to derive  $e$ . The first is to expand our idea of the slope of a line:

$$y = mx + b \quad (33)$$

We know slope is a ratio of  $\frac{\Delta x}{\Delta y}$  and for large  $x$  and  $y$  this can be expressed:

$$\frac{y_2 - y_1}{x_2 - x_1} \quad (34)$$

And we know  $y$  is really just  $f(x)$  so this becomes:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad (35)$$

But often we want to look at non-constant slopes... *i.e.* where the size of slope depends on where you are in the function. Here it's useful to think about what happens when  $x_2$  and  $x_1$  are very close together, *i.e.* when  $x_2 \rightarrow x_1$ . This is the beginning of the calculus idea. But for now, we'll just use this idea to compute  $e^x$  to get some experience with it, and to learn about  $e$ .

**Exercise 24.** *Let's say we want to consider something that grows at  $2^t$  where  $t$  is time. What is the relationship between the growth rate of the function at a place and the value of the function at that place. Is there a function where this ratio is 1 : 1?*

Exercise 16 offers us one way to prove  $e$ . We can also think about  $e$  as doing something, then doing it more times but less each time. Let's consider a thing that's continuously growing... and measuring it at smaller units of time but more frequently, perhaps at a half unit of time. We would find that we have to multiply two terms:

$$\left(1 + \frac{1}{2}\right) \cdot \left(1 + \frac{1}{2}\right) \quad (36)$$

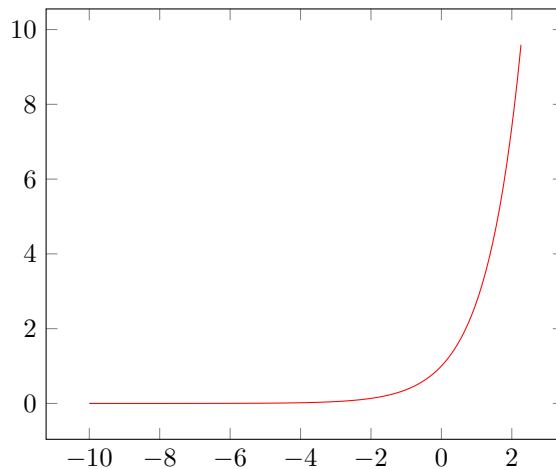
As we do this more and more, we'd get better approximations for  $e$  like:

$$\left(1 + \frac{1}{100}\right) \cdot \left(1 + \frac{1}{100}\right) \cdot \dots \cdot \left(1 + \frac{1}{100}\right) = \left(1 + \frac{1}{100}\right)^{100} \quad (37)$$

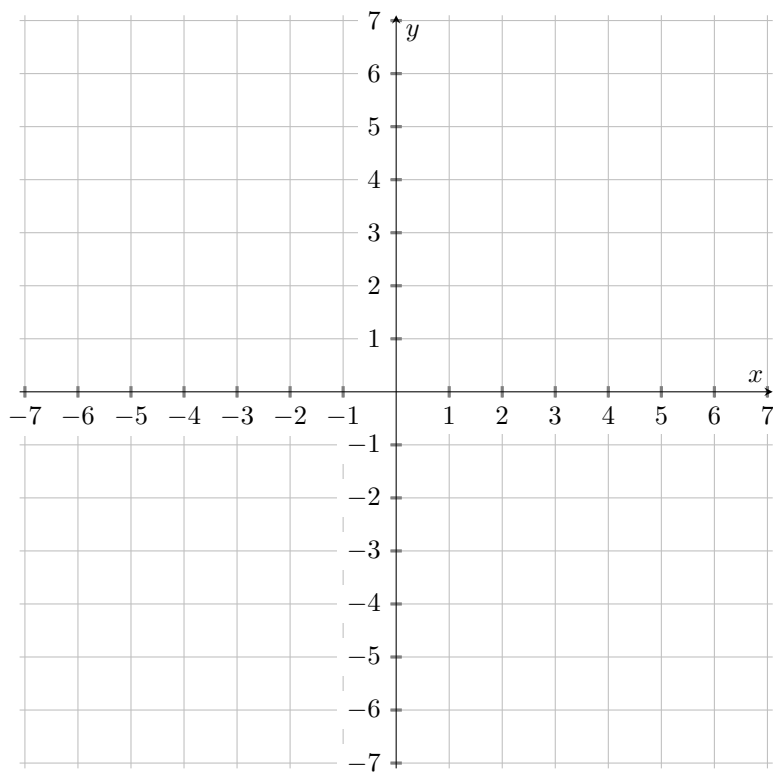
Throw this into the calculator and you'll find it's  $e$ . So in general:

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n \quad (38)$$

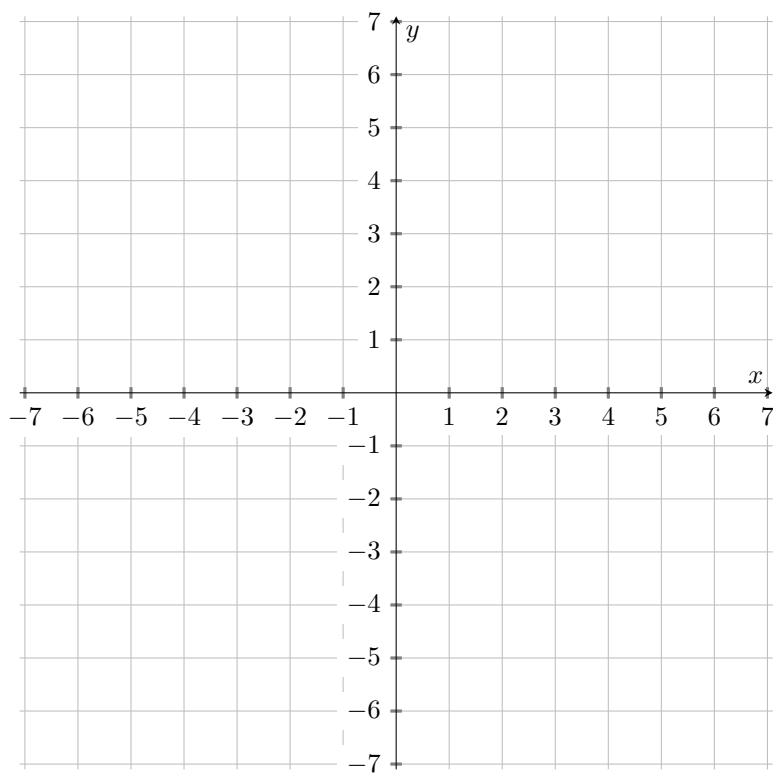
The graph of  $e^x$  is:



**Exercise 25.** Graph  $x^2$ . Graph  $2^x$ .



**Exercise 26.** Graph  $\log_2 x$ . Graph  $\ln$ . Graph  $\log_{10} x$ .



**Exercise 27.** Find  $\log_{10} 1$

**Exercise 28.** Find  $\log_{10} 10^7$

**Exercise 29.** Find the approximate value of  $\log_2 10^3$  rounded to the nearest whole number. No calculator! (Think about powers of 2.)

**Exercise 30.** Find  $e^{\ln \pi}$

**Exercise 31.** Find  $\log_{10} 1,000,000,000$

**Exercise 32.** Simplify  $\frac{a^3 b^7}{a^{-4} b^4}$

**Exercise 33.** Simplify  $\frac{x^{2(z+8)}}{x^{-z}}$

**Exercise 34.** Simplify  $\frac{8^3}{2^3 2^7}$

**Exercise 35.** Simplify  $(\frac{a^{1/3}}{b^{1/6}})^3$

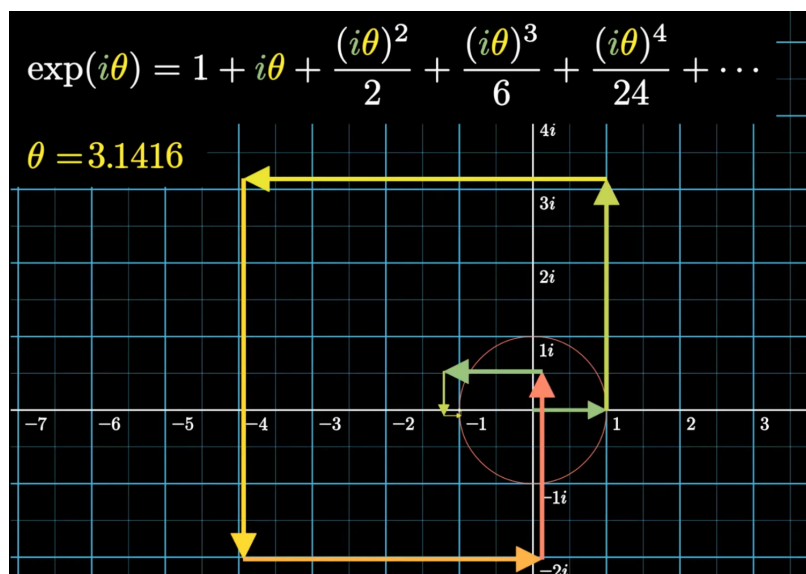
**Exercise 36.** Rewrite with quotient rule for logs:  $\ln \frac{\sqrt{3x-5}}{7}$

**Exercise 37.** Simplify  $\ln \frac{6}{e^2}$

**Exercise 38.** Find  $\log_5 \frac{1}{125}$

**Exercise 39.** Use change-of-base to rewrite this logarithm as a ratio of logarithms:  $\log_{1/2} x$ . Then graph the ratio and the original to verify equivalence.

**Exercise 40.** Assume  $\log \frac{a}{b} = m \log \frac{b}{a}$ . Find  $m$ .



### 3.3.1 More about $e$

We learned one way to think about exponents...

$$e^x = \underbrace{e \cdot e \cdot e}_{x \text{ times}}$$

but in finding a series formula for  $e$  we also learned another way to think about exponents:

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots \quad (39)$$

**Exercise 41.** Show that when you fully expand  $\exp(x) \cdot \exp(y)$ , each term has the form  $\frac{x^k y^m}{k!m!}$

**Exercise 42.** Show that when you expand  $\exp(x + y)$ , each term has the form  $\frac{1}{n!} \binom{n}{k} x^k y^{n-k}$ .

**Exercise 43.** Compare results above to explain why  $\exp(x + y) = \exp(x) \cdot \exp(y)$

### 3.3.2 Interest

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