1 Functions

1.1 Tues, Feb. 20: Domain and co-domain, graphing solutions to pairs of linear equations

A function is a defined relationship between two things. These can be inputs/outputs, two numbers, a number and an idea, etc. The first thing is called the "domain" and the second thing is the "co-domain":

$$F : A \to B \tag{1}$$

The function is said to be one-to-one (injective) if every $a \in A$ maps to a unique b ∈ B, *i.e.* no two $a \in A$ map to the same $b \in B$, and "onto" if for every $b \in B$ there is an $a \in A$ that maps to it. The former function is said to be an "injection" and the latter a "surjection". A function that is both is a "bijection," or a correspondence.

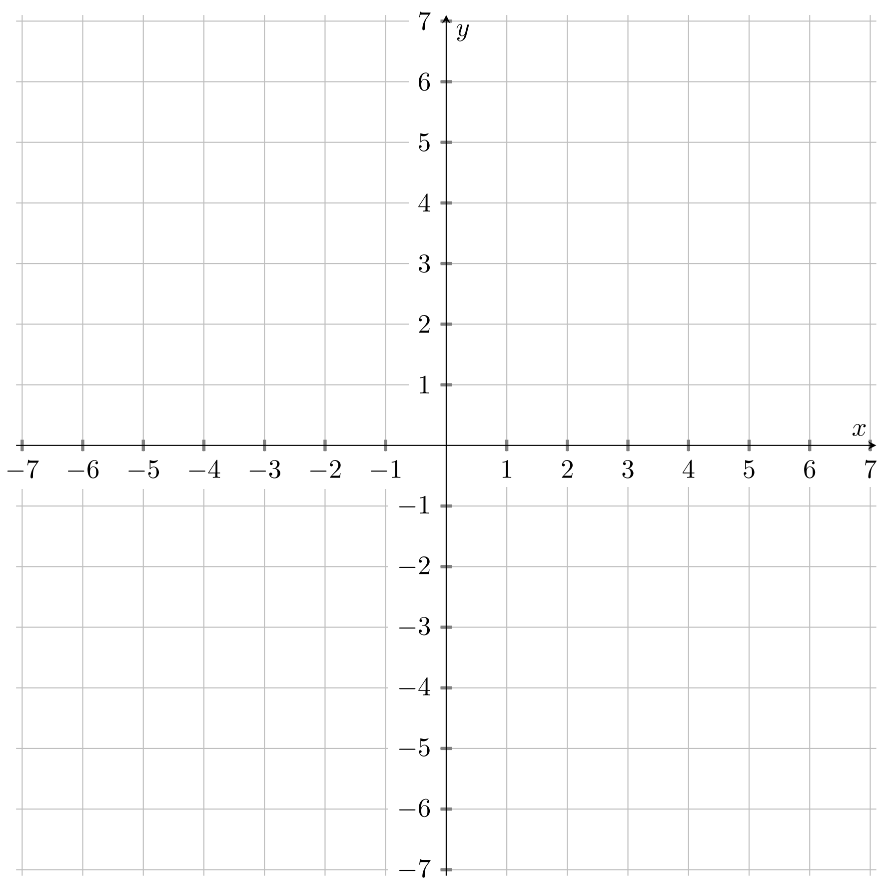
Exercise 1. *For the counting numbers* $1, 2, 3, ...$, *not including halves or fractional numbers, name a function that is one-to-one, onto, and both one-to-one and onto.*

The "linear" or line function is fundamental for relating two numbers with a constant relationship in the Cartesian plane. You probably know about the Cartesian plane from algebra... it's just the $x - y$ coordinate graph.

Pairs of these functions can be solved visually with graphing or with algebraic rearrangement. Each pair of linear equations will have exactly one solution (x, y) that satisfies both equations, and this point will be the intersection of the two lines, whereas many (x, y) will solve a single linear equation in the form:

$$y = f(x) = mx + b \tag{2}$$

Exercise 2. *Graph the points* $(6, 7)$ *and* $(-4, -8)$. *Find the slope of the line that runs through these points and its equation. Find another point on this line. Find its intercepts. Find the equation of a line perpendicular to this line, and the equation for a line parallel to it.*



It can be helpful to think about where the equation $y = mx + b$ comes from:

$$y = \underbrace{m}_{\text{the ratio of x and y}} x + \underbrace{b}_{\text{a constant value}}$$

It can be helpful to think about where m comes from:

$$\frac{y}{x} = \frac{y}{x} \rightarrow y = (\underbrace{\frac{y}{x}}_{\text{m}})x \tag{3}$$

We know slope is a ratio of $\frac{\Delta x}{\Delta y}$ and for any x and y this can be expressed:

$$\frac{y_2 - y_1}{x_2 - x_1} \tag{4}$$

1.1.1 Graphing a line and finding the equation of a line with any two points that lie on it

We can draw the graph for a line equation $y = mx + b$ by:

- identifying the b-value on the y-axis... (hint: this is found by imagining x to be 0)
- drawing from that point the line slope given by m

We can also find a line equation from any two points on the graph $(x_2, y_2)$and (x_1, y_1) by reversing the process:

- finding m, or the slope with $\frac{y_2 - y_2}{x_2 - x_1}$
- substituting one of our points for x and y and the value we just found for m to solve for b

We can also find the distance between any two points in a graph with the help of our old friend, Pythagoras.

Exercise 3. *Find the distance between the points* $(-2, -3)$ *and* $(3, 5)$ *with the Pythagorean theorem. Write this down for your notes as the "distance formula". Find their midpoint. Write down the midpoint formula for your notes. (Hint: it is like taking an average.)*

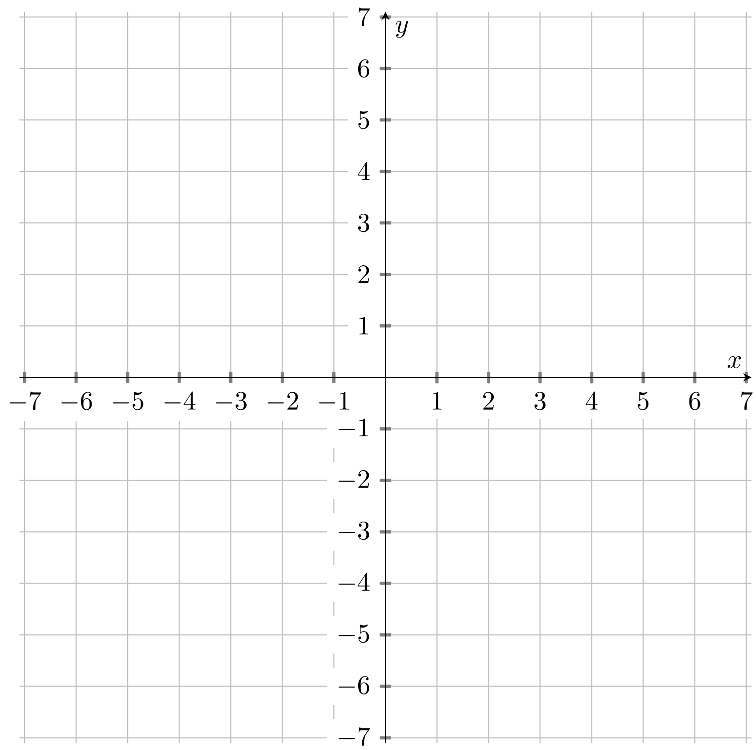
1.1.2 Inverse functions

It's often useful to think of the "opposite" of a function. If we have a function that takes some A and maps them to B, we might want to think about the "undo" function that takes B and turns them into A again. We call this the inverse function. We can define it this way:

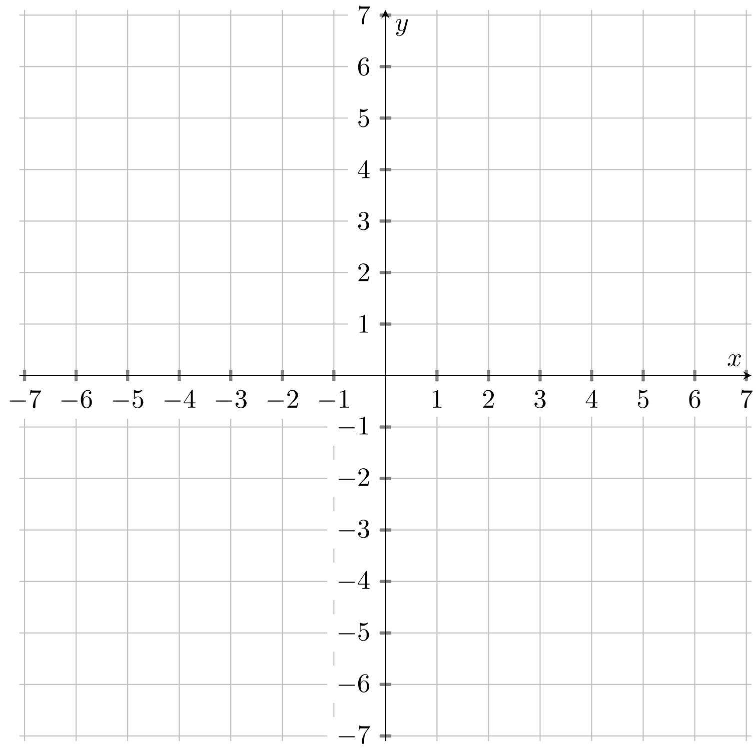
$$f^{-1}(f(x)) = x \tag{5}$$

And remember that $f(x) = y$ so this is the same as saying that $f(x) = x$ and $f(y) = y$. What are the properties of these kinds of functions? If a function f has some domain and range then for the inverse function f^{-1} the domain and range will be switched. In other words, the output values will become the input values and vice versa. A helpful trick is to remember that inverse functions are reflections about the line $y = x$.

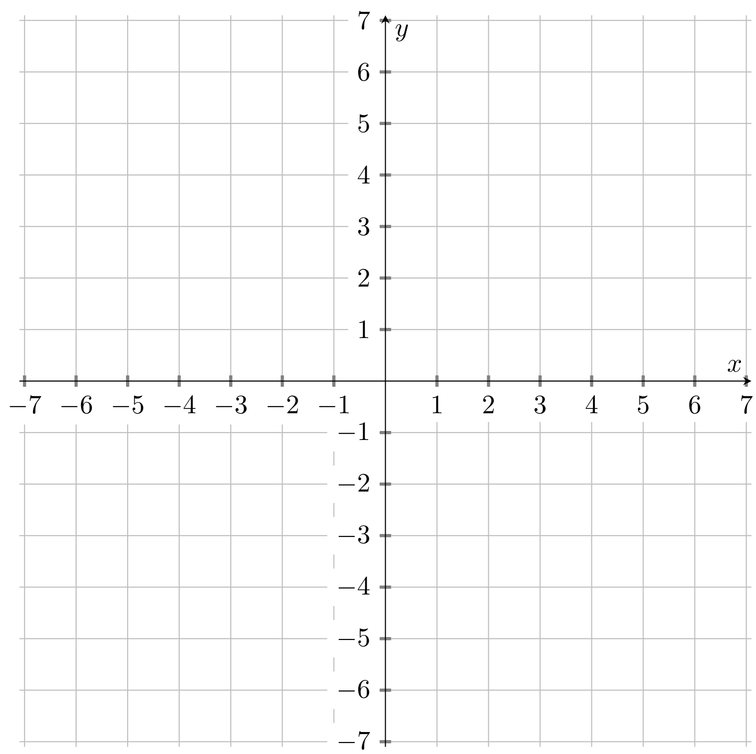
Exercise 4. *Graph $f(x) = 2x + 1$ and find its inverse.*



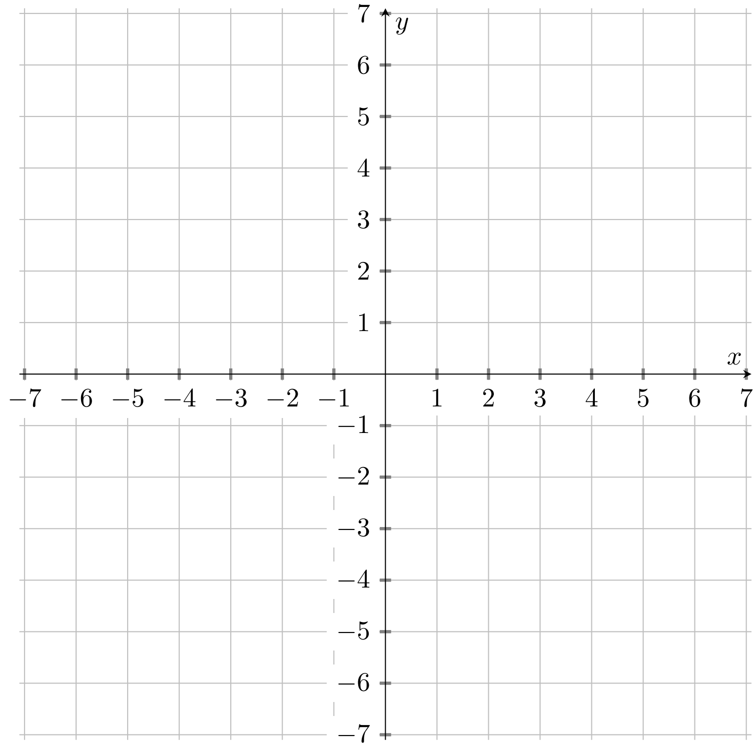
Exercise 5. *Graph $f(x) = \sqrt{x+1}$ and find its inverse. Note that while the domain of the inverse function will be larger than the range of the initial function, it must be restricted according to the original to match that function in a perfect reflection about $x = y$.*



Exercise 6. *Graph $f(x) = 1/x$. Where is the function undefined? What is the behavior of the function at its big values of x and y. Find the inverse of this function. (This might be a trick question. What is the way to find a function's inverse in general?)*



Exercise 7. *Graph* $x^2 - 5x - 6 = 0$*. Then solve. Solve first with guessing, then with completing the square, then with the quadratic formula.*



Exercise 8. *Write in standard form using complex conjuates:*

$$\frac{2+3i}{4-2i} \tag{6}$$

Exercise 9. *Write in standard form:*

$$12i + \frac{25}{5i^2} + i^7 \tag{7}$$

1.2 Thurs, Feb. 22: Shapes of common functions

Consider the functions:

$$f(x) = c \tag{8}$$

$$f(x) = x^2 \tag{9}$$

$$f(x) = x^3 \tag{10}$$

$$x^2 + y^2 = 1 \tag{11}$$

(Note: You may have heard of the vertical line test. Does this pass the vertical line test? If not a function, what is it? Are there other ways we could describe a circle that would pass the test?)

$$f(x) = |x| \tag{12}$$

$$f(x) = \sqrt{x} \tag{13}$$

$$f(x) = \frac{1}{x} \tag{14}$$

$$f(x) = \frac{1}{x^2} \tag{15}$$

What is the basic shape of each of these functions? How do their compositions with other functions affect their shapes?

Exercise 10. *Let's invent some points and functions and work on translating between the two. Use the method of* 2^n *steps to find points on the graph with the brute force method.*

1.3 Fri, Feb. 23: Shapes... derivation of π

Exercise 11. *What is the perimeter of a circle with radius 4?*

Exercise 12. *What is the volume of a sphere with diameter 6?*

Exercise 13. *What is the volume of a regular pyramid with side length 3?*

Exercise 14. *Imagine a sphere inside a cube. The sphere touches the cube once on all six sides. What is the volume between the shapes?*

Exercise 15. *Why is* π 3.14*? Can you come up with a proof for the value for* π*? Can you come up with a proof for the area of a circle?*

Exercise 16. *Write down all the shape equations for surface and volume for future reference... (You can look these up on the internet. Try to have about 10.)*