Running Lecture Outline: Calculus of Variations

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1 Functions

1.1 Tues, Feb. 20: Domain and co-domain, graphing solutions to pairs of linear equations

A function is a defined relationship between two things. These can be inputs/outputs, two numbers, a number and an idea, etc. The first thing is called the "domain" and the second thing is the "co-domain":

$$F : A \to B \tag{1}$$

The function is said to be one-to-one (injective) if every $a \in A$ maps to a unique b ∈ B, *i.e.* no two $a \in A$ map to the same $b \in B$, and "onto" if for every $b \in B$ there is an $a \in A$ that maps to it. The former function is said to be an "injection" and the latter a "surjection". A function that is both is a "bijection," or a correspondence.

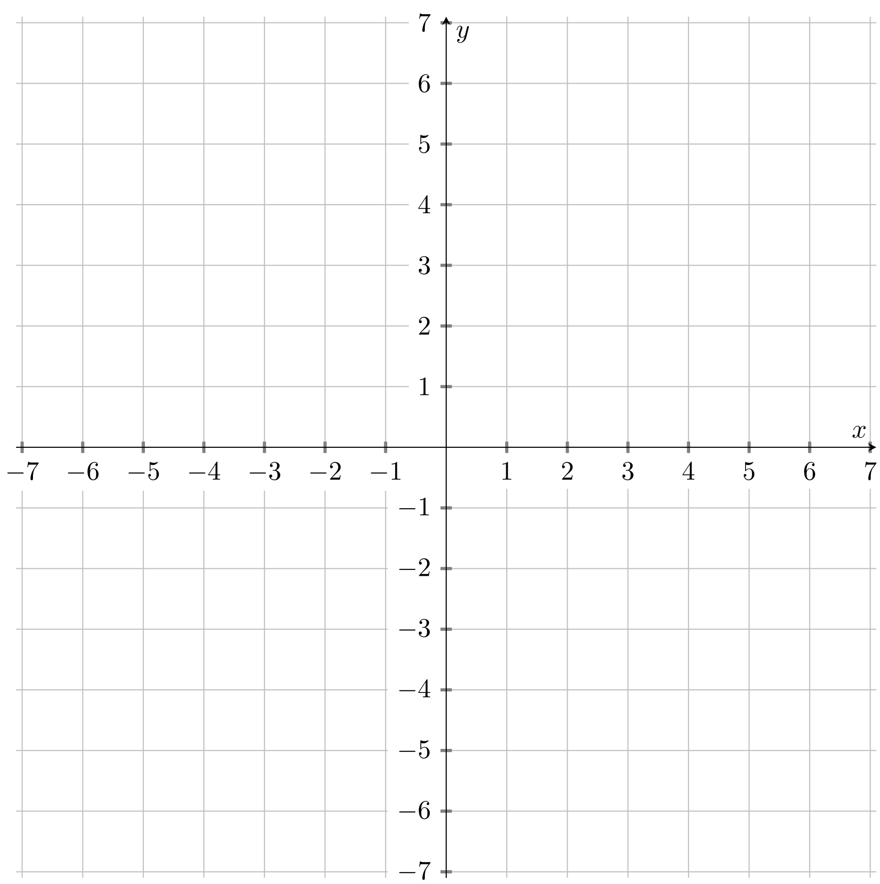
Exercise 1. *For the counting numbers* $1, 2, 3, ...,$ *not including halves or fractional numbers, name a function that is one-to-one, onto, and both one-to-one and onto.*

The "linear" or line function is fundamental for relating two numbers with a constant relationship in the Cartesian plane. You probably know about the Cartesian plane from algebra... it's just the $x - y$ coordinate graph.

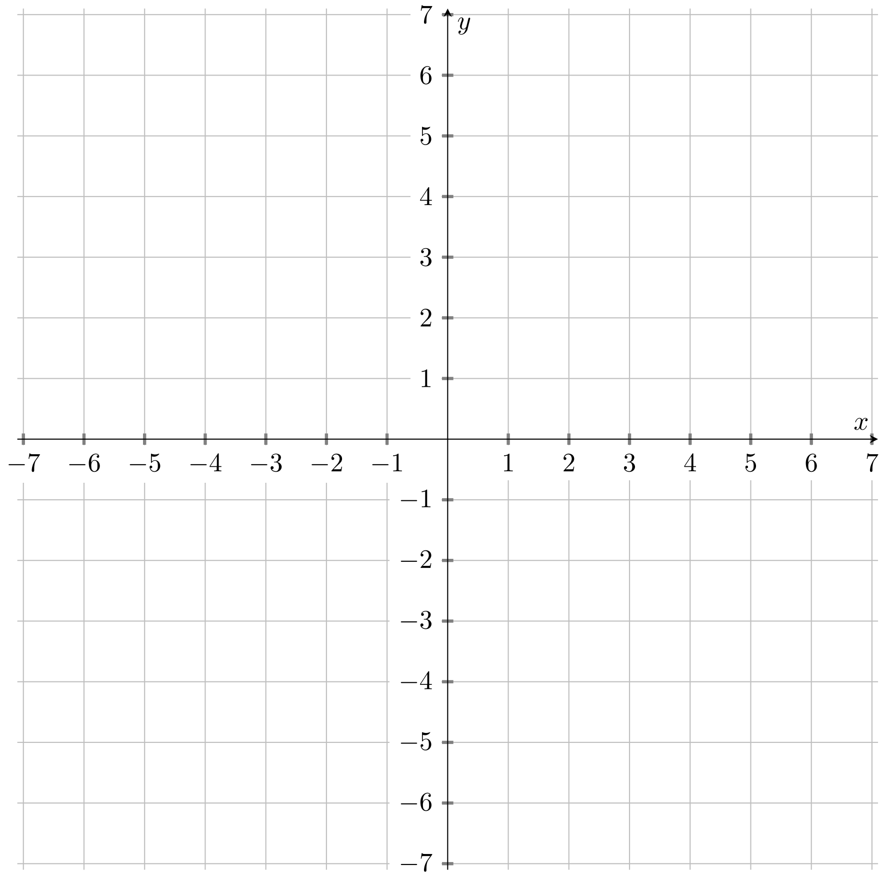
Pairs of these functions can be solved visually with graphing or with algebraic rearrangement. Each pair of linear equations will have exactly one solution (x, y) that satisfies both equations, and this point will be the intersection of the two lines, whereas many (x, y) will solve a single linear equation in the form:

$$y = f(x) = mx + b \tag{2}$$

Exercise 2. *Graph the points* $(6, 7)$ *and* $(-4, -8)$*. Find the slope of the line that runs through these points and its equation. Find another point on this line. Find its intercepts. Find the equation of a line perpendicular to this line, and the equation for a line parallel to it.*

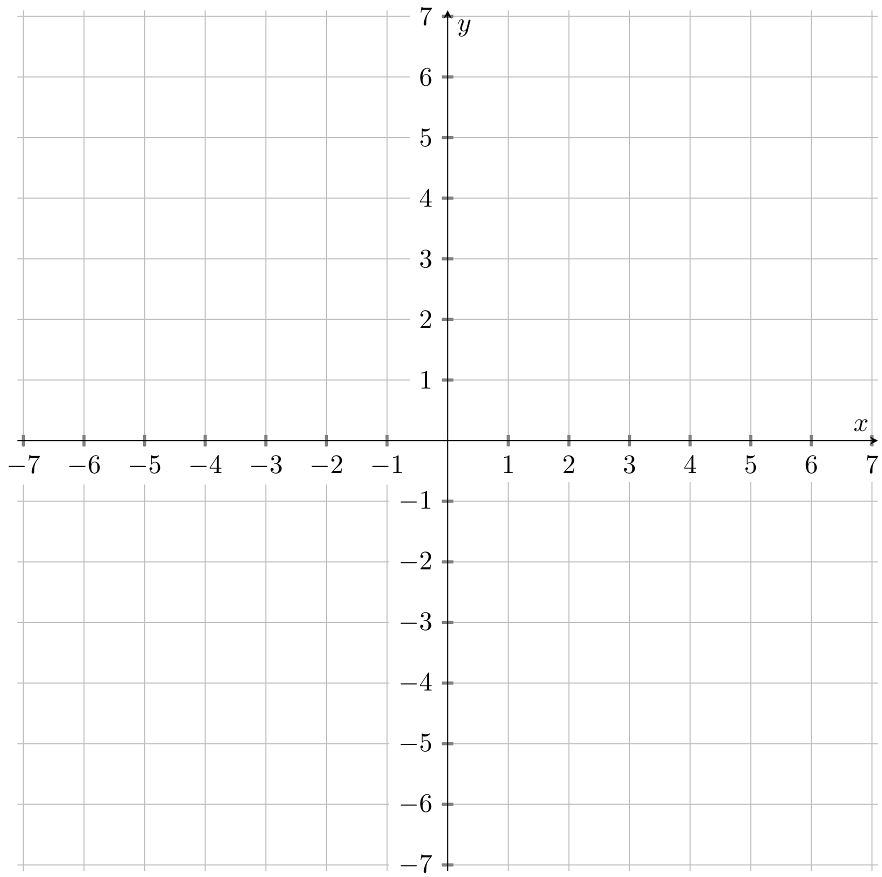


Exercise 3. *Graph* $f(x) = 1/x$. *Where is the function undefined? What is the behavior of the function at its big values of* x *and* y. *Find the inverse of this function. (This might be a trick question. What is the way to find a function's inverse in general?)*



Exercise 4. *Find the distance between the points* $(-2, -3)$ *and* $(3, 5)$ *with the Pythagorean theorem. Write this down for your notes as the "distance formula". Find their midpoint. Write down the midpoint formula for your notes. (Hint: it is like taking an average.)*

Exercise 5. *Graph $x^2 - 5x - 6 = 0$. Then solve. Solve first with guessing, then with completing the square, then with the quadratic formula.*



Exercise 6. *Write in standard form:*

$$\frac{2+3i}{4-2i} \tag{3}$$

1.2 Thurs, Feb. 22: Shapes of common functions

Consider the functions:

$$f(x) = c \tag{4}$$

$$f(x) = x^2 \tag{5}$$

$$f(x) = x^3 \tag{6}$$

$$x^2 + y^2 = 1 \tag{7}$$

(Note: You may have heard of the vertical line test. Does this pass the vertical line test? If not a function, what is it? Are there other ways we could describe a circle that would pass the test?)

$$f(x) = |x| \tag{8}$$

$$f(x) = \sqrt{x} \tag{9}$$

$$f(x) = \frac{1}{x} \tag{10}$$

$$f(x) = \frac{1}{x^2} \tag{11}$$

What is the basic shape of each of these functions? How do their compositions with other functions affect their shapes?

Exercise 7. *Let's invent some points and functions and work on translating between the two. Use the method of* 2^n *steps to find points on the graph with the brute force method.*

1.3 Fri, Feb. 23: Shapes... derivation of π

Exercise 8. *What is the perimeter of a circle with radius 4?*

Exercise 9. *What is the volume of a sphere with diameter 6?*

Exercise 10. *What is the volume of a regular pyramid with side length 3?*

Exercise 11. *Imagine a sphere inside a cube. The sphere touches the cube once on all six sides. What is the volume between the shapes?*

Exercise 12. *Why is* π 3.14*? Can you come up with a proof for the value for* π*? Can you come up with a proof for the area of a circle?*

Exercise 13. *Write down all the shape equations for surface and volume for future reference... (You can look these up on the internet. Try to have about 10.)*

x

2 Polynomials

2.1 Tues, Feb. 27: General form of quadratics

Polynomials are important in mathematics and life. It's worth thinking about why polynomials appear so often... I don't expect that you fully understand, but I do want us to spend some time thinking about how they relate important ideas about space and distance. Does this connection seem familiar? It is related to the two interpretations (graphic, algebraic) of linear equations in the form $y = mx + b$.

> Why are quadratic polynomials useful? There seem to be two different but interacting reasons. The first is that quadratic functions of a real variable are always either convex or concave and therefore have a unique maximum or minimum. The second is that quadratic functions are intimately related to bilinear forms and therefore can be accessed using linear algebra. —Paul Siegel, *Mathoverflow*

Polynomials have the form:

$$a_n x^n + \ldots + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{12}$$

Where the "poly" means many and the "nomial" refers to terms... or things added to each other. A polynomial with only one term is called a mononomial, one with two is called a binomial... and the highest order (or power) term designates whether it is quadratic, cubic, etc. The behavior of a polynomial is largely determined by its first term. Why is this? Think about properties of exponents. Is there a way we could say that x^5 is more "powerful" than x? This gives us the "leading coefficient test". And tells us how to graph polynomials. (Weird behavior often happens between the zeros, not outside of them). We begin investigating polynomials by thinking about quadratics. The standard form of a quadratic is:

$$f(x) = a(x - h)^2 + k \tag{13}$$

How could we derive this form from (12)? In this form, (h, k) will be the vertex. What does f look like if $a > 0$? If $a < 0$? Will the vertex be a min or max? For $ax^2 + bx + c = 0$, where a,b,c $\in \mathbb{R}$ and $a \neq 0$, $h = \frac{-b}{2a}$ and $k = f(h) = f(\frac{-b}{2a})$.

Exercise 14. *Write $f(x) = 2x^2 + 8x + 7$ in standard form by factoring around x, then working on the $x^2 + x$ term and inventing a constant value to make the factorization work... then adding that value to k.*

2.2 Thurs, Feb. 29: Complex roots of polynomial equations

"I think complex numbers are a little too advanced for humanity at this point... I don't think humans can understand them yet." —Nick, *a friend of mine and math researcher at UT*

2.3 Fri, Mar. 1: Review, graphing polynomials... higher order polynomials, division and the rational root test

3 Things of special interest: Exponentials, Parametric Equations

3.1 Tues, Mar. 5: Expansion of b^x

We often want to consider powers of a number where:

$$b^x = \underbrace{b \cdot b \cdot b}_{x \text{ times}} = y$$

From this idea we can derive "rules" for exponents.

$$x^0 = 1 \tag{14}$$

$$x^a \cdot x^b = x^{a+b} \tag{15}$$

$$(x^a)^b = x^{a \cdot b} \tag{16}$$

$$\frac{x^a}{x^b} = x^{a-b} \tag{17}$$

And we can think about when the numerator is just 1 so that:

$$\frac{1}{x^b} = x^{-b} \tag{18}$$

$$(\frac{x}{y})^a = \frac{x^a}{y^a} \tag{19}$$

$$x^{a/b} = \sqrt[b]{x^a} \tag{20}$$

Often students (including myself when I was learning these formulas) will get stuck on the idea of a fractional power $x^{1/2}$... how could you multiply a number by itself *less* than one time? But try not to think about this and instead think how division undoes multiplication...

Exercise 15. *Consider how fractional exponents work by comparing* $2^{3/3} = \sqrt[3]{2^3}$.

3.2 Thurs, Mar. 7: Logarithms

The logarithm language rewrites this in the format:

$$\log_b y = x \tag{21}$$

Where:

$$b^x = y \tag{22}$$

I've used the variables x and y in a consistent manner here, so you can copy between the formulas. But usually logs will be written such that the dependent variable, y, is the exponent... making them the inverse of b^x. The change of base format is very useful and helpful to consider as we understand logs:

$$\log_a x = \frac{\log_b x}{\log_b a} \tag{23}$$

Which we can prove by observing that:

$$y \log_b x = \log_b x^y \tag{24}$$

This is the power property of logs, and there are also properties for products

$$\log_a uv = \log_a u + \log_a v \tag{25}$$

...and quotients:

$$\log_a \frac{u}{v} = \log_a u - \log_a v \tag{26}$$

Another important identity that we should think about is:

$$b^{\log_b x} = x \tag{27}$$

3.3 Fri, Mar. 8: Deriving e

There are two ways to derive e. The first is to expand our idea of the slope of a line:

$$y = mx + b \tag{28}$$

We know slope is a ratio of $\frac{\Delta x}{\Delta y}$ and for large x and y this can be expressed:

$$\frac{y_2 - y_1}{x_2 - x_1} \tag{29}$$

And we know y is really just $f(x)$ so this becomes:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{30}$$

But often we want to look at non-constant slopes... *i.e.* where the size of slope depends on where you are in the function. Here it's useful to think about what happens when x_2 and x_1 are very close together, *i.e.* when $x_2 \to x_1$. This is the beginning of the calculus idea. But for now, we'll just use this idea to compute e^x to get some experience with it, and to learn about e.

Exercise 16. *Let's say we want to consider something that grows at 2^t where t is time. What is the relationship between the growth rate of the function at a place and the value of the function at that place. Is there a function where this ratio is $1:1$?*

Exercise 16 offers us one way to prove e. We can also think about e as doing something, then doing it more times but less each time. Let's consider a thing that's continuously growing... and measuring it at smaller units of time but more frequently, perhaps at a half unit of time. We would find that we have to multiply two terms:

$$(1 + \frac{1}{2}) \cdot (1 + \frac{1}{2}) \tag{31}$$

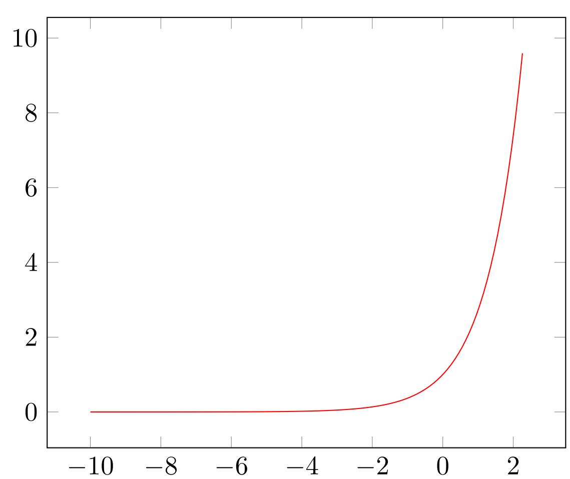
As we do this more and more, we'd get better approximations for e like:

$$(1 + \frac{1}{100}) \cdot (1 + \frac{1}{100}) \cdot ... \cdot (1 + \frac{1}{100}) = (1 + \frac{1}{100})^{100} \tag{32}$$

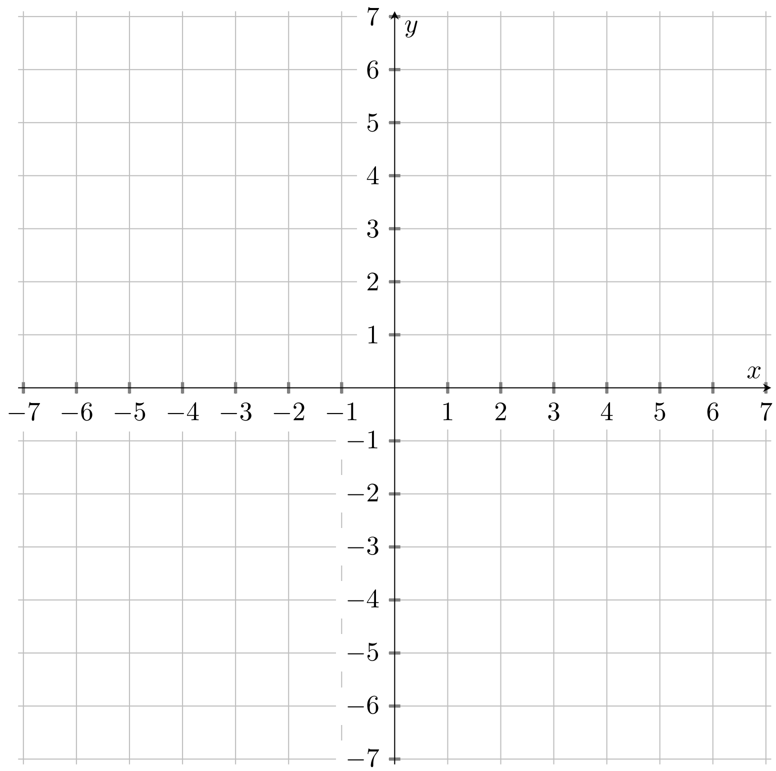
Throw this into the calculator and you'll find it's e. So in general:

$$e^x = \lim_{x \to 2} f(x) = (1 + \frac{1}{n})^n \tag{33}$$

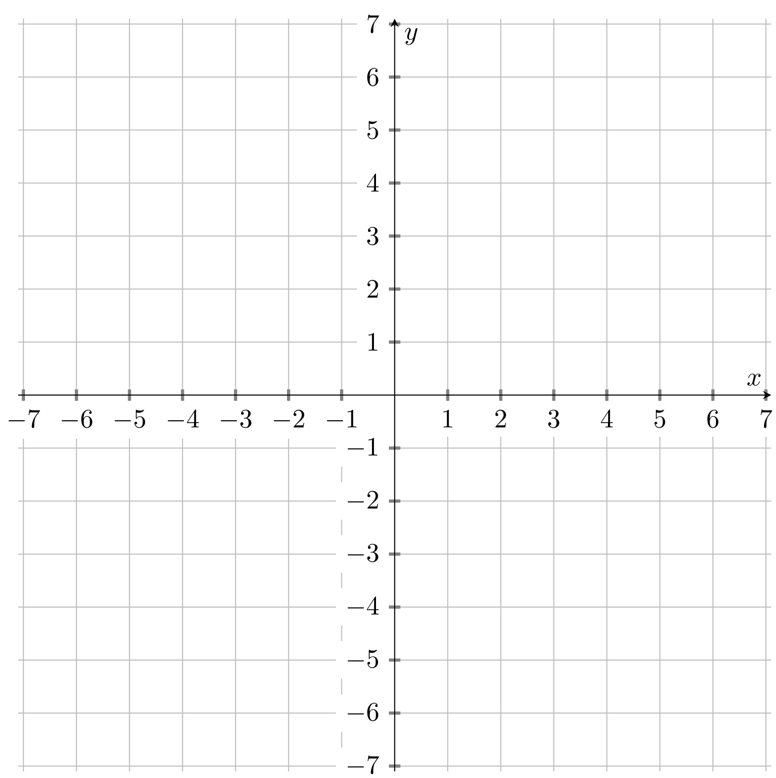
The graph of e^x is:



Exercise 17. *Graph* x^2. *Graph* 2^x.



Exercise 18. *Graph* $\log_2 x$. *Graph* ln. *Graph* $\log_{10} x$.



Exercise 19. *Find* $\log_{10} 1$

Exercise 20. *Find* $\log_{10} 10^7$

Exercise 21. *Find the approximate value of* $\log_2 10^3$ *rounded to the nearest whole number. No calculator! (Think about powers of 2.)*

Exercise 22. *Find* $e^{\ln \pi}$

Exercise 23. *Find* $\log_{10} 1,000,000,000$

Exercise 24. *Simplify* $\frac{a^3 b^7}{a^{-4} b^4}$

Exercise 25. *Simplify* $\frac{x^{2(z+8)}}{x^{-z}}$

Exercise 26. *Simplify* $\frac{8^3}{2^3 2^7}$

Exercise 27. *Simplify* $(\frac{a^{1/3}}{b^{1/6}})^3$

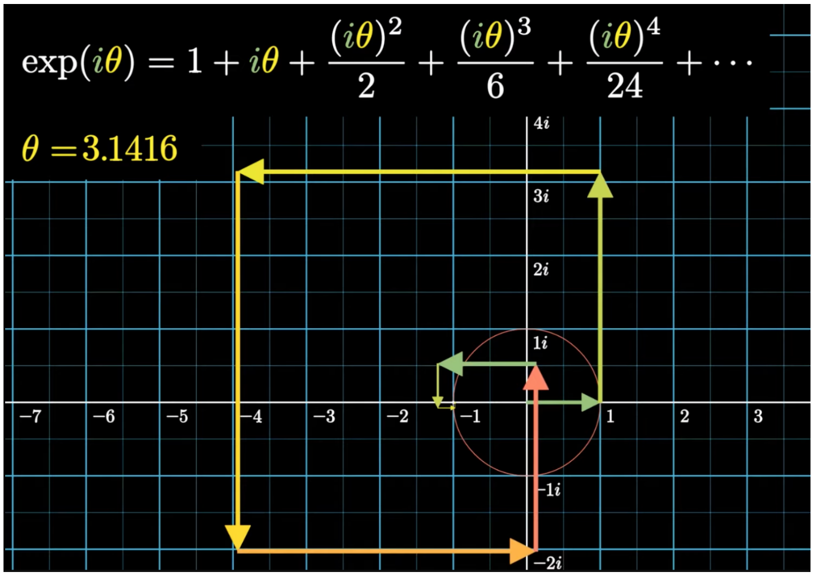
Exercise 28. *Rewrite with quotient rule for logs:* $\ln \frac{\sqrt{3x-5}}{7}$

Exercise 29. *Simplify* $\ln \frac{6}{e^2}$

Exercise 30. *Find* $\log_5 \frac{1}{125}$

Exercise 31. *Use change-of-base to rewrite this logarithm as a ratio of logarithms:* $\log_{1/2} x$*. Then graph the ratio and the original to verify equivalence.*

Exercise 32. *Assume* $\log \frac{a}{b} = m \log \frac{b}{a}$*. Find m.*



3.4 More about e

We learned one way to think about exponents...

$$e^x = \underbrace{e \cdot e \cdot e}_{x \text{ times}}$$

but in finding a series formula for e we also learned another way to think about exponents:

$$exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} ... \tag{34}$$

Exercise 33. *Show that when you fully expand $exp(x) \cdot exp(y)$, each term has the form $\frac{x^k y^m}{k!m!}$*

Exercise 34. *Show that when you expand $exp(x+y)$, each term has the form $\frac{1}{n!}\binom{n}{k} x^k y^{n-k}$.*

Exercise 35. *Compare results above to explain why $exp(x+y) = exp(x) \cdot exp(y)$*

3.5 Interest

x

4 Trigonometry and Euclid

4.1 Tues., Mar. 19

4.2 Thurs., Mar. 21

4.3 Fri., Mar. 22

5 Systems of equations

6 "The Matrix"

7 Sequences and series 1.0... polar coordinates smuggled in here

8 Limits

9 The derivative, derivation of $\ln(x)$ and other important derivatives

10 Integrals... the definite and the indefinite. The calculus approach to things you already know.

11 The idea of diff eq.

12 Sequences, Infinite series... Fourier, etc.