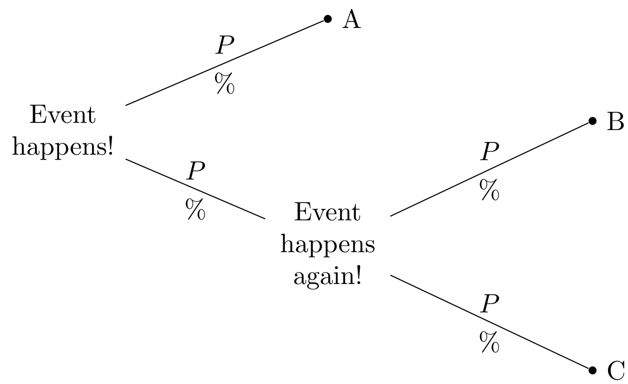
4 Probability and Combinatorics



4.1 What is probability?

What is probability? There's an interesting philosophical question in why we have *probability* at all. If the universe is just physical objects moving in predictable ways, why is there uncertainty? We can think about a few situations that might give rise to probability:

- Complexity *i.e.* two-joint pendulum
- Lack of knowledge
- Inherent randomness *i.e.* sub-atomic
- Prediction

We begin thinking about probability by examining the number of *ways something can happen*. This is called the "sample space" which technically consists of the set of all the ways a thing could happen, and a function that describes the likelihood of each of those things happening. The probability of an event $P(E_1)$ is defined as the number of times that specific event happens, divided by the number of times any event happens. The sum of the likelihood or probability of each event must be 1.

Exercise 1. *If the probabilities of four events $P(A, B, C, D) = 1$ and $P(A) = .1$, $P(B) = .23$, $P(C) = 0\ldots$ what is $P(D)$?*

Exercise 2. *What is the probability that two dice are rolled and the sum of the numbers rolled is greater than 4?*

4.1.1 Permutations with repetition

If we are allowed to make a choice several times, and continue choosing things we've chosen in the past, we call this a "permutation where repetition is allowed". Think of choosing the combination for a padlock. We may have three numbers, each ranging from 0 to 9 that we are allowed to use in setting our combination. Here the ways we can choose is given as:

$$10\ \cdot 10\ \cdot 10\ \cdot 10 = 10,000 \tag{1}$$

So if we choose from n things k times, the number of ways we can make this selection is:

$$n \cdot n \cdot n ... = n^k \tag{2}$$

Exercise 3. *How many ways can you set a locker room combination lock where there are three two-digit numbers that can each be chosen from 0 to 99?*

In the last case, the order the numbers are chosen in matters... because 21-11-57 is different from 1 1-21-57. But what if we're choosing teammates for a soccer match where the team "Ralph, Spud, Skeeter" is no different from the team "Spud, Skeeter, Ralph"?

4.1.2 Permutations without repetition

What if we don't allow repetition? Or if the pool of available things to be selected is depleted with each selection? Then we would have:

$$n! \tag{3}$$

Exercise 4. *How many ways can you arrange 10 books on a shelf?*

But what if we're not arranging *all* of the books but just some. We'd need a way of subtracting the rest of the multiplication steps, *i.e.* doing repeated division by all the options for choosing objects we're not choosing.

$$(n-1) \cdot (n-2) \cdot ... \cdot (n-k) = \frac{n!}{n-k!} =_n P_k \tag{4}$$

For n things chosen k at a time.

Exercise 5. *We can also consider an interesting variation on this theme. How many ways can you seat 3 people in a row so that in each arrangement one would have a different person on your left and right? How many ways can you seat 3 people in a circle such that in each arrangement you would have a different person to your left and right?*

4.1.3 Combinations

If order doesn't matter, we have a different case. How could we think about this conceptually to make a formula? One way is to consider dividing by the permutations for order... so since there are $n \cdot (n-1) \cdot (n-2)$... ways of arranging the 1^{st}, 2^{nd}, 3^{rd} places, etc., we can divide by this quantity to find the ways of combining things where order doesn't matter. This is expressed as a "combination" and the formula is:

$$\binom{n}{k} =_n C_k = \frac{n!}{k!(n-k)!} \tag{5}$$

Why? Let's derive this...

Exercise 6. *Write down all ways of combining A, B, and C in groups of 2 where repetition is not allowed and order matters. Then cross out all those that have the same combination of letters. What is the ratio of the new to the original number of groups?*

$$\begin{array}{ll} \text{AB} & AC \\ \text{BA} & BC \\ \text{CA} & CB \end{array}$$

So we can take number of permutations, and divide by the factorial of the number of groups to get the number of *unique* combinations.

4.1.4 Combinations with repetition

We can also think about a last case, combinations with repetition, but I think it's a little outside the scope of the short time we have to derive it... so I'll just present it for completeness but don't worry about it.

$$\binom{k+n-1}{k} =_n C_k = \frac{(k+n-1)!}{k!(n-1)!} \tag{6}$$

Exercise 7. *How many ways can you pick a team of 4 from 6 players?*

Exercise 8. *How many ways can you make a two-scoop ice cream cone from 8 flavors?*

Exercise 9. *How many ways could you form a committee of 10 where 5 must be male, 5 female, from a student body of 100 where 63 students are female and 37 are male?*

Exercise 10. *How many ways can you arrange a classroom of 16 desks where there are two pairs of students who must always sit next to each other?*

Exercise 11. *How many permutations are there of the word "mississippi"?*

The last exercise shows us that where elements of a set are repeated, and you want to find the number of *unique* orderings, you have to apply a slightly different formula than the ordinary one for permutations... what is it?

You must divide by the factorial of the number of times each non-unique element is found in the set, and do this for each non-unique element.

4.2 Note on Bayes' Theorem

$$P(\mathbf{B}) \cdot P(A|\mathbf{B}) = P(A) \cdot P(\mathbf{B}|A) \tag{7}$$

$$P(A|\mathbf{B}) = P(A)\frac{P(\mathbf{B}|A)}{P(\mathbf{B})} \tag{8}$$

Exercise 12. *Assume the probability of having tuberculosis (TB) is 0.0005, and a test for TB is 99% accurate. What is the probability one has TB if one tests positive for the disease?*

Exercise 13. *Bonus! Use Python to find all the permutations in the word "SKIN":*

```
from itertools import permutations

txt = ["".join(_) for _ in permutations('SKIN')]

print(txt)
```