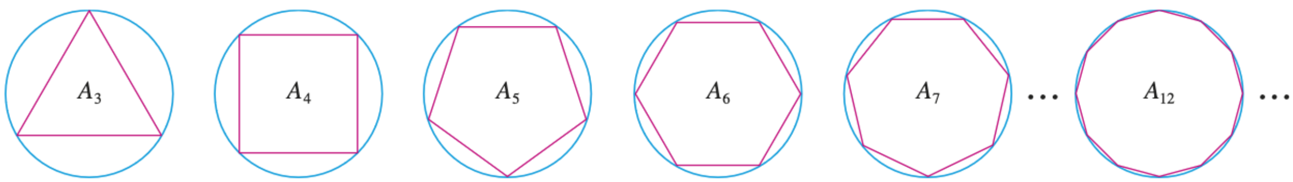
2 Rethinking functions with Limits

We can turn now from algebra and geometry review towards some of the ideas we can really call "calculus". At this point I think it'd be good to take some of Stewart's diagnostic tests for calculus found at the beginning of his book *Transcendentals*, available online. This will notify you of potential weak areas and let us know where to build extra algebra and geometry knowledge as we move on.

2.1 Two parts of calculus: change and the effects of change

We've already been introduced to limits through our investigation of π, where 3.14 is the number approached by taking successively better approximations of a circle by a polygon with more sides:



So

$$\pi r^2 = \lim_{n\to\infty} A_n \tag{49}$$

2.1.1 Motivating question: how often should interest compound? e^x!

If we think about a bank, we might imagine someone offering us to hold our money in exchange for paying us a small fee to use the money while we're not. Let's say this fee is 8% a year.

What would the bank owe us after a year? If we loaned 100$, this would be:

$$100\$ \cdot 1.08 = 108\$ \tag{50}$$

But there's nothing special about a year, instead of giving us 8$ at the end of the year, the bank could give us the $\frac{8}{4}$\$ after $\frac{1}{4}$ of the year, and do so 4 times throughout the year.

Exercise 135. *Which is better? A bank that compounds interest annually or one that compounds $\frac{1}{4}$ of interest quarterly?*

In general, the formula for the current amount of money for money earning interest at an interest rate of r that is compounded n times a year for some years t from an initial deposit of P is:

$$M = P(1 + \frac{r}{n})^{(nt)} \tag{51}$$

Exercise 136. *If you deposit 3,000$ in a bank that pays 9% interest and compounds twice a year, how much money would you have after 14 years?*

Exercise 137. *Would a bank that compounds every day, or even every second, be far better than a bank that compounds annually or only a little better? Compute some examples to prove this.*

Now recall our formula for e:

$$e^x = \lim_{x\to\infty} f(x) = (1 + \frac{1}{n})^n \tag{52}$$

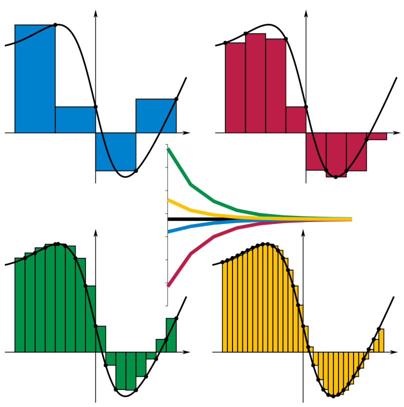
By visual inspection it's clear this is just the continuously compounding interest case!

2.1.2 Integrals with Riemann sums

In the same way a limit gets more specific with the idea of *rate*, the integral gets more specific with the idea of a *rate applied over time*. How do things accumulate?

$$\frac{Quantity}{Time} \cdot Time = Quantity \tag{53}$$

A fundamental calculus technique is to first answer a given problem with an approximation, then refine that approximation to make it better, then use limits in the refining process to find the exact answer. That is exactly what we will do here to develop a technique to find the area of more complicated regions.



When computing a Riemann sum, note that for each rectangle, length is...

$$x_1, x_2, x_3, ...x_n \tag{54}$$

And the heights are...

$$x_1^2, x_2^2, x_3^2, ...x_n^2 \tag{55}$$

Exercise 138. *Let's compute $y = x^2$ over the interval $[0, 2]$. Then find $\int_0^2 x^2\,dx$ and compare.*

In general a Riemann sum is:

$$A = \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx \tag{56}$$

Riemann sums are all calculated by partitioning the region into some sort of shapes (rectangles, trapezoids, parabolas, or cubics — sometimes infinitesimally small) that together form a region that is similar to the region being measured, then calculating the area for each of these shapes, and finally adding all of these small areas together. This approach can be used to find a numerical approximation for a definite integral even if the fundamental theorem of calculus does not make it easy to find a closed-form solution.

Exercise 139. *Let's explore left and right Riemann sums with this example. Find the Riemann sum of the function defined in the following table. (First draw its graph.)*

x	1	4	7	10
$f(x)$	6	8	3	5

So the choice of a starting place matters! Can we think of any better tessellations to approximate this figure?

Exercise 140. *Use a Riemann sum to integrate $sin(x)$. Is there an easier way to do this? Let's consider what's happening on the interval $[0, 2\pi]$. As always, begin by graphing or recalling the $sin(x)$ graph.*

$$\int_0^{2\pi} sin(x)\,dx$$

2.2 Limits

We've also already been introduced to the idea of limits through learning point-slope linear equations in the form $y = mx + b$. If we investigate m we find:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{57}$$

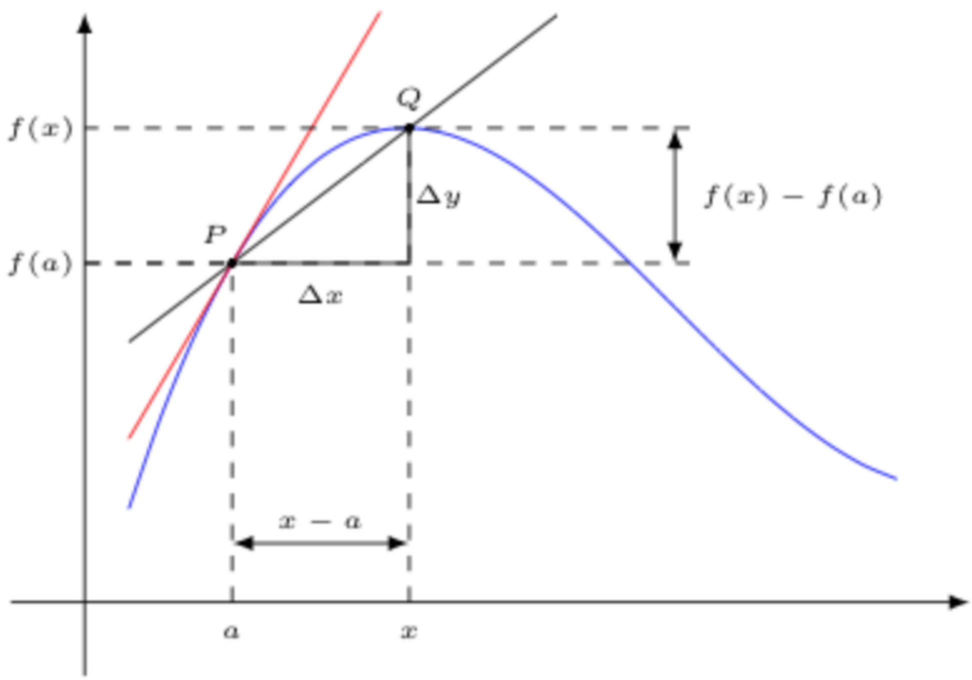
And we know y is really just $f(x)$ so this becomes:

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{58}$$

And for a smaller interval this is:

$$m = \lim_{x \to a} f(x) = \frac{f(x) - f(a)}{x - a} \tag{59}$$

What does this look like?



We can also consider a new variable, and set $x_1 = a$ and $x_2 = a + h$, considering when $h \lim 0$:
This gives:

$$\lim_{h \to 0} f(x) = \frac{f(a+h) - f(a)}{h} \tag{60}$$

Exercise 141. *Use the $h \to 0$ limit definition of derivative to find the derivative of $V(t) = 3 - 14t$.*

Exercise 142. *Use the $h \to 0$ limit definition of derivative to find the derivative of $f(x) = \frac{5}{x}$.*

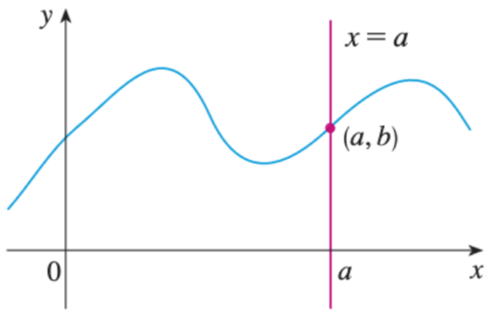
Exercise 143 (More advanced but only b/c the computation is messy.)**.** *Use the $h \to 0$ limit definition of derivative to find the derivative of $f(x) = \frac{x+1}{x+2}$.*

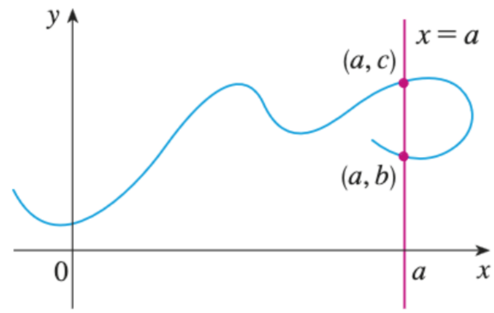
Exercise 144. *For $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$ evaluate $\frac{f(a+h)-f(a)}{h}$.*

2.3 Continuity

2.3.1 Domain restriction

It's useful to think about general facts about functions before thinking too hard about limits. Remember a function cannot map the same x to more than one unique y:





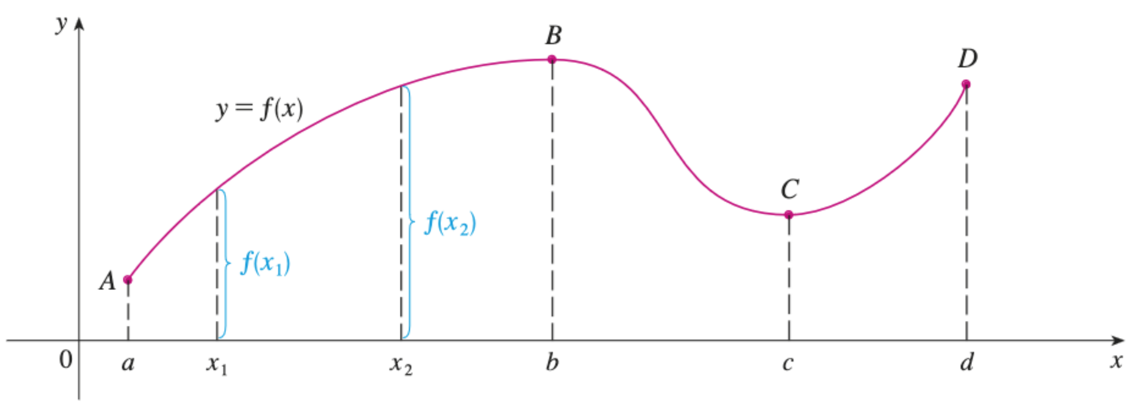
We investigate piecewise functions with the vertical line test by thinking about $x = y^2 - 2$.

Exercise 145. *Graph $x = y^2 - 2$ by factoring, and think about domain restrictions required to make it satisfy the definition of a function.*

2.3.2 Increasing or decreasing?

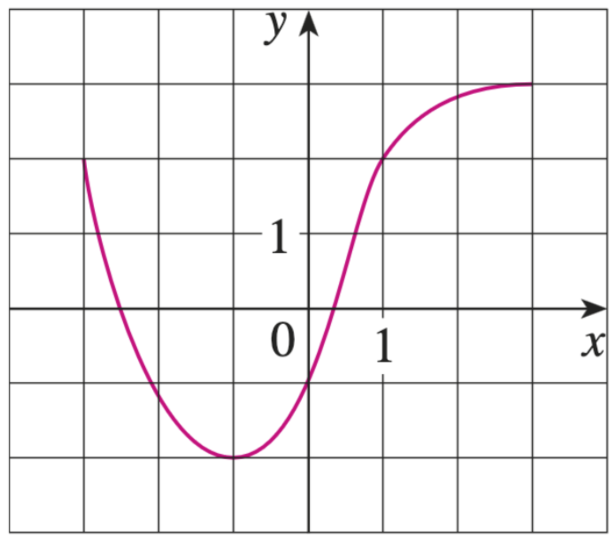
We also often want to know if a function is *increasing*, *decreasing* or neither...

Exercise 146. *For the function described in the image below, on what interval(s) is it increasing, decreasing, or neither?*



We can formalize this to say that a function is:

- If $f(x_1) < f(x_2)$ for $x_1 < X_2$
- If $f(x_1) > f(x_2)$ for $x_1 < X_2$



Exercise 147. *For the function described in the image above:*

- *What is* $f(-1)$*?*
- *What is* $f(2)$*?*
- *For what* x *is* $f(x) = 2$*?*
- *Find all* x *such that* $f(x) = 0$
- *What are domain and range of* f*?*
- *For slope,* m*, where is* $m = 0$*?*
- *On what intervals is* f *increasing, decreasing or neither?*

2.3.3 Limits and continuity

We say that a function f is *continuous* if

$$\lim_{x \to a} f(x) = f(a) \tag{61}$$

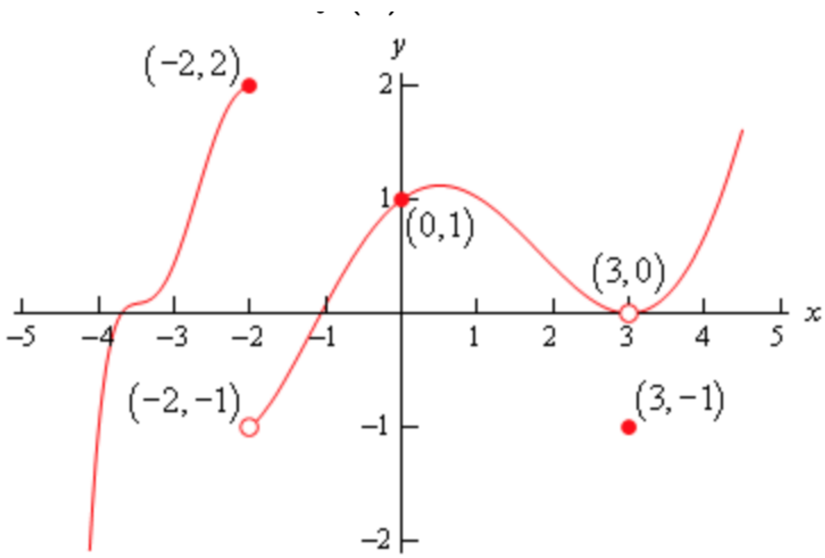
Exercise 148. *Is* $y = x^3$ *continuous?*

Exercise 149. *Is* $\frac{4t+10}{t^2-2t-15}$ *continuous? Why?*

And it is continuous on an interval if it is continuous on every point in the interval. Continuity also composes... which we will find in this example.

Exercise 150. *Evaluate* $\lim_{x \to 0} e^{sin(x)}$.

Exercise 151. *For the function f defined by the graph below, determine if $f(x)$ is continuous at $x = -2$, $x = 0$, and $x = 3$.*



Exercise 152. *I've given the graph of $f(x) = sin(\frac{1}{x})$ below. Graph it yourself to zoom in! Is $f(x)$ continuous at $x = 0$?*