Appendix B: Equilibria and Stability Analysis

B1. Equilibrium Points

Plasmid-free equilibrium

Setting $P = 0$, the plasmid-free equilibrium satisfies:

$$\frac{dF}{dt} = r(1-s)\left(1-\frac{F}{K}\right)F - \mu F = 0$$

Solving yields:

$$1 - \frac{F^*}{K} = \frac{\mu}{r(1-s)}$$

$$F^* = K\left(1 - \frac{\mu}{r(1-s)}\right)$$

which exists provided:

$$\mu < r(1-s)$$

Thus, the plasmid-free equilibrium is:

$$(F^*, P^*) = \left(K\left(1 - \frac{\mu}{r(1-s)}\right), 0\right)$$

Plasmid-free equilibrium

Setting $F = 0$, the plasmid-bearing equilibrium satisfies:

$$\frac{dP}{dt} = r(1-c)\left(1-\frac{P}{K}\right)P - \mu P - \delta P = 0$$

Solving yields:

$$1 - \frac{P^*}{K} = \frac{\mu + \delta}{r(1-c)}$$

$$P^* = K\left(1 - \frac{\mu + \delta}{r(1-c)}\right)$$

which exists provided:

$$\mu + \delta < r(1-c)$$

Thus, the plasmid-bearing equilibrium is:

$$(F^*, P^*) = \left(0, K\left(1 - \frac{\mu + \delta}{r(1-c)}\right)\right)$$

B2. Jacobian Matrix

Let $x = (F, P)^T$, the Jacobian is:

$$J(F,P) = \begin{pmatrix} \frac{\partial \dot{F}}{\partial F} & \frac{\partial \dot{F}}{\partial P} \\ \frac{\partial \dot{P}}{\partial F} & \frac{\partial \dot{P}}{\partial P} \end{pmatrix}$$

For the first row:

$$\dot{F} = r(1-s)F - \frac{r(1-s)}{K}F^2 - \frac{r(1-s)}{K}FP - \mu F + \delta P - \beta FP$$

with partial derivatives:

$$\frac{\partial \dot{F}}{\partial F} = r(1-s)\left(1 - \frac{2F+P}{K}\right) - \mu - \beta P$$

$$\frac{\partial \dot{F}}{\partial P} = -\frac{r(1-s)}{K}F + \delta - \beta F$$

For the second row:

$$\dot{P} = r(1-c)P - \frac{r(1-c)}{K}P^2 - \frac{r(1-c)}{K}FP - (\mu+\delta)P + \beta FP$$

with partial derivatives:

$$\frac{\partial \dot{P}}{\partial F} = -\frac{r(1-c)}{K}P + \beta P$$

$$\frac{\partial \dot{P}}{\partial P} = r(1-c)\left(1 - \frac{2P+F}{K}\right) - (\mu+\delta) + \beta F$$

Thus, the full Jacobian:

$$J(F,P) = \begin{pmatrix} r(1-s)\left(1 - \frac{2F+P}{K}\right) - \mu - \beta P & -\frac{r(1-s)}{K}F + \delta - \beta F \\ -\frac{r(1-c)}{K}P + \beta P & r(1-c)\left(1 - \frac{2P+F}{K}\right) - (\mu+\delta) + \beta F \end{pmatrix}$$

B3. Plasmid Invasion Criterion

To assess plasmid invasion, the Jacobian is evaluated at the plasmid-free equilibrium. The eigenvalue governing plasmid growth is given by:

$$\lambda_{invade} = r(1-c)\left(1 - \frac{F^*}{K}\right) - (\mu+\delta) + \beta F^*$$

Substitute F^* yields:

$$\lambda_{invade} = \beta K\left(1 - \frac{\mu}{r(1-s)}\right) - \delta - \mu\frac{c}{1-s}$$

Plasmids can invade if $\lambda_{invade} > 0$.

B4. Critical Transfer Rate

Setting $\lambda_{invade} = 0$ yields the critical conjugation rate required for invasion:

$$\beta_{crit} = \frac{\delta + \mu \frac{c}{1-s}}{K\left(1 - \frac{\mu}{r(1-s)}\right)}$$

When $s = 0$, this reduces to:

$$\beta_{crit} = \frac{\delta + \mu c}{K\left(1 - \frac{\mu}{r}\right)}$$

The critical transfer rate can be decomposed into additive components:

$$\beta_{crit} = \frac{1}{K}\left(\frac{\delta}{\left(1 - \frac{\mu}{r(1-s)}\right)} + \frac{\mu}{\left(1 - \frac{\mu}{r(1-s)}\right)} \frac{c}{1-s}\right)$$

This expression shows that segregational loss and plasmid cost independently increase the invasion threshold, with both effects amplified as the system approaches the demographic feasibility boundary $\mu \to r(1-s)$.

Defining as dimensionless ratios:

$$C = \frac{\mu c}{\delta(1-s)}, \qquad D = \frac{\mu}{r(1-s)}$$

the invasion threshold can be written compactly as:

$$\beta_{crit} = \frac{\delta}{K} \frac{1 + C}{1 - D}$$