
Appendix B: Equilibria and Stability Analysis

B.1 Equilibria Derivations

Plasmid-free only equilibrium (F^*, P^*) where $P^* = 0$

$$\frac{dF}{dt} = r(1-s) \left(1 - \frac{F}{K}\right) F - \mu F = 0$$

Factor F :

$$F \left[r(1-s) \left(1 - \frac{F}{K}\right) - \mu \right] = r(1-s) \left(1 - \frac{F^*}{K}\right) - \mu = 0$$

Solve for F^* :

$$1 - \frac{F^*}{K} = \frac{\mu}{r(1-s)}$$
$$F^* = K \left(1 - \frac{\mu}{r(1-s)}\right)$$

Thus, plasmid-free equilibrium:

$$(F^*, P^*) = \left(K \left(1 - \frac{\mu}{r(1-s)}\right), 0 \right)$$

And existence condition:

$$\mu < r(1-s)$$

Plasmid-bearing only equilibrium (F^*, P^*) where $F^* = 0$

$$\frac{dF}{dt} = r(1-c) \left(1 - \frac{P}{K}\right) P - \mu P - \delta P = 0$$

Factor P :

$$P \left[r(1-c) \left(1 - \frac{P}{K}\right) - \mu + \delta \right] = r(1-c) \left(1 - \frac{P^*}{K}\right) - \mu + \delta = 0$$

Solve for P^* :

$$1 - \frac{P^*}{K} = \frac{\mu + \delta}{r(1-c)}$$
$$P^* = K \left(1 - \frac{\mu + \delta}{r(1-c)}\right)$$

Thus, plasmid-bearing equilibrium:

$$(F^*, P^*) = \left(0, \left(1 - \frac{\mu + \delta}{r(1-c)}\right) \right)$$

And existence condition:

$$\mu + \delta < r(1-c)$$

B.2 Jacobian Matrix

The Jacobian is the matrix of the first derivatives:

$$J(F, P) = \begin{pmatrix} \frac{\partial \dot{F}}{\partial F} & \frac{\partial \dot{F}}{\partial P} \\ \frac{\partial \dot{P}}{\partial F} & \frac{\partial \dot{P}}{\partial P} \end{pmatrix}$$

The \dot{F} equation:

$$\dot{F} = r(1-s)F - \frac{r(1-s)}{K}F^2 - \frac{r(1-s)}{K}FP - \mu F + \delta P - \beta FP$$

With respect to F :

$$\frac{\partial \dot{F}}{\partial F} = r(1-s) - 2\frac{r(1-s)}{K}F - \frac{r(1-s)}{K}P - \mu - \beta P$$

$$\frac{\partial \dot{F}}{\partial F} = r(1-s) \left(1 - \frac{2F+P}{K}\right) - \mu - \beta P$$

With respect to P :

$$\frac{\partial \dot{F}}{\partial P} = -\frac{r(1-s)}{K}F + \delta - \beta F$$

The \dot{P} equation:

$$\dot{P} = r(1-c)P - \frac{r(1-c)}{K}P^2 - \frac{r(1-c)}{K}FP - (\mu + \delta)P + \beta FP$$

With respect to F :

$$\frac{\partial \dot{P}}{\partial F} = -\frac{r(1-c)}{K}P + \delta P$$

With respect to P :

$$\frac{\partial \dot{P}}{\partial P} = r(1-c) - 2\frac{r(1-c)}{K}P - \frac{r(1-c)}{K}F - (\mu + \delta) + \beta F$$

$$\frac{\partial \dot{P}}{\partial P} = r(1-c) \left(1 - \frac{2P+F}{K}\right) - (\mu + \delta) + \beta F$$

Thus, the full Jacobian:

$$J(F, P) = \begin{pmatrix} r(1-s) \left(1 - \frac{2F+P}{K}\right) - \mu - \beta P & -\frac{r(1-s)}{K}F + \delta - \beta F \\ -\frac{r(1-c)}{K}P + \delta P & r(1-c) \left(1 - \frac{2P+F}{K}\right) - (\mu + \delta) + \beta F \end{pmatrix}$$

B.3 Invasion Eigenvalue

To test whether plasmids can invade a plasmid-free population, evaluate the Jacobian at the plasmid-free equilibrium $(F^*, P^*) = (K(1 - \mu/r(1 - s)), 0)$:

The lower-right entry (effect on P) is the invasion eigenvalue:

$$\lambda_{invade} = \partial \dot{P} / \partial P|_{(F^*, 0)} = r(1 - c) \left(1 - \frac{F^*}{K}\right) - (\mu + \delta) + \beta F^*$$

Substitute $F^* = K(1 - \mu/r(1 - s))$:

$$\lambda_{invade} = r(1 - c) \frac{\mu}{r(1 - s)} - (\mu + \delta) + \beta K \left(1 - \frac{\mu}{r(1 - s)}\right)$$

Simplified:

$$\lambda_{invade} = \frac{r(1 - c)\mu}{r(1 - s)} - \mu - \delta + \beta K \left(1 - \frac{\mu}{r(1 - s)}\right)$$

$$\lambda_{invade} = \beta K \left(1 - \frac{\mu}{r(1 - s)}\right) - \delta - \mu \frac{c}{1 - s}$$

Plasmids can invade if $\lambda_{invade} > 0$.

B.4 Critical Values for Plasmid Invasion

Set $\lambda_{invade} = 0$ to solve for the critical transfer rate β_{crit} :

$$\beta_{crit} = \frac{\delta + \mu \frac{c}{1 - s}}{\left(1 - \frac{\mu}{r(1 - s)}\right)}$$

When $s = 0$:

$$\beta_{crit} = \frac{\delta + \mu c}{1 - \frac{\mu}{r}}$$

This is the minimum conjugation rate required for plasmids to successfully invade a plasmid-free population.

Set $\lambda_{invade} = 0$ to solve for the critical selective pressure s_{crit} :

$$s_{crit} = \frac{\mu \left(\frac{\beta K}{r} + c\right)}{\beta K - \delta}$$

This is minimum selective pressure needed for plasmids to provide a significant advantage over plasmid-free cells and invade the population.

B.5 Qualitative Interpretation of the Critical Transfer Rate

The critical transfer rate can be decomposed into additive components:

$$\beta_{crit} = \frac{\delta}{\left(1 - \frac{\mu}{r(1-s)}\right)} + \frac{\mu}{\left(1 - \frac{\mu}{r(1-s)}\right)} \frac{c}{1-s}$$

This form makes explicit that segregational loss (δ) and plasmid cost (c) independently increase the invasion threshold, with both contributions amplified by the demographic factor:

$$\frac{1}{1 - \frac{\mu}{r(1-s)}}$$

As host mortality approaches the effective growth rate ($r(1-s)$), the required transfer rate diverges.

Dividing by the plasmid cost yields:

$$\frac{\beta_{crit}}{c} = \frac{\delta}{\left(1 - \frac{\mu}{r(1-s)}\right)} + \frac{\mu}{(1-s)\left(1 - \frac{\mu}{r(1-s)}\right)}$$

In the absence of segregational loss ($\delta = 0$), plasmid invasion requires the transfer rate to exceed:

$$\beta_{crit} > \left(\frac{\mu}{(1-s)\left(1 - \frac{\mu}{r(1-s)}\right)} \right) c$$

demonstrating that conjugation must scale linearly with plasmid cost even when loss is negligible.

Similarly, normalising by the plasmid loss rate gives:

$$\frac{\beta_{crit}}{\delta} = \frac{1}{\left(1 - \frac{\mu}{r(1-s)}\right)} + \frac{\mu}{\left(1 - \frac{\mu}{r(1-s)}\right)} \frac{c}{\delta(1-s)}$$

When plasmid cost is small relative to loss ($c \ll \delta$), the invasion threshold is dominated by segregational loss, whereas high-cost plasmids require substantially higher transfer rates even when loss is low.

This can be written as dimensionless ratios:

$$C = \frac{\mu c}{\delta(1-s)}, D = \frac{\mu}{r(1-s)}$$

$$\beta_{crit} = \frac{\delta}{1-D} (1+C)$$

This shows that the invasion threshold increases linearly with the relative plasmid cost C and diverges as the system approaches the demographic feasibility boundary $D \rightarrow 1$, which corresponds to loss of the plasmid-free equilibrium.