# Appendix B: Equilibria and Stability Analysis

## B.1 Equilibria Derivations

Plasmid-free only equilibrium $(F^*, P^*)$ where $P^* = 0$

$$\frac{dF}{dt} = r(1-s)\left(1-\frac{F}{K}\right)F - \mu F = 0$$

Factor $F$:

$$F\left[r(1-s)\left(1-\frac{F}{K}\right) - \mu\right] = r(1-s)\left(1-\frac{F^*}{K}\right) - \mu = 0$$

Solve for $F^*$:

$$1-\frac{F^*}{K} = \frac{\mu}{r(1-s)}$$

$$F^* = K\left(1-\frac{\mu}{r(1-s)}\right)$$

Thus, plasmid-free equilibrium:

$$(F^*, P^*) = \left(K\left(1-\frac{\mu}{r(1-s)}\right), 0\right)$$

And existence condition:

$$\mu < r(1-s)$$

Plasmid-bearing only equilibrium $(F^*, P^*)$ where $F^* = 0$

$$\frac{dP}{dt} = r(1-c)\left(1-\frac{P}{K}\right)P - \mu P - \delta P = 0$$

Factor $P$:

$$P\left[r(1-c)\left(1-\frac{P}{K}\right) - \mu + \delta\right] = r(1-c)\left(1-\frac{P^*}{K}\right) - \mu + \delta = 0$$

Solve for $P^*$:

$$1-\frac{P^*}{K} = \frac{\mu+\delta}{r(1-c)}$$

$$P^* = K\left(1-\frac{\mu+\delta}{r(1-c)}\right)$$

Thus, plasmid-bearing equilibrium:

$$(F^*, P^*) = \left(0, K\left(1-\frac{\mu+\delta}{r(1-c)}\right)\right)$$

And existence condition:

$$\mu + \delta < r(1-c)$$

## B.2 Jacobian Matrix

The Jacobian is the matrix of the first derivatives:

$$J(F,P) = \begin{pmatrix} \frac{\partial \dot{F}}{\partial F} & \frac{\partial \dot{F}}{\partial P} \\ \frac{\partial \dot{P}}{\partial F} & \frac{\partial \dot{P}}{\partial P} \end{pmatrix}$$

The $\dot{F}$ equation:

$$\dot{F} = r(1-s)F - \frac{r(1-s)}{K}F^2 - \frac{r(1-s)}{K}FP - \mu F + \delta P - \beta FP$$

With respect to $F$:

$$\frac{\partial \dot{F}}{\partial F} = r(1-s) - 2\frac{r(1-s)}{K}F - \frac{r(1-s)}{K}P - \mu - \beta P$$

$$\frac{\partial \dot{F}}{\partial F} = r(1-s)\left(1 - \frac{2F+P}{K}\right) - \mu - \beta P$$

With respect to $P$:

$$\frac{\partial \dot{F}}{\partial P} = -\frac{r(1-s)}{K}F + \delta - \beta F$$

The $\dot{P}$ equation:

$$\dot{F} = r(1-c)P - \frac{r(1-c)}{K}P^2 - \frac{r(1-c)}{K}FP - (\mu+\delta)P + \beta FP$$

With respect to $F$:

$$\frac{\partial \dot{P}}{\partial F} = -\frac{r(1-c)}{K}P + \beta P$$

With respect to $P$:

$$\frac{\partial \dot{P}}{\partial P} = r(1-c) - 2\frac{r(1-c)}{K}P - \frac{r(1-c)}{K}F - (\mu+\delta) + \beta F$$

$$\frac{\partial \dot{P}}{\partial P} = r(1-c)\left(1 - \frac{2P+F}{K}\right) - (\mu+\delta) + \beta F$$

Thus, the full Jacobian:

$$J(F,P) = \begin{pmatrix} r(1-s)\left(1 - \frac{2F+P}{K}\right) - \mu - \beta P & -\frac{r(1-s)}{K}F + \delta - \beta F \\ -\frac{r(1-c)}{K}P + \beta P & r(1-c)\left(1 - \frac{2P+F}{K}\right) - (\mu+\delta) + \beta F \end{pmatrix}$$

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## B.3 Invasion Eigenvalue

To test whether plasmids can invade a plasmid-free population, evaluate the Jacobian at the plasmid-free equilibrium $(F^*, P^*) = (K(1 - \mu/r(1-s))),0)$:

The lower-right entry (effect on $P$) is the invasion eigenvalue:

$$\lambda_{invade} = \partial\dot{\mathrm{P}}/\partial P|_{(F^*,0)} = r(1-c)\left(1 - \frac{F^*}{K}\right) - (\mu + \delta) + \beta F^*$$

Substitute $F^* = K(1 - \mu/r(1-s))$:

$$\lambda_{invade} = r(1-c)\frac{\mu}{r(1-s)} - (\mu + \delta) + \beta K\left(1 - \frac{\mu}{r(1-s)}\right)$$

Simplified:

$$\lambda_{invade} = \frac{r(1-c)\mu}{r(1-s)} - \mu - \delta + \beta K\left(1 - \frac{\mu}{r(1-s)}\right)$$

$$\lambda_{invade} = \beta K\left(1 - \frac{\mu}{r(1-s)}\right) - \delta - \mu\frac{c}{1-s}$$

Plasmids can invade if $\lambda_{invade} > 0$.

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## B.4 Critical Values for Plasmid Invasion

Set $\lambda_{invade} = 0$ to solve for the critical transfer rate $\beta_{crit}$:

$$\beta_{crit} = \frac{\delta + \mu\frac{c}{1-s}}{K\left(1 - \frac{\mu}{r(1-s)}\right)}$$

When $s = 0$:

$$\beta_{crit} = \frac{\delta + \mu c}{K\left(1 - \frac{\mu}{r}\right)}$$

This is the minimum conjugation rate required for plasmids to successfully invade a plasmid-free population.

The critical transfer rate can be decomposed into additive components:

$$\beta_{crit} = \frac{1}{K}\left(\frac{\delta}{\left(1 - \frac{\mu}{r(1-s)}\right)} + \frac{\mu}{\left(1 - \frac{\mu}{r(1-s)}\right)}\frac{c}{1-s}\right)$$

This form makes explicit that segregational loss ($\delta$) and plasmid cost ($c$) independently increase the invasion threshold, with both contributions amplified by the demographic factor:

$$\frac{1}{1 - \frac{\mu}{r(1-s)}}$$

As host mortality approaches the effective growth rate ($r(1-s)$), the required transfer rate diverges.

Dividing by the plasmid cost yields:

$$\frac{\beta_{crit}}{c} = \frac{1}{K}\left(\frac{\delta}{c\left(1-\frac{\mu}{r(1-s)}\right)} + \frac{\mu}{(1-s)\left(1-\frac{\mu}{r(1-s)}\right)}\right)$$

In the absence of segregational loss ($\delta = 0$), plasmid invasion requires the transfer rate to exceed:

$$\beta_{crit} > \frac{\mu c}{K(1-s)\left(1-\frac{\mu}{r(1-s)}\right)}$$

demonstrating that conjugation must scale linearly with plasmid cost even when loss is negligible.

Similarly, normalising by the plasmid loss rate gives:

$$\frac{\beta_{crit}}{\delta} = \frac{1}{K\left(1-\frac{\mu}{r(1-s)}\right)}\left(1+\frac{\mu c}{\delta(1-s)}\right)$$

When plasmid cost is small relative to loss ($c \ll \delta$), the invasion threshold is dominated by segregational loss, whereas high-cost plasmids require substantially higher transfer rates even when loss is low.

This can be written as dimensionless ratios:

$$C = \frac{\mu c}{\delta(1-s)}, D = \frac{\mu}{r(1-s)}$$

$$\beta_{crit} = \frac{\delta}{K}\frac{1+C}{1-D}$$

This shows that the invasion threshold increases linearly with the relative plasmid cost $C$ and diverges as the system approaches the demographic feasibility boundary $D \to 1$, which corresponds to loss of the plasmid-free equilibrium.