NLO DIS cross section with massive quarks

1 Longitudinal cross section

$$\sigma_L^{\mathrm{NLO}} = \sigma_L^{\mathrm{IC}} + \sigma_L^{q\bar{q}} + \sigma_L^{q\bar{q}g} \tag{1}$$

1.1 Dipole part

$$\sigma_L^{\mathrm{IC}} = 4N_c\alpha_{\mathrm{em}}4Q^2 \sum_f e_f^2 \int_0^1 \mathrm{d}z \int \mathrm{d}^2\mathbf{x}_{01} \int \mathrm{d}^2\mathbf{b}\, [z(1-z)]^2\, K_0(|\mathbf{x}_{01}|\kappa_z)^2 N_{01} \tag{2}$$

$$\begin{aligned}\sigma_L^{q\bar{q}} = {}& 4N_c\alpha_{\mathrm{em}}4Q^2 \sum_f e_f^2 \int_0^1 \mathrm{d}z \int \mathrm{d}^2\mathbf{x}_{01} \int \mathrm{d}^2\mathbf{b}\, [z(1-z)]^2 \\ &\times \frac{\alpha_{\mathrm{s}} C_F}{\pi} \left[K_0(|\mathbf{x}_{01}|\kappa_z)^2 \left\{ \frac{5}{2} - \frac{\pi^2}{3} + \ln^2\left(\frac{z}{1-z}\right) + \Omega_{\mathcal{V}}^L(\gamma;z) + L(\gamma;z) \right\} + K_0(|\mathbf{x}_{01}|\kappa_z) I_{\mathcal{V}}^L(z,\mathbf{x}_{01}) \right] N_{01}\end{aligned} \tag{3}$$

Here $\kappa_v = \sqrt{v(1-v)Q^2 + m_f^2}$, $N_{01}(Y) = \mathrm{Re}[1 - S(\mathbf{x}_{01}, Y)]$ is the dipole amplitude.

$$\begin{aligned} L(\gamma;z) = {}& \mathrm{Li}_2\left(\frac{1}{1-\frac{1}{2z}(1-\gamma)}\right) + \mathrm{Li}_2\left(\frac{1}{1-\frac{1}{2z}(1+\gamma)}\right) \\ &+ \mathrm{Li}_2\left(\frac{1}{1-\frac{1}{2(1-z)}(1-\gamma)}\right) + \mathrm{Li}_2\left(\frac{1}{1-\frac{1}{2(1-z)}(1+\gamma)}\right) \end{aligned} \tag{4}$$

$$\gamma = \sqrt{1 + \frac{4m_f^2}{Q^2}}. \tag{5}$$

$$\begin{aligned}\Omega_{\mathcal{V}}^L(\gamma;z) = {}& \frac{1}{2z}\left[\ln(1-z) + \gamma \ln\left(\frac{1+\gamma}{1+\gamma-2z}\right)\right] + \frac{1}{2(1-z)}\left[\ln(z) + \gamma\ln\left(\frac{1+\gamma}{1+\gamma-2(1-z)}\right)\right] \\ &+ \frac{1}{4z(1-z)}\left[(\gamma-1) + \frac{2m_f^2}{Q^2}\right] \ln\left(\frac{\overline{Q}^2 + m_f^2}{m_f^2}\right).\end{aligned} \tag{6}$$

$$I_{\mathcal{V}}^L(Q, m_f, z, \mathbf{x}_{01}) = \tilde{\mathcal{I}}_{\mathcal{V}_{(a)+(b)}}(z,\mathbf{x}_{01}) + \tilde{\mathcal{I}}_{\mathcal{V}_{(c)+(d)}}(z,\mathbf{x}_{01}), \tag{7}$$

with

$$\begin{aligned}\tilde{\mathcal{I}}_{\mathcal{V}_{(a)+(b)}}(z,\mathbf{x}_{01}) = {}& \int_0^1 \frac{\mathrm{d}\xi}{\xi}\left(-\frac{2\ln\xi}{1-\xi} + \frac{1+\xi}{2}\right) \\ &\times \left[2K_0\left(|\mathbf{x}_{01}|\kappa_z\right) - K_0\left(|\mathbf{x}_{01}|\sqrt{\kappa_z^2 + \frac{(1-z)\xi}{1-\xi}m_f^2}\right) - K_0\left(|\mathbf{x}_{01}|\sqrt{\kappa_z^2 + \frac{z\xi}{1-\xi}m_f^2}\right)\right]\end{aligned} \tag{8}$$

and

$$
\begin{aligned}
\tilde{\mathcal{I}}_{\mathcal{V}_{(c)+(d)}}(z,\mathbf{x}_{01}) =& m_f^2 \int_0^1 \mathrm{d}\xi \int_0^1 \mathrm{d}x \\
&\Bigg\{ \left[K_0\left(|\mathbf{x}_{01}|\kappa_z\right) - K_0\left(|\mathbf{x}_{01}|\sqrt{\frac{\kappa_z^2}{1-x} + \kappa(z,x,\xi)}\right)\right] \\
&\qquad \times \frac{C_m^L(z,x,\xi)}{(1-\xi)(1-x)\left[x(1-\xi)+\frac{\xi}{1-z}\right]\left[\frac{x}{1-x}\kappa_z^2 + \kappa(z,x,\xi)\right]} \\
&+ \left[K_0\left(|\mathbf{x}_{01}|\kappa_z\right) - K_0\left(|\mathbf{x}_{01}|\sqrt{\frac{\kappa_z^2}{1-x} + \kappa(1-z,x,\xi)}\right)\right] \\
&\qquad \times \frac{C_m^L(1-z,x,\xi)}{(1-\xi)(1-x)\left[x(1-\xi)+\frac{\xi}{z}\right]\left[\frac{x}{1-x}\kappa_z^2 + \kappa(1-z,x,\xi)\right]} \Bigg\},
\end{aligned} \tag{9}
$$

The last two rows are the same as the first two with the substitution $z \leftrightarrow 1-z$. The coefficient C_m^L reads

$$
C_m^L(z,x,\xi) = \frac{z^2(1-\xi)}{1-z}\left[-\xi^2 + x(1-\xi)\frac{1+(1-\xi)\left(1+\frac{z\xi}{1-z}\right)}{x(1-\xi)+\frac{\xi}{1-z}}\right], \tag{10}
$$

and κ is defined as

$$
\kappa(z,x,\xi) = \frac{\xi m_f^2}{(1-\xi)(1-x)\left[x(1-\xi)+\frac{\xi}{1-z}\right]}\left[\xi(1-x) + x\left(1 - \frac{z(1-\xi)}{1-z}\right)\right]. \tag{11}
$$

JP: Tämä $I_{\mathcal{V}}^L$-lauseke on vähän pidempi kuin transverssiletterissä oleva, mutta ehkä soveltuu paremmin numeeriseen implementointiin? Alternatively, we can write

$$
\begin{aligned}
&\tilde{\mathcal{I}}_{\mathcal{V}_{(c)+(d)}}(z,\mathbf{x}_{01}) = -m_f^2 \int_0^z \frac{\mathrm{d}\chi}{1-\chi}\int_0^\infty \frac{\mathrm{d}u}{(u+1)^2}\frac{1}{\kappa_\chi^2} \\
&\times \left\{ \left[\frac{2\chi(u+1)}{u} + \frac{(2z-1)\chi(z-\chi)}{z(1-z)} - \frac{u(z-\chi)^2}{z(1-z)(u+1)}\right]\left[K_0\left(|\mathbf{x}_{01}|\sqrt{\kappa_z^2 + u\frac{1-z}{1-\chi}\kappa_\chi^2}\right) - K_0\left(|\mathbf{x}_{01}|\kappa_z\right)\right]\right\} \\
&+ (z \leftrightarrow 1-z).
\end{aligned} \tag{12}
$$

Again, the last row is the same as the first but with the substitution $z \leftrightarrow 1-z$.

1.2 $q\bar{q}g$ part

$$
\begin{aligned}
&\sigma_L^{q\bar{q}g} = 4N_c\alpha_{\mathrm{em}}4Q^2\frac{\alpha_{\mathrm{s}}C_F}{\pi}\sum_f e_f^2 \int \mathrm{d}^2\mathbf{x}_{01}\int \mathrm{d}^2\mathbf{x}_{20}\int \mathrm{d}^2\mathbf{b}\int_0^1 \mathrm{d}z_0 \int_{z_{2,\min}}^{1-z_0}\mathrm{d}z_2 \\
&\times \frac{1}{z_2}\Bigg\{ z_1^2\left[2z_0(z_0+z_2)+z_2^2\right]\left\{\frac{|\mathbf{x}_{20}|^2}{64}\left[\mathcal{G}_{(\mathrm{k})}^{(1;2)}\right]^2 N_{012} - \frac{e^{-|\mathbf{x}_{20}|^2/(|\mathbf{x}_{01}|^2 e^{\gamma_E})}}{|\mathbf{x}_{20}|^2}K_0\left(|\mathbf{x}_{01}|\sqrt{\overline{Q}_{(\mathrm{k})}^2 + m_f^2}\right)^2 N_{01}\right\} \\
&+ z_0^2\left[2z_1(z_1+z_2)+z_2^2\right]\left\{\frac{|\mathbf{x}_{21}|^2}{64}\left[\mathcal{G}_{(\mathrm{l})}^{(1;2)}\right]^2 N_{012} - \frac{e^{-|\mathbf{x}_{21}|^2/(|\mathbf{x}_{01}|^2 e^{\gamma_E})}}{|\mathbf{x}_{21}|^2}K_0\left(|\mathbf{x}_{01}|\sqrt{\overline{Q}_{(\mathrm{l})}^2 + m_f^2}\right)^2 N_{01}\right\} \\
&-\frac{1}{32}z_1 z_0\left[z_1(1-z_1)+z_0(1-z_0)\right]\left(\mathbf{x}_{20}\cdot\mathbf{x}_{21}\right)\mathcal{G}_{(\mathrm{k})}^{(1;2)}\mathcal{G}_{(\mathrm{l})}^{(1;2)} N_{012} \\
&+\frac{m_f^2}{16}z_2^4\left[\frac{z_1}{z_0+z_2}\mathcal{G}_{(\mathrm{k})}^{(1;1)} - \frac{z_0}{z_1+z_2}\mathcal{G}_{(\mathrm{l})}^{(1;1)}\right]^2 N_{012}\Bigg\}
\end{aligned} \tag{13}
$$

Here $N_{012} = \mathrm{Re}[1 - S_{012}]$.

$$
\begin{aligned}
\omega_{(\mathrm{k})} &= \frac{z_0 z_2}{z_1(z_0+z_2)^2}, & \omega_{(\mathrm{l})} &= \frac{z_1 z_2}{z_0(z_1+z_2)^2}, \\
\overline{Q}^2_{(\mathrm{k})} &= z_1(1-z_1)Q^2, & \overline{Q}^2_{(\mathrm{l})} &= z_0(1-z_0)Q^2, \\
\lambda_{(\mathrm{k})} &= \frac{z_1 z_2}{z_0}, & \lambda_{(\mathrm{l})} &= \frac{z_0 z_2}{z_1}, \\
\mathbf{x}_{2;(\mathrm{k})} &= \mathbf{x}_{20}, & \mathbf{x}_{2;(\mathrm{l})} &= \mathbf{x}_{21} \\
\mathbf{x}_{3;(\mathrm{k})} &= \mathbf{x}_{0+2;1}, & \mathbf{x}_{3;(\mathrm{l})} &= \mathbf{x}_{0;1+2}
\end{aligned}
\tag{14}
$$

$$
\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j, \qquad \mathbf{x}_{n+m;p} = -\mathbf{x}_{p;n+m} = \frac{z_n \mathbf{x}_n + z_m \mathbf{x}_m}{z_n + z_m} - \mathbf{x}_p. \tag{15}
$$

$$
\begin{aligned}
\mathcal{G}^{(a;b)}_{(\mathrm{x})} &= \int_0^\infty \frac{\mathrm{d}u}{u^a} \exp\!\left(-u\left[\overline{Q}^2_{(\mathrm{x})} + m_f^2\right] - \frac{|\mathbf{x}_{3;(\mathrm{x})}|^2}{4u}\right) \int_0^{u/\omega_{(\mathrm{x})}} \frac{\mathrm{d}t}{t^b} \exp\!\left(-t\omega_{(\mathrm{x})}\lambda_{(\mathrm{x})} m_f^2 - \frac{|\mathbf{x}_{2;(\mathrm{x})}|^2}{4t}\right) \\
&= \int_0^\infty \mathrm{d}u \int_0^{u/\omega_{(\mathrm{x})}} \mathrm{d}t \, \frac{1}{u^a t^b} g_{(\mathrm{x})}(t,u)
\end{aligned}
\tag{16}
$$

where

$$
g_{(\mathrm{x})}(t,u) = \exp\!\left(-u\left[\overline{Q}^2_{(\mathrm{x})} + m_f^2\right] - \frac{|\mathbf{x}_{3;(\mathrm{x})}|^2}{4u}\right) \exp\!\left(-t\omega_{(\mathrm{x})}\lambda_{(\mathrm{x})} m_f^2 - \frac{|\mathbf{x}_{2;(\mathrm{x})}|^2}{4t}\right) \tag{17}
$$

$$
\begin{aligned}
|\mathbf{x}_{3;(\mathrm{k})}|^2 &= \frac{z_0^2}{(z_0+z_2)^2}\mathbf{x}_{20}^2 + \mathbf{x}_{21}^2 - 2\frac{z_0}{z_0+z_2}\mathbf{x}_{20}\cdot\mathbf{x}_{21} \\
|\mathbf{x}_{3;(\mathrm{l})}|^2 &= \mathbf{x}_{20}^2 + \frac{z_1^2}{(z_1+z_2)^2}\mathbf{x}_{21}^2 - 2\frac{z_1}{z_1+z_2}\mathbf{x}_{20}\cdot\mathbf{x}_{21}
\end{aligned}
\tag{18}
$$

Yet another way to do this is to write

$$
\begin{aligned}
\mathcal{G}^{(1;2)}_{(\mathrm{x})} &= \int_0^\infty \frac{\mathrm{d}u}{u} \exp\!\left(-u\left[\overline{Q}^2_{(\mathrm{x})} + m_f^2\right] - \frac{|\mathbf{x}_{3;(\mathrm{x})}|^2}{4u}\right) \int_0^{u/\omega_{(\mathrm{x})}} \frac{\mathrm{d}t}{t^2} \exp\!\left(-t\omega_{(\mathrm{x})}\lambda_{(\mathrm{x})} m_f^2 - \frac{|\mathbf{x}_{2;(\mathrm{x})}|^2}{4t}\right) \\
&= \frac{8}{|\mathbf{x}_{2;(\mathrm{x})}|^2} K_0\left(\sqrt{\overline{Q}^2_{(\mathrm{x})} + m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{x})}|^2 + \omega_{(\mathrm{x})}|\mathbf{x}_{2;(\mathrm{x})}|^2}\right) \\
&\quad + \int_0^\infty \frac{\mathrm{d}u}{u} \exp\!\left(-u\left[\overline{Q}^2_{(\mathrm{x})} + m_f^2\right] - \frac{|\mathbf{x}_{3;(\mathrm{x})}|^2}{4u}\right) \int_0^{u/\omega_{(\mathrm{x})}} \frac{\mathrm{d}t}{t^2} \exp\!\left(-\frac{|\mathbf{x}_{2;(\mathrm{x})}|^2}{4t}\right)\left[e^{-t\omega_{(\mathrm{x})}\lambda_{(\mathrm{x})}m_f^2} - 1\right] \\
&= \frac{8}{|\mathbf{x}_{2;(\mathrm{x})}|^2} K_0\left(\sqrt{\overline{Q}^2_{(\mathrm{x})} + m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{x})}|^2 + \omega_{(\mathrm{x})}|\mathbf{x}_{2;(\mathrm{x})}|^2}\right) + \int_0^\infty \frac{\mathrm{d}u}{u} \int_0^{u/\omega_{(\mathrm{x})}} \frac{\mathrm{d}t}{t^2} \overline{g}_{(\mathrm{x})}(t,u) \\
&= \frac{8}{|\mathbf{x}_{2;(\mathrm{x})}|^2} K_0\left(\sqrt{\overline{Q}^2_{(\mathrm{x})} + m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{x})}|^2 + \omega_{(\mathrm{x})}|\mathbf{x}_{2;(\mathrm{x})}|^2}\right) + \int_0^1 \mathrm{d}y_u \int_0^1 \mathrm{d}y_t \, \frac{1}{y_u^2} \frac{1}{\omega_{(\mathrm{x})} t^2_{(\mathrm{x})}} \overline{g}_{(\mathrm{x})}(t_{(\mathrm{x})},u)
\end{aligned}
\tag{19}
$$

where $u = (1-y_u)/y_u$, $t_{(\mathrm{x})} = y_t \times u/\omega_{(\mathrm{x})}$, and

$$
\overline{g}_{(\mathrm{x})}(t,u) = \exp\!\left(-u\left[\overline{Q}^2_{(\mathrm{x})} + m_f^2\right] - \frac{|\mathbf{x}_{3;(\mathrm{x})}|^2}{4u}\right) \exp\!\left(-\frac{|\mathbf{x}_{2;(\mathrm{x})}|^2}{4t}\right)\left[e^{-t\omega_{(\mathrm{x})}\lambda_{(\mathrm{x})}m_f^2} - 1\right]. \tag{20}
$$

This should be easier to integrate numerically, as the divergent part (the Bessel function) has been extracted from the integral. With this we can write the $q\bar{q}g$ part as:

$$
\sigma_L^{q\bar{q}g} = 4N_c\alpha_{\mathrm{em}}4Q^2\frac{\alpha_{\mathrm{s}}C_F}{\pi}\sum_f e_f^2 \int \mathrm{d}^2\mathbf{x}_{01}\int \mathrm{d}^2\mathbf{x}_{20}\int \mathrm{d}^2\mathbf{b}\int_0^1 \mathrm{d}z_0 \int_{z_{2,\min}}^{1-z_0} \mathrm{d}z_2\,(I_1 + I_2 + I_3) \tag{21}
$$

where the different parts I_i have a different amount of integrals. They can be written as:

$$
\begin{aligned}
I_1 =& \frac{1}{z_2}\Bigg\{\frac{1}{|\mathbf{x}_{20}|^2} z_1^2\left[2z_0(z_0+z_2)+z_2^2\right]\Bigg[K_0\left(\sqrt{\overline{Q}^2_{(\mathrm{k})}+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{20})|^2}\right)^2 N_{012}\\
&-\frac{e^{-|\mathbf{x}_{20}|^2/(|\mathbf{x}_{01}|^2 e^{\gamma_E})}}{|\mathbf{x}_{20}|^2} K_0\left(|\mathbf{x}_{01}|\sqrt{\overline{Q}^2_{(\mathrm{k})}+m_f^2}\right)^2 N_{01}\Bigg]\\
&+\frac{1}{|\mathbf{x}_{21}|^2} z_0^2\left[2z_1(z_1+z_2)+z_2^2\right]\Bigg[K_0\left(\sqrt{\overline{Q}^2_{(\mathrm{l})}+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{l})}|^2+\omega_{(\mathrm{l})}|\mathbf{x}_{21})|^2}\right)^2 N_{012}\\
&-\frac{e^{-|\mathbf{x}_{21}|^2/(|\mathbf{x}_{01}|^2 e^{\gamma_E})}}{|\mathbf{x}_{21}|^2} K_0\left(|\mathbf{x}_{01}|\sqrt{\overline{Q}^2_{(\mathrm{l})}+m_f^2}\right)^2 N_{01}\Bigg]\\
&-2z_1z_0\left[z_1(1-z_1)+z_0(1-z_0)\right]\frac{\mathbf{x}_{20}\cdot\mathbf{x}_{21}}{|\mathbf{x}_{20}|^2|\mathbf{x}_{21}|^2}\\
&\times K_0\left(\sqrt{\overline{Q}^2_{(\mathrm{k})}+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{20})|^2}\right) K_0\left(\sqrt{\overline{Q}^2_{(\mathrm{l})}+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{l})}|^2+\omega_{(\mathrm{l})}|\mathbf{x}_{21})|^2}\right) N_{012}\Bigg\}
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
I_2 =& \int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}t\, \frac{1}{z_2}\\
&\times\Bigg\{z_1^2\left[2z_0(z_0+z_2)+z_2^2\right]\frac{1}{4}\frac{1}{ut^2}\theta\left(\frac{u}{\omega_{(\mathrm{k})}}-t\right)\overline{g}_{(\mathrm{k})}(t,u)K_0\left(\sqrt{\overline{Q}^2_{(\mathrm{k})}+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{20})|^2}\right)N_{012}\\
&+z_0^2\left[2z_1(z_1+z_2)+z_2^2\right]\frac{1}{4}\frac{1}{ut^2}\theta\left(\frac{u}{\omega_{(\mathrm{l})}}-t\right)\overline{g}_{(\mathrm{l})}(t,u)K_0\left(\sqrt{\overline{Q}^2_{(\mathrm{l})}+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{l})}|^2+\omega_{(\mathrm{l})}|\mathbf{x}_{21})|^2}\right)N_{012}\\
&-\frac{1}{4}z_1z_0\left[z_1(1-z_1)+z_0(1-z_0)\right](\mathbf{x}_{20}\cdot\mathbf{x}_{21})\frac{1}{t^2u}\\
&\times\Bigg[\frac{1}{|\mathbf{x}_{21}|^2}\theta\left(\frac{u}{\omega_{(\mathrm{k})}}-t\right)\overline{g}_{(\mathrm{k})}(t,u)K_0\left(\sqrt{\overline{Q}^2_{(\mathrm{l})}+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{l})}|^2+\omega_{(\mathrm{l})}|\mathbf{x}_{21})|^2}\right)\\
&+\frac{1}{|\mathbf{x}_{20}|^2}\theta\left(\frac{u}{\omega_{(\mathrm{l})}}-t\right)\overline{g}_{(\mathrm{l})}(t,u)K_0\left(\sqrt{\overline{Q}^2_{(\mathrm{k})}+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{20})|^2}\right)\Bigg]N_{012}\Bigg\}\\
=& \int_0^1 \mathrm{d}y_u \int_0^1 \mathrm{d}y_t\, \frac{1}{z_2}\\
&\times\Bigg\{z_1^2\left[2z_0(z_0+z_2)+z_2^2\right]\frac{1}{4}\frac{1}{t^2_{(\mathrm{k})}}\frac{1}{\omega_{(\mathrm{k})}y_u^2}\overline{g}_{(\mathrm{k})}(t_{(\mathrm{k})},u)K_0\left(\sqrt{\overline{Q}^2_{(\mathrm{k})}+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{20})|^2}\right)N_{012}\\
&+z_0^2\left[2z_1(z_1+z_2)+z_2^2\right]\frac{1}{4}\frac{1}{t^2_{(\mathrm{l})}}\frac{1}{\omega_{(\mathrm{l})}y_u^2}\overline{g}_{(\mathrm{l})}(t_{(\mathrm{l})},u)K_0\left(\sqrt{\overline{Q}^2_{(\mathrm{l})}+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{l})}|^2+\omega_{(\mathrm{l})}|\mathbf{x}_{21})|^2}\right)N_{012}\\
&-\frac{1}{4}z_1z_0\left[z_1(1-z_1)+z_0(1-z_0)\right](\mathbf{x}_{20}\cdot\mathbf{x}_{21})\\
&\times\Bigg[\frac{1}{|\mathbf{x}_{21}|^2}\frac{1}{t^2_{(\mathrm{k})}}\frac{1}{\omega_{(\mathrm{k})}y_u^2}\overline{g}_{(\mathrm{k})}(t_{(\mathrm{k})},u)K_0\left(\sqrt{\overline{Q}^2_{(\mathrm{l})}+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{l})}|^2+\omega_{(\mathrm{l})}|\mathbf{x}_{21})|^2}\right)\\
&+\frac{1}{|\mathbf{x}_{20}|^2}\frac{1}{t^2_{(\mathrm{l})}}\frac{1}{\omega_{(\mathrm{l})}y_u^2}\overline{g}_{(\mathrm{l})}(t_{(\mathrm{l})},u)K_0\left(\sqrt{\overline{Q}^2_{(\mathrm{k})}+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{20})|^2}\right)\Bigg]N_{012}\Bigg\}
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
I_3 = &\int_0^\infty \mathrm{d}u_1 \int_0^\infty \mathrm{d}u_2 \int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2 \, \frac{1}{z_2} \\
&\times \Bigg\{ z_1^2 \left[2z_0(z_0+z_2)+z_2^2\right] \frac{|\mathbf{x}_{20}|^2}{64} \frac{1}{u_1 u_2 t_1^2 t_2^2} \theta\left(\frac{u_1}{\omega_{(\mathrm{k})}} - t_1\right) \theta\left(\frac{u_2}{\omega_{(\mathrm{k})}} - t_2\right) \overline{g}_{(\mathrm{k})}(t_1,u_1)\overline{g}_{(\mathrm{k})}(t_2,u_2) N_{012} \\
&+ z_0^2 \left[2z_1(z_1+z_2)+z_2^2\right] \frac{|\mathbf{x}_{21}|^2}{64} \frac{1}{u_1 u_2 t_1^2 t_2^2} \theta\left(\frac{u_1}{\omega_{(\mathrm{l})}} - t_1\right) \theta\left(\frac{u_2}{\omega_{(\mathrm{l})}} - t_2\right) \overline{g}_{(\mathrm{l})}(t_1,u_1)\overline{g}_{(\mathrm{l})}(t_2,u_2) N_{012} \\
&- \frac{1}{32} z_1 z_0 \left[z_1(1-z_1)+z_0(1-z_0)\right] (\mathbf{x}_{20}\cdot\mathbf{x}_{21}) \frac{1}{u_1 u_2 t_1^2 t_2^2} \theta\left(\frac{u_1}{\omega_{(\mathrm{k})}} - t_1\right) \theta\left(\frac{u_2}{\omega_{(\mathrm{l})}} - t_2\right) \overline{g}_{(\mathrm{k})}(t_1,u_1)\overline{g}_{(\mathrm{l})}(t_2,u_2) N_{012} \\
&+ \frac{m_f^2}{16} z_2^4 \frac{1}{u_1 u_2 t_1 t_2} \left[\frac{z_1}{z_0+z_2} \theta\left(\frac{u_1}{\omega_{(\mathrm{k})}} - t_1\right) g_{(\mathrm{k})}(t_1,u_1) - \frac{z_0}{z_1+z_2} \theta\left(\frac{u_1}{\omega_{(\mathrm{l})}} - t_1\right) g_{(\mathrm{l})}(t_1,u_1) \right] \\
&\times \left[\frac{z_1}{z_0+z_2} \theta\left(\frac{u_2}{\omega_{(\mathrm{k})}} - t_2\right) g_{(\mathrm{k})}(t_2,u_2) - \frac{z_0}{z_1+z_2} \theta\left(\frac{u_2}{\omega_{(\mathrm{l})}} - t_2\right) g_{(\mathrm{l})}(t_2,u_2) \right] N_{012} \Bigg\} \\
= &\int_0^1 \mathrm{d}y_{u1} \int_0^1 \mathrm{d}y_{u2} \int_0^1 \mathrm{d}y_{t1} \int_0^1 \mathrm{d}y_{t2} \, \frac{1}{z_2} \\
&\times \Bigg\{ z_1^2 \left[2z_0(z_0+z_2)+z_2^2\right] \frac{|\mathbf{x}_{20}|^2}{64} \frac{1}{t_{1;(\mathrm{k})}^2 t_{2;(\mathrm{k})}^2} \frac{1}{\omega_{(\mathrm{k})} y_{u1}^2} \frac{1}{\omega_{(\mathrm{k})} y_{u2}^2} \overline{g}_{(\mathrm{k})}(t_{1;(\mathrm{k})},u_1)\overline{g}_{(\mathrm{k})}(t_{2;(\mathrm{k})},u_2) N_{012} \\
&+ z_0^2 \left[2z_1(z_1+z_2)+z_2^2\right] \frac{|\mathbf{x}_{21}|^2}{64} \frac{1}{t_{1;(\mathrm{l})}^2 t_{2;(\mathrm{l})}^2} \frac{1}{\omega_{(\mathrm{l})} y_{u1}^2} \frac{1}{\omega_{(\mathrm{l})} y_{u2}^2} \overline{g}_{(\mathrm{l})}(t_{1;(\mathrm{l})},u_1)\overline{g}_{(\mathrm{l})}(t_{2;(\mathrm{l})},u_2) N_{012} \\
&- \frac{1}{32} z_1 z_0 \left[z_1(1-z_1)+z_0(1-z_0)\right] (\mathbf{x}_{20}\cdot\mathbf{x}_{21}) \frac{1}{t_{1;(\mathrm{k})}^2 t_{2;(\mathrm{l})}^2} \frac{1}{\omega_{(\mathrm{k})} y_{u1}^2} \frac{1}{\omega_{(\mathrm{l})} y_{u2}^2} \overline{g}_{(\mathrm{k})}(t_{1;(\mathrm{k})},u_1)\overline{g}_{(\mathrm{l})}(t_{2;(\mathrm{l})},u_2) N_{012} \\
&+ \frac{m_f^2}{16} z_2^4 \frac{1}{y_{t1} y_{t2} u_1 u_2 y_{u1}^2 y_{u2}^2} \Bigg[\left(\frac{z_1}{z_0+z_2}\right)^2 g_{(\mathrm{k})}(t_{1;(\mathrm{k})},u_1) g_{(\mathrm{k})}(t_{2;(\mathrm{k})},u_2) \\
&+ \left(\frac{z_0}{z_1+z_2}\right)^2 g_{(\mathrm{l})}(t_{1;(\mathrm{l})},u_1) g_{(\mathrm{l})}(t_{2;(\mathrm{l})},u_2) - 2\frac{z_0}{z_1+z_2}\frac{z_1}{z_0+z_2} g_{(\mathrm{k})}(t_{1;(\mathrm{k})},u_1) g_{(\mathrm{l})}(t_{2;(\mathrm{l})},u_2) \Bigg] \Bigg\}
\end{aligned} \tag{24}
$$

where we also changed the integration variables: $t_{i;(\mathrm{x})} = y_{ti} \times u_i/\omega_{(\mathrm{x})}$ and $u = (1-y_u)/y_u$. This way we also get rid of the step functions.

2 Transverse cross section

$$
\sigma_T^{\mathrm{NLO}} = \sigma_T^{\mathrm{IC}} + \sigma_T^{q\bar{q}} + \sigma_T^{q\bar{q}g} \tag{25}
$$

2.1 Dipole part

$$
\sigma_T^{\mathrm{IC}} = 4N_c\alpha_{\mathrm{em}} \sum_f e_f^2 \int_0^1 \mathrm{d}z \int \mathrm{d}^2\mathbf{x}_{01} \int \mathrm{d}^2\mathbf{b} \left\{ \left[z^2+(1-z)^2\right] \left[\kappa_z K_1(|\mathbf{x}_{01}|\kappa_z)\right]^2 + m_f^2 K_0(|\mathbf{x}_{01}|\kappa_z)^2 \right\} N_{01} \tag{26}
$$

$$
\begin{aligned}
\sigma_T^{q\bar{q}} = &4N_c\alpha_{\mathrm{em}} \sum_f e_f^2 \int_0^1 \mathrm{d}z \int \mathrm{d}^2\mathbf{x}_{01} \int \mathrm{d}^2\mathbf{b} \\
&\times \frac{\alpha_{\mathrm{s}} C_F}{\pi} \left[K_1(|\mathbf{x}_{01}|\kappa_z)\kappa_z \left\{ \left[z^2+(1-z)^2\right] f_{\mathcal{V}}^T + \frac{2z-1}{2} f_{\mathcal{N}}^T \right\} + m_f^2 K_0(|\mathbf{x}_{01}|\kappa_z) f_{\mathcal{VMS}}^T \right] N_{01}
\end{aligned} \tag{27}
$$

where

$$
f_{\mathcal{V}}^T = \left\{ \frac{5}{2} - \frac{\pi^2}{3} + \ln^2\left(\frac{z}{1-z}\right) + \Omega_{\mathcal{V}}^T(\gamma;z) + L(\gamma;z) \right\} \kappa_z K_1(|\mathbf{x}_{01}|\kappa_z) + \tilde{I}_{\mathcal{V}}^T \tag{28}
$$

$$
f_{\mathcal{N}}^T = \Omega_{\mathcal{N}}^T \kappa_z K_1(|\mathbf{x}_{01}|\kappa_z) + \tilde{I}_{\mathcal{N}}^T \tag{29}
$$

$$f^T_{\mathcal{VMS}} = \left\{3 - \frac{\pi^2}{3} + \ln^2\left(\frac{z}{1-z}\right) + \Omega^T_{\mathcal{V}}(\gamma; z) + L(\gamma; z)\right\} K_0(|\mathbf{x}_{01}|\kappa_z) + \tilde{I}^T_{\mathcal{VMS}} \tag{30}$$

$$\begin{aligned}\Omega^T_{\mathcal{N}}(\gamma; z) =& \frac{1+z-2z^2}{z}\left[\ln(1-z) + \gamma\ln\left(\frac{1+\gamma}{1+\gamma-2z}\right)\right] \\ &+ \frac{1-z}{z}\left[\left(\frac{1}{2}+z\right)(\gamma-1) - \frac{m_f^2}{Q^2}\right]\ln\left(\frac{\overline{Q}^2 + m_f^2}{m_f^2}\right) - [z \leftrightarrow 1-z]\end{aligned} \tag{31}$$

$$\begin{aligned}\Omega^T_{\mathcal{V}}(\gamma; z) =& \left(1 + \frac{1}{2z}\right)\left[\ln(1-z) + \gamma\ln\left(\frac{1+\gamma}{1+\gamma-2z}\right)\right] \\ &- \frac{1}{2z}\left[\left(z+\frac{1}{2}\right)(1-\gamma) + \frac{m_f^2}{Q^2}\right]\ln\left(\frac{\overline{Q}^2 + m_f^2}{m_f^2}\right) + [z \leftrightarrow 1-z]\end{aligned} \tag{32}$$

$$\begin{aligned}\tilde{I}^T_{\mathcal{N}} =& \frac{2(1-z)}{z}\int_0^z \mathrm{d}\chi \int_0^\infty \frac{\mathrm{d}u}{(u+1)^3}\left\{\left[(2+u)uz + u^2\chi\right]\sqrt{\kappa_z^2 + u\frac{(1-z)}{1-\chi}\kappa_\chi^2}\,K_1\left(|\mathbf{x}_{01}|\sqrt{\kappa_z^2 + u\frac{1-z}{1-\chi}\kappa_\chi^2}\right)\right. \\ &\left.+\frac{m_f^2}{\kappa_\chi^2}\left(\frac{z}{1-z} + \frac{\chi}{1-\chi}[u - 2z - 2u\chi]\right)\left[\sqrt{\kappa_z^2 + u\frac{1-z}{1-\chi}\kappa_\chi^2}\,K_1\left(|\mathbf{x}_{01}|\sqrt{\kappa_z^2 + u\frac{1-z}{1-\chi}\kappa_\chi^2}\right) - \kappa_z K_1(|\mathbf{x}_{01}|\kappa_z)\right]\right\} \\ &- [z \leftrightarrow 1-z]\end{aligned} \tag{33}$$

$$\tilde{I}^T_{\mathcal{V}} = \tilde{I}^T_{\mathcal{V}1} + \tilde{I}^T_{\mathcal{V}2} \tag{34}$$

$$\begin{aligned}\tilde{I}^T_{\mathcal{V}1} =& \int_0^1 \mathrm{d}\xi\left[\frac{1}{\xi}\left(\frac{2\ln\xi}{1-\xi} - \frac{1+\xi}{2}\right)\left\{\sqrt{\kappa_z^2 + \frac{\xi(1-z)}{1-\xi}m_f^2}\,K_1\left(|\mathbf{x}_{01}|\sqrt{\kappa_z^2 + \frac{\xi(1-z)}{(1-\xi)}m_f^2}\right) - \kappa_z K_1(|\mathbf{x}_{01}|\kappa_z)\right\}\right. \\ &\left.- \left(\frac{\ln\xi}{(1-\xi)^2} + \frac{z}{1-\xi} + \frac{z}{2}\right)\frac{(1-z)m_f^2}{\sqrt{\kappa_z^2 + \frac{\xi(1-z)}{1-\xi}m_f^2}}K_1\left(|\mathbf{x}_{01}|\sqrt{\kappa_z^2 + \frac{\xi(1-z)}{1-\xi}m_f^2}\right)\right] + [z \leftrightarrow 1-z]\end{aligned} \tag{35}$$

$$\begin{aligned}\tilde{I}^T_{\mathcal{V}2} =& \int_0^z \mathrm{d}\chi\int_0^\infty \mathrm{d}u\left[-\frac{1}{1-\chi}\frac{1}{u(u+1)}\frac{m_f^2}{\kappa_\chi^2}\left[2\chi + \left(\frac{u}{1+u}\right)^2\frac{1}{z}(z-\chi)(1-2\chi)\right]\right. \\ &\times\left\{\sqrt{\kappa_z^2 + u\frac{1-z}{1-\chi}\kappa_\chi^2}\,K_1\left(|\mathbf{x}_{01}|\sqrt{\kappa_z^2 + u\frac{1-z}{1-\chi}\kappa_\chi^2}\right) - \kappa_z K_1(|\mathbf{x}_{01}|\kappa_z)\right\} \\ &- \frac{1}{(1-\chi)^2}\frac{1}{u+1}(z-\chi)\left[1 - \frac{2u}{1+u}(z-\chi) + \left(\frac{u}{1+u}\right)^2\frac{1}{z}(z-\chi)^2\right] \\ &\left.\times\frac{m_f^2}{\sqrt{\kappa_z^2 + u\frac{1-z}{1-\chi}\kappa_\chi^2}}K_1\left(|\mathbf{x}_{01}|\sqrt{\kappa_z^2 + u\frac{1-z}{1-\chi}\kappa_\chi^2}\right)\right] + [z \leftrightarrow 1-z]\end{aligned} \tag{36}$$

$$\tilde{I}^T_{\mathcal{VMS}} = \tilde{I}^T_{\mathcal{VMS}1} + \tilde{I}^T_{\mathcal{VMS}2} \tag{37}$$

$$\tilde{I}^T_{\mathcal{VMS}1} = \int_0^1 \mathrm{d}\xi \left[\frac{1}{\xi}\left(\frac{2\ln\xi}{1-\xi} - \frac{1+\xi}{2}\right)\left\{ K_0\left(|\mathbf{x}_{01}|\sqrt{\kappa_z^2 + \frac{\xi(1-z)}{(1-\xi)}m_f^2}\right) - K_0(|\mathbf{x}_{01}|\kappa_z)\right\} \right.$$
$$\left. + \left(-\frac{3}{2}\frac{1-z}{1-\xi} + \frac{1-z}{2}\right) K_0\left(|\mathbf{x}_{01}|\sqrt{\kappa_z^2 + \frac{\xi(1-z)}{1-\xi}m_f^2}\right)\right] + [z \leftrightarrow 1-z] \tag{38}$$

$$\tilde{I}^T_{\mathcal{VMS}2} = \int_0^z \mathrm{d}\chi \int_0^\infty \mathrm{d}u \left[\frac{1}{1-\chi}\frac{1}{(u+1)^2}\left[-z - \frac{u}{1+u}\frac{z+u\chi}{z}(\chi-(1-z))\right] K_0\left(|\mathbf{x}_{01}|\sqrt{\kappa_z^2 + u\frac{1-z}{1-\chi}\kappa_\chi^2}\right) \right.$$
$$+ \frac{1}{(u+1)^3}\left\{\frac{\kappa_z^2}{\kappa_\chi^2}\left[1 + u\frac{\chi(1-\chi)}{z(1-z)}\right] - \frac{m_f^2}{\kappa_\chi^2}\frac{\chi}{1-\chi}\left[2\frac{(1+u)^2}{u} + \frac{u}{z(1-z)}(z-\chi)^2\right]\right\}$$
$$\left. \times \left\{K_0\left(|\mathbf{x}_{01}|\sqrt{\kappa_z^2 + u\frac{1-z}{1-\chi}\kappa_\chi^2}\right) - K_0(|\mathbf{x}_{01}|\kappa_z)\right\}\right] + [z \leftrightarrow 1-z] \tag{39}$$

We can rewrite Eq. (27) by dividing it into parts with different amounts of integrals:

$$\begin{aligned}
\sigma_T^{q\bar{q}} &= 4N_c\alpha_{\mathrm{em}}\sum_f e_f^2 \int_0^1 \mathrm{d}z \int \mathrm{d}^2\mathbf{x}_{01}\int \mathrm{d}^2\mathbf{b}\,\frac{\alpha_{\mathrm{s}}C_F}{\pi} \\
&\times \left[[K_1(|\mathbf{x}_{01}|\kappa_z)\kappa_z]^2 \left\{ \left[z^2 + (1-z)^2\right]\left\{\frac{5}{2} - \frac{\pi^2}{3} + \ln^2\left(\frac{z}{1-z}\right) + \Omega^T_{\mathcal{V}}(\gamma;z) + L(\gamma;z)\right\} + \frac{2z-1}{2}\Omega^T_{\mathcal{N}}\right\} \right. \\
&+ m_f^2\left[K_0(|\mathbf{x}_{01}|\kappa_z)\right]^2\left\{3 - \frac{\pi^2}{3} + \ln^2\left(\frac{z}{1-z}\right) + \Omega^T_{\mathcal{V}}(\gamma;z) + L(\gamma;z)\right\} \\
&+ K_1(|\mathbf{x}_{01}|\kappa_z)\kappa_z\left\{\left[z^2+(1-z)^2\right]\tilde{I}^T_{\mathcal{V}1}\right\} + m_f^2 K_0(|\mathbf{x}_{01}|\kappa_z)\tilde{I}^T_{\mathcal{VMS}1} \\
&\left. + K_1(|\mathbf{x}_{01}|\kappa_z)\kappa_z\left\{\left[z^2+(1-z)^2\right]\tilde{I}^T_{\mathcal{V}2} + \frac{2z-1}{2}\tilde{I}^T_{\mathcal{N}}\right\} + m_f^2 K_0(|\mathbf{x}_{01}|\kappa_z)\tilde{I}^T_{\mathcal{VMS}2}\right] N_{01} \\
&= \sigma^{q\bar{q}}_{T0} + \sigma^{q\bar{q}}_{T1} + \sigma^{q\bar{q}}_{T2}
\end{aligned} \tag{40}$$

where

$$\begin{aligned}
\sigma_{T0}^{q\bar{q}} &= 4N_c\alpha_{\mathrm{em}}\sum_f e_f^2 \int_0^1 \mathrm{d}z \int \mathrm{d}^2\mathbf{x}_{01}\int \mathrm{d}^2\mathbf{b}\,\frac{\alpha_{\mathrm{s}}C_F}{\pi} \\
&\times \left[[K_1(|\mathbf{x}_{01}|\kappa_z)\kappa_z]^2 \left\{ \left[z^2 + (1-z)^2\right]\left\{\frac{5}{2} - \frac{\pi^2}{3} + \ln^2\left(\frac{z}{1-z}\right) + \Omega^T_{\mathcal{V}}(\gamma;z) + L(\gamma;z)\right\} + \frac{2z-1}{2}\Omega^T_{\mathcal{N}}\right\} \right. \\
&\left. + m_f^2\left[K_0(|\mathbf{x}_{01}|\kappa_z)\right]^2\left\{3 - \frac{\pi^2}{3} + \ln^2\left(\frac{z}{1-z}\right) + \Omega^T_{\mathcal{V}}(\gamma;z) + L(\gamma;z)\right\}\right] N_{01}
\end{aligned} \tag{41}$$

$$\begin{aligned}
\sigma_{T1}^{q\bar{q}} &= 4N_c\alpha_{\mathrm{em}}\sum_f e_f^2 \int_0^1 \mathrm{d}z \int \mathrm{d}^2\mathbf{x}_{01}\int \mathrm{d}^2\mathbf{b}\,\frac{\alpha_{\mathrm{s}}C_F}{\pi} \\
&\times \left[K_1(|\mathbf{x}_{01}|\kappa_z)\kappa_z\left\{\left[z^2+(1-z)^2\right]\tilde{I}^T_{\mathcal{V}1}\right\} + m_f^2 K_0(|\mathbf{x}_{01}|\kappa_z)\tilde{I}^T_{\mathcal{VMS}1}\right] N_{01}
\end{aligned} \tag{42}$$

$$\begin{aligned}
\sigma_{T2}^{q\bar{q}} &= 4N_c\alpha_{\mathrm{em}}\sum_f e_f^2 \int_0^1 \mathrm{d}z \int \mathrm{d}^2\mathbf{x}_{01}\int \mathrm{d}^2\mathbf{b}\,\frac{\alpha_{\mathrm{s}}C_F}{\pi} \\
&\times \left[K_1(|\mathbf{x}_{01}|\kappa_z)\kappa_z\left\{\left[z^2+(1-z)^2\right]\tilde{I}^T_{\mathcal{V}2} + \frac{2z-1}{2}\tilde{I}^T_{\mathcal{N}}\right\} + m_f^2 K_0(|\mathbf{x}_{01}|\kappa_z)\tilde{I}^T_{\mathcal{VMS}2}\right] N_{01}
\end{aligned} \tag{43}$$

The term $\sigma_{T0}^{q\bar{q}}$ has zero additional integrals, the term $\sigma_{T1}^{q\bar{q}}$ contains the integral $\int_0^1 \mathrm{d}\xi$, and the last term $\sigma_{T2}^{q\bar{q}}$ contains the double integral $\int_0^z \mathrm{d}\chi \int_0^\infty \mathrm{d}u \ldots + \int_0^{1-z} \mathrm{d}\chi \int_0^\infty \mathrm{d}u \ldots$. Note that the exchange $z \leftrightarrow 1-z$ also affects the upper limit in the χ integral.

2.2 $q\bar{q}g$ part

The following variables are used:

$$
\begin{aligned}
\omega_{(\mathrm{j})} &= \frac{z_0 z_2}{z_1(z_0+z_2)^2}, & \omega_{(\mathrm{k})} &= \frac{z_1 z_2}{z_0(z_1+z_2)^2},\\
\overline{Q}^2_{(\mathrm{j})} &= z_1(1-z_1)Q^2, & \overline{Q}^2_{(\mathrm{k})} &= z_0(1-z_0)Q^2,\\
\lambda_{(\mathrm{j})} &= \frac{z_1 z_2}{z_0}, & \lambda_{(\mathrm{k})} &= \frac{z_0 z_2}{z_1},\\
\mathbf{x}_{2;(\mathrm{j})} &= \mathbf{x}_{20}, & \mathbf{x}_{2;(\mathrm{k})} &= \mathbf{x}_{21},\\
\mathbf{x}_{3;(\mathrm{j})} &= \mathbf{x}_{0+2;1} = \mathbf{x}_{21} - \frac{z_0}{z_0+z_2}\mathbf{x}_{20},\\
\mathbf{x}_{3;(\mathrm{k})} &= \mathbf{x}_{0;1+2} = -\mathbf{x}_{20} + \frac{z_1}{z_1+z_2}\mathbf{x}_{21}
\end{aligned}
\tag{44}
$$

Note that these differ from the longitudinal! (k → j,l → k) Some useful relations:

$$
\begin{aligned}
|\mathbf{x}_{3;(\mathrm{j})}|^2 &= \frac{z_0^2}{(z_0+z_2)^2}\mathbf{x}_{20}^2 + \mathbf{x}_{21}^2 - 2\frac{z_0}{z_0+z_2}\mathbf{x}_{20}\cdot\mathbf{x}_{21}\\
|\mathbf{x}_{3;(\mathrm{k})}|^2 &= \mathbf{x}_{20}^2 + \frac{z_1^2}{(z_1+z_2)^2}\mathbf{x}_{21}^2 - 2\frac{z_1}{z_1+z_2}\mathbf{x}_{20}\cdot\mathbf{x}_{21}\\
\mathbf{x}_{2;(\mathrm{j})}\cdot\mathbf{x}_{3;(\mathrm{j})} &= \mathbf{x}_{20}\cdot\mathbf{x}_{21} - \frac{z_0}{z_0+z_2}|\mathbf{x}_{20}|^2\\
\mathbf{x}_{2;(\mathrm{k})}\cdot\mathbf{x}_{3;(\mathrm{k})} &= -\mathbf{x}_{20}\cdot\mathbf{x}_{21} + \frac{z_1}{z_1+z_2}|\mathbf{x}_{21}|^2\\
\mathbf{x}_{3;(\mathrm{j})}\cdot\mathbf{x}_{3;(\mathrm{k})} &= \frac{z_0}{z_0+z_2}|\mathbf{x}_{20}|^2 + \frac{z_1}{z_1+z_2}|\mathbf{x}_{21}|^2 - \left[1 + \frac{z_0 z_1}{(z_0+z_2)(z_1+z_2)}\right]\mathbf{x}_{20}\cdot\mathbf{x}_{21}\\
\mathbf{x}_{2;(\mathrm{j})}\cdot\mathbf{x}_{3;(\mathrm{k})} &= -|\mathbf{x}_{20}|^2 + \frac{z_1}{z_1+z_2}\mathbf{x}_{20}\cdot\mathbf{x}_{21}\\
\mathbf{x}_{2;(\mathrm{k})}\cdot\mathbf{x}_{3;(\mathrm{j})} &= |\mathbf{x}_{21}|^2 - \frac{z_0}{z_0+z_2}\mathbf{x}_{20}\cdot\mathbf{x}_{21}
\end{aligned}
\tag{45}
$$

$$
\sigma_T^{q\bar{q}g} = \sigma_T\big|_{q\bar{q}g}^{|(\mathrm{j})|^2+|\mathrm{k}|^2} + \sigma_T\big|_{q\bar{q}g}^{|(\mathrm{j})|^2_m+|\mathrm{k}|^2_m} + \sigma_T\big|_{q\bar{q}g}^{\mathrm{F}} + \sigma_T\big|_{q\bar{q}g}^{\mathrm{F}_m}
\tag{46}
$$

$$
\begin{aligned}
\sigma_T\big|_{q\bar{q}g}^{|(\mathrm{j})|^2+|\mathrm{k}|^2} &= 4N_c\alpha_{\mathrm{em}}\left(\frac{\alpha_{\mathrm{s}}C_F}{\pi}\right)\sum_f e_f^2\int\mathrm{d}^2\mathbf{x}_{01}\int\mathrm{d}^2\mathbf{x}_{20}\int\mathrm{d}^2\mathbf{b}\int_0^1\mathrm{d}z_0\int_{z_{2,\mathrm{min}}}^{1-z_0}\mathrm{d}z_2\\
&\times\frac{1}{z_2}\Bigg\{\frac{1}{(z_0+z_2)^2}\left[2z_0(z_0+z_2)+z_2^2\right]\left[1-2z_1(1-z_1)\right]\\
&\times\left[\frac{|\mathbf{x}_{3;(\mathrm{j})}|^2|\mathbf{x}_{2;(\mathrm{j})}|^2}{256}[\mathcal{G}^{(2;2)}_{(\mathrm{j})}]^2N_{012} - \frac{e^{-|\mathbf{x}_{2;(\mathrm{j})}|^2/(|\mathbf{x}_{01}|^2e^{\gamma_E})}}{|\mathbf{x}_{2;(\mathrm{j})}|^2}\left[\sqrt{\overline{Q}^2_{(\mathrm{j})}+m_f^2}K_1\left(|\mathbf{x}_{01}|\sqrt{\overline{Q}^2_{(\mathrm{j})}+m_f^2}\right)\right]^2N_{01}\right]\\
&+\frac{1}{(z_1+z_2)^2}\left[2z_1(z_1+z_2)+z_2^2\right]\left[1-2z_0(1-z_0)\right]\\
&\times\left[\frac{|\mathbf{x}_{3;(\mathrm{k})}|^2|\mathbf{x}_{2;(\mathrm{k})}|^2}{256}[\mathcal{G}^{(2;2)}_{(\mathrm{k})}]^2N_{012} - \frac{e^{-|\mathbf{x}_{2;(\mathrm{k})}|^2/(|\mathbf{x}_{01}|^2e^{\gamma_E})}}{|\mathbf{x}_{2;(\mathrm{k})}|^2}\left[\sqrt{\overline{Q}^2_{(\mathrm{k})}+m_f^2}K_1\left(|\mathbf{x}_{01}|\sqrt{\overline{Q}^2_{(\mathrm{k})}+m_f^2}\right)\right]^2N_{01}\right]\Bigg\}
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
\sigma_T\big|_{q\bar{q}g}^{|(\mathrm{j})|_m^2+|\mathrm{k}|_m^2} &= 4N_c\alpha_{\mathrm{em}}\left(\frac{\alpha_{\mathrm{s}}C_F}{\pi}\right)\sum_f e_f^2\int \mathrm{d}^2\mathbf{x}_{01}\int \mathrm{d}^2\mathbf{x}_{20}\int \mathrm{d}^2\mathbf{b}\int_0^1 \mathrm{d}z_0\int_{z_{2,\min}}^{1-z_0}\mathrm{d}z_2\\
&\times\frac{1}{z_2}m_f^2\Bigg\{\frac{1}{(z_0+z_2)^2}\left[2z_0(z_0+z_2)+z_2^2\right]\\
&\times\left[\frac{|\mathbf{x}_{2;(\mathrm{j})}|^2}{64}[\mathcal{G}_{(\mathrm{j})}^{(1;2)}]^2N_{012}-\frac{e^{-|\mathbf{x}_{2;(\mathrm{j})}|^2/(|\mathbf{x}_{01}|^2e^{\gamma_E})}}{|\mathbf{x}_{2;(\mathrm{j})}|^2}\left[K_0\left(|\mathbf{x}_{01}|\sqrt{\overline{Q}_{(\mathrm{j})}^2+m_f^2}\right)\right]^2N_{01}\right]\\
&+\frac{1}{(z_1+z_2)^2}\left[2z_1(z_1+z_2)+z_2^2\right]\\
&\times\left[\frac{|\mathbf{x}_{2;(\mathrm{k})}|^2}{64}[\mathcal{G}_{(\mathrm{k})}^{(1;2)}]^2N_{012}-\frac{e^{-|\mathbf{x}_{2;(\mathrm{k})}|^2/(|\mathbf{x}_{01}|^2e^{\gamma_E})}}{|\mathbf{x}_{2;(\mathrm{k})}|^2}\left[K_0\left(|\mathbf{x}_{01}|\sqrt{\overline{Q}_{(\mathrm{k})}^2+m_f^2}\right)\right]^2N_{01}\right]\Bigg\}
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
\sigma_T\big|_{q\bar{q}g}^{\mathrm{F}} &= 4N_c\alpha_{\mathrm{em}}\left(\frac{\alpha_{\mathrm{s}}C_F}{\pi}\right)\sum_f e_f^2\int \mathrm{d}^2\mathbf{x}_{01}\int \mathrm{d}^2\mathbf{x}_{20}\int \mathrm{d}^2\mathbf{b}\int_0^1 \mathrm{d}z_0\int_{z_{2,\min}}^{1-z_0}\mathrm{d}z_2\\
&\times\frac{1}{z_2}\frac{1}{2}\Bigg\{\frac{1}{64(z_0+z_2)(z_1+z_2)}\Big\{z_2(z_0-z_1)^2\left[\left(\mathbf{x}_{3;(\mathrm{j})}\cdot\mathbf{x}_{2;(\mathrm{j})}\right)\left(\mathbf{x}_{3;(\mathrm{k})}\cdot\mathbf{x}_{2;(\mathrm{k})}\right)-\left(\mathbf{x}_{3;(\mathrm{j})}\cdot\mathbf{x}_{2;(\mathrm{k})}\right)\left(\mathbf{x}_{3;(\mathrm{k})}\cdot\mathbf{x}_{2;(\mathrm{j})}\right)\right]\\
&-\left[z_1(z_0+z_2)+z_0(z_1+z_2)\right]\left[z_0(z_0+z_2)+z_1(z_1+z_2)\right]\left(\mathbf{x}_{2;(\mathrm{j})}\cdot\mathbf{x}_{2;(\mathrm{k})}\right)\left(\mathbf{x}_{3;(\mathrm{j})}\cdot\mathbf{x}_{3;(\mathrm{k})}\right)\Big\}\mathcal{G}_{(\mathrm{j})}^{(2;2)}\mathcal{G}_{(\mathrm{k})}^{(2;2)}\\
&-\frac{(z_0+z_2)z_1z_2}{16(z_1+z_2)^2}\left(\mathbf{x}_{2;(\mathrm{j})}\cdot\mathbf{x}_{3;(\mathrm{j})}\right)\mathcal{G}_{(\mathrm{j})}^{(2;2)}\mathcal{H}_{(\mathrm{k})}+\frac{(z_1+z_2)z_0z_2}{16(z_0+z_2)^2}\left(\mathbf{x}_{2;(\mathrm{k})}\cdot\mathbf{x}_{3;(\mathrm{k})}\right)\mathcal{G}_{(\mathrm{k})}^{(2;2)}\mathcal{H}_{(\mathrm{j})}\\
&-\frac{z_0^2z_1z_2}{16(z_0+z_2)^3}\left(\mathbf{x}_{2;(\mathrm{j})}\cdot\mathbf{x}_{3;(\mathrm{j})}\right)\mathcal{G}_{(\mathrm{j})}^{(2;2)}\mathcal{H}_{(\mathrm{j})}+\frac{z_1^2z_0z_2}{16(z_1+z_2)^3}\left(\mathbf{x}_{2;(\mathrm{k})}\cdot\mathbf{x}_{3;(\mathrm{k})}\right)\mathcal{G}_{(\mathrm{k})}^{(2;2)}\mathcal{H}_{(\mathrm{k})}\\
&+\frac{(z_0z_2)^2}{8(z_0+z_2)^4}[\mathcal{H}_{(\mathrm{j})}]^2+\frac{(z_1z_2)^2}{8(z_1+z_2)^4}[\mathcal{H}_{(\mathrm{k})}]^2\Bigg\}N_{012}
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
\sigma_T|^{\mathrm{F}_m}_{q\bar{q}g} =& 4N_c\alpha_{\mathrm{em}}\left(\frac{\alpha_{\mathrm{s}}C_F}{\pi}\right)\sum_f e_f^2\int \mathrm{d}^2\mathbf{x}_{01}\int \mathrm{d}^2\mathbf{x}_{20}\int \mathrm{d}^2\mathbf{b}\int_0^1 \mathrm{d}z_0\int_{z_{2,\min}}^{1-z_0}\mathrm{d}z_2\\
&\times\frac{1}{z_2}\frac{m_f^2}{2}\Bigg\{\frac{z_2^4}{64(z_0+z_2)^4}\left[4z_1(z_1-1)+2\right]|\mathbf{x}_{3;(\mathrm{j})}|^2[\mathcal{G}^{(2;1)}_{(\mathrm{j})}]^2-\frac{z_0z_1z_2^2}{16(z_0+z_2)^3}\left(\mathbf{x}_{3;(\mathrm{j})}\cdot\mathbf{x}_{2;(\mathrm{j})}\right)\mathcal{G}^{(1;2)}_{(\mathrm{j})}\mathcal{G}^{(2;1)}_{(\mathrm{j})}\\
&+\frac{z_2^4}{64(z_1+z_2)^4}\left[4z_0(z_0-1)+2\right]|\mathbf{x}_{3;(\mathrm{k})}|^2[\mathcal{G}^{(2;1)}_{(\mathrm{k})}]^2+\frac{z_0z_1z_2^2}{16(z_1+z_2)^3}\left(\mathbf{x}_{3;(\mathrm{k})}\cdot\mathbf{x}_{2;(\mathrm{k})}\right)\mathcal{G}^{(1;2)}_{(\mathrm{k})}\mathcal{G}^{(2;1)}_{(\mathrm{k})}\\
&-\frac{1}{32(z_0+z_2)(z_1+z_2)}\left[(2z_0+z_2)(2z_1+z_2)+z_2^2\right]\left(\mathbf{x}_{2;(\mathrm{j})}\cdot\mathbf{x}_{2;(\mathrm{k})}\right)\mathcal{G}^{(1;2)}_{(\mathrm{j})}\mathcal{G}^{(1;2)}_{(\mathrm{k})}\\
&+\frac{z_2^4}{32(z_0+z_2)^2(z_1+z_2)^2}\left[(2z_0+z_2)(2z_1+z_2)+z_2^2\right]\left(\mathbf{x}_{3;(\mathrm{j})}\cdot\mathbf{x}_{3;(\mathrm{k})}\right)\mathcal{G}^{(2;1)}_{(\mathrm{j})}\mathcal{G}^{(2;1)}_{(\mathrm{k})}\\
&+\frac{m_f^2}{16}\frac{2z_2^4}{(z_0+z_2)^4}[\mathcal{G}^{(1;1)}_{(\mathrm{j})}]^2+\frac{m_f^2}{16}\frac{2z_2^4}{(z_1+z_2)^4}[\mathcal{G}^{(1;1)}_{(\mathrm{k})}]^2\\
&-\frac{(z_0z_2)^2}{16(z_0+z_2)(z_1+z_2)^2}\left(\mathbf{x}_{2;(\mathrm{j})}\cdot\mathbf{x}_{3;(\mathrm{k})}\right)\mathcal{G}^{(1;2)}_{(\mathrm{j})}\mathcal{G}^{(2;1)}_{(\mathrm{k})}+\frac{(z_1z_2)^2}{16(z_0+z_2)^2(z_1+z_2)}\left(\mathbf{x}_{3;(\mathrm{j})}\cdot\mathbf{x}_{2;(\mathrm{k})}\right)\mathcal{G}^{(1;2)}_{(\mathrm{k})}\mathcal{G}^{(2;1)}_{(\mathrm{j})}\\
&-\frac{z_0z_1z_2^2}{16(z_0+z_2)^3}\left(\mathbf{x}_{2;(\mathrm{j})}\cdot\mathbf{x}_{3;(\mathrm{j})}\right)\mathcal{G}^{(1;1)}_{(\mathrm{j})}\mathcal{G}^{(2;2)}_{(\mathrm{j})}+\frac{z_0z_1z_2^2}{16(z_1+z_2)^3}\left(\mathbf{x}_{2;(\mathrm{k})}\cdot\mathbf{x}_{3;(\mathrm{k})}\right)\mathcal{G}^{(1;1)}_{(\mathrm{k})}\mathcal{G}^{(2;2)}_{(\mathrm{k})}\\
&-\frac{(z_0+z_2)z_2^2}{16(z_1+z_2)^2}\left(\mathbf{x}_{2;(\mathrm{j})}\cdot\mathbf{x}_{3;(\mathrm{j})}\right)\mathcal{G}^{(1;1)}_{(\mathrm{k})}\mathcal{G}^{(2;2)}_{(\mathrm{j})}+\frac{(z_1+z_2)z_2^2}{16(z_0+z_2)^2}\left(\mathbf{x}_{2;(\mathrm{k})}\cdot\mathbf{x}_{3;(\mathrm{k})}\right)\mathcal{G}^{(1;1)}_{(\mathrm{j})}\mathcal{G}^{(2;2)}_{(\mathrm{k})}\\
&+\frac{z_0z_2^3}{4(z_0+z_2)^4}\mathcal{H}_{(\mathrm{j})}\mathcal{G}^{(1;1)}_{(\mathrm{j})}+\frac{z_1z_2^3}{4(z_1+z_2)^4}\mathcal{H}_{(\mathrm{k})}\mathcal{G}^{(1;1)}_{(\mathrm{k})}\Bigg\}N_{012}
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
\mathcal{H}_{(\mathrm{x})}=&\int_0^\infty\frac{\mathrm{d}u}{u^2}\exp\Big\{-u\left[\overline{Q}^2_{(\mathrm{x})}+m_f^2(1+\lambda_{(\mathrm{x})})\right]\Big\}\exp\Big\{-\frac{|\mathbf{x}_{3;(\mathrm{x})}|^2+\omega_{(\mathrm{x})}|\mathbf{x}_{2;(\mathrm{x})}|^2}{4u}\Big\}\\
=&4\sqrt{\frac{\overline{Q}^2_{(\mathrm{x})}+m_f^2(1+\lambda_{(\mathrm{x})})}{|\mathbf{x}_{3;(\mathrm{x})}|^2+\omega_{(\mathrm{x})}|\mathbf{x}_{2;(\mathrm{x})}|^2}}K_1\left(\sqrt{\left(\overline{Q}^2_{(\mathrm{x})}+m_f^2(1+\lambda_{(\mathrm{x})})\right)\left(|\mathbf{x}_{3;(\mathrm{x})}|^2+\omega_{(\mathrm{x})}|\mathbf{x}_{2;(\mathrm{x})}|^2\right)}\right)
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
\mathcal{G}^{(2;2)}_{(\mathrm{x})}=&\int_0^\infty\frac{\mathrm{d}u}{u^2}\exp\left(-u\left[\overline{Q}^2_{(\mathrm{x})}+m_f^2\right]-\frac{|\mathbf{x}_{3;(\mathrm{x})}|^2}{4u}\right)\int_0^{u/\omega_{(\mathrm{x})}}\frac{\mathrm{d}t}{t^2}\exp\left(-t\omega_{(\mathrm{x})}\lambda_{(\mathrm{x})}m_f^2-\frac{|\mathbf{x}_{2;(\mathrm{x})}|^2}{4t}\right)\\
=&\frac{16}{|\mathbf{x}_{2;(\mathrm{x})}|^2}\sqrt{\frac{\overline{Q}^2_{(\mathrm{x})}+m_f^2}{|\mathbf{x}_{3;(\mathrm{x})}|^2+\omega_{(\mathrm{x})}|\mathbf{x}_{2;(\mathrm{x})}|^2}}K_1\left(\sqrt{\overline{Q}^2_{(\mathrm{x})}+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{x})}|^2+\omega_{(\mathrm{x})}|\mathbf{x}_{2;(\mathrm{x})}|^2}\right)\\
&+\int_0^\infty\frac{\mathrm{d}u}{u^2}\int_0^{u/\omega_{(\mathrm{x})}}\frac{\mathrm{d}t}{t^2}\overline{g}_{(\mathrm{x})}(t,u)\\
=&\frac{16}{|\mathbf{x}_{2;(\mathrm{x})}|^2}\sqrt{\frac{\overline{Q}^2_{(\mathrm{x})}+m_f^2}{|\mathbf{x}_{3;(\mathrm{x})}|^2+\omega_{(\mathrm{x})}|\mathbf{x}_{2;(\mathrm{x})}|^2}}K_1\left(\sqrt{\overline{Q}^2_{(\mathrm{x})}+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{x})}|^2+\omega_{(\mathrm{x})}|\mathbf{x}_{2;(\mathrm{x})}|^2}\right)\\
&+\int_0^1\mathrm{d}y_u\int_0^1\mathrm{d}y_t\,\frac{1}{y_u^2}\frac{1}{u\omega_{(\mathrm{x})}t^2_{(\mathrm{x})}}\overline{g}_{(\mathrm{x})}(t_{(\mathrm{x})},u)
\end{aligned}
\tag{52}
$$

We will again divide these into parts with a different amount of integrals. Eq. (52) will be used to write the functions in a more convergent form.

$$
\begin{aligned}
\sigma_T|_{q\bar{q}g}^{|(\mathrm{j})|^2+|\mathrm{k}|^2} =& 4N_c\alpha_{\mathrm{em}}\frac{\alpha_{\mathrm{s}}C_F}{\pi}\sum_f e_f^2\int \mathrm{d}^2\mathbf{x}_{01}\int \mathrm{d}^2\mathbf{x}_{20}\int \mathrm{d}^2\mathbf{b}\int_0^1 \mathrm{d}z_0\int_{z_{2,\min}}^{1-z_0}\mathrm{d}z_2 \\
&\times\left(I_1^T|_{q\bar{q}g}^{|(\mathrm{j})|^2+|\mathrm{k}|^2}+I_2^T|_{q\bar{q}g}^{|(\mathrm{j})|^2+|\mathrm{k}|^2}+I_3^T|_{q\bar{q}g}^{|(\mathrm{j})|^2+|\mathrm{k}|^2}\right)
\end{aligned}
\tag{53}
$$

where

$$
\begin{aligned}
I_1^T|_{q\bar{q}g}^{|(\mathrm{j})|^2+|\mathrm{k}|^2} &= \frac{1}{z_2}\Bigg\{\frac{1}{(z_0+z_2)^2}\left[2z_0(z_0+z_2)+z_2^2\right]\left[1-2z_1(1-z_1)\right] \\
&\times\frac{\overline{Q}^2_{(\mathrm{j})}+m_f^2}{|\mathbf{x}_{2;(\mathrm{j})}|^2}\Bigg[\frac{|\mathbf{x}_{3;(\mathrm{j})}|^2}{|\mathbf{x}_{3;(\mathrm{j})}|^2+\omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}\left[K_1\left(\sqrt{|\mathbf{x}_{3;(\mathrm{j})}|^2+\omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}\sqrt{\overline{Q}^2_{(\mathrm{j})}+m_f^2}\right)\right]^2 N_{012} \\
&-e^{-|\mathbf{x}_{2;(\mathrm{j})}|^2/(|\mathbf{x}_{01}|^2e^{\gamma_E})}\left[K_1\left(|\mathbf{x}_{01}|\sqrt{\overline{Q}^2_{(\mathrm{j})}+m_f^2}\right)\right]^2 N_{01}\Bigg] \\
&+\frac{1}{(z_1+z_2)^2}\left[2z_1(z_1+z_2)+z_2^2\right]\left[1-2z_0(1-z_0)\right] \\
&\times\frac{\overline{Q}^2_{(\mathrm{k})}+m_f^2}{|\mathbf{x}_{2;(\mathrm{k})}|^2}\Bigg[\frac{|\mathbf{x}_{3;(\mathrm{k})}|^2}{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}\left[K_1\left(\sqrt{\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{k})}|^2}\sqrt{\overline{Q}^2_{(\mathrm{k})}+m_f^2}\right)\right]^2 N_{012} \\
&-e^{-|\mathbf{x}_{2;(\mathrm{k})}|^2/(|\mathbf{x}_{01}|^2e^{\gamma_E})}\left[K_1\left(|\mathbf{x}_{01}|\sqrt{\overline{Q}^2_{(\mathrm{k})}+m_f^2}\right)\right]^2 N_{01}\Bigg]\Bigg\}
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
I_2^T|_{q\bar{q}g}^{|(\mathrm{j})|^2+|\mathrm{k}|^2} &= \int_0^1\mathrm{d}y_u\int_0^1\mathrm{d}y_t\,\frac{1}{z_2}N_{012} \\
&\times\Bigg\{\frac{1}{y_u^2}\frac{1}{u\omega_{(\mathrm{j})}t_{(\mathrm{j})}^2}\frac{1}{(z_0+z_2)^2}\left[2z_0(z_0+z_2)+z_2^2\right]\left[1-2z_1(1-z_1)\right] \\
&\times\overline{g}_{(\mathrm{j})}(t_{(\mathrm{j})},u)\frac{|\mathbf{x}_{3;(\mathrm{j})}|^2}{8}\sqrt{\frac{\overline{Q}^2_{(\mathrm{j})}+m_f^2}{|\mathbf{x}_{3;(\mathrm{j})}|^2+\omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}}K_1\left(\sqrt{\overline{Q}^2_{(\mathrm{j})}+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{j})}|^2+\omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}\right) \\
&+\frac{1}{y_u^2}\frac{1}{u\omega_{(\mathrm{k})}t_{(\mathrm{k})}^2}\frac{1}{(z_1+z_2)^2}\left[2z_1(z_1+z_2)+z_2^2\right]\left[1-2z_0(1-z_0)\right] \\
&\times\overline{g}_{(\mathrm{k})}(t_{(\mathrm{k})},u)\frac{|\mathbf{x}_{3;(\mathrm{k})}|^2}{8}\sqrt{\frac{\overline{Q}^2_{(\mathrm{k})}+m_f^2}{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}}K_1\left(\sqrt{\overline{Q}^2_{(\mathrm{k})}+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}\right)\Bigg\}
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
I_3^T|_{q\bar{q}g}^{|(\mathrm{j})|^2+|\mathrm{k}|^2} &= \int_0^1\mathrm{d}y_{u1}\int_0^1\mathrm{d}y_{t1}\int_0^1\mathrm{d}y_{u2}\int_0^1\mathrm{d}y_{t2}\,\frac{1}{z_2}N_{012} \\
&\times\Bigg\{\frac{1}{y_{u1}^2}\frac{1}{u_1\omega_{(\mathrm{j})}t_{1;(\mathrm{j})}^2}\frac{1}{y_{u2}^2}\frac{1}{u_2\omega_{(\mathrm{j})}t_{2;(\mathrm{j})}^2}\frac{1}{(z_0+z_2)^2}\left[2z_0(z_0+z_2)+z_2^2\right]\left[1-2z_1(1-z_1)\right] \\
&\times\frac{|\mathbf{x}_{3;(\mathrm{j})}|^2|\mathbf{x}_{2;(\mathrm{j})}|^2}{256}\overline{g}_{(\mathrm{j})}(t_{1;(\mathrm{j})},u_1)\overline{g}_{(\mathrm{j})}(t_{2;(\mathrm{j})},u_2) \\
&+\frac{1}{y_{u1}^2}\frac{1}{u_1\omega_{(\mathrm{k})}t_{1;(\mathrm{k})}^2}\frac{1}{y_{u2}^2}\frac{1}{u_2\omega_{(\mathrm{k})}t_{2;(\mathrm{k})}^2}\frac{1}{(z_1+z_2)^2}\left[2z_1(z_1+z_2)+z_2^2\right]\left[1-2z_0(1-z_0)\right] \\
&\times\frac{|\mathbf{x}_{3;(\mathrm{k})}|^2|\mathbf{x}_{2;(\mathrm{k})}|^2}{256}\overline{g}_{(\mathrm{k})}(t_{1;(\mathrm{k})},u_1)\overline{g}_{(\mathrm{k})}(t_{2;(\mathrm{k})},u_2)\Bigg\}
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
\sigma_T\big|_{q\bar{q}g}^{|(\mathrm{j})|_m^2+|\mathrm{k}|_m^2} =& 4N_c\alpha_{\mathrm{em}}\frac{\alpha_{\mathrm{s}}C_F}{\pi}\sum_f e_f^2\int \mathrm{d}^2\mathbf{x}_{01}\int \mathrm{d}^2\mathbf{x}_{20}\int \mathrm{d}^2\mathbf{b}\int_0^1 \mathrm{d}z_0\int_{z_{2,\min}}^{1-z_0}\mathrm{d}z_2 \\
&\times\left(I_1^T\big|_{q\bar{q}g}^{|(\mathrm{j})|_m^2+|\mathrm{k}|_m^2}+I_2^T\big|_{q\bar{q}g}^{|(\mathrm{j})|_m^2+|\mathrm{k}|_m^2}+I_3^T\big|_{q\bar{q}g}^{|(\mathrm{j})|_m^2+|\mathrm{k}|_m^2}\right)
\end{aligned}
\tag{57}
$$

where

$$
\begin{aligned}
I_1^T\big|_{q\bar{q}g}^{|(\mathrm{j})|_m^2+|\mathrm{k}|_m^2} =& \frac{1}{z_2}m_f^2\Bigg\{\frac{1}{(z_0+z_2)^2}\left[2z_0(z_0+z_2)+z_2^2\right] \\
&\times\frac{1}{|\mathbf{x}_{2;(\mathrm{j})}|^2}\Bigg[\left[K_0\left(\sqrt{\overline{Q}_{(\mathrm{j})}^2+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{j})}|^2+\omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}\right)\right]^2 N_{012} \\
&-e^{-|\mathbf{x}_{2;(\mathrm{j})}|^2/(|\mathbf{x}_{01}|^2e^{\gamma_E})}\left[K_0\left(|\mathbf{x}_{01}|\sqrt{\overline{Q}_{(\mathrm{j})}^2+m_f^2}\right)\right]^2 N_{01}\Bigg] \\
&+\frac{1}{(z_1+z_2)^2}\left[2z_1(z_1+z_2)+z_2^2\right] \\
&\times\frac{1}{|\mathbf{x}_{2;(\mathrm{k})}|^2}\Bigg[\left[K_0\left(\sqrt{\overline{Q}_{(\mathrm{k})}^2+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}\right)\right]^2 N_{012} \\
&-e^{-|\mathbf{x}_{2;(\mathrm{k})}|^2/(|\mathbf{x}_{01}|^2e^{\gamma_E})}\left[K_0\left(|\mathbf{x}_{01}|\sqrt{\overline{Q}_{(\mathrm{k})}^2+m_f^2}\right)\right]^2 N_{01}\Bigg]\Bigg\}
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
&I_2^T\big|_{q\bar{q}g}^{|(\mathrm{j})|_m^2+|\mathrm{k}|_m^2} = \int_0^1\mathrm{d}y_u\int_0^1\mathrm{d}y_t\,\frac{1}{z_2}N_{012}m_f^2 \\
&\Bigg\{\frac{1}{y_u^2\omega_{(\mathrm{j})}t_{(\mathrm{j})}^2}\frac{1}{(z_0+z_2)^2}\left[2z_0(z_0+z_2)+z_2^2\right]\frac{1}{4}\overline{g}_{(\mathrm{j})}(t_{(\mathrm{j})},u_2)K_0\left(\sqrt{\overline{Q}_{(\mathrm{j})}^2+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{j})}|^2+\omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}\right) \\
&+\frac{1}{y_u^2\omega_{(\mathrm{k})}t_{(\mathrm{k})}^2}\frac{1}{(z_1+z_2)^2}\left[2z_1(z_1+z_2)+z_2^2\right]\frac{1}{4}\overline{g}_{(\mathrm{k})}(t_{(\mathrm{k})},u_2)K_0\left(\sqrt{\overline{Q}_{(\mathrm{k})}^2+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}\right)\Bigg\}
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
&I_3^T\big|_{q\bar{q}g}^{|(\mathrm{j})|_m^2+|\mathrm{k}|_m^2} = \int_0^1\mathrm{d}y_{u1}\int_0^1\mathrm{d}y_{t1}\int_0^1\mathrm{d}y_{u2}\int_0^1\mathrm{d}y_{t2}\,\frac{1}{z_2}N_{012}m_f^2 \\
&\Bigg\{\frac{1}{y_{u1}^2}\frac{1}{\omega_{(\mathrm{j})}t_{1;(\mathrm{j})}^2}\frac{1}{y_{u2}^2}\frac{1}{\omega_{(\mathrm{j})}t_{2;(\mathrm{j})}^2}\frac{1}{(z_0+z_2)^2}\left[2z_0(z_0+z_2)+z_2^2\right]\frac{|\mathbf{x}_{2;(\mathrm{j})}|^2}{64}\overline{g}_{(\mathrm{j})}(t_{1;(\mathrm{j})},u_1)\overline{g}_{(\mathrm{j})}(t_{2;(\mathrm{j})},u_2) \\
&+\frac{1}{y_{u1}^2}\frac{1}{\omega_{(\mathrm{k})}t_{1;(\mathrm{k})}^2}\frac{1}{y_{u2}^2}\frac{1}{\omega_{(\mathrm{k})}t_{2;(\mathrm{k})}^2}\frac{1}{(z_1+z_2)^2}\left[2z_1(z_1+z_2)+z_2^2\right]\frac{|\mathbf{x}_{2;(\mathrm{k})}|^2}{64}\overline{g}_{(\mathrm{k})}(t_{1;(\mathrm{k})},u_1)\overline{g}_{(\mathrm{k})}(t_{2;(\mathrm{k})},u_2)\Bigg\}
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
\sigma_T\big|_{q\bar{q}g}^{\mathrm{F}} =& 4N_c\alpha_{\mathrm{em}}\frac{\alpha_{\mathrm{s}}C_F}{\pi}\sum_f e_f^2\int \mathrm{d}^2\mathbf{x}_{01}\int \mathrm{d}^2\mathbf{x}_{20}\int \mathrm{d}^2\mathbf{b}\int_0^1 \mathrm{d}z_0\int_{z_{2,\min}}^{1-z_0}\mathrm{d}z_2 \\
&\times\left(I_1^T\big|_{q\bar{q}g}^{\mathrm{F}}+I_2^T\big|_{q\bar{q}g}^{\mathrm{F}}+I_3^T\big|_{q\bar{q}g}^{\mathrm{F}}\right)
\end{aligned}
\tag{61}
$$

where

$$
\begin{aligned}
I_1^T|_{q\bar{q}g}^{\mathrm{F}} = \frac{1}{z_2}\frac{1}{2}\Bigg\{ & \frac{4}{(z_0+z_2)(z_1+z_2)} \left\{ z_2(z_0-z_1)^2 \left[\left(\mathbf{x}_{3;(\mathrm{j})}\cdot\mathbf{x}_{2;(\mathrm{j})}\right)\left(\mathbf{x}_{3;(\mathrm{k})}\cdot\mathbf{x}_{2;(\mathrm{k})}\right) - \left(\mathbf{x}_{3;(\mathrm{j})}\cdot\mathbf{x}_{2;(\mathrm{k})}\right)\left(\mathbf{x}_{3;(\mathrm{k})}\cdot\mathbf{x}_{2;(\mathrm{j})}\right)\right]\right. \\
& - \left[z_1(z_0+z_2)+z_0(z_1+z_2)\right]\left[z_0(z_0+z_2)+z_1(z_1+z_2)\right]\left(\mathbf{x}_{2;(\mathrm{j})}\cdot\mathbf{x}_{2;(\mathrm{k})}\right)\left.\left(\mathbf{x}_{3;(\mathrm{j})}\cdot\mathbf{x}_{3;(\mathrm{k})}\right)\right\} \\
& \times \frac{1}{|\mathbf{x}_{2;(\mathrm{j})}|^2}\sqrt{\frac{\overline{Q}_{(\mathrm{j})}^2+m_f^2}{|\mathbf{x}_{3;(\mathrm{j})}|^2+\omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}}K_1\left(\sqrt{\overline{Q}_{(\mathrm{j})}^2+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{j})}|^2+\omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}\right) \\
& \times \frac{1}{|\mathbf{x}_{2;(\mathrm{k})}|^2}\sqrt{\frac{\overline{Q}_{(\mathrm{k})}^2+m_f^2}{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}}K_1\left(\sqrt{\overline{Q}_{(\mathrm{k})}^2+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}\right) \\
& - \frac{(z_0+z_2)z_1z_2}{(z_1+z_2)^2}\left(\mathbf{x}_{2;(\mathrm{j})}\cdot\mathbf{x}_{3;(\mathrm{j})}\right)\mathcal{H}_{(\mathrm{k})}\frac{1}{|\mathbf{x}_{2;(\mathrm{j})}|^2}\sqrt{\frac{\overline{Q}_{(\mathrm{j})}^2+m_f^2}{|\mathbf{x}_{3;(\mathrm{j})}|^2+\omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}}K_1\left(\sqrt{\overline{Q}_{(\mathrm{j})}^2+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{j})}|^2+\omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}\right) \\
& + \frac{(z_1+z_2)z_0z_2}{(z_0+z_2)^2}\left(\mathbf{x}_{2;(\mathrm{k})}\cdot\mathbf{x}_{3;(\mathrm{k})}\right)\mathcal{H}_{(\mathrm{j})}\frac{1}{|\mathbf{x}_{2;(\mathrm{k})}|^2}\sqrt{\frac{\overline{Q}_{(\mathrm{k})}^2+m_f^2}{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}}K_1\left(\sqrt{\overline{Q}_{(\mathrm{k})}^2+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}\right) \\
& - \frac{z_0^2z_1z_2}{(z_0+z_2)^3}\left(\mathbf{x}_{2;(\mathrm{j})}\cdot\mathbf{x}_{3;(\mathrm{j})}\right)\mathcal{H}_{(\mathrm{j})}\frac{1}{|\mathbf{x}_{2;(\mathrm{j})}|^2}\sqrt{\frac{\overline{Q}_{(\mathrm{j})}^2+m_f^2}{|\mathbf{x}_{3;(\mathrm{j})}|^2+\omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}}K_1\left(\sqrt{\overline{Q}_{(\mathrm{j})}^2+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{j})}|^2+\omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}\right) \\
& + \frac{z_1^2z_0z_2}{(z_1+z_2)^3}\left(\mathbf{x}_{2;(\mathrm{k})}\cdot\mathbf{x}_{3;(\mathrm{k})}\right)\mathcal{H}_{(\mathrm{k})}\frac{1}{|\mathbf{x}_{2;(\mathrm{k})}|^2}\sqrt{\frac{\overline{Q}_{(\mathrm{k})}^2+m_f^2}{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}}K_1\left(\sqrt{\overline{Q}_{(\mathrm{k})}^2+m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2+\omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}\right) \\
& + \frac{(z_0z_2)^2}{8(z_0+z_2)^4}[\mathcal{H}_{(\mathrm{j})}]^2 + \frac{(z_1z_2)^2}{8(z_1+z_2)^4}[\mathcal{H}_{(\mathrm{k})}]^2 \Bigg\} N_{012}
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
I_2^T\big|_{q\bar{q}g}^{\mathrm{F}} = & \int_0^1 \mathrm{d}y_u \int_0^1 \mathrm{d}y_t \, \frac{1}{z_2}\frac{1}{2} N_{012} \\
& \times \Bigg\{ \frac{1}{4(z_0+z_2)(z_1+z_2)} \left\{ z_2(z_0-z_1)^2 \left[\left(\mathbf{x}_{3;(\mathrm{j})} \cdot \mathbf{x}_{2;(\mathrm{j})}\right) \left(\mathbf{x}_{3;(\mathrm{k})} \cdot \mathbf{x}_{2;(\mathrm{k})}\right) - \left(\mathbf{x}_{3;(\mathrm{j})} \cdot \mathbf{x}_{2;(\mathrm{k})}\right) \left(\mathbf{x}_{3;(\mathrm{k})} \cdot \mathbf{x}_{2;(\mathrm{j})}\right) \right] \right. \\
& \left. - \left[z_1(z_0+z_2) + z_0(z_1+z_2) \right] \left[z_0(z_0+z_2) + z_1(z_1+z_2) \right] \left(\mathbf{x}_{2;(\mathrm{j})} \cdot \mathbf{x}_{2;(\mathrm{k})}\right) \left(\mathbf{x}_{3;(\mathrm{j})} \cdot \mathbf{x}_{3;(\mathrm{k})}\right) \right\} \\
& \times \Bigg\{ \frac{1}{y_u^2} \frac{1}{u\omega_{(\mathrm{k})} t_{(\mathrm{k})}^2} \overline{g}_{(\mathrm{k})}(t_{(\mathrm{k})},u) \frac{1}{|\mathbf{x}_{2;(\mathrm{j})}|^2} \sqrt{\frac{\overline{Q}_{(\mathrm{j})}^2 + m_f^2}{|\mathbf{x}_{3;(\mathrm{j})}|^2 + \omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}} K_1 \left(\sqrt{\overline{Q}_{(\mathrm{j})}^2 + m_f^2} \sqrt{|\mathbf{x}_{3;(\mathrm{j})}|^2 + \omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2} \right) \\
& + \frac{1}{y_u^2} \frac{1}{u\omega_{(\mathrm{j})} t_{(\mathrm{j})}^2} \overline{g}_{(\mathrm{j})}(t_{(\mathrm{j})},u) \frac{1}{|\mathbf{x}_{2;(\mathrm{k})}|^2} \sqrt{\frac{\overline{Q}_{(\mathrm{k})}^2 + m_f^2}{|\mathbf{x}_{3;(\mathrm{k})}|^2 + \omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}} K_1 \left(\sqrt{\overline{Q}_{(\mathrm{k})}^2 + m_f^2} \sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2 + \omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{k})}|^2} \right) \Bigg\} \\
& - \frac{(z_0+z_2)z_1z_2}{16(z_1+z_2)^2} \left(\mathbf{x}_{2;(\mathrm{j})} \cdot \mathbf{x}_{3;(\mathrm{j})}\right) \mathcal{H}_{(\mathrm{k})} \frac{1}{y_u^2} \frac{1}{u\omega_{(\mathrm{j})} t_{(\mathrm{j})}^2} \overline{g}_{(\mathrm{j})}(t_{(\mathrm{j})},u) \\
& + \frac{(z_1+z_2)z_0z_2}{16(z_0+z_2)^2} \left(\mathbf{x}_{2;(\mathrm{k})} \cdot \mathbf{x}_{3;(\mathrm{k})}\right) \mathcal{H}_{(\mathrm{j})} \frac{1}{y_u^2} \frac{1}{u\omega_{(\mathrm{k})} t_{(\mathrm{k})}^2} \overline{g}_{(\mathrm{k})}(t_{(\mathrm{k})},u) \\
& - \frac{z_0^2 z_1 z_2}{16(z_0+z_2)^3} \left(\mathbf{x}_{2;(\mathrm{j})} \cdot \mathbf{x}_{3;(\mathrm{j})}\right) \mathcal{H}_{(\mathrm{j})} \frac{1}{y_u^2} \frac{1}{u\omega_{(\mathrm{j})} t_{(\mathrm{j})}^2} \overline{g}_{(\mathrm{j})}(t_{(\mathrm{j})},u) \\
& + \frac{z_1^2 z_0 z_2}{16(z_1+z_2)^3} \left(\mathbf{x}_{2;(\mathrm{k})} \cdot \mathbf{x}_{3;(\mathrm{k})}\right) \mathcal{H}_{(\mathrm{k})} \frac{1}{y_u^2} \frac{1}{u\omega_{(\mathrm{k})} t_{(\mathrm{k})}^2} \overline{g}_{(\mathrm{k})}(t_{(\mathrm{k})},u) \Bigg\}
\end{aligned} \tag{63}
$$

$$
\begin{aligned}
I_3^T\big|_{q\bar{q}g}^{\mathrm{F}} = & \int_0^1 \mathrm{d}y_{u1} \int_0^1 \mathrm{d}y_{t1} \int_0^1 \mathrm{d}y_{u2} \int_0^1 \mathrm{d}y_{t2} \, \frac{1}{z_2}\frac{1}{2} N_{012} \\
& \times \frac{1}{y_{u1}^2} \frac{1}{u_1\omega_{(\mathrm{j})} t_{1;(\mathrm{j})}^2} \frac{1}{y_{u2}^2} \frac{1}{u_2\omega_{(\mathrm{k})} t_{2;(\mathrm{k})}^2} \overline{g}_{(\mathrm{j})}(t_{1;(\mathrm{j})},u_1) \overline{g}_{(\mathrm{k})}(t_{2;(\mathrm{k})},u_2) \\
& \times \frac{1}{64(z_0+z_2)(z_1+z_2)} \left\{ z_2(z_0-z_1)^2 \left[\left(\mathbf{x}_{3;(\mathrm{j})} \cdot \mathbf{x}_{2;(\mathrm{j})}\right) \left(\mathbf{x}_{3;(\mathrm{k})} \cdot \mathbf{x}_{2;(\mathrm{k})}\right) - \left(\mathbf{x}_{3;(\mathrm{j})} \cdot \mathbf{x}_{2;(\mathrm{k})}\right) \left(\mathbf{x}_{3;(\mathrm{k})} \cdot \mathbf{x}_{2;(\mathrm{j})}\right) \right] \right. \\
& \left. - \left[z_1(z_0+z_2) + z_0(z_1+z_2) \right] \left[z_0(z_0+z_2) + z_1(z_1+z_2) \right] \left(\mathbf{x}_{2;(\mathrm{j})} \cdot \mathbf{x}_{2;(\mathrm{k})}\right) \left(\mathbf{x}_{3;(\mathrm{j})} \cdot \mathbf{x}_{3;(\mathrm{k})}\right) \right\}
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
\sigma_T\big|_{q\bar{q}g}^{\mathrm{F}_m} = & 4N_c\alpha_{\mathrm{em}} \frac{\alpha_{\mathrm{s}} C_F}{\pi} \sum_f e_f^2 \int \mathrm{d}^2\mathbf{x}_{01} \int \mathrm{d}^2\mathbf{x}_{20} \int \mathrm{d}^2\mathbf{b} \int_0^1 \mathrm{d}z_0 \int_{z_{2,\min}}^{1-z_0} \mathrm{d}z_2 \\
& \times \left(I_2^T\big|_{q\bar{q}g}^{\mathrm{F}_m} + I_3^T\big|_{q\bar{q}g}^{\mathrm{F}_m} \right)
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
I_1^T\big|_{q\bar{q}g}^{\mathrm{F}_m} = & \frac{1}{z_2} \frac{m_f^2}{2} N_{012} \\
& \times \Bigg\{ - \frac{1}{32(z_0+z_2)(z_1+z_2)} \left[(2z_0+z_2)(2z_1+z_2) + z_2^2 \right] \left(\mathbf{x}_{2;(\mathrm{j})} \cdot \mathbf{x}_{2;(\mathrm{k})}\right) \\
& \times \frac{8}{|\mathbf{x}_{2;(\mathrm{j})}|^2} K_0 \left(\sqrt{\overline{Q}_{(\mathrm{j})}^2 + m_f^2} \sqrt{|\mathbf{x}_{3;(\mathrm{j})}|^2 + \omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2} \right) \frac{8}{|\mathbf{x}_{2;(\mathrm{k})}|^2} K_0 \left(\sqrt{\overline{Q}_{(\mathrm{k})}^2 + m_f^2} \sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2 + \omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2} \right) \Bigg\}
\end{aligned} \tag{66}
$$

$$
\begin{aligned}
I_2^T\big|_{q\bar{q}g}^{\mathrm{F}_m} &= \int_0^1 \mathrm{d}y_{u1} \int_0^1 \mathrm{d}y_{t1}\, \frac{1}{z_2} \frac{m_f^2}{2} N_{012} \\
&\times \Bigg\{ -\frac{z_0 z_1 z_2^2}{16(z_0+z_2)^3} \left(\mathbf{x}_{3;(\mathrm{j})} \cdot \mathbf{x}_{2;(\mathrm{j})}\right) \frac{u}{y_u^2 \omega_{(\mathrm{j})}} \frac{1}{u^2 t_{(\mathrm{j})}} g_{(\mathrm{j})}(t_{(\mathrm{j})}, u) \frac{8}{|\mathbf{x}_{2;(\mathrm{j})}|^2} K_0\left(\sqrt{\overline{Q}_{(\mathrm{j})}^2 + m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{j})}|^2 + \omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}\right) \\
&+ \frac{z_0 z_1 z_2^2}{16(z_1+z_2)^3} \left(\mathbf{x}_{3;(\mathrm{k})} \cdot \mathbf{x}_{2;(\mathrm{k})}\right) \frac{u}{y_u^2 \omega_{(\mathrm{k})}} \frac{1}{u^2 t_{(\mathrm{k})}} g_{(\mathrm{k})}(t_{(\mathrm{k})}, u) \frac{8}{|\mathbf{x}_{2;(\mathrm{k})}|^2} K_0\left(\sqrt{\overline{Q}_{(\mathrm{k})}^2 + m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2 + \omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}\right) \\
&- \frac{1}{32(z_0+z_2)(z_1+z_2)} \left[(2z_0+z_2)(2z_1+z_2) + z_2^2\right] \left(\mathbf{x}_{2;(\mathrm{j})} \cdot \mathbf{x}_{2;(\mathrm{k})}\right) \\
&\times \Bigg\{ \frac{u}{y_u^2 \omega_{(\mathrm{k})}} \frac{1}{u t_{(\mathrm{k})}^2} \overline{g}_{(\mathrm{k})}(t_{(\mathrm{k})}, u) \frac{8}{|\mathbf{x}_{2;(\mathrm{j})}|^2} K_0\left(\sqrt{\overline{Q}_{(\mathrm{j})}^2 + m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{j})}|^2 + \omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}\right) \\
&+ \frac{u}{y_u^2 \omega_{(\mathrm{j})}} \frac{1}{u t_{(\mathrm{j})}^2} \overline{g}_{(\mathrm{j})}(t_{(\mathrm{j})}, u) \frac{8}{|\mathbf{x}_{2;(\mathrm{k})}|^2} K_0\left(\sqrt{\overline{Q}_{(\mathrm{k})}^2 + m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2 + \omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}\right) \Bigg\} \\
&- \frac{(z_0 z_2)^2}{16(z_0+z_2)(z_1+z_2)^2} \left(\mathbf{x}_{2;(\mathrm{j})} \cdot \mathbf{x}_{3;(\mathrm{k})}\right) \frac{u}{y_u^2 \omega_{(\mathrm{k})}} \frac{1}{u^2 t_{(\mathrm{k})}} g_{(\mathrm{k})}(t_{(\mathrm{k})}, u) \\
&\times \frac{8}{|\mathbf{x}_{2;(\mathrm{j})}|^2} K_0\left(\sqrt{\overline{Q}_{(\mathrm{j})}^2 + m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{j})}|^2 + \omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}\right) \\
&+ \frac{(z_1 z_2)^2}{16(z_0+z_2)^2(z_1+z_2)} \left(\mathbf{x}_{3;(\mathrm{j})} \cdot \mathbf{x}_{2;(\mathrm{k})}\right) \frac{u}{y_u^2 \omega_{(\mathrm{j})}} \frac{1}{u^2 t_{(\mathrm{j})}} g_{(\mathrm{j})}(t_{(\mathrm{j})}, u) \\
&\times \frac{8}{|\mathbf{x}_{2;(\mathrm{k})}|^2} K_0\left(\sqrt{\overline{Q}_{(\mathrm{k})}^2 + m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2 + \omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}\right) \\
&- \frac{z_0 z_1 z_2^2}{16(z_0+z_2)^3} \left(\mathbf{x}_{2;(\mathrm{j})} \cdot \mathbf{x}_{3;(\mathrm{j})}\right) \frac{u}{y_u^2 \omega_{(\mathrm{j})}} \frac{1}{u t_{(\mathrm{j})}} g_{(\mathrm{j})}(t_{(\mathrm{j})}, u) \\
&\times \frac{16}{|\mathbf{x}_{2;(\mathrm{j})}|^2} \sqrt{\frac{\overline{Q}_{(\mathrm{j})}^2 + m_f^2}{|\mathbf{x}_{3;(\mathrm{j})}|^2 + \omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}} K_1\left(\sqrt{\overline{Q}_{(\mathrm{j})}^2 + m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{j})}|^2 + \omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}\right) \\
&+ \frac{z_0 z_1 z_2^2}{16(z_1+z_2)^3} \left(\mathbf{x}_{2;(\mathrm{k})} \cdot \mathbf{x}_{3;(\mathrm{k})}\right) \frac{u}{y_u^2 \omega_{(\mathrm{k})}} \frac{1}{u t_{(\mathrm{k})}} g_{(\mathrm{k})}(t_{(\mathrm{k})}, u) \\
&\times \frac{16}{|\mathbf{x}_{2;(\mathrm{k})}|^2} \sqrt{\frac{\overline{Q}_{(\mathrm{k})}^2 + m_f^2}{|\mathbf{x}_{3;(\mathrm{k})}|^2 + \omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}} K_1\left(\sqrt{\overline{Q}_{(\mathrm{k})}^2 + m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2 + \omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}\right) \\
&- \frac{(z_0+z_2) z_2^2}{16(z_1+z_2)^2} \left(\mathbf{x}_{2;(\mathrm{j})} \cdot \mathbf{x}_{3;(\mathrm{j})}\right) \frac{u}{y_u^2 \omega_{(\mathrm{k})}} \frac{1}{u t_{(\mathrm{k})}} g_{(\mathrm{k})}(t_{(\mathrm{k})}, u) \\
&\times \frac{16}{|\mathbf{x}_{2;(\mathrm{j})}|^2} \sqrt{\frac{\overline{Q}_{(\mathrm{k})}^2 + m_f^2}{|\mathbf{x}_{3;(\mathrm{j})}|^2 + \omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}} K_1\left(\sqrt{\overline{Q}_{(\mathrm{j})}^2 + m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{j})}|^2 + \omega_{(\mathrm{j})}|\mathbf{x}_{2;(\mathrm{j})}|^2}\right) \\
&+ \frac{(z_1+z_2) z_2^2}{16(z_0+z_2)^2} \left(\mathbf{x}_{2;(\mathrm{k})} \cdot \mathbf{x}_{3;(\mathrm{k})}\right) \frac{u}{y_u^2 \omega_{(\mathrm{j})}} \frac{1}{u t_{(\mathrm{j})}} g_{(\mathrm{j})}(t_{(\mathrm{j})}, u) \\
&\times \frac{16}{|\mathbf{x}_{2;(\mathrm{k})}|^2} \sqrt{\frac{\overline{Q}_{(\mathrm{k})}^2 + m_f^2}{|\mathbf{x}_{3;(\mathrm{k})}|^2 + \omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}} K_1\left(\sqrt{\overline{Q}_{(\mathrm{k})}^2 + m_f^2}\sqrt{|\mathbf{x}_{3;(\mathrm{k})}|^2 + \omega_{(\mathrm{k})}|\mathbf{x}_{2;(\mathrm{k})}|^2}\right) \\
&+ \frac{z_0 z_2^3}{4(z_0+z_2)^4} \mathcal{H}_{(\mathrm{j})} \frac{u}{y_u^2 \omega_{(\mathrm{j})}} \frac{1}{u t_{(\mathrm{j})}} g_{(\mathrm{j})}(t_{(\mathrm{j})}, u) + \frac{z_1 z_2^3}{4(z_1+z_2)^4} \mathcal{H}_{(\mathrm{k})} \frac{u}{y_u^2 \omega_{(\mathrm{k})}} \frac{1}{u t_{(\mathrm{k})}} g_{(\mathrm{k})}(t_{(\mathrm{k})}, u) \Bigg\}
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
I_3^T|_{q\bar{q}g}^{\mathrm{F}_m} &= \int_0^1 \mathrm{d}y_{u1} \int_0^1 \mathrm{d}y_{t1} \int_0^1 \mathrm{d}y_{u2} \int_0^1 \mathrm{d}y_{t2}\, \frac{1}{z_2} \frac{m_f^2}{2} N_{012} \\
&\times \Bigg\{ \frac{z_2^4}{64(z_0+z_2)^4} \left[4z_1(z_1-1)+2\right] |\mathbf{x}_{3;(\mathrm{j})}|^2 \frac{u_1}{y_{u1}^2\omega_{(\mathrm{j})}} \frac{1}{u_1^2 t_{1;(\mathrm{j})}} \frac{u_2}{y_{u2}^2\omega_{(\mathrm{j})}} \frac{1}{u_2^2 t_{2;(\mathrm{j})}} g_{(\mathrm{j})}(t_{1;(\mathrm{j})},u_1) g_{(\mathrm{j})}(t_{2;(\mathrm{j})},u_2) \\
&- \frac{z_0 z_1 z_2^2}{16(z_0+z_2)^3} \left(\mathbf{x}_{3;(\mathrm{j})} \cdot \mathbf{x}_{2;(\mathrm{j})}\right) \frac{u_1}{y_{u1}^2\omega_{(\mathrm{j})}} \frac{1}{u_1 t_{1;(\mathrm{j})}^2} \frac{u_2}{y_{u2}^2\omega_{(\mathrm{j})}} \frac{1}{u_2^2 t_{2;(\mathrm{j})}} \overline{g}_{(\mathrm{j})}(t_{1;(\mathrm{j})},u_1) g_{(\mathrm{j})}(t_{2;(\mathrm{j})},u_2) \\
&+ \frac{z_2^4}{64(z_1+z_2)^4} \left[4z_0(z_0-1)+2\right] |\mathbf{x}_{3;(\mathrm{k})}|^2 \frac{u_1}{y_{u1}^2\omega_{(\mathrm{k})}} \frac{1}{u_1^2 t_{1;(\mathrm{k})}} \frac{u_2}{y_{u2}^2\omega_{(\mathrm{k})}} \frac{1}{u_2^2 t_{2;(\mathrm{k})}} g_{(\mathrm{k})}(t_{1;(\mathrm{k})},u_1) g_{(\mathrm{k})}(t_{2;(\mathrm{k})},u_2) \\
&+ \frac{z_0 z_1 z_2^2}{16(z_1+z_2)^3} \left(\mathbf{x}_{3;(\mathrm{k})} \cdot \mathbf{x}_{2;(\mathrm{k})}\right) \frac{u_1}{y_{u1}^2\omega_{(\mathrm{k})}} \frac{1}{u_1 t_{1;(\mathrm{k})}^2} \frac{u_2}{y_{u2}^2\omega_{(\mathrm{k})}} \frac{1}{u_2^2 t_{2;(\mathrm{k})}} \overline{g}_{(\mathrm{k})}(t_{1;(\mathrm{k})},u_1) g_{(\mathrm{k})}(t_{2;(\mathrm{k})},u_2) \\
&- \frac{1}{32(z_0+z_2)(z_1+z_2)} \left[(2z_0+z_2)(2z_1+z_2)+z_2^2\right] \left(\mathbf{x}_{2;(\mathrm{j})} \cdot \mathbf{x}_{2;(\mathrm{k})}\right) \\
&\times \frac{u_1}{y_{u1}^2\omega_{(\mathrm{j})}} \frac{1}{u_1 t_{1;(\mathrm{j})}^2} \frac{u_2}{y_{u2}^2\omega_{(\mathrm{k})}} \frac{1}{u_2 t_{2;(\mathrm{k})}^2} \overline{g}_{(\mathrm{j})}(t_{1;(\mathrm{j})},u_1) \overline{g}_{(\mathrm{k})}(t_{2;(\mathrm{k})},u_2) \\
&+ \frac{z_2^4}{32(z_0+z_2)^2(z_1+z_2)^2} \left[(2z_0+z_2)(2z_1+z_2)+z_2^2\right] \left(\mathbf{x}_{3;(\mathrm{j})} \cdot \mathbf{x}_{3;(\mathrm{k})}\right) \\
&\times \frac{u_1}{y_{u1}^2\omega_{(\mathrm{j})}} \frac{1}{u_1^2 t_{1;(\mathrm{j})}} \frac{u_2}{y_{u2}^2\omega_{(\mathrm{k})}} \frac{1}{u_2^2 t_{2;(\mathrm{k})}} g_{(\mathrm{j})}(t_{1;(\mathrm{j})},u_1) g_{(\mathrm{k})}(t_{2;(\mathrm{k})},u_2) \\
&+ \frac{m_f^2}{16} \frac{2z_2^4}{(z_0+z_2)^4} \frac{u_1}{y_{u1}^2\omega_{(\mathrm{j})}} \frac{1}{u_1 t_{1;(\mathrm{j})}} \frac{u_2}{y_{u2}^2\omega_{(\mathrm{j})}} \frac{1}{u_2 t_{2;(\mathrm{j})}} g_{(\mathrm{j})}(t_{1;(\mathrm{j})},u_1) g_{(\mathrm{j})}(t_{2;(\mathrm{j})},u_2) \\
&+ \frac{m_f^2}{16} \frac{2z_2^4}{(z_1+z_2)^4} \frac{u_1}{y_{u1}^2\omega_{(\mathrm{k})}} \frac{1}{u_1 t_{1;(\mathrm{k})}} \frac{u_2}{y_{u2}^2\omega_{(\mathrm{k})}} \frac{1}{u_2 t_{2;(\mathrm{k})}} g_{(\mathrm{k})}(t_{1;(\mathrm{k})},u_1) g_{(\mathrm{k})}(t_{2;(\mathrm{k})},u_2) \\
&- \frac{(z_0 z_2)^2}{16(z_0+z_2)(z_1+z_2)^2} \left(\mathbf{x}_{2;(\mathrm{j})} \cdot \mathbf{x}_{3;(\mathrm{k})}\right) \frac{u_1}{y_{u1}^2\omega_{(\mathrm{j})}} \frac{1}{u_1 t_{1;(\mathrm{j})}^2} \frac{u_2}{y_{u2}^2\omega_{(\mathrm{k})}} \frac{1}{u_2^2 t_{2;(\mathrm{k})}} \overline{g}_{(\mathrm{j})}(t_{1;(\mathrm{j})},u_1) g_{(\mathrm{k})}(t_{2;(\mathrm{k})},u_2) \\
&+ \frac{(z_1 z_2)^2}{16(z_0+z_2)^2(z_1+z_2)} \left(\mathbf{x}_{3;(\mathrm{j})} \cdot \mathbf{x}_{2;(\mathrm{k})}\right) \frac{u_1}{y_{u1}^2\omega_{(\mathrm{k})}} \frac{1}{u_1 t_{1;(\mathrm{k})}^2} \frac{u_2}{y_{u2}^2\omega_{(\mathrm{j})}} \frac{1}{u_2^2 t_{2;(\mathrm{j})}} \overline{g}_{(\mathrm{k})}(t_{1;(\mathrm{k})},u_1) g_{(\mathrm{j})}(t_{2;(\mathrm{j})},u_2) \\
&- \frac{z_0 z_1 z_2^2}{16(z_0+z_2)^3} \left(\mathbf{x}_{2;(\mathrm{j})} \cdot \mathbf{x}_{3;(\mathrm{j})}\right) \frac{u_1}{y_{u1}^2\omega_{(\mathrm{j})}} \frac{1}{u_1 t_{1;(\mathrm{j})}} \frac{u_2}{y_{u2}^2\omega_{(\mathrm{j})}} \frac{1}{u_2^2 t_{2;(\mathrm{j})}^2} g_{(\mathrm{j})}(t_{1;(\mathrm{j})},u_1) \overline{g}_{(\mathrm{j})}(t_{2;(\mathrm{j})},u_2) \\
&+ \frac{z_0 z_1 z_2^2}{16(z_1+z_2)^3} \left(\mathbf{x}_{2;(\mathrm{k})} \cdot \mathbf{x}_{3;(\mathrm{k})}\right) \frac{u_1}{y_{u1}^2\omega_{(\mathrm{k})}} \frac{1}{u_1 t_{1;(\mathrm{k})}} \frac{u_2}{y_{u2}^2\omega_{(\mathrm{k})}} \frac{1}{u_2^2 t_{2;(\mathrm{k})}^2} g_{(\mathrm{k})}(t_{1;(\mathrm{k})},u_1) \overline{g}_{(\mathrm{k})}(t_{2;(\mathrm{k})},u_2) \\
&- \frac{(z_0+z_2) z_2^2}{16(z_1+z_2)^2} \left(\mathbf{x}_{2;(\mathrm{j})} \cdot \mathbf{x}_{3;(\mathrm{j})}\right) \frac{u_1}{y_{u1}^2\omega_{(\mathrm{k})}} \frac{1}{u_1 t_{1;(\mathrm{k})}} \frac{u_2}{y_{u2}^2\omega_{(\mathrm{j})}} \frac{1}{u_2^2 t_{2;(\mathrm{j})}^2} g_{(\mathrm{k})}(t_{1;(\mathrm{k})},u_1) \overline{g}_{(\mathrm{j})}(t_{2;(\mathrm{j})},u_2) \\
&+ \frac{(z_1+z_2) z_2^2}{16(z_0+z_2)^2} \left(\mathbf{x}_{2;(\mathrm{k})} \cdot \mathbf{x}_{3;(\mathrm{k})}\right) \frac{u_1}{y_{u1}^2\omega_{(\mathrm{j})}} \frac{1}{u_1 t_{1;(\mathrm{j})}} \frac{u_2}{y_{u2}^2\omega_{(\mathrm{k})}} \frac{1}{u_2^2 t_{2;(\mathrm{k})}^2} g_{(\mathrm{j})}(t_{1;(\mathrm{j})},u_1) \overline{g}_{(\mathrm{k})}(t_{2;(\mathrm{k})},u_2) \Bigg\}
\end{aligned}
\tag{68}
$$

For the numerical integration we want to sum I_1, I_2, I_3 contributions together and then integrate.

3 A better way to implement $q\bar{q}g$

We can do one integral in the $\mathcal{G}$ functions:

$$\begin{aligned}
\mathcal{G}^{(a;b)}_{(\mathrm{x})} &= \int_0^\infty \frac{\mathrm{d}u}{u^a} \exp\!\left(-u\left[\overline{Q}^2_{(\mathrm{x})} + m_f^2\right] - \frac{|\mathbf{x}_{3;(\mathrm{x})}|^2}{4u}\right) \int_0^{u/\omega_{(\mathrm{x})}} \frac{\mathrm{d}t}{t^b} \exp\!\left(-t\omega_{(\mathrm{x})}\lambda_{(\mathrm{x})} m_f^2 - \frac{|\mathbf{x}_{2;(\mathrm{x})}|^2}{4t}\right) \\
&= \int_0^1 \frac{\mathrm{d}y}{y^{\frac{1}{2}(2-a+b)}} 2^{a+b-1} \omega^{b-1}_{(\mathrm{x})} \left(\frac{y|\mathbf{x}_{3;(\mathrm{x})}|^2 + \omega_{(\mathrm{x})}|\mathbf{x}_{2;(\mathrm{x})}|^2}{y\lambda_{(\mathrm{x})} m_f^2 + \overline{Q}^2_{(\mathrm{x})} + m_f^2}\right)^{\frac{1}{2}(2-a-b)} \\
&\quad \times K_{a+b-2}\left(\sqrt{\frac{1}{y}\left(y\lambda_{(\mathrm{x})} m_f^2 + \overline{Q}^2_{(\mathrm{x})} + m_f^2\right)\left(y|\mathbf{x}_{3;(\mathrm{x})}|^2 + \omega_{(\mathrm{x})}|\mathbf{x}_{2;(\mathrm{x})}|^2\right)}\right)
\end{aligned} \tag{69}$$