single numbers stone, harper 2006

43.X PFPL

$$
\begin{array}{rcll}
\textsf{BinOp} & \oplus & ::= & \texttt{Product} \mid \texttt{Sum} \mid \texttt{Arrow} \\
\textsf{Kind} & \kappa & ::= & \texttt{Ty} \mid \texttt{KHole} \mid \texttt{S}(\tau) \\
\textsf{ConstantTypes} & c & ::= & \texttt{Int} \mid \texttt{Float} \mid \texttt{Bool} \\
\textsf{UserHTyp} & \hat{\tau} & ::= & c \mid \hat{\tau}_1 \oplus \hat{\tau}_2 \mid \texttt{list}(\hat{\tau}) \mid (\!|\,|\!)^u \mid (\!|\hat{\tau}|\!)^u \\
\textsf{InternalHTyp} & \tau & ::= & c \mid \tau_1 \oplus \tau_2 \mid \texttt{list}(\tau) \mid (\!|\,|\!)^u \mid (\!|\tau|\!)^u \mid t \\
\textsf{TypeVars} & t & & \\
\textsf{TypePattern} & \rho & ::= & t \mid (\!|\,|\!) \mid (\!|t|\!) \\
\textsf{UserExpression} & e & ::= & \texttt{type}\ \rho\ \texttt{=}\ \hat{\tau}\ \texttt{in}\ e \mid \textit{elided} \\
\textsf{InternalExpression} & \tau & ::= & \texttt{type}\ \rho\ \texttt{=}\ \tau : \kappa\ \texttt{in}\ d \mid \textit{elided}
\end{array}
$$

$\boxed{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2}$ κ_1 is a consistent subkind of κ_2

$$
\texttt{KCHoleL}\ \frac{}{\Delta; \Phi \vdash \texttt{KHole} \lesssim \kappa}
\qquad
\texttt{KCHoleR}\ \frac{}{\Delta; \Phi \vdash \kappa \lesssim \texttt{KHole}}
\qquad
\texttt{KCRespectEquiv}\ \frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2}
$$

$$
\texttt{KCSubsumption}\ \frac{\Delta; \Phi \vdash \tau \Leftarrow \texttt{Ty}}{\Delta; \Phi \vdash \texttt{S}(\tau) \lesssim \texttt{Ty}}
$$
7

$\boxed{t\ \texttt{valid}}$ t is a valid type variable

t is valid if it is not a builtin-type or keyword, begins with an alpha char or underscore, and only contains alphanumeric characters, underscores, and primes.

$\boxed{\Delta; \Phi \vdash \kappa\ \texttt{kind}}$ κ forms a kind

$$
\texttt{KFTy}\ \frac{}{\Delta; \Phi \vdash \texttt{Ty}\ \texttt{kind}}
\qquad
\texttt{KFHole}\ \frac{}{\Delta; \Phi \vdash \texttt{KHole}\ \texttt{kind}}
\qquad
\texttt{KFSing}\ \frac{\Delta; \Phi \vdash \tau \Leftarrow \texttt{Ty}}{\Delta; \Phi \vdash \texttt{S}(\tau)\ \texttt{kind}}
$$
43.2a

$\boxed{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}$ κ_1 is equivalent to κ_2

$$\texttt{KERefl}\quad \frac{}{\Delta;\Phi \vdash \kappa \equiv \kappa}$$

$$\texttt{KESymm}\quad \frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_2 \equiv \kappa_1}$$

$$\texttt{KETrans}\quad \frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta;\Phi \vdash \kappa_2 \equiv \kappa_3}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_3}$$

$$\texttt{KESingEquiv}\quad \frac{\Delta;\Phi \vdash \tau_1 \equiv \tau_2}{\Delta;\Phi \vdash \texttt{S}(\tau_1) \equiv \texttt{S}(\tau_2)}$$

$\boxed{\Delta;\Phi \vdash \tau \Rightarrow \kappa}$ τ synthesizes kind κ

would need to define labelled singletons for KHole

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$$\texttt{KSConst}\quad \frac{}{\Delta;\Phi \vdash c \Rightarrow \texttt{S}(c)}$$

$$\texttt{KSVar}\quad \frac{t : \kappa \in \Phi}{\Delta;\Phi \vdash t \Rightarrow \texttt{S}_{\kappa}(t)}$$

$$\texttt{KSUVar}\quad \frac{t \notin \mathsf{dom}(\Phi)}{\Delta;\Phi \vdash t \Rightarrow \texttt{KHole}}$$

$$\texttt{KSBinOp}\quad \frac{\Delta;\Phi \vdash \tau_1 \Leftarrow \cancel{\texttt{S}(\tau_1)}\ \text{Ty} \qquad \Delta;\Phi \vdash \tau_2 \Leftarrow \cancel{\texttt{S}(\tau_2)}\ \text{Ty}}{\Delta;\Phi \vdash \tau_1 \oplus \tau_2 \Rightarrow \texttt{S}(\tau_1 \oplus \tau_2)}$$

$$\texttt{KSList}\quad \frac{\Delta;\Phi \vdash \tau \Leftarrow \cancel{\texttt{S}(\tau)}\ \text{Ty}}{\Delta;\Phi \vdash \texttt{list}(\tau) \Rightarrow \texttt{S}(\texttt{list}(\tau))}$$

$$\texttt{KSEHole}\quad \frac{u :: \kappa \in \Delta}{\Delta;\Phi \vdash (\!|\,|\!)^u \Rightarrow \kappa}$$

$$\texttt{KSNEHole}\quad \frac{u :: \kappa \in \Delta \qquad \Delta;\Phi \vdash \tau \Rightarrow \kappa'}{\Delta;\Phi \vdash (\!|\tau|\!)^u \Rightarrow \kappa}$$

$\boxed{\Delta;\Phi \vdash \tau \Leftarrow \kappa}$ τ analyzes against kind κ

$$\texttt{KAASubsume}\quad \frac{\Phi \vdash \tau \Rightarrow \kappa' \qquad \Delta;\Phi \vdash \kappa' \lesssim \kappa}{\Delta;\Phi \vdash \tau \Leftarrow \kappa}$$

$$\text{KAVar}\quad \frac{t :: \kappa_1 \in \Phi \qquad \kappa_1 \lesssim \kappa}{\Delta;\Phi \vdash t \Leftarrow \kappa}$$

$\boxed{\Delta; \Phi \vdash \tau_1 \equiv \tau_2}$ τ_1 is equivalent to τ_2

$$\texttt{KCESymm}\quad \frac{\Delta; \Phi \vdash \tau_1 \equiv \tau_2}{\Delta; \Phi \vdash \tau_2 \equiv \tau_1}$$

$$\texttt{KCETrans}\quad \frac{\Delta; \Phi \vdash \tau_1 \equiv \tau_2 \qquad \Delta; \Phi \vdash \tau_2 \equiv \tau_3}{\Delta; \Phi \vdash \tau_1 \equiv \tau_3}$$

$$\texttt{KCESingEquiv}\quad \frac{\Delta; \Phi \vdash \tau_1 \Leftarrow \texttt{S}(\tau_2)}{\Delta; \Phi \vdash \tau_1 \equiv \tau_2}$$

$$\text{KCERefl}\quad \frac{}{\Delta; \Phi \vdash \tau \equiv \tau}$$

$$\cancel{\texttt{KCEConst}\quad \frac{}{\Delta; \Phi \vdash c \equiv c}}$$

$$\cancel{\texttt{KCEVar}\quad \frac{t : \kappa \in \Phi}{\Delta; \Phi \vdash t \equiv t}}$$

$$\texttt{KCEBinOp}\quad \frac{\Delta; \Phi \vdash \tau_1 \equiv \tau_2 \qquad \Delta; \Phi \vdash \tau_3 \equiv \tau_4}{\Delta; \Phi \vdash \tau_1 \oplus \tau_3 \equiv \tau_2 \oplus \tau_4}$$

$$\texttt{KCEList}\quad \frac{\Delta; \Phi \vdash \tau_1 \equiv \tau_2}{\Delta; \Phi \vdash \texttt{list}(\tau_1) \equiv \texttt{list}(\tau_2)}$$

$$\texttt{KCEEHole}\quad \frac{u :: \kappa \in \Delta}{\Delta; \Phi \vdash (\!|\,|\!)^u \equiv (\!|\,|\!)^u}$$

* if $u::\kappa \notin \Delta$ KCERefl says OK

$$\texttt{KCENEHole}\quad \frac{u :: \kappa \in \Delta \qquad \Delta; \Phi \vdash \tau \Leftarrow \kappa'}{\Delta; \Phi \vdash (\!|\tau|\!)^u \equiv (\!|\tau|\!)^u}$$

* sim

$\boxed{\Delta; \Phi \vdash \tau :: \kappa}$ τ is well formed at kind κ

$$\frac{t :: \kappa \in \Phi}{\Delta; \Phi \vdash t :: \kappa}\; 17$$

$$\frac{\Delta; \Phi \vdash \tau :: Ty}{\Delta; \Phi \vdash \text{list}(\tau) :: Ty}$$

$$\frac{\Delta; \Phi \vdash \tau :: \kappa_1 \qquad \Delta; \Phi \vdash \kappa_1 \lesssim \kappa}{\Delta; \Phi \vdash \tau :: \kappa}\; 43.2d$$

$$\boxed{\frac{\Delta; \Phi \vdash \tau :: Ty}{\Delta; \Phi \vdash \tau :: S(\tau)}\; \begin{matrix}23\\43.2b\end{matrix}}$$

$$\frac{u :: \kappa \in \Delta}{\Delta; \Phi \vdash (\!|\,|\!)^u :: \kappa}$$

$$\frac{}{\Delta; \Phi \vdash c :: Ty}\; 16$$

$$\frac{\Delta; \Phi \vdash u :: \kappa \in \Delta \qquad \Delta; \Phi \vdash \tau :: \kappa_1}{\Delta; \Phi \vdash (\!|\tau|\!)^u :: \kappa}$$

$$\boxed{\frac{\Delta; \Phi \vdash \tau :: \kappa}{\Delta; \Phi \vdash \tau :: S_\kappa(\tau)}\; \text{replaces } 23?}$$

$$\frac{\Delta; \Phi \vdash \tau_1 :: Ty \qquad \Delta; \Phi \vdash \tau_2 :: Ty}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 :: Ty}$$

$$\frac{t \notin dom(\Phi)}{\Delta; \Phi \vdash t :: KHole}$$

$$S_{Ty}(\tau) := S(\tau)$$
$$S_{S(\tau_1)}(\tau) := S(\tau)$$
$$S_{KHole}(\tau) := KHole$$

$\boxed{\Phi \vdash \hat{\tau} \Rightarrow \kappa \rightsquigarrow \tau \dashv \Delta}$ $\hat{\tau}$ synthesizes kind κ and elaborates to τ

$$\frac{}{\Phi \vdash c \Rightarrow \mathtt{S}(c) \rightsquigarrow c \dashv \cdot}\ \texttt{TElabSConst}$$

$$\frac{\Phi \vdash \hat{\tau}_1 \Leftarrow \mathtt{Ty} \rightsquigarrow \tau_1 \dashv \Delta_1 \qquad \Phi \vdash \hat{\tau}_2 \Leftarrow \mathtt{Ty} \rightsquigarrow \tau_2 \dashv \Delta_2}{\Phi \vdash \hat{\tau}_1 \oplus \hat{\tau}_2 \Rightarrow \mathtt{S}(\tau_1 \oplus \tau_2) \rightsquigarrow \tau_1 \oplus \tau_2 \dashv \Delta_1 \cup \Delta_2}\ \texttt{TElabSBinOp}$$

$$\frac{\Phi \vdash \hat{\tau} \Leftarrow \mathtt{Ty} \rightsquigarrow \tau \dashv \Delta}{\Phi \vdash \mathtt{list}(\hat{\tau}) \Rightarrow \mathtt{S}(\mathtt{list}(\tau)) \rightsquigarrow \mathtt{list}(\tau) \dashv \Delta}\ \texttt{TElabSList} \qquad \frac{t : \kappa \in \Phi}{\Phi \vdash t \Rightarrow \mathtt{S}(t) \rightsquigarrow t \dashv \cdot}\ \texttt{TElabSVar}$$

$$\frac{t \notin \mathsf{dom}(\Phi)}{\Phi \vdash t \Rightarrow \mathtt{KHole} \rightsquigarrow (\!|t|\!)^u \dashv u :: \mathtt{KHole}}\ \texttt{TElabSUVar} \qquad \frac{}{\Phi \vdash (\!|\,|\!)^u \Rightarrow \mathtt{KHole} \rightsquigarrow (\!|\,|\!)^u \dashv u :: \mathtt{KHole}}\ \texttt{TElabSHole}$$

$$\frac{\Phi \vdash \hat{\tau} \Rightarrow \kappa \rightsquigarrow \tau \dashv \Delta}{\Phi \vdash (\!|\hat{\tau}|\!)^u \Rightarrow \mathtt{KHole} \rightsquigarrow (\!|\tau|\!)^u \dashv \Delta, u :: \mathtt{KHole}}\ \texttt{TElabSNEHole}$$

$\boxed{\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau \dashv \Delta}$ $\hat{\tau}$ analyzes against kind κ_1 and elaborates to τ

$$\frac{\hat{\tau} \neq (\!|\,|\!)^u \qquad \hat{\tau} \neq (\!|\hat{\tau}'|\!)^u \qquad \Phi \vdash \hat{\tau} \Rightarrow \kappa' \rightsquigarrow \tau \dashv \Delta \qquad \Delta; \Phi \vdash \kappa' \lesssim \kappa}{\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau \dashv \Delta}\ \texttt{TElabASubsume}$$

$$\frac{}{\Phi \vdash (\!|\,|\!)^u \Leftarrow \kappa \rightsquigarrow (\!|\,|\!)^u \dashv u :: \kappa}\ \texttt{TElabAEHole} \qquad \frac{\Phi \vdash \hat{\tau} \Rightarrow \kappa' \rightsquigarrow \tau \dashv \Delta}{\Phi \vdash (\!|\hat{\tau}|\!)^u \Leftarrow \kappa \rightsquigarrow (\!|\tau|\!)^u \dashv \Delta, u :: \kappa}\ \texttt{TElabANEHole}$$

$\boxed{\Phi_1 \vdash \tau : \kappa \rhd \rho \dashv \Phi_2}$ ρ matches against $\tau : \kappa$ extending Φ if necessary

$$\frac{t\ \mathtt{valid}}{\Phi \vdash \tau : \kappa \rhd t \dashv \Phi, t :: \kappa}\ \texttt{RESVar} \qquad \frac{}{\Phi \vdash \tau : \kappa \rhd (\!|\,|\!) \dashv \Phi}\ \texttt{RESEHole} \qquad \frac{\neg(t\ \mathtt{valid})}{\Phi \vdash \tau : \kappa \rhd (\!|t|\!) \dashv \Phi}\ \texttt{RESVarHole}$$

$\boxed{\Gamma; \Phi \vdash e \Rightarrow \hat{\tau} \rightsquigarrow d \dashv \Delta}$ e synthesizes type τ and elaborates to d

ESDefine

$$\frac{\Phi_1 \vdash \hat{\tau} \Rightarrow \kappa \rightsquigarrow \tau \dashv \Delta_1 \qquad \Phi_1 \vdash \tau : \kappa \rhd \rho \dashv \Phi_2 \qquad \Gamma; \Phi_2 \vdash e \Rightarrow \tau_1 \rightsquigarrow d \dashv \Delta_2}{\Gamma; \Phi_1 \vdash \texttt{type}\ \rho\ \texttt{=}\ \hat{\tau}\ \texttt{in}\ e \Rightarrow \tau_1 \rightsquigarrow \texttt{type}\ \rho\ \texttt{=}\ \tau : \kappa\ \texttt{in}\ d \dashv \Delta_1 \cup \Delta_2}$$

$\boxed{\Delta; \Gamma; \Phi \vdash d : \tau}$ d is assigned type τ

DEDefine

$$\frac{\Phi_1 \vdash \tau_1 : \kappa \rhd \rho \dashv \Phi_2 \qquad \Delta; \Gamma; \Phi_2 \vdash d : \tau_2}{\Delta; \Gamma; \Phi_1 \vdash \texttt{type}\ \rho\ \texttt{=}\ \tau_1 : \kappa\ \texttt{in}\ d : \tau_2}$$

Theorem 1 (Well-Kinded Elaboration)
(1) If $\Phi \vdash \hat{\tau} \Rightarrow \kappa \rightsquigarrow \tau \dashv \Delta$ *then* $\Delta; \Phi \vdash \tau \Rightarrow \kappa$
(2) If $\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau \dashv \Delta$ *then* $\Delta; \Phi \vdash \tau \Leftarrow \kappa$

This is like the Typed Elaboration theorem in the POPL19 paper.

Theorem 2 (Elaborability)
(1) $\exists \Delta$ *s.t. if* $\Delta; \Phi \vdash \tau \Rightarrow \kappa$ *then* $\exists \hat{\tau}$ *such that* $\Phi \vdash \hat{\tau} \Rightarrow \kappa \rightsquigarrow \tau \dashv \Delta$
(2) $\exists \Delta$ *s.t. if* $\Delta; \Phi \vdash \tau \Leftarrow \kappa$ *then* $\exists \hat{\tau}$ *such that* $\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau \dashv \Delta$

This is similar but a little different from Elaborability theorem in the POPL19 paper. Choose the Δ that is emitted from elaboration and then there's an $\hat{\tau}$ that elaborates to any of the τ forms. Elaborability and Well-Kinded Elaboration implies we can just rely on the elaboration forms for the premises of any rules that demand kind synthesis/analysis.

Theorem 3 (Type Elaboration Unicity)
(1) If $\Phi \vdash \hat{\tau} \Rightarrow \kappa_1 \rightsquigarrow \tau_1 \dashv \Delta_1$ *and* $\Phi \vdash \hat{\tau} \Rightarrow \kappa_2 \rightsquigarrow \tau_2 \dashv \Delta_2$ *then* $\kappa_1 = \kappa_2$, $\tau_1 = \tau_2$, $\Delta_1 = \Delta_2$
(2) If $\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau_1 \dashv \Delta_1$ *and* $\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau_2 \dashv \Delta_2$ *then* $\tau_1 = \tau_2$, $\Delta_1 = \Delta_2$

This is like the Elaboration Unicity theorem in the POPL19 paper.

Theorem 4 (Kind Synthesis Precision)
If $\Delta; \Phi \vdash \tau \Rightarrow \kappa_1$ *and* $\Delta; \Phi \vdash \tau \Leftarrow \kappa_2$ *then* $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$

Kind Synthesis Precision says that synthesis finds the most precise kappa possible for a given input type. This is somewhat trivial, but interesting to note because it means we can expect singletons wherever possible.

Theorem 5.1 (Kind Analysis Soundness)

If $\Delta; \Phi \vdash \tau \Leftarrow \kappa$ then $\Delta; \Phi \vdash \tau :: \kappa$

Theorem 5.2 (Kind Analysis Completeness)

If $\Delta; \Phi \vdash \tau :: \kappa$ then $\Delta; \Phi \vdash \tau \Leftarrow \kappa$

5.1 Induction on complexity

Base cases

$c :: Ty \;\&\; \kappa = Ty \Rightarrow Ty \lesssim Ty \Rightarrow c :: \kappa$
or

$c \Leftarrow \kappa \Rightarrow c \Rightarrow \kappa' \;\&\; \kappa' \lesssim \kappa \Rightarrow \kappa' = S(c) \Rightarrow S(c) \lesssim \kappa \Rightarrow S(c) \equiv \kappa \Rightarrow c :: S(c) \;\&\; S(c) \leq \kappa \Rightarrow c :: \kappa$

$t \Leftarrow \kappa \Rightarrow t \Rightarrow \kappa' \;\&\; \kappa' \lesssim \kappa \Rightarrow \kappa' = S(t) \;\&\; t :: \kappa'' \in \Phi \Rightarrow t :: \kappa'' \Rightarrow \kappa = Ty \Rightarrow t :: Ty \Rightarrow t :: \kappa$

or

$\kappa \equiv S(t) \Rightarrow \kappa \lesssim S(t) \Rightarrow$

$\underset{\sim}{t} \Leftarrow \kappa$

$()D^{u} \Leftarrow \kappa$