Typechecking a Simplified MIL

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```
git clone https://github.com/RobertHarper/TILT-Compiler
cd ./TILT-Compiler
cd ./Doc
lualatex miltheory.tex
# code.sty is at ../Apps/Twelf/tex/code.sty
```

1 Simplified MIL

1.1 Abstract Syntax

$$
\begin{array}{rcll}
A & ::= & T & \text{kind of types} \\
 & \mid & S(M) & \text{singleton kind} \\
 & \mid & \Pi x{:}A_1.A_2 & \text{function kind} \\
 & \mid & \Sigma x{:}A_1.A_2 & \text{dependent pair kind}
\end{array}
$$

$$
\begin{array}{rcll}
M & ::= & b & \text{base type} \\
 & \mid & x & \text{constructor variable} \\
 & \mid & \lambda x{:}A.M & \text{constructor function} \\
 & \mid & M\,M' & \text{constructor application} \\
 & \mid & \langle M, M' \rangle & \text{constructor pair} \\
 & \mid & \pi_i M & \text{constructor projection}
\end{array}
$$

1.2 Static Semantics

Well-Formed Context $\boxed{\Gamma \vdash \textbf{ok}}$

$$\frac{}{\bullet \vdash \text{ok}} \tag{1}$$

$$\frac{\Gamma \vdash A \qquad x \notin \text{BV}(\Gamma)}{\bullet, x{:}A \vdash \text{ok}} \tag{2}$$

Well-Formed Kind $\boxed{\Gamma \vdash A}$

$$\frac{\Gamma \vdash \text{ok}}{\Gamma \vdash T} \tag{3}$$

$$\frac{\Gamma \vdash M : T}{\Gamma \vdash S(M)} \tag{4}$$

$$\frac{\Gamma \vdash A_1 \qquad \Gamma, x{:}A_1 \vdash A_2}{\Gamma \vdash \Pi x{:}A_1.A_2} \tag{5}$$

$$\frac{\Gamma \vdash A_1 \qquad \Gamma, x{:}A_1 \vdash A_2}{\Gamma \vdash \Sigma x{:}A_1.A_2} \tag{6}$$

Subkinding $\boxed{\Gamma \vdash A \preceq A'}$

$$\frac{\Gamma \vdash \mathrm{ok}}{\Gamma \vdash T \preceq T} \tag{7}$$

$$\frac{\Gamma \vdash M_1 \equiv M_2 : T}{\Gamma \vdash S(M_1) \preceq S(M_2)} \tag{8}$$

$$\frac{\Gamma \vdash S(M)}{\Gamma \vdash S(M) \preceq T} \tag{9}$$

$$\frac{\Gamma \vdash A_1' \preceq A_1 \qquad \Gamma, x{:}A_1' \vdash A_2 \preceq A_2'}{\Gamma \vdash \Pi x{:}A_1.A_2 \preceq \Pi x{:}A_1'.A_2'} \tag{10}$$

$$\frac{\Gamma \vdash A_1 \preceq A_1' \quad \Gamma, x{:}A_1 \vdash A_2 \preceq A_2'}{\Gamma \vdash \Sigma x{:}A_1.A_2 \preceq \Sigma x{:}A_1'.A_2'} \tag{11}$$

Well-Formed Constructor $\boxed{\Gamma \vdash M : A}$

$$\frac{\Gamma \vdash \mathrm{ok}}{\Gamma \vdash b : T} \tag{12}$$

$$\frac{\Gamma \vdash \mathrm{ok}}{\Gamma \vdash x : \Gamma(x)} \tag{13}$$

$$\frac{\Gamma \vdash A_1 \qquad \Gamma, x{:}A_1 \vdash M : A_2}{\Gamma \vdash \lambda x{:}A_1.M : \Pi x{:}A_1.A_2} \tag{14}$$

$$\frac{\Gamma \vdash M : \Pi x{:}A_1.A_2 \qquad \Gamma \vdash M_1 : A_1}{\Gamma \vdash M\, M_1 : \{x \mapsto M_1\}A_2} \tag{15}$$

$$\frac{\Gamma \vdash M : \Sigma x{:}A_1.A_2}{\Gamma \vdash \pi_1 M : A_1} \tag{16}$$

$$\frac{\Gamma \vdash M : \Sigma x{:}A_1.A_2}{\Gamma \vdash \pi_2 M : \{x \mapsto \pi_1 M\}A_2} \tag{17}$$

$$\frac{\Gamma \vdash M_1 : A_1 \qquad \Gamma, x{:}A_1 \vdash A_2 \qquad \Gamma \vdash M_2 : \{x \mapsto M_1\}A_2}{\Gamma \vdash \langle M_1, M_2 \rangle : \Sigma x{:}A_1.A_2} \tag{18}$$

$$\frac{\Gamma \vdash M : T}{\Gamma \vdash M : S(M)} \tag{19}$$

$$\frac{\Gamma \vdash M : \Sigma x{:}A_1.A_2 \qquad \Gamma \vdash \pi_1 M : A_1'}{\Gamma \vdash M : \Sigma x{:}A_1'.A_2} \tag{20}$$

$$\frac{\Gamma \vdash M : \Sigma x{:}A_1.A_2 \qquad \Gamma \vdash \pi_2 M : A_2'}{\Gamma \vdash M : \Sigma x{:}A_1.A_2'} \tag{21}$$

$$\frac{\Gamma \vdash M : \Pi x{:}A_1.A_2 \qquad \Gamma, x{:}A_1 \vdash M\,x : A_2'}{\Gamma \vdash M : \Pi x{:}A_1.A_2'} \tag{22}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash A \preceq A'}{\Gamma \vdash M : A'} \tag{23}$$

Constructor Equivalence $\boxed{\Gamma \vdash M \equiv M' : A}$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash M \equiv M : A} \tag{24}$$

$$\frac{\Gamma \vdash M' \equiv M : A}{\Gamma \vdash M \equiv M' : A} \tag{25}$$

$$\frac{\Gamma \vdash M \equiv M' : A \qquad \Gamma \vdash M' \equiv M'' : A}{\Gamma \vdash M \equiv M'' : A} \tag{26}$$

$$\frac{\Gamma \vdash M : S(M')}{\Gamma \vdash M \equiv M' : T} \tag{27}$$

$$\frac{\Gamma \vdash A_1 \equiv A_1' \qquad \Gamma, x{:}A_1 \vdash M \equiv M' : A_2}{\Gamma \vdash \lambda x{:}A_1.M \equiv \lambda x{:}A_1'.M' : \Pi x{:}A_1.A_2} \tag{28}$$

$$\frac{\Gamma \vdash M \equiv M' : \Sigma x{:}A_1.A_2}{\Gamma \vdash \pi_1 M \equiv \pi_1 M' : A_1} \tag{29}$$

$$\frac{\Gamma \vdash M \equiv M' : \Sigma x{:}A_1.A_2}{\Gamma \vdash \pi_2 M \equiv \pi_2 M' : \{x {\mapsto} \pi_1 M\}A_2} \tag{30}$$

$$\frac{\Gamma \vdash M_1 \equiv M_1' : A_1 \qquad \Gamma, x{:}A_1 \vdash A_2 \qquad \Gamma \vdash M_2 \equiv M_2' : \{x {\mapsto} M_1\}A_2}{\Gamma \vdash \langle M_1, M_2 \rangle \equiv \langle M_1', M_2' \rangle : \Sigma x{:}A_1.A_2} \tag{31}$$

$$\frac{\Gamma \vdash M \equiv M' : \Pi x{:}A_1.A_2 \qquad \Gamma \vdash M_1 \equiv M_1' : A_1}{\Gamma \vdash M\,M_1 \equiv M'\,M_1' : \{x {\mapsto} M_1\}A_2} \tag{32}$$

$$\frac{\Gamma \vdash M : \Pi x{:}A_1.A_2}{\Gamma \vdash \lambda x{:}A_1.(M\,x) \equiv M : \Pi x{:}A_1.A_2} \tag{33}$$

$$\frac{\Gamma \vdash \lambda x{:}A_2.M : \Pi x{:}A_2.A \qquad \Gamma \vdash M_2 : A_2}{\Gamma \vdash (\lambda x{:}A_2.M)\,M_2 \equiv \{x {\mapsto} M_2\}M : \{x {\mapsto} M_2\}A} \tag{34}$$

$$\frac{\Gamma \vdash \langle M_1, M_2 \rangle : \Sigma x{:}A_1.A_2}{\Gamma \vdash \pi_1 \langle M_1, M_2 \rangle \equiv M_1 : A_1} \tag{35}$$

$$\frac{\Gamma, x{:}A_1 \vdash A_2 \qquad \Gamma \vdash \langle M_1, M_2\rangle : \Sigma x{:}A_1.A_2}{\Gamma \vdash \pi_2\langle M_1, M_2\rangle \equiv M_2 : \{x{\mapsto}M_1\}A_2} \tag{36}$$

$$\frac{\Gamma \vdash M : \Sigma x{:}A_1.A_2}{\Gamma \vdash M \equiv \langle \pi_1 M, \pi_2 M\rangle : \Sigma x{:}A_1.A_2} \tag{37}$$

$$\frac{\Gamma \vdash M_1 \equiv M_2 : A \qquad \Gamma \vdash A \preceq A'}{\Gamma \vdash M_1 \equiv M_2 : A'} \tag{38}$$

2 Algorithm for Constructor Equivalence

2.1 Elimination Contexts

$$\begin{array}{rcl} E & ::= & \bullet \\ & \mid & E\,M \\ & \mid & \pi_1 E \\ & \mid & \pi_2 E \end{array}$$

2.2 Equivalence algorithm

$\Gamma \vdash E[(\lambda x{:}A.M)\, M'] \Downarrow V$	if $\Gamma \vdash E[\{x{\mapsto}M'\}M] \Downarrow V$
$\Gamma \vdash E[\pi_1\langle M_1, M_2\rangle] \Downarrow V$	if $\Gamma \vdash E[M_1] \Downarrow V$
$\Gamma \vdash E[\pi_2\langle M_1, M_2\rangle] \Downarrow V$	if $\Gamma \vdash E[M_2] \Downarrow V$
$\Gamma \vdash E[x] \Downarrow V$	if $\Gamma \vdash E[x] \Uparrow S(M)$ and $\Gamma \vdash M \Downarrow V$
$\Gamma \vdash V \Downarrow V$	
$\Gamma \vdash x \Uparrow \Gamma(x)$	
$\Gamma \vdash \pi_1 E[x] \Uparrow A_1$	if $\Gamma \vdash E[x] \Uparrow \Sigma y{:}A_1.A_2$
$\Gamma \vdash \pi_2 E[x] \Uparrow \{y{\mapsto}\pi_1 E[x]\}A_2$	if $\Gamma \vdash E[x] \Uparrow \Sigma y{:}A_1.A_2$
$\Gamma \vdash E[x]\, M \Uparrow \{y{\mapsto}M\}A_2$	if $\Gamma \vdash E[x] \Uparrow \Pi y{:}A_1.A_2$
$\Gamma \vdash M \iff M' : T$	if $\Gamma \vdash M \Downarrow V$, $\Gamma \vdash M' \Downarrow V'$, and $\Gamma \vdash V : T \sim V' : T$
$\Gamma \vdash M \iff M' : S(M'')$	always
$\Gamma \vdash M \iff M' : \Pi x{:}A.A'$	if $\Gamma, x{:}A \vdash M\, x \iff M'\, x : A'$
$\Gamma \vdash M \iff M' : \Sigma x{:}A_1.A_2$	if $\Gamma \vdash \pi_1 M \iff \pi_1 M' : A_1$, $\Gamma \vdash \pi_2 M \iff \pi_2 M' : \{x{\mapsto}\pi_1 M\}A_2$, and $\Gamma \vdash \pi_2 M \iff \pi_2 M' : \{x{\mapsto}\pi_1 M'\}A_2$
$\Gamma \vdash b : T \sim b : T$	
$\Gamma \vdash y : \Gamma(y) \sim y : \Gamma(y)$	
$\Gamma \vdash (E[y])\, M : A \sim (E'[y])\, M' : A'$	if $\Gamma \vdash E[y] : \Pi x{:}A_1.A_2 \sim E'[y] : \Pi x{:}A_1'.A_2'$, andalso $A = \{x{\mapsto}M\}A_2$, $A' = \{x{\mapsto}M'\}A_2'$, $\Gamma \vdash M_1 \iff M_2 : A_1$, and $\Gamma \vdash M_1 \iff M_2 : A_1'$.
$\Gamma \vdash \pi_1(E[y]) : A \sim \pi_1(E[y]) : A'$	if $\Gamma \vdash E[y] : \Sigma x{:}A_1.A_2 \sim E'[y] : \Sigma x{:}A_1'.A_2'$, and $A = A_1$ and $A' = A_1'$
$\Gamma \vdash \pi_2(E[y]) : A \sim \pi_2(E'[y]) : A'$	if $\Gamma \vdash E[y] : \Sigma x{:}A_1.A_2 \sim E'[y] : \Sigma x{:}A_1'.A_2'$, $A = \{x{\mapsto}\pi_1(E[y])\}A_2$, and $A' = \{x{\mapsto}\pi_1(E'[y])\}A_2'$
$\Gamma \vdash T \iff T$	always
$\Gamma \vdash S(M_1) \iff S(M_2)$	if $\Gamma \vdash M_1 \iff M_2 : T$
$\Gamma \vdash \Pi x{:}A_1.B_1 \iff \Pi x{:}A_2.B_2$	if $\Gamma \vdash A_1 \iff A_2$ and $\Gamma, x{:}A_1 \vdash B_1 \iff B_2$
$\Gamma \vdash \Sigma x{:}A_1.B_1 \iff \Sigma x{:}A_2.B_2$	if $\Gamma \vdash A_1 \iff A_2$ and $\Gamma, x{:}A_1 \vdash B_1 \iff B_2$
$\Gamma \vdash A$ norm	if $\Gamma \vdash A$ and $\Gamma \vdash A \iff A$
$\Gamma \vdash M$ norm A	if $\Gamma \vdash M : A$ and $\Gamma \vdash M \iff M : A$
$\Gamma \vdash M$ norm* A	if $\Gamma \vdash M : A \sim M : A$

3 Correctness

- A type $[\Delta]$ iff A sub A $[\Delta]$.
- A sub A' $[\Delta]$ iff
  1. $\Delta \vdash A \preceq A'$
  2. – $A = T$ and $A' = T$
     – Or, $A' = S(M)$ and $A' = T$ and $\Delta \vdash M$ norm T
     – Or, $A = S(M)$ and $A' = S(M')$ and $\Delta \vdash M \iff M' : T$
     – Or, $A = \Pi x{:}A_1.A_2$ and $A' = \Pi x{:}A'_1.A'_2$ and A'_1 sub A_1 $[\Delta]$ and $\forall \Delta' \supseteq \Delta$ if N_1 is N_2 in A'_1 $[\Delta']$ then $\{x{\mapsto}N_1\}A_2$ sub $\{x{\mapsto}N_2\}A'_2$ $[\Delta']$.
     – Or, $A = \Sigma x{:}A_1.A_2$ and $A' = \Sigma x{:}A'_1.A'_2$ and A_1 sub A'_1 $[\Delta]$ and $\forall \Delta' \supseteq \Delta$ if N_1 is N_2 in A_1 $[\Delta']$ then $\{x{\mapsto}N_1\}A_2$ sub $\{x{\mapsto}N_2\}A'_2$ $[\Delta']$.
- A_1 is A_2 $[\Delta]$ iff A_1 sub A_2 $[\Delta]$ and A_2 sub A_1 $[\Delta]$.
- M in A $[\Delta]$ iff M is M in A $[\Delta]$.
- M_1 is M_2 in A $[\Delta]$ iff
  1. A type $[\Delta]$
  2. $\Delta \vdash M_1 \equiv M_2 : A$
  3. – $A = T$ and $\Delta \vdash M_1 \iff M_2 : T$.
     – Or, $A = S(N)$, $\Delta \vdash M_1 \iff N : T$, and $\Delta \vdash M_2 \iff N : T$.
     – Or, $A = \Pi x{:}A_1.A_2$ and $\forall \Delta' \supseteq \Delta$ if M'_1 is M'_2 in A_1 $[\Delta']$ then $M_1\,M'_1$ is $M_2\,M'_2$ in $\{x{\mapsto}M'_1\}A_2$ $[\Delta']$.
     – Or, $A = \Sigma x{:}A_1.A_2$ and $\pi_1 M_1$ is $\pi_1 M_2$ in A_1 $[\Delta]$ and $\pi_2 M_1$ is $\pi_2 M_2$ in $\{x{\mapsto}\pi_1 M_1\}A_2$ $[\Delta]$ (and hence $\pi_2 M_1$ is $\pi_2 M_2$ in $\{x{\mapsto}\pi_1 M_2\}A_2$ $[\Delta]$).
- γ_1 is γ_2 in Γ $[\Delta]$ if
  1. $\Gamma \vdash$ ok
  2. $\forall x \in \operatorname{dom}\Gamma.\ \gamma_1 x$ is $\gamma_2 x$ in $\hat{\gamma_1}(\Gamma x)$ $[\Delta]$ and $\forall x \in \operatorname{dom}\Gamma.\ \gamma_1 x$ is $\gamma_2 x$ in $\hat{\gamma_2}(\Gamma x)$ $[\Delta]$.
- γ in γ $[\Delta]\Gamma$ if γ is γ in Γ $[\Delta]$.

4 Proofs

Lemma 1

1. *If $\Gamma \vdash A$ then $\Gamma \vdash \_ \iff \_ : A$ is a partial equivalence relation on well-formed terms of type A.*
2. *$\Delta \vdash \_ \iff \_$ is a partial equivalence relation on well-formed types.*
3. *If $\Gamma \vdash M : A \sim M' : A'$ then $\Gamma \vdash M' : A' \sim M : A$ and $\Gamma \vdash M$ norm* A. If $\Gamma \vdash M : A \sim M' : A'$ and $\Gamma \vdash M' : A' \sim M'' : A''$ then $\Gamma \vdash M : A \sim M'' : A''$.*
4. *If A type $[\Delta]$ then $\_$ is $\_$ in A $[\Delta]$ is a partial equivalence relation on well-formed terms of type A.*
5. *If $\Delta \vdash$ ok then $\_$ sub $\_$ $[\Delta]$ is a (reflexive) partial order on well-formed types.*
6. *If $\Gamma \vdash$ ok and $\Delta \vdash$ ok then $\_$ is $\_$ in Γ $[\Delta]$ is a partial equivalence relation on environments.*

Proof: Straightforward induction. QED

Lemma 2 (Semantic Subtyping)
If A' sub A $[\Delta]$ *and* M_1 is M_2 in A' $[\Delta]$ *then* M_1 is M_2 in A $[\Delta]$.

Proof: By induction on A'. QED

Lemma 3
If $\Gamma \vdash M : A \sim M' : A'$ *then* $\Gamma \vdash M \Uparrow A$ *and* $\Gamma \vdash M' \Uparrow A'$.

Lemma 4
If $\Gamma \vdash M \Uparrow A$ *then* $\Gamma \vdash M : A$.

Lemma 5 (Correctness)

1. *If* $\Gamma \vdash M_1 : A$, $\Gamma \vdash M_2 : A$, *and* $\Gamma \vdash M_1 \iff M_2 : A$ *then* $\Gamma \vdash M_1 \equiv M_2 : A$.
2. *If* $\Gamma \vdash$ *ok and* $\Gamma \vdash M : A \sim M' : A'$ *then* $\Gamma \vdash A \equiv A'$ *and* $\Gamma \vdash M \equiv M : A$.

Theorem 1
If $\Delta \vdash A$ *and* A type $[\Delta]$ *then*

1. $\Delta \vdash A$ *norm*
2. *If* $\Delta \vdash E[y] : A \sim E'[y] : A'$ *and* A is A' $[\Delta]$, *then* $E[y]$ is $E'[y]$ in A $[\Delta]$.
3. *If* M_1 is M_2 in A $[\Delta]$ *then* $\Delta \vdash M_1 \iff M_2 : A$.

Proof: By induction on A.
Assume $\Delta \vdash A$ and A type $[\Delta]$.

- Case: $A = T$.
  1. $\Delta \vdash T$ norm by definition.
  2. By inversion, $A' = T$. Then $\Delta \vdash E[y] \Downarrow E[y]$ because $\Delta \vdash E[y] \Uparrow T$ and similarly, $\Delta \vdash E'[y] \Downarrow E'[y]$. Thus $\Delta \vdash E[y] : T \sim E'[y] : T$ implies $\Delta \vdash E[y] \iff E'[y] : T$. By Lemma 5 we have $\Delta \vdash E[y] \equiv E[y] : T$. Therefore $E[y]$ is $E'[y]$ in T $[\Delta]$.
  3. Follows directly from the definition of M_1 is M_2 in T $[\Delta]$.

- Case: $A = S(N)$.
  1. Because $S(N)$ type $[\Delta]$ we have $\Delta \vdash N$ norm T and therefore $\Delta \vdash S(N)$ norm.
  2. By inversion, $A' = S(N')$ with $\Delta \vdash N \iff N' : T$, so by Lemma 1 $\Delta \vdash N$ norm T and $\Delta \vdash N'$ norm T.

     By Lemma 3, $E[y]$ is a head expansion of N and $E'[y]$ is a head expansion of N'; since algorithmic equivalence is clearly closed under head expansion of normalizing expressions, we have $\Delta \vdash E[y] \iff N : T$ and $\Delta \vdash E'[y] \iff N' : T$.

     Finally, by Lemma 5 we have $\Delta \vdash N \equiv N' : S(N)$. Therefore, $E[y]$ is $E'[y]$ in $S(N)$ $[\Delta]$.
  3. Follows from the definition of M_1 is M_2 in $S(N)$ $[\Delta]$ and Lemma 1.

- Case: $A = \Pi x{:}A_1.A_2$.
  1. $\Pi x{:}A_1.A_2$ type $[\Delta]$ implies A_1 type $[\Delta]$. This further implies that $\Delta \vdash A_1$, and by IH(1) that $\Delta \vdash A_1$ norm. By Lemma 1 and IH(2) we have x in A_1 $[\Delta, x{:}A_1]$, so that A_2 type $[\Delta, x{:}A_1]$. By IH(1), $\Delta, x{:}A_1 \vdash A_2$ norm; therefore $\Delta \vdash \Pi x{:}A_1.A_2$ norm.

2. By Lemma 5, $\Delta \vdash E[y] \equiv E'[y] : A$.

   Let $\Delta' \supseteq \Delta$ be a well-formed context, and assume that M_1 is M_1' in A_1 $[\Delta']$. By MONOTONICITY and inversion, $A' = \Pi x{:}A_1'.A_2'$ where A_1 is A_1' $[\Delta']$ and $\{x{\mapsto}M_1\}A_2$ is $\{x{\mapsto}M_1'\}A_2'$ $[\Delta']$. Thus M_1 is M_1' in A_1' $[\Delta']$. By IH(3), $\Delta' \vdash M_1 \iff M_1' : A_1$ and $\Delta' \vdash M_1 \iff M_1' : A_1'$; hence $\Delta' \vdash (E[y])\, M_1 : \{x{\mapsto}M_1\}A_2 \sim (E'[y])\, M_1' : \{x{\mapsto}M_1'\}A_2'$. By IH(2), $(E[y])\, M_1$ is $(E'[y])\, M_1'$ in $\{x{\mapsto}M_1\}A_2$ $[\Delta']$.

   Therefore, $E[y]$ is $E'[y]$ in $\Pi x{:}A_1.A_2$ $[\Delta']$.

3. Assume M_1 is M_2 in $\Pi x{:}A_1.A_2$ $[\Delta]$. Since A_1 type $[\Delta]$, by IH(2) we have x in A_1 $[\Delta, x{:}A_1]$. Then $M_1\, x$ is $M_2\, x$ in A_2 $[\Delta, x{:}A_1]$. By IH(3), $\Delta, x{:}A_1 \vdash M_1\, x \iff M_2\, x : A_2$. Therefore, $\Delta \vdash M_1 \iff M_2 : \Pi x{:}A_1.A_2$.

- Case: $A = \Sigma x{:}A_1.A_2$.

  1. If A type $[\Delta]$ then A_1 type $[\Delta]$, so by IH(1) we have $\Delta \vdash A_1$ norm. By IH(2) we know that x in A_1 $[\Delta, x{:}A_1]$ and hence A_2 type $[\Delta, x{:}A_1]$. Again by IH(1) we have $\Delta, x{:}A_1 \vdash A_2$ norm. Therefore $\Delta \vdash A$ norm.

  2. By Lemma 5, $\Delta \vdash E[y] \equiv E'[y] : A$.

     By inversion we must have $A' = \Sigma x{:}A_1'.A_2'$ with A_1 is A_1' $[\Delta]$ and $\{x{\mapsto}M_1\}A_2$ is $\{x{\mapsto}M_1'\}A_2'$ $[\Delta]$ whenever M_1 is M_1' in A_1 $[\Delta]$.

     First, by IH(3) $\Delta \vdash \pi_1 E[y] : A_1 \sim \pi_1 E'[y] : A_1'$ and A_1 is A_1' $[\Delta]$ imply $\pi_1 E[y]$ is $\pi_1 E'[y]$ in A_1 $[\Delta]$.

     Then $\Delta \vdash \pi_2 E[y] : \{x{\mapsto}\pi_1 E[y]\}A_2 \sim \pi_2 E'[y] : \{x{\mapsto}\pi_1 E'[y]\}A_2'$ and $\{x{\mapsto}M_1\}A_2$ is $\{x{\mapsto}M_1'\}A_2'$ $[\Delta]$. Thus by IH(3) again, we have $\pi_2 E[y]$ is $\pi_2 E'[y]$ in $\{x{\mapsto}\pi_1 E[y]\}A_2$ $[\Delta]$.

     Therefore, $E[y]$ is $E'[y]$ in A $[\Delta]$.

  3. Assume M_1 is M_2 in $\Sigma x{:}A_1.A_2$ $[\Delta]$. By IH(3) $\pi_1 M_1$ is $\pi_2 M_2$ in A_1 $[\Delta]$ implies $\Delta \vdash \pi_1 M_1 \iff \pi_1 M_2 : A_1$. Similarly, $\pi_2 M_1$ is $\pi_2 M_2$ in $\{x{\mapsto}\pi_1 M_1\}A_2$ $[\Delta]$ implies $\Delta \vdash \pi_2 M_1 \iff \pi_2 M_2 : \{x{\mapsto}\pi_1 M\}A_2$. Therefore, $\Delta \vdash M_1 \iff M_2 : A$.

QED

We define $M_1 \simeq M_2$ as the symmetric relation generated by:

- $E[\{x{\mapsto}M_1\}M] \simeq E[(\lambda x{:}A_1.M)\, M_1]$
- $E[M_1] \simeq E[\pi_1\langle M_1, M_2\rangle]$
- $E[M_2] \simeq E[\pi_2\langle M_1, M_2\rangle]$

Theorem 2
If $M_1 \simeq M_2$, M_1 in A $[\Delta]$, *and* $\Delta \vdash M_2 : A$ *then* M_1 is M_2 in A $[\Delta]$.

Proof: By induction on A.
Assume $M_1 \simeq M_2$, M_1 in A $[\Delta]$, and $\Delta \vdash M_2 : A$. By LEMMA, $\Delta \vdash M_1 \equiv M_2 : A$.

- Case: $A = T$.

  By Theorem 1, we have $\Delta \vdash M_1$ norm T. Thus $\Delta \vdash M_1 \Downarrow V$ and $\Delta \vdash V$ norm T. But clearly $\Delta \vdash M_2 \Downarrow V$ as well. Hence $\Delta \vdash M_1 \iff M_2 : T$ and therefore M_1 is M_2 in T $[\Delta]$.

- Case: $A = S(N)$.

  First, M_1 in A $[\Delta]$ implies $\Delta \vdash M_1 \iff N : T$. Exactly as in the previous case, $\Delta \vdash M_1 \iff M_2 : T$. By Lemma 1, $\Delta \vdash M_2 \iff N : T$, and therefore, M_1 is M_2 in $S(N)$ $[\Delta]$.

- Case: $T = \Pi x{:}A'.A''$.

  Let $\Delta' \supseteq \Delta$ be a valid context and assume M_1' is M_2' in A' $[\Delta']$. Then $M_1\,M_1' \simeq M_2\,M_2'$ so by IH, $M_1\,M_1'$ is $M_2\,M_2'$ in $\{x{\mapsto}M_1'\}A''$ $[\Delta]$.

  Therefore, M_1 is M_2 in $\Pi x{:}A'.A''$ $[\Delta]$.

- Case: $A = \Sigma x{:}A'.A''$.

  First, $\pi_1 M_1$ in A' $[\Delta]$ and $\pi_2 M_1$ in $\{x{\mapsto}\pi_1 M_1\}A'$ $[\Delta]$. By IH, $\pi_1 M_1$ is $\pi_1 M_2$ in A' $[\Delta]$ and $\pi_2 M_1$ is $\pi_2 M_2$ in $\{x{\mapsto}\pi_1 M_1\}A'$ $[\Delta]$.

  Therefore, M_1 is M_2 in $\Sigma x{:}A'.A''$ $[\Delta]$

QED

Theorem 3
Assume that $\Delta \vdash ok$ and that γ_1 is γ_2 in Γ $[\Delta]$.

1. *If $\Gamma \vdash A$ then $\hat{\gamma}_1 A$ sub $\hat{\gamma}_2 A$ $[\Delta]$ (hence $\hat{\gamma}_1 A$ is $\hat{\gamma}_2 A$ $[\Delta]$).*

2. *If $\Gamma \vdash A_1 \preceq A_2$ then $\hat{\gamma}_1 A_1$ sub $\hat{\gamma}_1 A_2$ $[\Delta]$.*

3. *If $\Gamma \vdash M : A$ then $\hat{\gamma}_1 M$ is $\hat{\gamma}_2 M$ in $\hat{\gamma}_1 A$ $[\Delta]$ (hence $\hat{\gamma}_1 M$ is $\hat{\gamma}_2 M$ in $\hat{\gamma}_2 A$ $[\Delta]$).*

4. *If $\Gamma \vdash M_1 \equiv M_2 : A$ then $\hat{\gamma}_1 M_1$ is $\hat{\gamma}_1 M_2$ in $\hat{\gamma}_1 A$ $[\Delta]$.*

Proof:

- Rule 3. T sub T $[\Delta]$ by definition.

- Rule 4. By IH, $\hat{\gamma}_1 M$ is $\hat{\gamma}_2 M$ in T $[\Delta]$. Thus $\Delta \vdash \hat{\gamma}_1 M \iff \hat{\gamma}_2 M : T$ and $\Delta \vdash \hat{\gamma}_1 M \equiv \hat{\gamma}_2 M : T$, so $\Delta \vdash S(\hat{\gamma}_1 M) \preceq S(\hat{\gamma}_2 M)$. Therefore $S(\hat{\gamma}_1 M)$ sub $S(\hat{\gamma}_2 M)$ $[\Delta]$.

- Rule 5. By IH, $\hat{\gamma}_1 A_1$ is $\hat{\gamma}_2 A_1$ $[\Delta]$. Let $\Delta' \supseteq \Delta$ and assume N_1 is N_2 in $\hat{\gamma}_1 A_1$ $[\Delta']$. (Then N_1 is N_2 in $\hat{\gamma}_2 A_1$ $[\Delta']$.) By IH, $\gamma_1[x{\mapsto}N_1]A_2$ is $\gamma_2[x{\mapsto}N_2]A_2$ $[\Delta']$. That is, $\{x{\mapsto}N_1\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ is $\{x{\mapsto}N_2\}(\gamma_2\widehat{[x{\mapsto}x]}A_2)$ $[\Delta']$. Therefore, $\Pi x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ sub $\Pi x{:}\hat{\gamma}_2 A_1.(\gamma_2\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$.

- Rule 6. By IH, $\hat{\gamma}_1 A_1$ is $\hat{\gamma}_2 A_1$ $[\Delta]$. Let $\Delta' \supseteq \Delta$ and assume N_1 is N_2 in $\hat{\gamma}_1 A_1$ $[\Delta']$. (Then N_1 is N_2 in $\hat{\gamma}_2 A_1$ $[\Delta']$.) By IH, $\gamma_1[x{\mapsto}N_1]A_2$ is $\gamma_2[x{\mapsto}N_2]A_2$ $[\Delta']$. That is, $\{x{\mapsto}N_1\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ is $\{x{\mapsto}N_2\}(\gamma_2\widehat{[x{\mapsto}x]}A_2)$ $[\Delta']$. Therefore, $\Sigma x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ sub $\Sigma x{:}\hat{\gamma}_2 A_1.(\gamma_2\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$.

- Rule 7. T sub T $[\Delta]$ by definition.

- Rule 8. By IH, $\hat{\gamma}_1 M_1$ is $\hat{\gamma}_1 M_2$ in T $[\Delta]$. Thus $\Delta \vdash \hat{\gamma}_1 M_1 \iff \hat{\gamma}_2 M_1 : T$ and $\Delta \vdash \hat{\gamma}_1 M_1 \equiv \hat{\gamma}_2 M : T$, so $\Delta \vdash S(\hat{\gamma}_1 M) \preceq S(\hat{\gamma}_2 M)$. Therefore $S(\hat{\gamma}_1 M)$ sub $S(\hat{\gamma}_2 M)$ $[\Delta]$.

- Rule 9. By IH, $S(\hat{\gamma}_1 M)$ type $[\Delta]$. Thus $\Delta \vdash \hat{\gamma}_1 M$ norm T and $\Delta \vdash \hat{\gamma}_1 M : T$. Therefore $\Delta \vdash S(\hat{\gamma}_1 M) \preceq T$, so $S(\hat{\gamma}_1 M)$ sub T $[\Delta]$.

- Rule 10. By IH, $\hat{\gamma}_1 A_1'$ sub $\hat{\gamma}_1 A_1$ $[\Delta]$. Let $\Delta' \supseteq \Delta$ and assume M is M' in $\hat{\gamma}_1 A_1'$ $[\Delta]$. By IH, $\hat{\gamma}_1[x{\mapsto}M]A_2$ sub $\hat{\gamma}_1[x{\mapsto}M']A_2'$ $[\Delta']$. That is, $\{x{\mapsto}M\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ sub $\{x{\mapsto}M'\}(\gamma_1\widehat{[x{\mapsto}x]}A_2')$ $[\Delta']$. Therefore, $\Pi x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ sub $\Pi x{:}\hat{\gamma}_1 A_1'.(\gamma_1\widehat{[x{\mapsto}x]}A_2')$ $[\Delta]$.

- Rule 11. By IH, $\hat{\gamma}_1 A_1$ sub $\hat{\gamma}_1 A_1'$ $[\Delta]$. Let $\Delta' \supseteq \Delta$ and assume M is M' in $\hat{\gamma}_1 A_1$ $[\Delta]$. By IH, $\hat{\gamma}_1[x{\mapsto}M]A_2$ sub $\hat{\gamma}_1[x{\mapsto}M']A_2'$ $[\Delta']$. That is, $\{x{\mapsto}M\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ sub $\{x{\mapsto}M'\}(\gamma_1\widehat{[x{\mapsto}x]}A_2')$ $[\Delta']$. Therefore, $\Sigma x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ sub $\Sigma x{:}\hat{\gamma}_1 A_1'.(\gamma_1\widehat{[x{\mapsto}x]}A_2')$ $[\Delta]$.

- Rule 12. b is b in T $[\Delta]$ by definition.
- Rule 13. By assumption, $\hat{\gamma}_1 x$ is $\hat{\gamma}_2 x$ in $\hat{\gamma}_1(\Gamma x)$ $[\Delta]$.
- Rule 14. By IH, $\hat{\gamma}_1 A_1$ is $\hat{\gamma}_2 A_1$ $[\Delta]$. By Theorem 1, x in $\hat{\gamma}_1 A_1$ $[\Delta, x{:}\hat{\gamma}_1 A_1]$, so by IH $\gamma_1\widehat{[x\mapsto x]}M$ is $\gamma_2\widehat{[x\mapsto x]}M$ in $\gamma_1\widehat{[x\mapsto x]}M$ $[\Delta, x{:}\hat{\gamma}_1 A_1]$. Thus $\Delta \vdash \lambda x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x\mapsto x]}M) \equiv \lambda x{:}\hat{\gamma}_2 A_1.(\gamma_2\widehat{[x\mapsto x]}M) : \Pi x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x\mapsto x]}A_2)$.

  Let $\Delta' \supseteq \Delta$ and M_2 is $\hat{\gamma}_1 A_1$ in Δ' $[M_1]$. By IH, $\{x\mapsto M_1\}(\gamma_1\widehat{[x\mapsto x]}M)$ is $\{x\mapsto M_2\}(\gamma_2\widehat{[x\mapsto x]}M)$ in $\{x\mapsto M_1\}(\gamma_1\widehat{[x\mapsto x]}A_2)$ $[\Delta']$. By Theorem 2 and Lemma 1, $(\lambda x{:}\hat{\gamma}_1 A_1.\gamma_1\widehat{[x\mapsto x]}M)\, M_1$ is $(\lambda x{:}\hat{\gamma}_2 A_1.\gamma_2\widehat{[x\mapsto x]}M)\, M_2$ in $\{x\mapsto M_1\}(\gamma_1\widehat{[x\mapsto x]}A_2)$ $[\Delta']$.

  Therefore, $\lambda x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x\mapsto x]}M)$ is $\lambda x{:}\hat{\gamma}_2 A_1.(\gamma_2\widehat{[x\mapsto x]}M)$ in $\Pi x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x\mapsto x]}A_2)$ $[\Delta']$.
- Rule 15. By IH we have $\hat{\gamma}_1 M$ is $\hat{\gamma}_2 M$ in $\Pi x{:}\hat{\gamma}_1 A_1.\gamma_1\widehat{[x\mapsto x]}A_2$ $[\Delta]$ and $\hat{\gamma}_1 M_1$ is $\hat{\gamma}_2 M_1$ in $\hat{\gamma}_1 A_1$ $[\Delta]$. Thus $(\hat{\gamma}_1 M)\,(\hat{\gamma}_1 M_1)$ is $(\hat{\gamma}_2 M)\,(\hat{\gamma}_1 M_1)$ in $\hat{\gamma}_1(\{x\mapsto M_1\}A_2)$ $[\Delta]$. Further, since $\hat{\gamma}_2 M$ in $\Pi x{:}\hat{\gamma}_1 A_1.\gamma_1\widehat{[x\mapsto x]}A_2$ $[\Delta]$, we have $(\hat{\gamma}_2 M)\,(\hat{\gamma}_1 M_1)$ is $(\hat{\gamma}_2 M)\,(\hat{\gamma}_2 M_1)$ in $\hat{\gamma}_1(\{x\mapsto M_1\}A_2)$ $[\Delta]$. By symmetry and transivity, $(\hat{\gamma}_1 M)\,(\hat{\gamma}_1 M_1)$ is $(\hat{\gamma}_2 M)\,(\hat{\gamma}_2 M_1)$ in $\hat{\gamma}_1(\{x\mapsto M_1\}A_2)$ $[\Delta]$.
- Rule 16. By IH we have $\hat{\gamma}_1 M$ is $\hat{\gamma}_2 M$ in $\Sigma x{:}\hat{\gamma}_1 A_1.\gamma_1\widehat{[x\mapsto x]}A_2$ $[\Delta]$. Therefore, $\pi_1(\hat{\gamma}_1 M)$ is $\pi_1(\hat{\gamma}_2 M)$ in $\hat{\gamma}_1 A_1$ $[\Delta]$.
- Rule 17. By IH we have $\hat{\gamma}_1 M$ is $\hat{\gamma}_2 M$ in $\Sigma x{:}\hat{\gamma}_1 A_1.\gamma_1\widehat{[x\mapsto x]}A_2$ $[\Delta]$. Therefore, $\pi_2(\hat{\gamma}_1 M)$ is $\pi_2(\hat{\gamma}_2 M)$ in $\{x\mapsto\hat{\gamma}_1(\pi_1 M)\}(\gamma_1\widehat{[x\mapsto x]}A_2)$ $[\Delta]$, or equivalently $\hat{\gamma}_1(\pi_2 M)$ is $\hat{\gamma}_2(\pi_2 M)$ in $\hat{\gamma}_1(\{x\mapsto\pi_1 M\}A_2)$ $[\Delta]$.
- Rule 18. By IH we have $\hat{\gamma}_1 M_1$ is $\hat{\gamma}_2 M_1$ in $\hat{\gamma}_1 A_1$ $[\Delta]$ and $\hat{\gamma}_1 M_2$ is well-formed. By Theorem 2, we have $\hat{\gamma}_1(\pi_1\langle M_1, M_2\rangle)$ is $\hat{\gamma}_2(\pi_1\langle M_1, M_2\rangle)$ in $\hat{\gamma}_1 A_1$ $[\Delta]$. Again by IH we have $\hat{\gamma}_1 M_2$ is $\hat{\gamma}_2 M_2$ in $\{x\mapsto\hat{\gamma}_1 M_1\}(\gamma_1\widehat{[x\mapsto x]}A_2)$ $[\Delta]$. Again by Theorem 2, $\hat{\gamma}_1(\pi_1\langle M_1, M_2\rangle)$ is $\hat{\gamma}_1 M_1$ in $\hat{\gamma}_1 A_1$ $[\Delta]$. Thus, we may apply the IH to get $\hat{\gamma}_1\widehat{[x\mapsto\hat{\gamma}_1 M_1]}A_2$ sub $\hat{\gamma}_1\widehat{[x\mapsto\hat{\gamma}_1(\pi_1\langle M_1, M_2\rangle)]}A_2$ $[\Delta]$. Applying Theorem 2 one last time, we have $\hat{\gamma}_1(\pi_2\langle M_1, M_2\rangle)$ is $\hat{\gamma}_2(\pi_2\langle M_1, M_2\rangle)$ in $\{x\mapsto\hat{\gamma}_1 M_1\}(\gamma_1\widehat{[x\mapsto x]}A_2)$ $[\Delta]$. Therefore, $\hat{\gamma}_1 M_2$ is $\hat{\gamma}_2 M_2$ in $\{x\mapsto\hat{\gamma}_1(\pi_1\langle M_1, M_2\rangle)\}(\gamma_1\widehat{[x\mapsto x]}A_2)$ $[\Delta]$.
- Rule 19. By IH, $\hat{\gamma}_1 M$ is $\hat{\gamma}_2 M$ in T $[\Delta]$. Thus $\Delta \vdash \hat{\gamma}_1 M$ norm T, $\Delta \vdash \hat{\gamma}_1 M : T$, $\Delta \vdash \hat{\gamma}_1 M$, and $\Delta \vdash \hat{\gamma}_1 M : S(\hat{\gamma}_1 M)$, and so $\hat{\gamma}_1 M$ in $S(\hat{\gamma}_1 M)$ $[\Delta]$.

  Furthermore, we have $\Delta \vdash \hat{\gamma}_2 M$ norm T, $\Delta \vdash \hat{\gamma}_1 M \iff \hat{\gamma}_2 M : T$, $\Delta \vdash \hat{\gamma}_1 M \equiv \hat{\gamma}_2 M : T$, $\Delta \vdash S(\hat{\gamma}_1 M) \equiv S(\hat{\gamma}_2 M)$, and $\Delta \vdash \hat{\gamma}_2 M : S(\hat{\gamma}_1 M)$. Therefore $\hat{\gamma}_2 M$ in $S(\hat{\gamma}_1 M)$ $[\Delta]$ and so $\hat{\gamma}_1 M$ is $\hat{\gamma}_2 M$ in $S(\hat{\gamma}_1 M)$ $[\Delta]$.
- Rule 20. By IH $\hat{\gamma}_1 M$ is $\hat{\gamma}_2 M$ in $\Sigma x{:}(\hat{\gamma}_1 A_1).(\gamma_1\widehat{[x\mapsto x]}A_2)$ $[\Delta]$ and $\hat{\gamma}_1\pi_1 M$ is $\hat{\gamma}_2\pi_1 M$ in $\hat{\gamma}_1 A_1'$ $[\Delta]$. Thus $\pi_2\hat{\gamma}_1 M$ is $\pi_2\hat{\gamma}_2 M$ in $\{x\mapsto\pi_1\hat{\gamma}_1 M\}A_2$ $[\Delta]$. Therefore $\hat{\gamma}_1 M$ is $\hat{\gamma}_2 M$ in $\Sigma x{:}(\hat{\gamma}_1 A_1').(\gamma_1\widehat{[x\mapsto x]}A_2)$ $[\Delta]$ and
- Rule 21. By IH,$\hat{\gamma}_1 M$ is $\hat{\gamma}_2 M$ in $\Sigma x{:}(\hat{\gamma}_1 A_1).(\gamma_1\widehat{[x\mapsto x]}A_2)$ $[\Delta]$, and so $\pi_1\hat{\gamma}_1 M$ is $\pi_1\hat{\gamma}_2 M$ in $\hat{\gamma}_1 A_1$ $[\Delta]$. Again by IH, $\hat{\gamma}_1\pi_2 M$ is $\hat{\gamma}_2\pi_2 M$ in $\hat{\gamma}_1\{x\mapsto\pi_1 M\}A_2'$ $[\Delta]$. Therefore $\hat{\gamma}_1 M$ is $\hat{\gamma}_2 M$ in $\Sigma x{:}(\hat{\gamma}_1 A_1).(\gamma_1\widehat{[x\mapsto x]}A_2')$ $[\Delta]$.
- Rule 22. Assume $\Delta' \supseteq \Delta$ is well-formed, and let M_1 in $\hat{\gamma}_1 A_1$ $[\Delta']$. By IH, $\hat{\gamma}_1\widehat{[x\mapsto M_1]}(M\,x)$ is $\hat{\gamma}_2\widehat{[x\mapsto M_1]}(M\,x)$ in $\hat{\gamma}_1\widehat{[x\mapsto M_1]}A_2'$ $[\Delta]$. That is, $(\hat{\gamma}_1 M)\,M_1$ is $(\hat{\gamma}_2 M)\,M_1$ in $\widehat{[x\mapsto M_1]}(\gamma_1\widehat{[x\mapsto x]}A_2')$ $[\Delta]$. Therefore, $\hat{\gamma}_1 M$ is $\hat{\gamma}_2 M$ in $\Pi x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x\mapsto x]}A_2')$ $[\Delta]$.

- Rule 23 By IH we have $\hat{\gamma}_1 M$ is $\hat{\gamma}_2 M$ in $\hat{\gamma}_1 A$ $[\Delta]$ and $\hat{\gamma}_1 A$ sub $\hat{\gamma}_1 A'$ $[\Delta]$. Therefore, $\hat{\gamma}_1 M$ is $\hat{\gamma}_2 M$ in $\hat{\gamma}_1 A'$ $[\Delta]$.

- Rule 24 By IH we have $\hat{\gamma}_1 M$ is $\hat{\gamma}_1 M$ in $\hat{\gamma}_1 A$ $[\Delta]$.,

- Rule 25 By symmetry of semantic equivalence.

- Rule 26 By transitivity of semantic equivalence.

- Rule 27 By IH we have $\hat{\gamma}_1 M$ in $S(\hat{\gamma}_1 M')$ $[\Delta]$. Thus $\Delta \vdash \hat{\gamma}_1 M \equiv \hat{\gamma}_1 M' : T$ and $\Delta \vdash \hat{\gamma}_1 M \iff \hat{\gamma}_1 M' : T$. Therefore $\hat{\gamma}_1 M$ is $\hat{\gamma}_1 M'$ in T $[\Delta]$.

- Rule 28 Let the valid world $\Delta' \supseteq \Delta$ and M_1 in $\hat{\gamma}_1 A_1$ $[\Delta']$ be given. By IH, $\{x{\mapsto}M_1\}(\gamma_1\widehat{[x{\mapsto}x]}M)$ is $\{x{\mapsto}M_1\}(\gamma_1\widehat{[x{\mapsto}x]}M')$ in $\{x{\mapsto}M_1\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta']$. By Theorem 2, we have $(\lambda x{:}\hat{\gamma}_1 A_1.M)\,M_1$ is $(\lambda x{:}\hat{\gamma}_1 A_1.M')\,M_1$ in $\{x{\mapsto}M_1\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta']$. Therefore, $\lambda x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x{\mapsto}x]}M)$ is $\lambda x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x{\mapsto}x]}M')$ in $\Pi x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$.

- Rule 29 By IH, $\hat{\gamma}_1 M$ is $\hat{\gamma}_1 M'$ in $\Sigma x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$. Therefore, $\pi_1\hat{\gamma}_1 M$ is $\pi_1\hat{\gamma}_1 M'$ in $\hat{\gamma}_1 A_1$ $[\Delta]$.

- Rule 30 By IH, $\hat{\gamma}_1 M$ is $\hat{\gamma}_1 M'$ in $\Sigma x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$. Therefore, $\pi_2\hat{\gamma}_1 M$ is $\pi_2\hat{\gamma}_1 M'$ in $\{x{\mapsto}\pi_1\hat{\gamma}_1 M\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$.

- Rule 31. By IH, $\hat{\gamma}_1 M_1$ is $\hat{\gamma}_1 M_1'$ in $\hat{\gamma}_1 A_1$ $[\Delta]$ and $\hat{\gamma}_1 M_2$ is $\hat{\gamma}_1 M_2'$ in $\{x{\mapsto}\hat{\gamma}_1 M_1\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$. By Theorem 2, $\hat{\gamma}_1\pi_1\langle M_1, M_2\rangle$ is $\hat{\gamma}_1\pi_1\langle M_1', M_2'\rangle$ in $\hat{\gamma}_1 A_1$ $[\Delta]$, $\hat{\gamma}_1\pi_2\langle M_1, M_2\rangle$ is $\hat{\gamma}_1\pi_2\langle M_1', M_2'\rangle$ in $\{x{\mapsto}\hat{\gamma}_1 M_1\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$, and $\hat{\gamma}_1 M_1$ is $\hat{\gamma}_1\pi_1\langle M_1, M_2\rangle$ in $\hat{\gamma}_1 A_1$ $[\Delta]$. Thus by IH again $\hat{\gamma}_1[x{\mapsto}\hat{\gamma}_1\pi_1\langle M_1, M_2\rangle]A_2$ sub $\hat{\gamma}_1[x{\mapsto}\hat{\gamma}_1 M_1]A_2$ $[\Delta]$ Thus $\hat{\gamma}_1\pi_2\langle M_1, M_2\rangle$ is $\hat{\gamma}_1\pi_2\langle M_1', M_2'\rangle$ in $\{x{\mapsto}\hat{\gamma}_1\pi_1\langle M_1, M_2\rangle\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$, and so $\hat{\gamma}_1\langle M_1, M_2\rangle$ is $\hat{\gamma}_1\langle M_1', M_2'\rangle$ in $\Sigma x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$.

- Rule 32. By IH, $\hat{\gamma}_1 M$ is $\hat{\gamma}_1 M'$ in $\Pi x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$ and $\hat{\gamma}_1 M_1$ is $\hat{\gamma}_1 M_1'$ in $\hat{\gamma}_1 A_1$ $[\Delta]$. Therefore, $(\hat{\gamma}_1 M)\,(\hat{\gamma}_1 M_1)$ is $(\hat{\gamma}_1 M')\,(\hat{\gamma}_1 M_1')$ in $\{x{\mapsto}\hat{\gamma}_1 M_1\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$.

- Rule 33. Let the well-formed $\Delta' \supseteq \Delta$ and M_1 in $\hat{\gamma}_1 A_1$ $[\Delta']$ be given. By IH and this assumption, $\hat{\gamma}_1 M\, M_1$ in $\{x{\mapsto}\hat{\gamma}_1 M_1\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta']$. But $\hat{\gamma}_1 M\, M_1 = \{x{\mapsto}M_1\}(\hat{\gamma}_1 M\, x)$, so by Theorem 2 we have $(\lambda x{:}\hat{\gamma}_1 A_1.\hat{\gamma}_1 M\, x)\,M_1$ is $\hat{\gamma}_1 M\, M_1$ in $\{x{\mapsto}M_1\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta']$. Therefore $\lambda x{:}\hat{\gamma}_1 A_1.\hat{\gamma}_1 M\, x$ is $\hat{\gamma}_1 M$ in $\Pi x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$.

- Rule 34. By IH, $\hat{\gamma}_1 M_1$ in $\hat{\gamma}_1 A_1$ $[\Delta]$. Thus again by IH we have $\gamma_1[\widehat{x{\mapsto}\hat{\gamma}_1} M_1]M$ in $\gamma_1[\widehat{x{\mapsto}\hat{\gamma}_1} M_1]A_2$ $[\Delta]$. By Theorem 2 we have $(\lambda x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x{\mapsto}x]}M))\,M_1$ is $\{x{\mapsto}\hat{\gamma}_1 M_1\}(\gamma_1\widehat{[x{\mapsto}x]}M)$ in $\{x{\mapsto}\hat{\gamma}_1 M_1\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$.

- Rule 35. By IH, $\langle\hat{\gamma}_1 M_1, \hat{\gamma}_1 M_2\rangle$ in $\Sigma x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$. By Theorem 2 we have $\pi_1\langle\hat{\gamma}_1 M_1, \hat{\gamma}_1 M_2\rangle$ is $\hat{\gamma}_1 M_1$ in $\hat{\gamma}_1 A_1$ $[\Delta]$.

- Rule 36. By IH, $\langle\hat{\gamma}_1 M_1, \hat{\gamma}_1 M_2\rangle$ in $\Sigma x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$. Thus $\pi_1\langle\hat{\gamma}_1 M_1, \hat{\gamma}_1 M_2\rangle$ in $\hat{\gamma}_1 A_1$ $[\Delta]$ and $\pi_2\langle\hat{\gamma}_1 M_1, \hat{\gamma}_1 M_2\rangle$ in $\{x{\mapsto}\pi_1\langle\hat{\gamma}_1 M_1, \hat{\gamma}_1 M_2\rangle\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$. By Theorem 2 we have $\pi_1\langle\hat{\gamma}_1 M_1, \hat{\gamma}_1 M_2\rangle$ is $\hat{\gamma}_1 M_1$ in $\hat{\gamma}_1 A_1$ $[\Delta]$ and $\pi_2\langle\hat{\gamma}_1 M_1, \hat{\gamma}_1 M_2\rangle$ is $\hat{\gamma}_1 M_2$ in $\{x{\mapsto}\pi_1\langle\hat{\gamma}_1 M_1, \hat{\gamma}_1 M_2\rangle\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$. From the IH we get $\{x{\mapsto}\pi_1\langle\hat{\gamma}_1 M_1, \hat{\gamma}_1 M_2\rangle\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ sub $\{x{\mapsto}\hat{\gamma}_1 M_1\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$, and the desired conclusion follows.

- Rule 37. By IH we have $\hat{\gamma}_1 M$ in $\Pi x{:}\hat{\gamma}_1 A_1.(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$. Thus $\pi_1\hat{\gamma}_1 M$ in $\hat{\gamma}_1 A_1$ $[\Delta]$ and $\pi_2\hat{\gamma}_1 M$ in $\{x{\mapsto}\pi_1\hat{\gamma}_1 M\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$. By Theorem 2, $\pi_1\hat{\gamma}_1 M$ is $\pi_1\langle\pi_1\hat{\gamma}_1 M, \pi_2\hat{\gamma}_1 M\rangle$ in $\hat{\gamma}_1 A_1$ $[\Delta]$ and $\pi_2\hat{\gamma}_1 M$ is $\pi_2\langle\pi_2\hat{\gamma}_1 M, \pi_2\hat{\gamma}_1 M\rangle$ in $\{x{\mapsto}\pi_1\hat{\gamma}_1 M\}(\gamma_1\widehat{[x{\mapsto}x]}A_2)$ $[\Delta]$. Therefore, $\hat{\gamma}_1 M$ is $\hat{\gamma}_1\langle\pi_1\hat{\gamma}_1 M, \pi_2\hat{\gamma}_1 M\rangle$ in $\hat{\gamma}_1(\Pi x{:}A_1.A_2)$ $[\Delta]$.

- Rule 38. By IH we have $\hat{\gamma}_1 M_1$ is $\hat{\gamma}_1 M_2$ in $\hat{\gamma}_1 A$ $[\Delta]$ and $\hat{\gamma}_1 A$ sub $\hat{\gamma}_1 A'$ $[\Delta]$. By Lemma 2, $\hat{\gamma}_1 M_1$ is $\hat{\gamma}_1 M_2$ in $\hat{\gamma}_1 A'$ $[\Delta]$

QED

Corollary 1

1. *If* $\Gamma \vdash A$ *then* A type $[\Gamma]$.

2. *If* $\Gamma \vdash A \preceq A'$ *then* A sub A' $[\Gamma]$.

3. *If* $\Gamma \vdash M : A$ *then* M in A $[\Gamma]$.

4. *If* $\Gamma \vdash M \equiv M' : A$ *then* M is M' in A $[\Gamma]$.

Theorem 4

1. *If* $\vdash \Gamma_1 \equiv \Gamma_2$, $\Gamma_1 \vdash M_1$ *norm* A_1, $\Gamma_2 \vdash M_2$ *norm* A_2, *and* $\Gamma_1 \vdash A_1 \equiv A_2$, *then* $\Gamma \vdash M \iff M_2 : A_1$ *is decidable.*

2. *If* $\vdash \Gamma_1 \equiv \Gamma_2$, $\Gamma_1 \vdash M$ *norm** A, *and* $\Gamma_2 \vdash M'$ *norm** A', *then* $\Gamma \vdash M : A \sim M' : A'$ *is decidable.*

Proof: By induction on the proof that $\Gamma_1 \vdash M_1$ norm A_1 or $\Gamma_1 \vdash M_1$ norm* A_1 respectively. Note that for there to be any chance for the algorithmic equivalence to hold, the corresponding proofs for $\Gamma_2 \vdash M_2$ norm A_2 and $\Gamma_2 \vdash M_2$ norm* A_2 must use the same final step. QED