

single numbers

43.X

stone, harper 2006

PFPL

$\text{BinOp } \oplus ::= \text{Product} \mid \text{Sum} \mid \text{Arrow}$   
 $\text{Kind } \kappa ::= \text{Ty} \mid \text{KHole} \mid \text{S}(\tau)$   
 $\text{ConstantTypes } c ::= \text{Int} \mid \text{Float} \mid \text{Bool}$   
 $\text{UserHTyp } \hat{\tau} ::= c \mid \hat{\tau}_1 \oplus \hat{\tau}_2 \mid \text{list}(\hat{\tau}) \mid \langle \rangle^u \mid \langle \hat{\tau} \rangle^u$   
 $\text{InternalHTyp } \tau ::= c \mid \tau_1 \oplus \tau_2 \mid \text{list}(\tau) \mid \langle \rangle^u \mid \langle \tau \rangle^u \mid \underline{t}$   
 $\text{TypeVars } t$   
 $\text{TypePattern } \rho ::= t \mid \langle \rangle \mid \langle t \rangle$   
 $\text{UserExpression } e ::= \text{type } \rho = \hat{\tau} \text{ in } e \mid \text{elided}$   
 $\text{InternalExpression } \tau ::= \text{type } \rho = \tau : \kappa \text{ in } d \mid \text{elided}$

$\boxed{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2}$   $\kappa_1$  is a consistent subkind of  $\kappa_2$

KCHoleL

$\frac{}{\Delta; \Phi \vdash \text{KHole} \lesssim \kappa}$

KCHoleR

$\frac{}{\Delta; \Phi \vdash \kappa \lesssim \text{KHole}}$

KCRespectEquiv

$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2}$

KCSubsumption

$\frac{\Delta; \Phi \vdash \tau \not\Leftarrow \text{Ty}}{\Delta; \Phi \vdash \text{S}(\tau) \lesssim \text{Ty}}$  7

$\boxed{t \text{ valid}}$   $t$  is a valid type variable

$t$  is valid if it is not a builtin-type or keyword, begins with an alpha char or underscore, and only contains alphanumeric characters, underscores, and primes.

$\boxed{\Delta; \Phi \vdash \kappa \text{ kind}}$   $\kappa$  forms a kind

KFTy

$\frac{}{\Delta; \Phi \vdash \text{Ty} \text{ kind}}$

KFHole

$\frac{}{\Delta; \Phi \vdash \text{KHole} \text{ kind}}$

KFSing

$\frac{\Delta; \Phi \vdash \tau \not\Leftarrow \text{Ty}}{\Delta; \Phi \vdash \text{S}(\tau) \text{ kind}}$  43.2a

$\boxed{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}$   $\kappa_1$  is equivalent to  $\kappa_2$

$$\begin{array}{c} \text{KESym} \\ \frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_2 \equiv \kappa_1} \end{array} \quad \text{KETrans} \quad \frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \quad \Delta; \Phi \vdash \kappa_2 \equiv \kappa_3}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3}$$

$$\text{KESingEquiv} \quad \frac{\Delta; \Phi \vdash \tau_1 \equiv \tau_2}{\Delta; \Phi \vdash S(\tau_1) \equiv S(\tau_2)}$$

$\boxed{\Delta; \Phi \vdash \tau \Rightarrow \kappa}$   $\tau$  synthesizes kind  $\kappa$

would need to define labelled singletors for KHole  
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$$\begin{array}{c} \text{KSConst} \\ \frac{}{\Delta; \Phi \vdash c \Rightarrow S(c)} \end{array} \quad \text{KSVar} \quad \frac{t : \kappa \in \Phi}{\Delta; \Phi \vdash t \Rightarrow S(t)} \quad \text{KSUVar} \quad \frac{t \notin \text{dom}(\Phi)}{\Delta; \Phi \vdash t \Rightarrow \text{KHole}}$$

$$\text{KSBinOp} \quad \frac{\Delta; \Phi \vdash \tau_1 \Leftarrow S(\tau_1) \quad \Delta; \Phi \vdash \tau_2 \Leftarrow S(\tau_2)}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 \Rightarrow S(\tau_1 \oplus \tau_2)}$$

$$\text{KSList} \quad \frac{\Delta; \Phi \vdash \tau \Leftarrow S(\tau)}{\Delta; \Phi \vdash \text{list}(\tau) \Rightarrow S(\text{list}(\tau))} \quad \text{KSEHole} \quad \frac{u :: \kappa \in \Delta}{\Delta; \Phi \vdash \llbracket \tau \rrbracket^u \Rightarrow \kappa}$$

$$\text{KSNEHole} \quad \frac{u :: \kappa \in \Delta \quad \Delta; \Phi \vdash \tau \Rightarrow \kappa'}{\Delta; \Phi \vdash \llbracket \tau \rrbracket^u \Rightarrow \kappa}$$

$\boxed{\Delta; \Phi \vdash \tau \Leftarrow \kappa}$   $\tau$  analyzes against kind  $\kappa$

$$\text{KAASubsume} \quad \frac{\Phi \vdash \tau \Rightarrow \kappa' \quad \Delta; \Phi \vdash \kappa' \lesssim \kappa}{\Delta; \Phi \vdash \tau \Leftarrow \kappa}$$

$$\text{KAVar} \quad \frac{t :: \kappa_1 \in \Phi \quad \kappa_1 \lesssim \kappa}{\Delta; \Phi \vdash t \Leftarrow \kappa}$$

$\Delta; \Phi \vdash \tau_1 \equiv \tau_2$   $\tau_1$  is equivalent to  $\tau_2$

$$\frac{\text{KCESymm} \quad \Delta; \Phi \vdash \tau_1 \equiv \tau_2}{\Delta; \Phi \vdash \tau_2 \equiv \tau_1}$$

$$\frac{\text{KCETrans} \quad \Delta; \Phi \vdash \tau_1 \equiv \tau_2 \quad \Delta; \Phi \vdash \tau_2 \equiv \tau_3}{\Delta; \Phi \vdash \tau_1 \equiv \tau_3}$$

$$\frac{\text{KCESingEquiv} \quad \Delta; \Phi \vdash \tau_1 \not\equiv S(\tau_2)}{\Delta; \Phi \vdash \tau_1 \equiv \tau_2}$$

$$\frac{\text{KCERefl} \quad \tau :: k}{\Delta; \Phi \vdash \tau \equiv \tau}$$

$$\frac{\text{KCEConst} \quad c \equiv c}{\Delta; \Phi \vdash c \equiv c}$$

$$\frac{\text{KCEVar} \quad t : \kappa \in \Phi}{\Delta; \Phi \vdash t \equiv t}$$

$$\frac{\text{KCEBinOp} \quad \Delta; \Phi \vdash \tau_1 \equiv \tau_2 \quad \Delta; \Phi \vdash \tau_3 \equiv \tau_4}{\Delta; \Phi \vdash \tau_1 \oplus \tau_3 \equiv \tau_2 \oplus \tau_4}$$

$$\frac{\text{KCEList} \quad \Delta; \Phi \vdash \tau_1 \equiv \tau_2}{\Delta; \Phi \vdash \text{list}(\tau_1) \equiv \text{list}(\tau_2)}$$

$$\frac{\text{KCEEHole} \quad u :: \kappa \in \Delta}{\Delta; \Phi \vdash (\bigoplus)^u \equiv (\bigoplus)^u}$$

\* if  $u :: \kappa \notin \Delta$   
KCERefl says ok

$$\frac{\text{KCENEHole} \quad u :: \kappa \in \Delta \quad \Delta; \Phi \vdash \tau \not\equiv \kappa'}{\Delta; \Phi \vdash (\bigoplus)^\tau \equiv (\bigoplus)^\tau}$$

\* sim

$$\Delta; \Phi \vdash \tau :: k$$

$\tau$  is well formed at kind  $k$

$$\frac{t :: k \in \Xi}{\Delta; \Phi \vdash t :: k} \text{ 17}$$

$$\frac{\Delta; \Phi \vdash \tau :: \tau_y}{\Delta; \Phi \vdash \text{list}(\tau) :: \tau_y}$$

$$\frac{\Delta; \Phi \vdash \tau :: k_1 \quad \Delta; \Phi \vdash k_1 \leq k}{\Delta; \Phi \vdash \tau :: k} \text{ 43.2d}$$

$$\frac{\Delta; \Phi \vdash \tau :: \tau_y \quad 23}{\Delta; \Phi \vdash \tau :: S(\tau)} \text{ 43.2b}$$

$$\frac{u :: k \in \Delta}{\Delta; \Phi \vdash (\bigoplus)^u :: k}$$

$$\frac{\Delta; \Phi \vdash u :: k \in \Delta \quad \Delta; \Phi \vdash \tau :: k_1}{\Delta; \Phi \vdash (\bigoplus)^\tau :: k}$$

$$\frac{\Delta; \Phi \vdash \tau :: k}{\Delta; \Phi \vdash \tau :: S_k(\tau)} \text{ replaces 23?}$$

$$\frac{}{\Delta; \Phi \vdash c :: \tau_y} \text{ 16}$$

$$\frac{\Delta; \Phi \vdash \tau_1 :: \tau_y \quad \Delta; \Phi \vdash \tau_2 :: \tau_y}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 :: \tau_y}$$

$$\frac{t \notin \text{dom}(\Xi)}{\Delta; \Phi \vdash t :: k_{\text{Hole}}}$$

$$S_{\tau_y}(\tau) := S(\tau)$$

$$S_{S(\tau_y)}(\tau) := S(\tau)$$

$$S_{k_{\text{Hole}}}(\tau) := k_{\text{Hole}}$$

$\boxed{\Phi \vdash \hat{\tau} \Rightarrow \kappa \rightsquigarrow \tau \dashv \Delta}$   $\hat{\tau}$  synthesizes kind  $\kappa$  and elaborates to  $\tau$

TElabSConst

$$\frac{}{\Phi \vdash c \Rightarrow S(c) \rightsquigarrow c \dashv \cdot}$$

TElabSBinOp

$$\frac{\Phi \vdash \hat{\tau}_1 \Leftarrow \text{Ty} \rightsquigarrow \tau_1 \dashv \Delta_1 \quad \Phi \vdash \hat{\tau}_2 \Leftarrow \text{Ty} \rightsquigarrow \tau_2 \dashv \Delta_2}{\Phi \vdash \hat{\tau}_1 \oplus \hat{\tau}_2 \Rightarrow S(\tau_1 \oplus \tau_2) \rightsquigarrow \tau_1 \oplus \tau_2 \dashv \Delta_1 \cup \Delta_2}$$

TElabSList

$$\frac{\Phi \vdash \hat{\tau} \Leftarrow \text{Ty} \rightsquigarrow \tau \dashv \Delta}{\Phi \vdash \text{list}(\hat{\tau}) \Rightarrow S(\text{list}(\tau)) \rightsquigarrow \text{list}(\tau) \dashv \Delta}$$

TElabSVar

$$\frac{t : \kappa \in \Phi}{\Phi \vdash t \Rightarrow S(t) \rightsquigarrow t \dashv \cdot}$$

TElabSUNVar

$$\frac{t \notin \text{dom}(\Phi)}{\Phi \vdash t \Rightarrow \text{KHole} \rightsquigarrow \langle t \rangle^u \dashv u :: \text{KHole}}$$

TElabSHole

$$\frac{}{\Phi \vdash \langle \rangle^u \Rightarrow \text{KHole} \rightsquigarrow \langle \rangle^u \dashv u :: \text{KHole}}$$

TElabSNEHole

$$\frac{\Phi \vdash \hat{\tau} \Rightarrow \kappa \rightsquigarrow \tau \dashv \Delta}{\Phi \vdash \langle \hat{\tau} \rangle^u \Rightarrow \text{KHole} \rightsquigarrow \langle \tau \rangle^u \dashv \Delta, u :: \text{KHole}}$$

$\boxed{\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau \dashv \Delta}$   $\hat{\tau}$  analyzes against kind  $\kappa_1$  and elaborates to  $\tau$

TElabASubsume

$$\frac{\hat{\tau} \neq \langle \rangle^u \quad \hat{\tau} \neq \langle \hat{\tau}' \rangle^u \quad \Phi \vdash \hat{\tau} \Rightarrow \kappa' \rightsquigarrow \tau \dashv \Delta \quad \Delta; \Phi \vdash \kappa' \lesssim \kappa}{\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau \dashv \Delta}$$

TElabAEHole

$$\frac{}{\Phi \vdash \langle \rangle^u \Leftarrow \kappa \rightsquigarrow \langle \rangle^u \dashv u :: \kappa}$$

TElabANEHole

$$\frac{\Phi \vdash \hat{\tau} \Rightarrow \kappa' \rightsquigarrow \tau \dashv \Delta}{\Phi \vdash \langle \hat{\tau} \rangle^u \Leftarrow \kappa \rightsquigarrow \langle \tau \rangle^u \dashv \Delta, u :: \kappa}$$

$\boxed{\Phi_1 \vdash \tau : \kappa \triangleright \rho \dashv \Phi_2}$   $\rho$  matches against  $\tau : \kappa$  extending  $\Phi$  if necessary

RESVar

$$\frac{t \text{ valid}}{\Phi \vdash \tau : \kappa \triangleright t \dashv \Phi, t :: \kappa}$$

RESEHole

$$\frac{}{\Phi \vdash \tau : \kappa \triangleright \langle \rangle \dashv \Phi}$$

RESVarHole

$$\frac{\neg(t \text{ valid})}{\Phi \vdash \tau : \kappa \triangleright \langle t \rangle \dashv \Phi}$$

$\boxed{\Gamma; \Phi \vdash e \Rightarrow \hat{\tau} \rightsquigarrow d \dashv \Delta}$   $e$  synthesizes type  $\tau$  and elaborates to  $d$

**ESDefine**

$$\frac{\begin{array}{c} \Phi_1 \vdash \hat{\tau} \Rightarrow \kappa \rightsquigarrow \tau \dashv \Delta_1 \\ \Phi_1 \vdash \tau : \kappa \triangleright \rho \dashv \Phi_2 \quad \Gamma; \Phi_2 \vdash e \Rightarrow \tau_1 \rightsquigarrow d \dashv \Delta_2 \end{array}}{\Gamma; \Phi_1 \vdash \mathbf{type} \ \rho = \hat{\tau} \ \mathbf{in} \ e \Rightarrow \tau_1 \rightsquigarrow \mathbf{type} \ \rho = \tau : \kappa \ \mathbf{in} \ d \dashv \Delta_1 \cup \Delta_2}$$

$\boxed{\Delta; \Gamma; \Phi \vdash d : \tau}$   $d$  is assigned type  $\tau$

**DEDefine**

$$\frac{\Phi_1 \vdash \tau_1 : \kappa \triangleright \rho \dashv \Phi_2 \quad \Delta; \Gamma; \Phi_2 \vdash d : \tau_2}{\Delta; \Gamma; \Phi_1 \vdash \mathbf{type} \ \rho = \tau_1 : \kappa \ \mathbf{in} \ d : \tau_2}$$

### Theorem 1 (Well-Kinded Elaboration)

- (1) If  $\Phi \vdash \hat{\tau} \Rightarrow \kappa \rightsquigarrow \tau \dashv \Delta$  then  $\Delta; \Phi \vdash \tau \Rightarrow \kappa$
- (2) If  $\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau \dashv \Delta$  then  $\Delta; \Phi \vdash \tau \Leftarrow \kappa$

This is like the Typed Elaboration theorem in the POPL19 paper.

### Theorem 2 (Elaborability)

- (1)  $\exists \Delta$  s.t. if  $\Delta; \Phi \vdash \tau \Rightarrow \kappa$  then  $\exists \hat{\tau}$  such that  $\Phi \vdash \hat{\tau} \Rightarrow \kappa \rightsquigarrow \tau \dashv \Delta$
- (2)  $\exists \Delta$  s.t. if  $\Delta; \Phi \vdash \tau \Leftarrow \kappa$  then  $\exists \hat{\tau}$  such that  $\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau \dashv \Delta$

This is similar but a little different from Elaborability theorem in the POPL19 paper. Choose the  $\Delta$  that is emitted from elaboration and then there's an  $\hat{\tau}$  that elaborates to any of the  $\tau$  forms. Elaborability and Well-Kinded Elaboration implies we can just rely on the elaboration forms for the premises of any rules that demand kind synthesis/analysis.

### Theorem 3 (Type Elaboration Unicity)

- (1) If  $\Phi \vdash \hat{\tau} \Rightarrow \kappa_1 \rightsquigarrow \tau_1 \dashv \Delta_1$  and  $\Phi \vdash \hat{\tau} \Rightarrow \kappa_2 \rightsquigarrow \tau_2 \dashv \Delta_2$  then  $\kappa_1 = \kappa_2$ ,  $\tau_1 = \tau_2$ ,  $\Delta_1 = \Delta_2$
- (2) If  $\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau_1 \dashv \Delta_1$  and  $\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau_2 \dashv \Delta_2$  then  $\tau_1 = \tau_2$ ,  $\Delta_1 = \Delta_2$

This is like the Elaboration Unicity theorem in the POPL19 paper.

#### Theorem 4 (Kind Synthesis Precision)

If  $\Delta; \Phi \vdash \tau \Rightarrow \kappa_1$  and  $\Delta; \Phi \vdash \tau \Leftarrow \kappa_2$  then  $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$

Kind Synthesis Precision says that synthesis finds the most precise kappa possible for a given input type. This is somewhat trivial, but interesting to note because it means we can expect singletons wherever possible.

#### Theorem 5.1 (Kind Analysis Soundness)

If  $\Delta; \Phi \vdash \tau \Leftarrow \kappa$  then  $\Delta; \Phi \vdash \tau :: \kappa$

#### Theorem 5.2 (Kind Analysis Completeness)

If  $\Delta; \Phi \vdash \tau :: \kappa$  then  $\Delta; \Phi \vdash \tau \Leftarrow \kappa$

5.1 Induction on complexity

Base cases

$$c :: \tau_g \text{ \& } \kappa = \tau_g \Rightarrow \tau_g \leq \tau_g \Rightarrow c :: \kappa$$

or

$$c \Leftarrow \kappa \Rightarrow c \Rightarrow \kappa' \text{ \& } \kappa' \leq \kappa \Rightarrow \kappa' = S(c) \Rightarrow S(c) \lesssim \kappa \Rightarrow S(c) \equiv \kappa \Rightarrow c :: S(c) \text{ \& } S(c) \leq \kappa \Rightarrow c :: \kappa$$

$$t \Leftarrow \kappa \Rightarrow t \Rightarrow \kappa' \text{ \& } \kappa' \leq \kappa \Rightarrow \kappa' = S(t) \text{ \& } t :: \kappa' \leq \kappa \Rightarrow t :: \kappa'' \Rightarrow \kappa = \tau_g \Rightarrow t :: \tau_g \Rightarrow t :: \kappa$$

$$\underline{t} \Leftarrow \kappa$$

or

$$\kappa \in S(t) \Rightarrow \kappa \leq S(t) \Rightarrow$$

$$\text{()D}'' \Leftarrow \kappa$$