some notes of mine

hejohns' notes

hejohns

December 21, 2022

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Papers/Articles

2012 - Soare, *Formalism and intuition in computability*

Tags: computability,history

Date: 2022-12-17

up to p.9

f

1981 - Kleene, *Origins of recursive function theory*

Tags: logic,computability,history

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need to read last 10 pages

nice review by Steward Shapiro, 1990

λ-defineable = Church, recursive = Gödel, Herbrand, computable = Turing,

(although until 193?, recursive $\mapsto$ primitive recursive, for Gödel, now recursive $\mapsto$ Herbrand-Gödel general recursive)

1995 - Makiko Nakano, *Makiko's Diary*

Tags:

Date: December 21, 2022

Translated by Kazuko Smith

sticky rice (desserts) = ½sticky rice + ½normal

Presentations/Lectures

2007 - Bryan Cantrill, *Dtrace*

Tags:

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Link: recording

A

Dictionary

AFAI*

Tags: terminology

Date: 2022-12-21

As Far As I (Can Tell / Know / …)

Craig's Trick

Tags: logic,computability

Date: 2022-12-18

Theorem (Craig's Trick)
From Mathew (MATH 684): S ce set of sentences $\implies \exists S^*.S^*$ computable $\wedge$ they have the same theory.

(my terminology) a theory is computable $\iff$ it is ce [1]

ie a theory is computably axiomatized $\iff$ it is computably enumerably axiomatized

Proof. MATH 684: S ce, so you only have a listing of the sentences. We can make it strictly monotonic by relisting, but by adding a bunch of tautological noise or padding to each sentence (say, by conjuncting tautologies, assuming eg Gödel's prime factorization encoding) st the Gödel number is much larger. (Each sentence is relisted logically equivalently.) □

See: Theory

So ce theories are just as effective as computable ones, which is good news for axiom schemas. A priori, it's not clear that theories w/ axiom schemas are as effective as finitely axiomatizable ones, but intuitively, axiom schemas are of the same character, are "easily checkable". Craig's trick formally grounds this.
eg PA is as effective as PA^-. (although PA^- has nice utility for the Entscheidungsproblem.)

1: …sentences up to logical equivalence

Wikipedia has a similar sketch.

Somehow the proof itself doesn't feel intuitive, but the "intended use" of the theorem *is*.

Complete Lattice

Tags: order

Date: 2022-12-21

Definition (Complete Lattice)
TFAE

- lattice w/ all joins and meets $\implies$
- lattice w/ all joins, meets, top, bottom
- lattice w/ all joins [2]

See: Thin Yoneda Embedding

2: People tend to say that the reals are "*the* complete ordered field", *in the sense that every bounded (above) set has a supremum. Which implies that every bounded (below) set has a infimum. (NOTE: "has" has a very specific meaning that everyone learns to live with.)

* TODO: apparently the ambiguity of "complete" makes a difference, but afaik the following is itself correct.

Enumeration Operator

Tags: logic,computability

Date: 2022-12-18

From Rogers' 1967 *Theory of Recursive Functions*.

Definition (Enumeration Operator)
Each enumeration reduction witness z and B determine the A, so each z determines a enumeration operator $\Phi_z : 2^\omega \to 2^\omega$.

An archetypical example: In Gödel's incompleteness theorems, each computable enumeration of a theory (my sense) gives rise to an enumeration of the theory (the deductive closure). In Miller's terms, there is the "deducibility operator" D that gives for each axiom set B, $D(B)$, the set of consequences.

ie $\Phi_z(B) = A \iff A \leq_e X$ witnessed by z.

$A \equiv_e B \iff A \leq_e B \wedge B \leq_e A$

I'm thinking of the deducibility operator the whole time.

Theorem

- enumeration operators compose, by inspection
- $A \subseteq B \implies \Phi(A) \subseteq \Phi(B)$ (monotonicity)
- $x \in \Phi(B) \implies \exists C.C$ finite $\wedge C \subseteq B \wedge x \in \Phi(C)$ (continuity) [3]

3: which I'd call compactness

See: Dana Scott's graph model of λ-calculus

From Rogers' 1967 *Theory of Recursive Functions*.

Enumeration Reducibility

Tags: logic,computability

Date: 2022-12-18

This definition is not as nice as (many-)one or Turing reduction, but the idea is that we want to "effectively list A using any listing (computable or not) of B". Note that enumeration reductions "only use positive information about B, and produce only positive information about A; whereas Turing reductions use and produce both positive and negative information." (paraphrased from the introduction to Russell Miller's *Non-coding Enumeration Operators*.)

Definition (Enumeration Reduc(tion/ible))
$A \leq_e B \iff \exists z.\forall x.x \in A \leftrightarrow \exists u.\langle x, y\rangle \in W_z \wedge D_u \subseteq B$

where z is the Gödel code of the reduction witness, and D_u is the finite set associated with u as a canonical index (ie a tuple). [4]

4: The idea w/ u is that to list A while watching elements enter B, you should only need (to see) a finite amount of B to list a particular element $x \in A$.

Rogers' (really simple) examples:

- $\{2n | n \in \omega\} \leq_e \omega$
- A ce $\implies \forall B.A \leq_e B$

See: Enumeration Operator

Dana Scott's Graph Model

Tags: logic,λ-calculus

Date: 2022-12-18

Definition

$$[\![\lambda x.]\!] := \tag{0.1}$$
$$[\![e_1 e_2]\!] := \tag{0.2}$$

See: Enumeration Operator

Herbrand's Theorem

Tags: logic

Date: 2022-12-17

TODO: convert notes from Prof. Blass' November seminar Herbrand's Theorem

Knaster-Tarski

Tags: logic

Date: 2022-12-19

Theorem (Knaster-Tarski Fixpoint Theorem)
Every monotone function on a complete lattice has a complete lattice of fixpoints.

This theorem has many statements, and this is the easiest for me to remember. The complete lattice is often a powerset lattice.

Proof. Widely available. □

Example

- The deducibility operator is a monotone function on sets of sentences. [5]
  The bottom (least) fixpoint is the (deductively closed) empty theory. The top (greatest) fixpoint is the inconsistent theory, ie the set of all sentences.
  Consistency of the empty theory (by Gentzen's original cut elimination, or by existence of a model) says this complete lattice is nontrivial.
  Any consistent, computably axiomatizable (deductively closed) theory that proves more than the empty theory is an intermediate fixpoint– eg PA.
  Incompleteness says there is no intermediate fixpoint above PA that is complete, but there are at least $2^{\aleph_0}$ intermediate fixpoints above PA where we keep adding $Con(T)$ or $\neg Con(T)$. [6]

5: For simplicity, assume everything is about– and still true about– a fixed language of arithmetic.

6: Are there complete intermediate fixpoints?

See: Enumeration Operator, Dana Scott's Graph Model

Locus Solum

Tags:

Date: December 21, 2022

This is my version of Girard's dictionary.

Also afaik the most Girard paper there is

PA

Tags:

Date: December 21, 2022

Peano Arithmetic w/ induction. PA^{-} := PA \ induction

Realizability

Tags:

Date: December 21, 2022

This is how we can attach beamer presentations

Theory

Tags: logic

Date: 2022-12-18

An unfortunately ambiguous term, but you can usually figure it out from context, if it really matters.

I tend to use "Theory" to just mean a set of sentences, as in the Γ in the sequent $\Gamma \vdash$. So I see a finite set for "the theory of groups", and a finite set unioned w/ a schema for "Peano Arithmetic". (and the empty set for the Entscheidungsproblem.) Sometimes, people mean a deductively closed set of sentences.

TFAE

Tags: terminology

Date: 2022-12-21

The Following Are Equivalent

When there are multiple characterizations of the same thing. Typically followed by a bullet list.

Thin Yoneda Embedding

Tags: order

Date: 2022-12-21

NOTE: re name: iirc this is mentioned in Awodey in the Yoneda chapter

Theorem (see Stoy,*Denotational Semantics* 6.29. Theorem.)
Every partial order can be embedded in a complete lattice.

The partial order is often a lattice already. I suppose you'd have to ensure the embedding respects the lattice structure as well, but afaict this should happen. Similar situation w/ the Yoneda embedding.

Proof. alt.

- The presheaf category in the Yoneda embedding is morally a complete lattice.
- Dedekind cuts.
  ie map each object to the downards closure, and utilize the lattice structure of *Set*
  ie $\cup, \cap$ and reasonable $\top, \bot$

$\square$

A nice example from Stoy (starting ~p.89): (everything standard $\leq$ ordering) $\mathbb{Q}$ is a partial order. †$\mathbb{Q}$ can be completed by adding $\top = +\infty, \bot = -\infty$ and taking Dedekind cuts.

See: Complete Lattice

† actually $\mathbb{Q}$ is a total non-empty order, thus a lattice, but for the theorem statement, we'll describe it this way.

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