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A New PDF-SYM Method

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ABSTRACT

Regarding the justification of the new PDF-SYM method.

Keywords: weak lensing, selection bias, Fourier quad

1. FORMALISM

Taking into account the contribution of the source-noise crossing term, we get:

$$\begin{aligned} G_1 =& \int d^2\vec{k}\left\{\left(k_x^2-k_y^2\right)+\hat{g}_1\left[2k^2-\beta^2\left(k^4+k_x^4-6k_x^2k_y^2+k_y^4\right)\right]\right\}T(\vec{k})\left(|F_k|^2+F_kN_k^*+F_k^*N_k\right) \\ =& \Delta^2\sum_k A_k\left(|F_k|^2+F_kN_k^*+F_k^*N_k\right)=\Delta^2\sum_k A_k\left(|F_k|^2+2F_k^{Re}N_k^{Re}+2F_k^{Im}N_k^{Im}\right) \end{aligned} \tag{1}$$

$$\begin{aligned} G_2 =& \int d^2\vec{k}\left\{(2k_xk_y)+\hat{g}_2\left[2k^2-\beta^2\left(k^4-k_x^4+6k_x^2k_y^2-k_y^4\right)\right]\right\}T(\vec{k})\left(|F_k|^2+F_kN_k^*+F_k^*N_k\right) \\ =& \Delta^2\sum_k B_k\left(|F_k|^2+2F_k^{Re}N_k^{Re}+2F_k^{Im}N_k^{Im}\right) \end{aligned} \tag{2}$$

The PDF P_G of G_1 and G_2 is given by:

$$\begin{aligned} & P_G\left(G_1,G_2\right) \\ =& \left\{\prod_k\int\int dN_k^{Re}dN_k^{Im}P_N(N_k^{Re},N_k^{Im})\right\} \\ & \times\delta_D\left[G_1-\Delta^2\sum_k A_k\left(|F_k|^2+2F_k^{Re}N_k^{Re}+2F_k^{Im}N_k^{Im}\right)\right]\delta_D\left[G_2-\Delta^2\sum_k B_k\left(|F_k|^2+2F_k^{Re}N_k^{Re}+2F_k^{Im}N_k^{Im}\right)\right] \\ =& \frac{1}{(2\pi)^2}\left\{\prod_k\int\int dN_k^{Re}dN_k^{Im}P_N(N_k^{Re},N_k^{Im})\right\} \\ & \times\int\int dxdy\exp\left(ixG_1+iyG_2\right)\exp\left\{-2i\Delta^2\sum_k(xA_k+yB_k)\left(F_k^{Re}N_k^{Re}+F_k^{Im}N_k^{Im}\right)-i\Delta^2\sum_k(xA_k+yB_k)|F_k|^2\right\} \end{aligned} \tag{3}$$

Assume that the noise is mainly white noise, therefore $P_N(N_k^{Re},N_k^{Im})$ are independent for different values of k:

$$P_N(N_k^{Re},N_k^{Im})=\frac{1}{2\pi\sigma^2}\exp\left[-\frac{\left(N_k^{Re}\right)^2+\left(N_k^{Im}\right)^2}{2\sigma^2}\right] \tag{4}$$

On the other hand, we have:

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}\sigma}\int du\exp\left(-\frac{u^2}{2\sigma^2}-i\alpha u\right)=\frac{1}{\sqrt{2\pi}\sigma}\int du\exp\left(-\frac{u^2+2i\sigma^2\alpha u-\sigma^4\alpha^2}{2\sigma^2}-\frac{\sigma^2\alpha^2}{2}\right) \\ =& \frac{1}{\sqrt{2\pi}\sigma}\int du\exp\left(-\frac{(u+i\sigma^2\alpha)^2}{2\sigma^2}-\frac{\sigma^2\alpha^2}{2}\right)=\exp\left(-\frac{\sigma^2\alpha^2}{2}\right) \end{aligned} \tag{5}$$

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Therefore, we have:

$$\begin{aligned}
&P_G\left(G_1, G_2\right) \qquad (6)\\
=&\frac{1}{(2\pi)^2}\int\int dxdy\exp\left(ixG_1+iyG_2\right)\times\exp\left\{-2\sigma^2\Delta^4\sum_k(xA_k+yB_k)^2|F_k|^2-i\Delta^2\sum_k(xA_k+yB_k)|F_k|^2\right\}\\
=&\frac{1}{(2\pi)^2}\int\int dxdy\exp\left[ix\left(G_1-\Gamma_1\right)+iy\left(G_2-\Gamma_2\right)\right]\times\exp\left(-x^2U_1-2xyU_2-y^2U_3\right)\\
=&\frac{1}{(2\pi)^2}\frac{\pi}{\sqrt{U_1U_3-U_2^2}}\exp\left[-\frac{\left(G_1-\Gamma_1\right)^2U_3-2\left(G_1-\Gamma_1\right)\left(G_2-\Gamma_2\right)U_2+\left(G_2-\Gamma_2\right)^2U_1}{4(U_1U_3-U_2^2)}\right]
\end{aligned}$$

in which we have:

$$\begin{aligned}
&\left(\Gamma_1,\Gamma_2\right)=\Delta^2\sum_k|F_k|^2\left(A_k,B_k\right) \qquad (7)\\
&\left(U_1,U_2,U_3\right)=2\sigma^2\Delta^4\sum_k|F_k|^2\left(A_k^2,A_kB_k,B_k^2\right)\\
&\int\int dxdy\exp\left(-x^2C-2xyD-y^2E+ixA+iyB\right)\\
=&\int\int dxdy\exp\left[-x^2C-Cx(2yD-iA)/C-C(2yD-iA)^2/(2C)^2+C(2yD-iA)^2/(2C)^2-y^2E+iyB\right]\\
=&\sqrt{\frac{\pi}{C}}\int dy\exp\left[y^2(D^2/C-E)+iy(B-AD/C)-A^2/(4C)\right]\\
=&\sqrt{\frac{\pi}{C}}\int dy\exp\left\{\left(D^2/C-E\right)\left[y+i\frac{(B-AD/C)}{2(D^2/C-E)}\right]^2+(D^2/C-E)\frac{(B-AD/C)^2}{4(D^2/C-E)^2}-A^2/(4C)\right\}\\
=&\frac{\pi}{\sqrt{CE-D^2}}\exp\left\{\frac{C(B-AD/C)^2-A^2(D^2/C-E)}{4(D^2-EC)}\right\}=\frac{\pi}{\sqrt{CE-D^2}}\exp\left\{\frac{CB^2-2ABD+EA^2}{4(D^2-EC)}\right\}
\end{aligned}$$

For the 1-D PDF of G_1, we have:

$$\begin{aligned}
&\int_{-\infty}^{\infty}dG_2P_G\left(G_1,G_2\right) \qquad (8)\\
=&\frac{1}{(2\pi)^2}\frac{\pi}{\sqrt{U_1U_3-U_2^2}}\int_{-\infty}^{\infty}dG_2\exp\left[-\frac{\left(G_1-\Gamma_1\right)^2U_3-2\left(G_1-\Gamma_1\right)\left(G_2-\Gamma_2\right)U_2+\left(G_2-\Gamma_2\right)^2U_1}{4(U_1U_3-U_2^2)}\right]\\
=&\frac{1}{(2\pi)^2}\frac{\pi}{\sqrt{U_1U_3-U_2^2}}\int_{-\infty}^{\infty}dz\exp\left\{-\frac{U_1\left[z-\left(G_1-\Gamma_1\right)U_2/U_1\right]^2-U_1\left(G_1-\Gamma_1\right)^2\left(U_2/U_1\right)^2+\left(G_1-\Gamma_1\right)^2U_3}{4(U_1U_3-U_2^2)}\right\}\\
=&\frac{1}{(2\pi)^2}\frac{\pi\sqrt{\pi}}{\sqrt{U_1}}\exp\left\{-\frac{\left(G_1-\Gamma_1\right)^2}{4U_1}\right\}
\end{aligned}$$

$$\begin{aligned}
\Gamma_1=&\Delta^2\sum_kA_k|F_k|^2 \qquad (9)\\
=&\int d^2\vec{k}\left\{\left(k_x^2-k_y^2\right)+\hat{g}_1\left[2k^2-\beta^2\left(k^4+k_x^4-6k_x^2k_y^2+k_y^4\right)\right]\right\}T(\vec{k})|F_k|^2\\
=&\int d^2\vec{k}\left\{\left(k_x^2-k_y^2\right)+\hat{g}_1\left[2k^2-\beta^2\left(k^4+k_x^4-6k_x^2k_y^2+k_y^4\right)\right]\right\}\exp(-\beta^2k^2)|F^L(\vec{k})|^2\\
=&\int d^2\vec{k}\left\{\left(k_x^2-k_y^2\right)+\hat{g}_1\left[2k^2-\beta^2\left(k^4+k_x^4-6k_x^2k_y^2+k_y^4\right)\right]\right\}\exp(-\beta^2k^2)|F^S(\mathrm{M}^{-1}\vec{k})|^2
\end{aligned}$$

where

$$\mathrm{M}=\begin{bmatrix}1-g_1 & -g_2\\ -g_2 & 1+g_1\end{bmatrix}$$

. Let us assume that $\hat{g}_{1,2} = g_{1,2}$, i.e., the PDF has been symmetrized by the correct shear values. We then have:

$$\Gamma_1 = \int d^2\vec{k}\left\{k_x^2 - k_y^2 + 4\beta^2 g_2 k_x k_y (k_x^2 - k_y^2)\right\}\exp(-\beta^2 k^2)|F^S(\vec{k})|^2 \tag{10}$$

Similarly, we have:

$$\begin{aligned}U_1 &= 2\sigma^2\Delta^4\sum_k A_k^2|F_k|^2 \\ &= 2\sigma^2\Delta^2\int d^2\vec{k}\left\{\left(k_x^2-k_y^2\right)+\hat{g}_1\left[2k^2-\beta^2\left(k^4+k_x^4-6k_x^2k_y^2+k_y^4\right)\right]\right\}^2 T^2(\vec{k})|F_k|^2 \\ &= 2\sigma^2\Delta^2\int d^2\vec{k}\left\{\left(k_x^2-k_y^2\right)^2+2\hat{g}_1\left(k_x^2-k_y^2\right)\left[2k^2-\beta^2\left(k^4+k_x^4-6k_x^2k_y^2+k_y^4\right)\right]\right\}\exp(-2\beta^2k^2)\frac{|F^S(\mathrm{M}^{-1}\vec{k})|^2}{|W_{PSF}(\vec{k})|^2} \\ &= 2\sigma^2\Delta^2\int d^2\vec{k}\left\{\left(k_x^2-k_y^2\right)^2+8\beta^2g_2k_xk_y\left(k_x^2-k_y^2\right)^2\right\}\exp(-2\beta^2k^2)\frac{|F^S(\vec{k})|^2}{|W_{PSF}(\mathrm{M}\vec{k})|^2}\end{aligned} \tag{11}$$

The term $W_{PSF}(\mathrm{M}\vec{k})$ is the source of the bias, as it makes the 1-D PDF defined in eq.(16) asymmetric after averaging over intrinsicly isotropic galaxy images.

Assuming the ellipticity of the PSF is (e_1, e_2), and suppose the components are small, we have:

$$|W_{PSF}(\mathrm{M}\vec{k})|^2 = W(k^2) - W'[2(e_1+g_1)(k_x^2-k_y^2) + 4(e_2+g_2)k_xk_y] \tag{12}$$

$$|W_{PSF}^{-2}(\mathrm{M}\vec{k})|^2 = W^{-1} + W^{-2}W'[2(e_1+g_1)(k_x^2-k_y^2) + 4(e_2+g_2)k_xk_y] \tag{13}$$

For convenience in the rest of our discussion, we set $g_2 = 0$, and the ellipticity e_2 of the PSF is also zero, we have:

$$U_1 = 2\sigma^2\Delta^2\int d^2\vec{k}\exp(-2\beta^2k^2)|F^S(\vec{k})|^2\left[W^{-1}(k_x^2-k_y^2)^2 + 2W^{-2}W'(e_1+g_1)(k_x^2-k_y^2)^3\right] \tag{14}$$

We can rewrite the functions of Γ_1 and U_1 as:

$$\begin{aligned}\Gamma_1 &= \int kdk\int d\theta\exp(-\beta^2k^2)|F^S(k,\theta)|^2k^2\cos(2\theta) \\ U_1 &= 2\sigma^2\Delta^2\int kdk\int d\theta\exp(-2\beta^2k^2)|F^S(k,\theta)|^2\left[W^{-1}k^4\cos^2(2\theta)+2(e_1+g_1)W^{-2}W'k^6\cos^3(2\theta)\right] = U_{10}+\delta U\end{aligned} \tag{15}$$

For the 1-D PDF of G_1, we have:

$$\begin{aligned}\int_{-\infty}^{\infty} dG_2 P_G\left(G_1, G_2\right) &= \frac{1}{(2\pi)^2}\frac{\pi\sqrt{\pi}}{\sqrt{U_1}}\exp\left\{-\frac{\left(G_1-\Gamma_1\right)^2}{4U_1}\right\} \\ &\approx \frac{1}{(2\pi)^2}\frac{\pi\sqrt{\pi}}{\sqrt{U_{10}}}\left[1-\frac{\delta U}{2U_{10}}+\frac{\left(G_1-\Gamma_1\right)^2}{4U_{10}}\frac{\delta U}{U_{10}}\right]\exp\left\{-\frac{\left(G_1-\Gamma_1\right)^2}{4U_{10}}\right\}\end{aligned} \tag{16}$$

The terms proportional to δU, when averaged over galaxies, generate the asymmetry of P_G.

Now, let us consider the case of adding compensation terms to G_1 and G_2:

$$\begin{aligned}\Delta G_1 &= \int d^2\vec{k}\left(k_x^2-k_y^2\right)T'(\vec{k})\left(F_kN_k^* + F_k^*N_k\right) \\ \Delta G_2 &= \int d^2\vec{k}\left(2k_xk_y\right)T'(\vec{k})\left(F_kN_k^* + F_k^*N_k\right)\end{aligned} \tag{17}$$

Similarly, we can calculate the PDF of $\Delta G_{1,2}$:

$$\begin{aligned}&P_\Delta\left(\Delta G_1, \Delta G_2\right) \\ &= \frac{1}{(2\pi)^2}\frac{\pi}{\sqrt{V_1V_3-V_2^2}}\exp\left[-\frac{\Delta G_1^2V_3 - 2\Delta G_1\Delta G_2V_2 + \Delta G_2^2V_1}{4(V_1V_3-V_2^2)}\right]\end{aligned} \tag{18}$$

in which we have:

$$
\begin{aligned}
V_1 &= 2\sigma^2\Delta^4\sum_k \left(k_x^2-k_y^2\right)^2 T'^2(\vec{k})|F_k|^2 = 2\sigma^2\Delta^2\int d^2\vec{k}\left(k_x^2-k_y^2\right)^2 T'^2(\vec{k})|F_k|^2 \\
&= 2\sigma^2\Delta^2\int d^2\vec{k}\left(k_x^2-k_y^2\right)^2\exp(-2\beta^2k^2)|F^S(\mathrm{M}^{-1}\vec{k})|^2\frac{|W_{PSF}(\vec{k})|^2}{|W_{PSF}^{\perp}(\vec{k})|^4} \\
V_2 &= 2\sigma^2\Delta^2\int d^2\vec{k}2k_xk_y\left(k_x^2-k_y^2\right)\exp(-2\beta^2k^2)|F^S(\mathrm{M}^{-1}\vec{k})|^2\frac{|W_{PSF}(\vec{k})|^2}{|W_{PSF}^{\perp}(\vec{k})|^4} \\
V_3 &= 2\sigma^2\Delta^2\int d^2\vec{k}(2k_xk_y)^2\exp(-2\beta^2k^2)|F^S(\mathrm{M}^{-1}\vec{k})|^2\frac{|W_{PSF}(\vec{k})|^2}{|W_{PSF}^{\perp}(\vec{k})|^4}
\end{aligned}
\tag{19}
$$

in which $W_{PSF}^{\perp}$ refers to the PSF rotated by 90 degrees. According to eq.(12), we have:

$$
|W_{PSF}^{\perp}(\mathrm{M}\vec{k})|^2 = W(k^2) - W'[2(-e_1+g_1)(k_x^2-k_y^2)+4(-e_2+g_2)k_xk_y]
\tag{20}
$$

Again, set $g_2=e_2=0$, we get:

$$
|\frac{W_{PSF}^2(\mathrm{M}\vec{k})|^2}{|W_{PSF}^{\perp}(\mathrm{M}\vec{k})|^4} = W^{-1}+W^{-2}W'(-6e_1+2g_1)(k_x^2-k_y^2)
\tag{21}
$$

and

$$
\begin{aligned}
V_3 &= 2\sigma^2\Delta^2\int d^2\vec{k}\exp(-2\beta^2k^2)|F^S(\vec{k})|^2 \\
&\times\left[(2k_xk_y)^2+4\beta^2g_1(2k_xk_y)^2\left(k_x^2-k_y^2\right)\right]\left[W^{-1}+W^{-2}W'(-6e_1+2g_1)(k_x^2-k_y^2)\right]
\end{aligned}
\tag{22}
$$

When the compensation terms are rotated by 45 degrees and added to G_1 and G_2, the new PDF becomes:

$$
\begin{aligned}
&P_F(G_1,G_2) = \int dx\int dy P_G(G_1-x,G_2-y)P_\Delta(-y,x) \\
&= \frac{\pi^2}{(2\pi)^4}\frac{1}{\sqrt{U_1U_3-U_2^2}\sqrt{V_1V_3-V_2^2}}\int dx\int dy\exp\left\{-\frac{V_3y^2+V_1x^2+2V_2xy}{4(V_1V_3-V_2^2)}\right\} \\
&\times\exp\left\{-\frac{U_3(G_1-\Gamma_1-x)^2+U_1(G_2-\Gamma_2-y)^2-2U_2(G_1-\Gamma_1-x)(G_2-\Gamma_2-y)}{4(U_1U_3-U_2^2)}\right\}
\end{aligned}
\tag{23}
$$

The corresponding 1-D PDF can be written as:

$$
\begin{aligned}
&F(G_1) = \int dG_2P_F(G_1,G_2) \\
&= \frac{2\sqrt{\pi}\pi^2}{(2\pi)^4}\frac{1}{\sqrt{U_1}\sqrt{V_1V_3-V_2^2}}\int dx\int dy\exp\left\{-\frac{(G_1-\Gamma_1-x)^2}{4U_1}-\frac{V_3y^2+V_1x^2+2V_2xy}{4(V_1V_3-V_2^2)}\right\} \\
&= \frac{4\pi^3}{(2\pi)^4}\frac{1}{\sqrt{U_1}\sqrt{V_3}}\int dx\exp\left\{-\frac{(G_1-\Gamma_1-x)^2}{4U_1}-\frac{x^2}{4V_3}\right\} \\
&= \frac{8\pi^3\sqrt{\pi}}{(2\pi)^4}\frac{1}{\sqrt{U_1+V_3}}\exp\left\{-\frac{(G_1-\Gamma_1)^2}{4(U_1+V_3)}\right\}
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
U_1+V_3 &= 2\sigma^2\Delta^2\int kdk\int d\theta\exp(-2\beta^2k^2)|F^S(k,\theta)|^2 \\
&\times\left\{k^4W^{-1}+2g_1k^6W^{-1}\cos(2\theta)[\beta^2+W^{-1}W'-\beta^2\cos(4\theta)]+2e_1k^6\cos(6\theta)W^{-2}W'\right\} \\
&\times\left\{k^4W^{-1}+g_1k^6W^{-1}[(\beta^2+2W^{-1}W')\cos(2\theta)-\beta^2\cos(6\theta)]+2e_1k^6\cos(6\theta)W^{-2}W'\right\} \\
&= N+\delta_N
\end{aligned}
\tag{25}
$$

In the 1-D PDF given by $F(G_1)$, Γ_1 can be treated as a small quantity comparing to G_1, as the bias is important only for galaxies of very small signal-to-noise ratios. $U_1 + V_3$ is a scalar with a perturbation, made of quantities of spin-2 and spin-6. $F(G_1)$ can be expanded as:

$$F(G_1) \propto \frac{1}{\sqrt{N}} \left\{ 1 + \frac{G_1 \Gamma_1}{2N} \left(1 - \frac{\delta_N}{N} \right) + \frac{\delta_N}{N} \left(\frac{G_1^2 + \Gamma_1^2}{4N} - \frac{1}{2} \right) \right\} \exp\left(-\frac{G_1^2 + \Gamma_1^2}{4N} \right) \tag{26}$$

Neglect Γ_1^2, and average over an ensemble of galaxies of random rotations, we get:

$$\bar{F}(G_1) \propto \frac{1}{\sqrt{N}} \left\{ 1 - \frac{G_1}{2N^2} \langle \Gamma_1 \delta_N \rangle \right\} \exp\left(-\frac{G_1^2}{4N} \right) \tag{27}$$

The asymmetric part of $F(G_1)$ is still existent:

$$\bar{F}^{ASYM}(G_1) \propto -\frac{G_1}{2N^2\sqrt{N}} \langle \Gamma_1 \delta_N \rangle \exp\left(-\frac{G_1^2}{4N} \right) \tag{28}$$

However, we find that the term $\langle \Gamma_1 \delta_N \rangle$ does not contain contributions from the PSF ellipticity, which is now in a term with spin-6 in the expression of δ_N, therefore does not couple with Γ_1, a spin-2 quantity. In principle, there should still be some small amount of multiplicative bias on g_1, as δ_N does contain a spin-2 term from g_1. This magnitude of this bias is likely very small, as Γ_1 is already very small comparing to G_1. This fact can also be seen in the following way: if the PSF in $T'(k)$ is not rotated by 90 degree, there is some residual additive bias of order 10^{-4} when the PSF ellipticity is 0.1, because δ_N in this case contains a term of spin-2 from e_1. As g_1 is an order of magnitude smaller than 0.1, the shear bias introduced by eq.(28) must be of order 10^{-5}, which is not significant for our purpose.