

lensing potential

$$\Psi(\boldsymbol{\theta}, \chi) = \frac{2}{c^2} \int_0^\chi d\chi' \frac{f_K(\chi - \chi')}{f_K(\chi)f_K(\chi')} \Phi(f_K(\chi')\boldsymbol{\theta}, \chi') \quad (1)$$

Lensing matrix:

$$A = \delta_{ij} - \partial_i \partial_j \Psi = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix} \quad (2)$$

where

$$\begin{aligned} \gamma_1 &= \frac{1}{2}(\partial_1 \partial_1 - \partial_2 \partial_2) \Psi; & \gamma_2 &= \partial_1 \partial_2 \Psi, \\ \kappa &= \frac{1}{2}(\partial_1 \partial_1 + \partial_2 \partial_2) \Psi. \end{aligned} \quad (3)$$

$$\begin{aligned} \kappa &= \frac{1}{2}(\partial_1 \partial_1 + \partial_2 \partial_2) \int \tilde{\Psi}(\mathbf{l}) e^{i\mathbf{l} \cdot \boldsymbol{\theta}} d^2 l \\ &= -\frac{1}{2} \int (l_1^2 + l_2^2) \tilde{\Psi}(\mathbf{l}) e^{i\mathbf{l} \cdot \boldsymbol{\theta}} d^2 l \\ \gamma &= \gamma_1 + i\gamma_2 \\ &= \frac{1}{2}(\partial_1 \partial_1 - \partial_2 \partial_2 + 2i\partial_1 \partial_2) \int \tilde{\Psi}(\mathbf{l}) e^{i\mathbf{l} \cdot \boldsymbol{\theta}} d^2 l \\ &= -\frac{1}{2} \int (l_1^2 - l_2^2 + 2il_1 l_2) \tilde{\Psi}(\mathbf{l}) e^{i\mathbf{l} \cdot \boldsymbol{\theta}} d^2 l \\ \tilde{\kappa}(\mathbf{l}') &= -\frac{1}{2} \int (l_1^2 + l_2^2) \tilde{\Psi}(\mathbf{l}) e^{i\mathbf{l} \cdot \boldsymbol{\theta}} e^{-i\mathbf{l}' \cdot \boldsymbol{\theta}} d^2 l d^2 \theta \\ &= -\frac{1}{2} \int (l_1^2 + l_2^2) \tilde{\Psi}(\mathbf{l}) e^{i(\mathbf{l} - \mathbf{l}') \cdot \boldsymbol{\theta}} d^2 l d^2 \theta \\ &= -\frac{(2\pi)^2}{2} \int (l_1^2 + l_2^2) \tilde{\Psi}(\mathbf{l}) \delta^2(\mathbf{l} - \mathbf{l}') d^2 l \\ &= -\frac{(2\pi)^2}{2} (l_1'^2 + l_2'^2) \tilde{\Psi}(\mathbf{l}') \\ \tilde{\gamma}(\mathbf{l}') &= -\frac{1}{2} \int (l_1^2 - l_2^2 + 2il_1 l_2) \tilde{\Psi}(\mathbf{l}) e^{i\mathbf{l} \cdot \boldsymbol{\theta}} e^{-i\mathbf{l}' \cdot \boldsymbol{\theta}} d^2 l d^2 \theta \\ &= -\frac{(2\pi)^2}{2} (l_1'^2 - l_2'^2 + 2il_1' l_2') \tilde{\Psi}(\mathbf{l}') \\ &= -\frac{(2\pi)^2}{2} (l_1' + il_2')^2 \tilde{\Psi}(\mathbf{l}') \end{aligned} \quad (4)$$

Therefore, we have

$$\tilde{\gamma}(\mathbf{l}) = \frac{(l_1 + il_2)^2}{l_1^2 + l_2^2} \tilde{\kappa}(\mathbf{l}) = [\cos^2(\beta) - \sin^2(\beta) + 2i \cos(\beta) \sin(\beta)] \tilde{\kappa}(\mathbf{l}) = e^{2i\beta} \tilde{\kappa}(\mathbf{l}). \quad (5)$$

The power spectrum of  $\gamma$  equals that of  $\kappa$ , *i.e.*  $P_\gamma(\mathbf{l}) = P_\kappa(\mathbf{l})$ .

Relate  $\kappa(\boldsymbol{\theta}, \chi)$  to the overdensity  $\delta(f_K(\chi)\boldsymbol{\theta}, \chi)$ .

$$\begin{aligned}
\kappa(\boldsymbol{\theta}, \chi) &= \frac{1}{c^2} \int_0^\chi d\chi' \frac{f_K(\chi - \chi')}{f_K(\chi)f_K(\chi')} \left( \frac{\partial^2}{\partial \theta_1^2} + \frac{\partial^2}{\partial \theta_2^2} \right) \Phi(f_K(\chi')\boldsymbol{\theta}, \chi') \\
&= \frac{1}{c^2} \int_0^\chi d\chi' \frac{f_K(\chi - \chi')}{f_K(\chi)f_K(\chi')} f_K(\chi')^2 \left( \frac{\partial^2}{\partial (f_K(\chi')\theta_1)^2} + \frac{\partial^2}{\partial (f_K(\chi')\theta_2)^2} \right) \Phi(f_K(\chi')\boldsymbol{\theta}, \chi') \\
&= \frac{1}{c^2} \int_0^\chi d\chi' \frac{f_K(\chi - \chi')}{f_K(\chi)} f_K(\chi') \left( \frac{\partial^2}{\partial (f_K(\chi')\theta_1)^2} + \frac{\partial^2}{\partial (f_K(\chi')\theta_2)^2} + \frac{\partial^2}{\partial \chi'^2} \right) \Phi(f_K(\chi')\boldsymbol{\theta}, \chi') \\
&= \frac{1}{c^2} \int_0^\chi d\chi' \frac{f_K(\chi - \chi')}{f_K(\chi)} f_K(\chi') \nabla^2 \Phi(f_K(\chi')\boldsymbol{\theta}, \chi') \\
&= \frac{1}{c^2} \int_0^\chi d\chi' \frac{f_K(\chi - \chi')}{f_K(\chi)} f_K(\chi') 4\pi G a(\chi')^2 \bar{\rho} \delta(f_K(\chi')\boldsymbol{\theta}, \chi') \\
&= \frac{1}{c^2} \int_0^\chi d\chi' \frac{f_K(\chi - \chi')}{f_K(\chi)} f_K(\chi') 4\pi G a(\chi')^2 \bar{\rho}_0 a(\chi')^{-3} \delta(f_K(\chi')\boldsymbol{\theta}, \chi') \\
&= \frac{1}{c^2} \int_0^\chi d\chi' \frac{f_K(\chi - \chi')}{f_K(\chi)} f_K(\chi') 4\pi G a(\chi')^2 \rho_{crit} \Omega_{m0} a(\chi')^{-3} \delta(f_K(\chi')\boldsymbol{\theta}, \chi') \\
&= \frac{1}{c^2} \int_0^\chi \frac{d\chi'}{a(\chi')} \frac{f_K(\chi - \chi') f_K(\chi')}{f_K(\chi)} 4\pi G \frac{3H_0^2}{8\pi G} \Omega_{m0} \delta(f_K(\chi')\boldsymbol{\theta}, \chi') \\
&= \frac{3H_0^2 \Omega_{m0}}{2c^2} \int_0^\chi \frac{d\chi'}{a(\chi')} \frac{f_K(\chi - \chi') f_K(\chi')}{f_K(\chi)} \delta(f_K(\chi')\boldsymbol{\theta}, \chi')
\end{aligned} \tag{6}$$

The mean convergence

$$\begin{aligned}
\kappa(\boldsymbol{\theta}) &= \int_0^{\chi_{lim}} d\chi n(\chi) \kappa(\boldsymbol{\theta}, \chi) \\
&= \frac{3H_0^2 \Omega_{m0}}{2c^2} \int_0^{\chi_{lim}} d\chi n(\chi) \int_0^\chi \frac{d\chi'}{a(\chi')} \frac{f_K(\chi - \chi') f_K(\chi')}{f_K(\chi)} \delta(f_K(\chi')\boldsymbol{\theta}, \chi') \\
&= \frac{3H_0^2 \Omega_{m0}}{2c^2} \int_0^{\chi_{lim}} d\chi' \int_{\chi'}^{\chi_{lim}} d\chi \frac{n(\chi)}{a(\chi')} \frac{f_K(\chi - \chi') f_K(\chi')}{f_K(\chi)} \delta(f_K(\chi')\boldsymbol{\theta}, \chi') \\
&= \frac{3H_0^2 \Omega_{m0}}{2c^2} \int_0^{\chi_{lim}} d\chi \int_\chi^{\chi_{lim}} d\chi' \frac{n(\chi')}{a(\chi)} \frac{f_K(\chi' - \chi) f_K(\chi)}{f_K(\chi')} \delta(f_K(\chi)\boldsymbol{\theta}, \chi) \\
&= \frac{3H_0^2 \Omega_{m0}}{2c^2} \int_0^{\chi_{lim}} d\chi \frac{f_K(\chi)}{a(\chi)} \delta(f_K(\chi)\boldsymbol{\theta}, \chi) \int_\chi^{\chi_{lim}} d\chi' \frac{n(\chi')}{a(\chi)} \frac{f_K(\chi' - \chi)}{f_K(\chi')} \\
&= \frac{3H_0^2 \Omega_{m0}}{2c^2} \int_0^{\chi_{lim}} \frac{d\chi}{a(\chi)} g(\chi) f_K(\chi) \delta(f_K(\chi)\boldsymbol{\theta}, \chi)
\end{aligned} \tag{7}$$