

# A New PDF-SYM Method

HE KUN LI<sup>1</sup> AND JUN ZHANG<sup>\*1</sup>

<sup>1</sup>*Department of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, China*

## ABSTRACT

Regarding the justification of the new PDF-SYM method.

*Keywords:* weak lensing, selection bias, Fourier quad

## 1. FORMALISM

Taking into account the contribution of the source-noise crossing term, we get:

$$\begin{aligned} G_1 &= \int d^2\vec{k} \{ (k_x^2 - k_y^2) + \hat{g}_1 [2k^2 - \beta^2 (k^4 + k_x^4 - 6k_x^2 k_y^2 + k_y^4)] \} T(\vec{k}) (|F_k|^2 + F_k N_k^* + F_k^* N_k) \\ &= \Delta^2 \sum_k A_k (|F_k|^2 + F_k N_k^* + F_k^* N_k) = \Delta^2 \sum_k A_k (|F_k|^2 + 2F_k^{Re} N_k^{Re} + 2F_k^{Im} N_k^{Im}) \end{aligned} \quad (1)$$

$$\begin{aligned} G_2 &= \int d^2\vec{k} \{ (2k_x k_y) + \hat{g}_2 [2k^2 - \beta^2 (k^4 - k_x^4 + 6k_x^2 k_y^2 - k_y^4)] \} T(\vec{k}) (|F_k|^2 + F_k N_k^* + F_k^* N_k) \\ &= \Delta^2 \sum_k B_k (|F_k|^2 + 2F_k^{Re} N_k^{Re} + 2F_k^{Im} N_k^{Im}) \end{aligned} \quad (2)$$

The PDF  $P_G$  of  $G_1$  and  $G_2$  is given by:

$$\begin{aligned} &P_G(G_1, G_2) \\ &= \left\{ \prod_k \int \int dN_k^{Re} dN_k^{Im} P_N(N_k^{Re}, N_k^{Im}) \right\} \\ &\times \delta_D \left[ G_1 - \Delta^2 \sum_k A_k (|F_k|^2 + 2F_k^{Re} N_k^{Re} + 2F_k^{Im} N_k^{Im}) \right] \delta_D \left[ G_2 - \Delta^2 \sum_k B_k (|F_k|^2 + 2F_k^{Re} N_k^{Re} + 2F_k^{Im} N_k^{Im}) \right] \\ &= \frac{1}{(2\pi)^2} \left\{ \prod_k \int \int dN_k^{Re} dN_k^{Im} P_N(N_k^{Re}, N_k^{Im}) \right\} \\ &\times \int \int dx dy \exp(ixG_1 + iyG_2) \exp \left\{ -2i\Delta^2 \sum_k (xA_k + yB_k) (F_k^{Re} N_k^{Re} + F_k^{Im} N_k^{Im}) - i\Delta^2 \sum_k (xA_k + yB_k) |F_k|^2 \right\} \end{aligned} \quad (3)$$

Assume that the noise is mainly white noise, therefore  $P_N(N_k^{Re}, N_k^{Im})$  are independent for different values of  $k$ :

$$P_N(N_k^{Re}, N_k^{Im}) = \frac{1}{2\pi\sigma^2} \exp \left[ -\frac{(N_k^{Re})^2 + (N_k^{Im})^2}{2\sigma^2} \right] \quad (4)$$

On the other hand, we have:

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}\sigma} \int du \exp \left( -\frac{u^2}{2\sigma^2} - i\alpha u \right) = \frac{1}{\sqrt{2\pi}\sigma} \int du \exp \left( -\frac{u^2 + 2i\sigma^2\alpha u - \sigma^4\alpha^2}{2\sigma^2} - \frac{\sigma^2\alpha^2}{2} \right) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int du \exp \left( -\frac{(u + i\sigma^2\alpha)^2}{2\sigma^2} - \frac{\sigma^2\alpha^2}{2} \right) = \exp \left( -\frac{\sigma^2\alpha^2}{2} \right) \end{aligned} \quad (5)$$

Therefore, we have:

$$\begin{aligned}
& P_G(G_1, G_2) \\
&= \frac{1}{(2\pi)^2} \int \int dx dy \exp(ixG_1 + iyG_2) \times \exp \left\{ -2\sigma^2 \Delta^4 \sum_k (xA_k + yB_k)^2 |F_k|^2 - i\Delta^2 \sum_k (xA_k + yB_k) |F_k|^2 \right\} \\
&= \frac{1}{(2\pi)^2} \int \int dx dy \exp [ix(G_1 - \Gamma_1) + iy(G_2 - \Gamma_2)] \times \exp (-x^2 U_1 - 2xy U_2 - y^2 U_3) \\
&= \frac{1}{(2\pi)^2} \frac{\pi}{\sqrt{U_1 U_3 - U_2^2}} \exp \left[ -\frac{(G_1 - \Gamma_1)^2 U_3 - 2(G_1 - \Gamma_1)(G_2 - \Gamma_2) U_2 + (G_2 - \Gamma_2)^2 U_1}{4(U_1 U_3 - U_2^2)} \right]
\end{aligned} \tag{6}$$

in which we have:

$$\begin{aligned}
(\Gamma_1, \Gamma_2) &= \Delta^2 \sum_k |F_k|^2 (A_k, B_k) \\
(U_1, U_2, U_3) &= 2\sigma^2 \Delta^4 \sum_k |F_k|^2 (A_k^2, A_k B_k, B_k^2) \\
&\int \int dx dy \exp (-x^2 C - 2xy D - y^2 E + ix A + iy B) \\
&= \int \int dx dy \exp [-x^2 C - Cx(2yD - iA)/C - C(2yD - iA)^2/(2C)^2 + C(2yD - iA)^2/(2C)^2 - y^2 E + iyB] \\
&= \sqrt{\frac{\pi}{C}} \int dy \exp [y^2 (D^2/C - E) + iy(B - AD/C) - A^2/(4C)] \\
&= \sqrt{\frac{\pi}{C}} \int dy \exp \left\{ (D^2/C - E) \left[ y + i \frac{(B - AD/C)}{2(D^2/C - E)} \right]^2 + (D^2/C - E) \frac{(B - AD/C)^2}{4(D^2/C - E)^2} - A^2/(4C) \right\} \\
&= \frac{\pi}{\sqrt{CE - D^2}} \exp \left\{ \frac{C(B - AD/C)^2 - A^2(D^2/C - E)}{4(D^2 - EC)} \right\} = \frac{\pi}{\sqrt{CE - D^2}} \exp \left\{ \frac{CB^2 - 2ABD + EA^2}{4(D^2 - EC)} \right\}
\end{aligned} \tag{7}$$

For the 1-D PDF of  $G_1$ , we have:

$$\begin{aligned}
& \int_{-\infty}^{\infty} dG_2 P_G(G_1, G_2) \\
&= \frac{1}{(2\pi)^2} \frac{\pi}{\sqrt{U_1 U_3 - U_2^2}} \int_{-\infty}^{\infty} dG_2 \exp \left[ -\frac{(G_1 - \Gamma_1)^2 U_3 - 2(G_1 - \Gamma_1)(G_2 - \Gamma_2) U_2 + (G_2 - \Gamma_2)^2 U_1}{4(U_1 U_3 - U_2^2)} \right] \\
&= \frac{1}{(2\pi)^2} \frac{\pi}{\sqrt{U_1 U_3 - U_2^2}} \int_{-\infty}^{\infty} dz \exp \left\{ -\frac{U_1 [z - (G_1 - \Gamma_1) U_2/U_1]^2 - U_1 (G_1 - \Gamma_1)^2 (U_2/U_1)^2 + (G_1 - \Gamma_1)^2 U_3}{4(U_1 U_3 - U_2^2)} \right\} \\
&= \frac{1}{(2\pi)^2} \frac{\pi \sqrt{\pi}}{\sqrt{U_1}} \exp \left\{ -\frac{(G_1 - \Gamma_1)^2}{4U_1} \right\}
\end{aligned} \tag{8}$$

$$\begin{aligned}
\Gamma_1 &= \Delta^2 \sum_k A_k |F_k|^2 \\
&= \int d^2 \vec{k} \{ (k_x^2 - k_y^2) + \hat{g}_1 [2k^2 - \beta^2 (k^4 + k_x^4 - 6k_x^2 k_y^2 + k_y^4)] \} T(\vec{k}) |F_k|^2 \\
&= \int d^2 \vec{k} \{ (k_x^2 - k_y^2) + \hat{g}_1 [2k^2 - \beta^2 (k^4 + k_x^4 - 6k_x^2 k_y^2 + k_y^4)] \} \exp(-\beta^2 k^2) |F^L(\vec{k})|^2 \\
&= \int d^2 \vec{k} \{ (k_x^2 - k_y^2) + \hat{g}_1 [2k^2 - \beta^2 (k^4 + k_x^4 - 6k_x^2 k_y^2 + k_y^4)] \} \exp(-\beta^2 k^2) |F^S(M^{-1} \vec{k})|^2
\end{aligned} \tag{9}$$

where

$$M = \begin{bmatrix} 1 - g_1 & -g_2 \\ -g_2 & 1 + g_1 \end{bmatrix}$$

. Let us assume that  $\hat{g}_{1,2} = g_{1,2}$ , i.e., the PDF has been symmetrized by the correct shear values. We then have:

$$\Gamma_1 = \int d^2\vec{k} \{k_x^2 - k_y^2 + 4\beta^2 g_2 k_x k_y (k_x^2 - k_y^2)\} \exp(-\beta^2 k^2) |F^S(\vec{k})|^2 \quad (10)$$

Similarly, we have:

$$\begin{aligned} U_1 &= 2\sigma^2 \Delta^4 \sum_k A_k^2 |F_k|^2 \\ &= 2\sigma^2 \Delta^2 \int d^2\vec{k} \{ (k_x^2 - k_y^2) + \hat{g}_1 [2k^2 - \beta^2 (k^4 + k_x^4 - 6k_x^2 k_y^2 + k_y^4)] \}^2 T^2(\vec{k}) |F_k|^2 \\ &= 2\sigma^2 \Delta^2 \int d^2\vec{k} \left\{ (k_x^2 - k_y^2)^2 + 2\hat{g}_1 (k_x^2 - k_y^2) [2k^2 - \beta^2 (k^4 + k_x^4 - 6k_x^2 k_y^2 + k_y^4)] \right\} \exp(-2\beta^2 k^2) \frac{|F^S(M^{-1}\vec{k})|^2}{|W_{PSF}(\vec{k})|^2} \\ &= 2\sigma^2 \Delta^2 \int d^2\vec{k} \left\{ (k_x^2 - k_y^2)^2 + 8\beta^2 g_2 k_x k_y (k_x^2 - k_y^2)^2 \right\} \exp(-2\beta^2 k^2) \frac{|F^S(\vec{k})|^2}{|W_{PSF}(M\vec{k})|^2} \end{aligned} \quad (11)$$

The term  $W_{PSF}(M\vec{k})$  is the source of the bias, as it makes the 1-D PDF defined in eq.(16) asymmetric after averaging over intrinsically isotropic galaxy images.

Assuming the ellipticity of the PSF is  $(e_1, e_2)$ , and suppose the components are small, we have:

$$|W_{PSF}(M\vec{k})|^2 = W(k^2) - W'[2(e_1 + g_1)(k_x^2 - k_y^2) + 4(e_2 + g_2)k_x k_y] \quad (12)$$

$$|W_{PSF}^{-2}(M\vec{k})|^2 = W^{-1} + W^{-2}W'[2(e_1 + g_1)(k_x^2 - k_y^2) + 4(e_2 + g_2)k_x k_y] \quad (13)$$

For convenience in the rest of our discussion, we set  $g_2 = 0$ , and the ellipticity  $e_2$  of the PSF is also zero, we have:

$$U_1 = 2\sigma^2 \Delta^2 \int d^2\vec{k} \exp(-2\beta^2 k^2) |F^S(\vec{k})|^2 [W^{-1}(k_x^2 - k_y^2)^2 + 2W^{-2}W'(e_1 + g_1)(k_x^2 - k_y^2)^3] \quad (14)$$

We can rewrite the functions of  $\Gamma_1$  and  $U_1$  as:

$$\Gamma_1 = \int k dk \int d\theta \exp(-\beta^2 k^2) |F^S(k, \theta)|^2 k^2 \cos(2\theta) \quad (15)$$

$$U_1 = 2\sigma^2 \Delta^2 \int k dk \int d\theta \exp(-2\beta^2 k^2) |F^S(k, \theta)|^2 [W^{-1}k^4 \cos^2(2\theta) + 2(e_1 + g_1)W^{-2}W'k^6 \cos^3(2\theta)] = U_{10} + \delta U$$

For the 1-D PDF of  $G_1$ , we have:

$$\begin{aligned} \int_{-\infty}^{\infty} dG_2 P_G(G_1, G_2) &= \frac{1}{(2\pi)^2} \frac{\pi\sqrt{\pi}}{\sqrt{U_1}} \exp\left\{-\frac{(G_1 - \Gamma_1)^2}{4U_1}\right\} \\ &\approx \frac{1}{(2\pi)^2} \frac{\pi\sqrt{\pi}}{\sqrt{U_{10}}} \left[1 - \frac{\delta U}{2U_{10}} + \frac{(G_1 - \Gamma_1)^2}{4U_{10}} \frac{\delta U}{U_{10}}\right] \exp\left\{-\frac{(G_1 - \Gamma_1)^2}{4U_{10}}\right\} \end{aligned} \quad (16)$$

The terms proportional to  $\delta U$ , when averaged over galaxies, generate the asymmetry of  $P_G$ .

Now, let us consider the case of adding compensation terms to  $G_1$  and  $G_2$ :

$$\begin{aligned} \Delta G_1 &= \int d^2\vec{k} (k_x^2 - k_y^2) T'(\vec{k}) (F_k N_k^* + F_k^* N_k) \\ \Delta G_2 &= \int d^2\vec{k} (2k_x k_y) T'(\vec{k}) (F_k N_k^* + F_k^* N_k) \end{aligned} \quad (17)$$

Similarly, we can calculate the PDF of  $\Delta G_{1,2}$ :

$$\begin{aligned} P_{\Delta}(\Delta G_1, \Delta G_2) &= \frac{1}{(2\pi)^2} \frac{\pi}{\sqrt{V_1 V_3 - V_2^2}} \exp\left[-\frac{\Delta G_1^2 V_3 - 2\Delta G_1 \Delta G_2 V_2 + \Delta G_2^2 V_1}{4(V_1 V_3 - V_2^2)}\right] \end{aligned} \quad (18)$$

in which we have:

$$\begin{aligned}
V_1 &= 2\sigma^2 \Delta^4 \sum_k (k_x^2 - k_y^2)^2 T'^2(\vec{k}) |F_k|^2 = 2\sigma^2 \Delta^2 \int d^2 \vec{k} (k_x^2 - k_y^2)^2 T'^2(\vec{k}) |F_k|^2 \\
&= 2\sigma^2 \Delta^2 \int d^2 \vec{k} (k_x^2 - k_y^2)^2 \exp(-2\beta^2 k^2) |F^S(M^{-1}\vec{k})|^2 \frac{|W_{PSF}(\vec{k})|^2}{|W_{PSF}^\perp(\vec{k})|^4} \\
V_2 &= 2\sigma^2 \Delta^2 \int d^2 \vec{k} 2k_x k_y (k_x^2 - k_y^2) \exp(-2\beta^2 k^2) |F^S(M^{-1}\vec{k})|^2 \frac{|W_{PSF}(\vec{k})|^2}{|W_{PSF}^\perp(\vec{k})|^4} \\
V_3 &= 2\sigma^2 \Delta^2 \int d^2 \vec{k} (2k_x k_y)^2 \exp(-2\beta^2 k^2) |F^S(M^{-1}\vec{k})|^2 \frac{|W_{PSF}(\vec{k})|^2}{|W_{PSF}^\perp(\vec{k})|^4}
\end{aligned} \tag{19}$$

in which  $W_{PSF}^\perp$  refers to the PSF rotated by 90 degrees. According to eq.(12), we have:

$$|W_{PSF}^\perp(M\vec{k})|^2 = W(k^2) - W'[2(-e_1 + g_1)(k_x^2 - k_y^2) + 4(-e_2 + g_2)k_x k_y] \tag{20}$$

Again, set  $g_2 = e_2 = 0$ , we get:

$$\frac{|W_{PSF}^2(M\vec{k})|^2}{|W_{PSF}^\perp(M\vec{k})|^4} = W^{-1} + W^{-2}W'(-6e_1 + 2g_1)(k_x^2 - k_y^2) \tag{21}$$

and

$$\begin{aligned}
V_3 &= 2\sigma^2 \Delta^2 \int d^2 \vec{k} \exp(-2\beta^2 k^2) |F^S(\vec{k})|^2 \\
&\times [(2k_x k_y)^2 + 4\beta^2 g_1 (2k_x k_y)^2 (k_x^2 - k_y^2)] [W^{-1} + W^{-2}W'(-6e_1 + 2g_1)(k_x^2 - k_y^2)]
\end{aligned} \tag{22}$$

When the compensation terms are rotated by 45 degrees and added to  $G_1$  and  $G_2$ , the new PDF becomes:

$$\begin{aligned}
P_F(G_1, G_2) &= \int dx \int dy P_G(G_1 - x, G_2 - y) P_\Delta(-y, x) \\
&= \frac{\pi^2}{(2\pi)^4} \frac{1}{\sqrt{U_1 U_3 - U_2^2} \sqrt{V_1 V_3 - V_2^2}} \int dx \int dy \exp \left\{ -\frac{V_3 y^2 + V_1 x^2 + 2V_2 xy}{4(V_1 V_3 - V_2^2)} \right\} \\
&\times \exp \left\{ -\frac{U_3(G_1 - \Gamma_1 - x)^2 + U_1(G_2 - \Gamma_2 - y)^2 - 2U_2(G_1 - \Gamma_1 - x)(G_2 - \Gamma_2 - y)}{4(U_1 U_3 - U_2^2)} \right\}
\end{aligned} \tag{23}$$

The corresponding 1-D PDF can be written as:

$$\begin{aligned}
F(G_1) &= \int dG_2 P_F(G_1, G_2) \\
&= \frac{2\sqrt{\pi}\pi^2}{(2\pi)^4} \frac{1}{\sqrt{U_1}\sqrt{V_1 V_3 - V_2^2}} \int dx \int dy \exp \left\{ -\frac{(G_1 - \Gamma_1 - x)^2}{4U_1} - \frac{V_3 y^2 + V_1 x^2 + 2V_2 xy}{4(V_1 V_3 - V_2^2)} \right\} \\
&= \frac{4\pi^3}{(2\pi)^4} \frac{1}{\sqrt{U_1}\sqrt{V_3}} \int dx \exp \left\{ -\frac{(G_1 - \Gamma_1 - x)^2}{4U_1} - \frac{x^2}{4V_3} \right\} \\
&= \frac{8\pi^3\sqrt{\pi}}{(2\pi)^4} \frac{1}{\sqrt{U_1 + V_3}} \exp \left\{ -\frac{(G_1 - \Gamma_1)^2}{4(U_1 + V_3)} \right\}
\end{aligned} \tag{24}$$

$$\begin{aligned}
U_1 + V_3 &= 2\sigma^2 \Delta^2 \int k dk \int d\theta \exp(-2\beta^2 k^2) |F^S(k, \theta)|^2 \\
&\times \{k^4 W^{-1} + 2g_1 k^6 W^{-1} \cos(2\theta) [\beta^2 + W^{-1}W' - \beta^2 \cos(4\theta)] + 2e_1 k^6 \cos(6\theta) W^{-2}W'\} \\
&\times \{k^4 W^{-1} + g_1 k^6 W^{-1} [(\beta^2 + 2W^{-1}W') \cos(2\theta) - \beta^2 \cos(6\theta)] + 2e_1 k^6 \cos(6\theta) W^{-2}W'\} \\
&= N + \delta_N
\end{aligned} \tag{25}$$

In the 1-D PDF given by  $F(G_1)$ ,  $\Gamma_1$  can be treated as a small quantity comparing to  $G_1$ , as the bias is important only for galaxies of very small signal-to-noise ratios.  $U_1 + V_3$  is a scalar with a perturbation, made of quantities of spin-2 and spin-6.  $F(G_1)$  can be expanded as:

$$F(G_1) \propto \frac{1}{\sqrt{N}} \left\{ 1 + \frac{G_1 \Gamma_1}{2N} \left( 1 - \frac{\delta_N}{N} \right) + \frac{\delta_N}{N} \left( \frac{G_1^2 + \Gamma_1^2}{4N} - \frac{1}{2} \right) \right\} \exp \left( -\frac{G_1^2 + \Gamma_1^2}{4N} \right) \quad (26)$$

Neglect  $\Gamma_1^2$ , and average over an ensemble of galaxies of random rotations, we get:

$$\bar{F}(G_1) \propto \frac{1}{\sqrt{N}} \left\{ 1 - \frac{G_1}{2N^2} \langle \Gamma_1 \delta_N \rangle \right\} \exp \left( -\frac{G_1^2}{4N} \right) \quad (27)$$

The asymmetric part of  $F(G_1)$  is still existent:

$$\bar{F}^{ASYM}(G_1) \propto -\frac{G_1}{2N^2 \sqrt{N}} \langle \Gamma_1 \delta_N \rangle \exp \left( -\frac{G_1^2}{4N} \right) \quad (28)$$

However, we find that the term  $\langle \Gamma_1 \delta_N \rangle$  does not contain contributions from the PSF ellipticity, which is now in a term with spin-6 in the expression of  $\delta_N$ , therefore does not couple with  $\Gamma_1$ , a spin-2 quantity. In principle, there should still be some small amount of multiplicative bias on  $g_1$ , as  $\delta_N$  does contain a spin-2 term from  $g_1$ . This magnitude of this bias is likely very small, as  $\Gamma_1$  is already very small comparing to  $G_1$ . This fact can also be seen in the following way: if the PSF in  $T'(k)$  is not rotated by 90 degree, there is some residual additive bias of order  $10^{-4}$  when the PSF ellipticity is 0.1, because  $\delta_N$  in this case contains a term of spin-2 from  $e_1$ . As  $g_1$  is an order of magnitude smaller than 0.1, the shear bias introduced by eq.(28) must be of order  $10^{-5}$ , which is not significant for our purpose.