Mechanism Design Project 5: Kempe's Universality Theorem

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Abstract

A gear-train mechanism made up of only Lego parts which can generate the specified curve given was created and verified. The design was theoretically/ computationally analyzed using MATLAB and SolidWorks. The design was also built and experimentally validated using only Lego parts. The five-leaf flower curve was created with a gear train that consisted of two 8 teeth Lego gears and one 40 tooth gear, with a 0.6-inch link arm protruding from the second 8 teeth gear. The specifications to the design and testing are discussed as well as the possible improvements that can be made for future experiments.

Introduction

Gear trains are mechanism composed of gears that offer a mechanical advantage for power transmission and precise movement sourcing from a single input. For this project we will focus on the specified movement that geartrains offer in relation to Kempe's theorem. Kempe's theorem describes the fact that any algebraic plane curve can be drawn using linkages. This idea can be mechanically advantageous as it creates a way for automation and the real-world unique curves to meet. The report is divided into the following sections: (1) a theoretical analysis and introduction to plane curves and geometrically tracing them, (2) a description and breakdown of our experimental apparatus and our procedure for verifying the design, (3) the results of the theoretical calculations, solid works model, and experimental drawings, (4) a comparison of results which have been obtained computationally and experimentally, and (5) a conclusion summarizing our final thoughts and a discussion of future improvements.

Background Work

Hypotrochoid by the shape combines hypocycloid (Figure 1a) and rose curves (Figure 1b).

(a) *(b)*

Figure 1. Hypocycloid (a). Rose curve(b). Courtesy https://mathworld.wolfram.com/

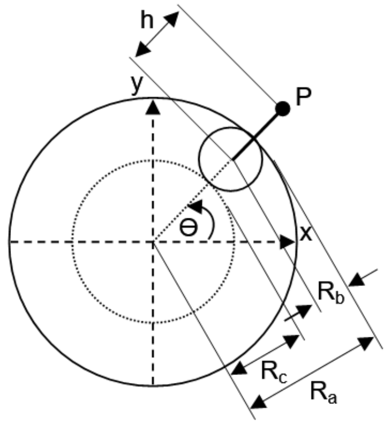

Figure 2. Hypotrochoid Theoretical Mechanism

We will use a theoretical mechanism Figure 2. Hypotrochoid is a type of roulette curve traced by point P, attached to circle with radius R_b. The number of cusps in the hypotrochoid is n, and the radius of the ring is $R_a = R_b * n$.
We assume that $h = 2R_b$
Therefore, the needed parameters are:

$$R_a = 5R_b, R_c = 3R_b, h = 2 \quad \textbf{[1]}$$

The parametric equations for the position of hypotrochoid:

$$x = (R_a - R_b)cos\theta + h * cos\left(\frac{a-b}{b}\theta\right) \quad \textbf{[2]}$$

$$y = (R_a - R_b)sin\theta - h * sin\left(\frac{a-b}{b}\theta\right) \quad \textbf{[3]}$$

We are constrained with gears of 8,12,16,20,24, and 40 teeth. Since we have 5 cusps, the unduloid (surface of revolution) R_b rotates 4 times around its axis for each revolution of R_c, gear ratio of 4:1. To obtain the needed gear ratio and keep the proper distances of parameters [1] we can build a geartrain Figure 3. On the main axis we place a 24 teeth gear, attached is 12 gear teeth $\rightarrow$ $1:2\ ratio$, distance from [0;0] $4.5R_b$to the center of 12 teeth gear. Compounded onto 12 teeth gear another 24 teeth gear, which in turn is again attached to the 12 teeth gear $\rightarrow$ $1:2\ ratio, overall\ 1:4\ ratio$, distance from [0;0] $9R_b$. Compounded onto this gear is a 20 teeth gear, attached in the opposite direction to another 20 teeth gear $\rightarrow$ $1:1\ ratio, overall\ 1:4\ ratio$, distance from [0;0] is $9R_b - 2 * 2.5R_b = 4R_b$, which is exact center we need to place the final gear R_b, and attach a bar h, with pin P. Now if we run the initial 24 teeth gear clock-wise, the gear $R_b$will rotate counter clock-wise, which will replicate the dynamics plotted in MATLAB code (Appendix).

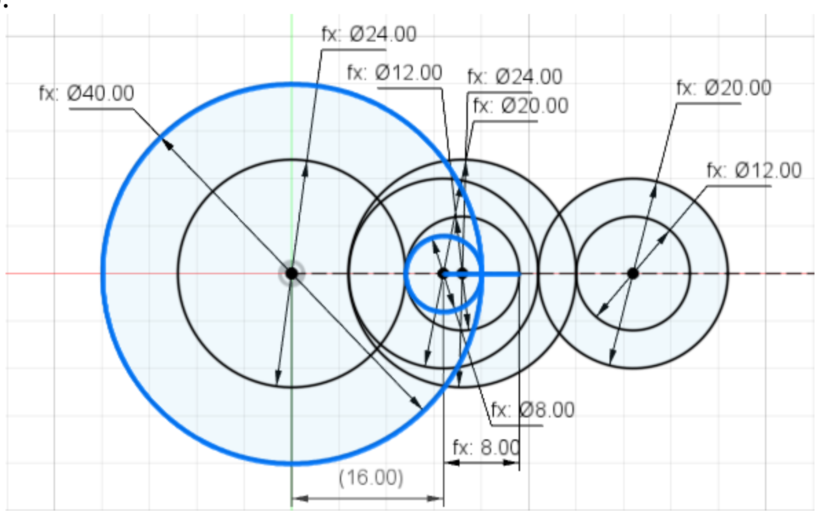

Figure 3. Gear-Train Mechanism (Autodesk Fusion 360 CAD Drawing)

The designed gear-train (Figure 3) is too complex to practically assemble and test using LEGO parts, therefore we reduced the design to a 40 teeth gear and 2X 8 teeth gears, with a link arm (see experimental apparatus).

Experimental Apparatus and Procedure
The experimental apparatus consists of one 40 tooth Lego Gear and two 8 teeth Lego gears, which re the important components of the mechanism. There is also one Lego axle which holds in place the 40-tooth gear in place and two Lego pins which allow the 8 teeth gear to rotate freely. Attached to the last gear is a trimmed piece of plastic measuring .6 inches acting as the link arm which will be drawing the curve. The axle holding the 40-tooth gear is supported by an axle brick which doesn't allow rotation. The whole mechanism is then supported by ten 2x4 bricks which allows it to rotate freely and not move in the z direction (creating only planar movement). The experimental design can be seen in figure 4 tracing the specified shape.
We also used a Video tracking software to trace the curves created by the gear-train. This was done as another way for verifying the design and to get data points to plot with the theoretical results which is shown in figure 5. The model was also created within Solid Works modeling and was tested using a built-in software (SolidWorks Simulation) to
trace out the output curve. Figure 4 shows the model in solid works that was created.

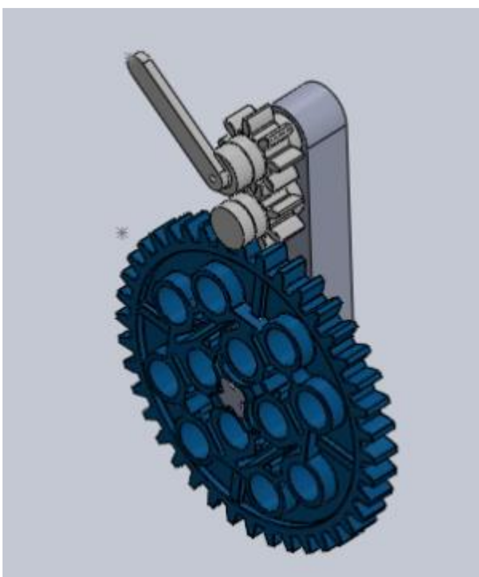

Figure 4. Solid works Model of the gear-Train

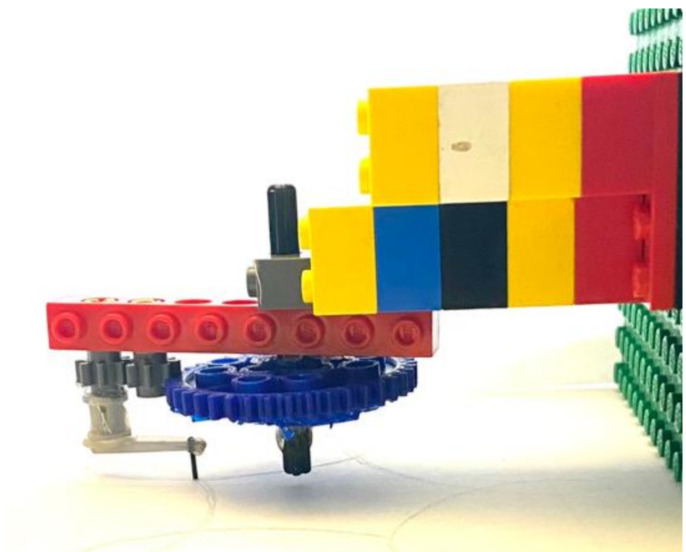

Figure 5. Experimental setup of the gear-train using Legos

Results
Theoretical
From given plot we assume that the overall length $(R_c + R_b + h)$ $L = 3R_b + R_b + 2R_b = 12\ LU$ Therefore $R_b = 2LU, R_c = 6LU, R_a = 10LU, h = 4LU$. Our result shows a replica of the given plot of the initial problem. The Experimental results (scaled to match LU [X/12.7-32;Y/12.7]) indicate very similar results to analytical.

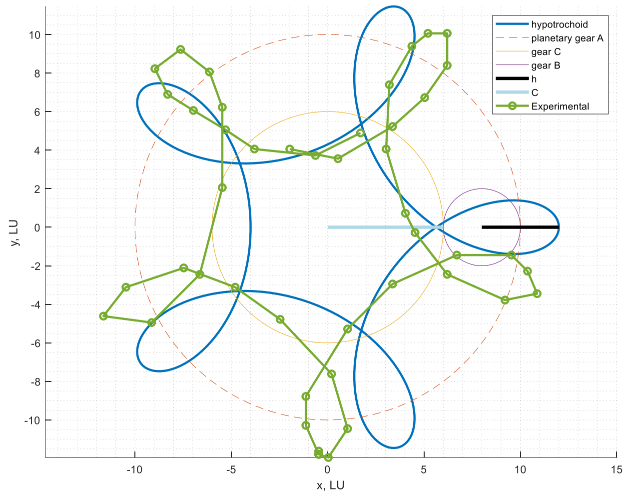

Figure 6. Graphed results from MATLAB (Blue) and Tracker Software (green)

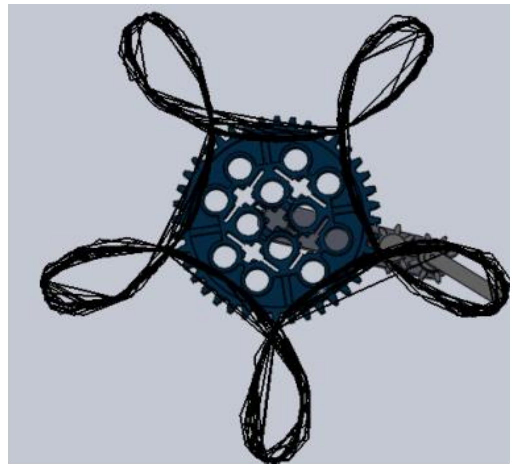

Figure 7. Solid Works simulation of the designed gear-train

<u>*Computationally*</u>
The results shown in figure 6 show the computational results done in MATLAB in blue of the gear- train setup that was chosen. The results can also be compared to figure 7 where the solid works simulation was completed using the same model.
<u>*Experimentally*</u>
The experimental results were collected in two different methods. The first method was by physically applying the mechanism on a sheet of paper and tracing the output created. This gave us the curve shown in figure 8 which was done in pencil. The second method for verifying the design is by using a tracker software which will track the movement of the link arm. Figure 6 shows the results of this experiment in green.

Interpretation
As shown in the results, the theoretical/ computational and experimental output curves all resemble the given curve that needed to be replicated. Referring to figure 6 we can see that the dimension of the curve is also fitting in the -10 to 10 LU range that was given to us. The solid works model shown in figure 7 also confirms the design as it produced the same curve that was predicted with the theoretically derived model. Some discrepancies are shown in figure 7 and this can be due to the fitment of the gears since they are modeled after Lego

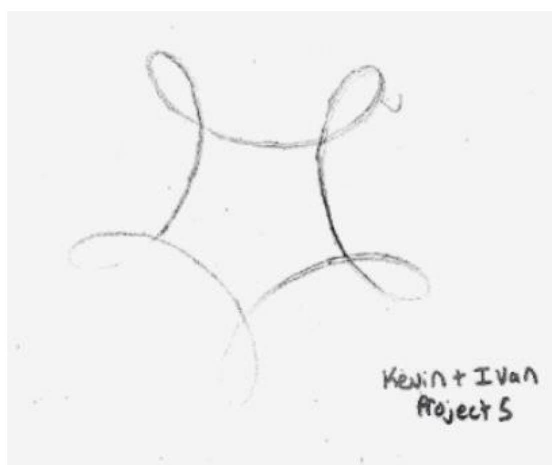

Figure 8. Lego Built Output curve

gears. Because of the loose-fitting gears, the software rendered some lines that seem to be straight and not curved as it should be. It can also be seen that the tracings aren't complete for figure 8. This is resulting from not having a perfectly flat surface to trace upon, leading to gaps in the tracing. Overall the discrepancies due to gear fitment and surface leveling does not invalidate the results in any way.

Conclusion

A design was created to draw the 5-leaf rose curve using only Lego parts. This was done by first by looking analytically and determining the correct radii of the gears and length of the link arm. After this we were then able to choose the correct gear ratio of 40 teeth to 8 teeth to create the 5 rotations needed to produce the given curve. The final gear-train mechanism consists of one 40 teeth gear connected to two 8 teeth gear, which is then connected to a link arm measured at 0.6 inches. This was the simplest design was able to successfully trace the given curve using only Lego parts. Another design was discovered which is shown in figure 3 which may have also worked but proved to be too complex to pursue as the goal was to find a basic gear-train system to produce the curves. The results shown does prove that the system was designed correctly. Although there arose some discrepancies during the testing of the system, these discrepancies do not invalidate the design but rather describes certain flaws found it the Lego pieces. Future experiments can be done using fully 3d printed models to assure that the gear fitment is accurate.

References

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[4] Saxena, A. (2011). Kempe's linkages and the Universality theorem. *Resonance, 16*(3), 220–237. https://doi.org/10.1007/s12045-011-0028-x

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Appendix

MATLAB Code.

```
clear
close all

phi = linspace (0, 2*pi, 360);
b = 2;a = b*5;
c = 6;
h=2*b;
%hypotrochoid calc
X = (a-b)*cos(phi)+h*cos((a-b)/b*phi);
Y = (a-b)*sin(phi)-h*sin((a-b)/b*phi);
%Gear b instantaneous position
Xb = b*cos(phi)+(c+b);
Yb = b*sin(phi);
%---plots---%
figure
grid minor; hold on;
for i = 1:1:length(phi)
    clf
    grid minor
    hold on
    axis equal
    plot (X,Y, 'DisplayName', 'hypotrochoid')
    plot(a*cos(phi), a*sin(phi),'--', 'DisplayName','planetary gear A')
    plot(c*cos(phi), c*sin(phi),'DisplayName', 'gear C')
        %%%dynamics%%
    plot(Xb*cos(phi(i))-Yb*sin(phi(i)),
Xb*sin(phi(i))+Yb*cos(phi(i)),'DisplayName','gear B')
    line ([(c+b)*cos(phi(i)), X(i)], [(c+b)*sin(phi(i)), Y(i)],'LineWidth',
3,'Color','black','DisplayName', 'h')
    line ([0, c*cos(phi(i))], [0, c*sin(phi(i))],'LineWidth',
3,'Color','#ADD8E6','DisplayName', 'C')
    pause(0.001)
    hold off
end
axis equal
ylabel('y, LU'); xlabel('x, LU')
hold off; legend('Location','northeast')
```