Pseudorandomness (IV)

601.642/442: Modern Cryptography

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Pseudorandomness: Next-Bit Test

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- **Next Bit Test**: for a truly random sequence of bits, it is not possible to predict the "next bit" in the sequence with probability better than 1/2 even given all previous bits of the sequence so far
- A sequence of bits *passes the next bit test* if no efficient adversary can predict "the next bit" in the sequence with probability better than 1/2 even given all previous bits of the sequence so far

Next-bit Unpredictability

Definition (Next-bit Unpredictability)

An ensemble of distributions $\{X_n\}$ over $\{0,1\}^{\ell(n)}$ is next-bit unpredictable if, for all $0 \leqslant i < \ell(n)$ and n.u. PPT $\mathcal{A}$, $\exists$ negligible function $\nu(\cdot)$ s.t.:

$$\Pr[t = t_1 \dots t_{\ell(n)} \leftarrow X_n : \mathcal{A}(t_1 \dots t_i) = t_{i+1}] \leqslant \frac{1}{2} + \nu(n)$$

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Theorem (Completeness of Next-bit Test)

If $\{X_n\}$ is next-bit unpredictable then $\{X_n\}$ is pseudorandom.

Next-bit Unpredictability $\Longrightarrow$ Pseudorandomness

$$H_n^{(i)} := \left\{ x \leftarrow X_n, u \leftarrow U_n : x_1 \dots x_i u_{i+1} \dots u_{\ell(n)} \right\}$$

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- Exercise: Do the reduction on your own

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    $\Rightarrow f(s)$ is uniformly distributed

PRG with 1-bit stretch

- Let $f : \{0,1\}^* \to \{0,1\}^*$ be a **OWP**
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- Proof Idea: Use next-bit unpredictability. Since first n bits of the output are uniformly distributed (since f is a permutation), any adversary for next-bit unpredictability with non-negligible advantage $\frac{1}{p(n)}$ must be predicting the $(n+1)$th bit with advantage $\frac{1}{p(n)}$. Build an adversary for hard-core predicate to get a contradiction.

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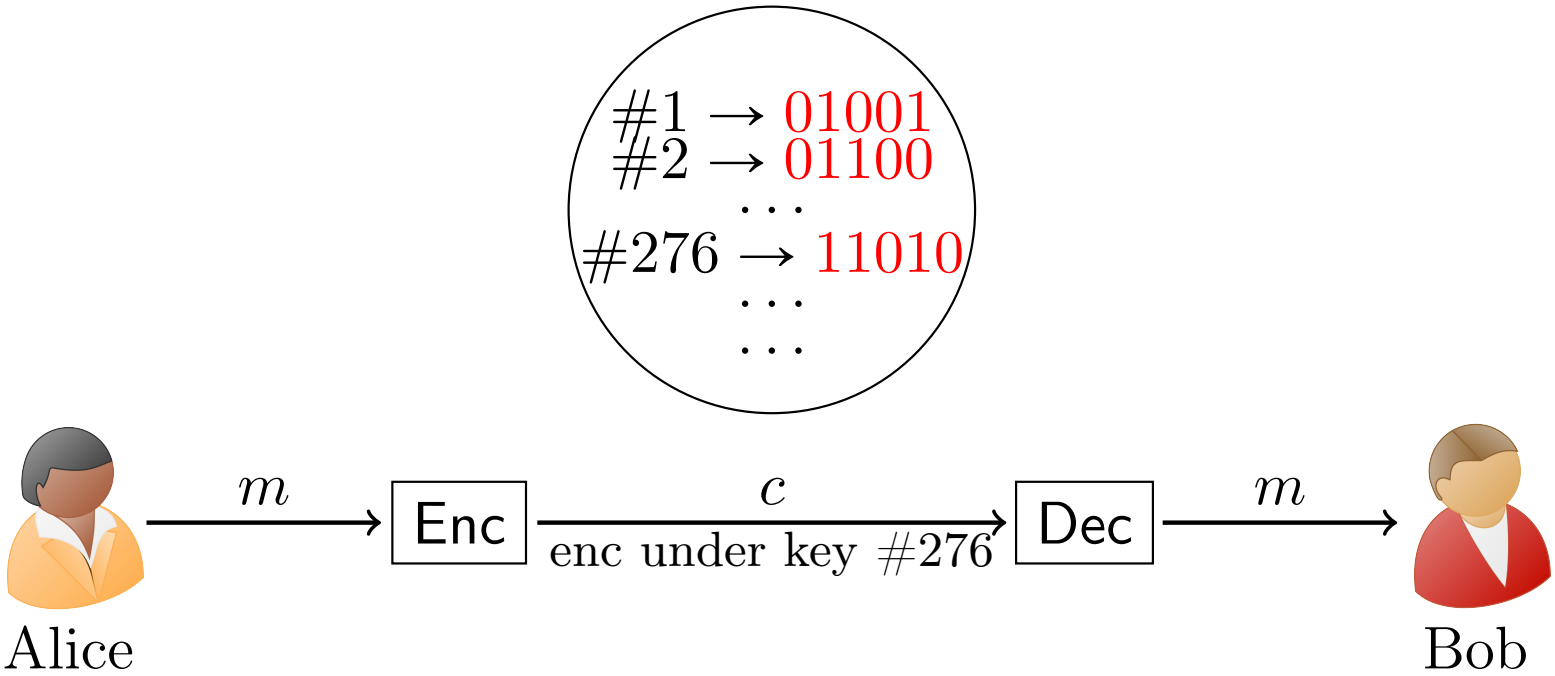
Idea: Functions that index exponentially long pseudorandom strings

Motivation

- Imagine if Alice and Bob had an exponential amount of shared randomness - not just a short key.
- They could split it up into λ-bit chunks and use each one as a one-time pad whenever they want to send an encrypted message of length λ.



- Although Alice publicly announces which location/chunk was used as each OTP key, Eve doesn't know the value at that location.

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- The size of the table for $F = 2^n \times n = n2^n$
- Total number of functions mapping n bits to n bits $= 2^{n2^n}$

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- M's output distribution identical to that of F.

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  - but actually needs much fewer bits to describe/store/query?

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- **Idea**: D can only query the function on inputs of its choice, and see the output.

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- PRF Security: No D can win with probability better than $1/2$.

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- **Easy to compute:** there is an efficient algorithm M such that $\forall k, x : M(k, x) = F_k(x)$.
- **Hard to distinguish:** for every non-uniform PPT D there exists a negligible function ν such that $\forall n \in \mathbb{N}$:

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- **Hard to distinguish:** for every non-uniform PPT D there exists a negligible function ν such that $\forall n \in \mathbb{N}$:

$$|\Pr[D \text{ wins } \mathsf{GuessGame}] - 1/2| \leqslant \nu(n).$$

where GuessGame is defined below

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Remark: note that for any b only one of the two functions is ever used.

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- **Simpler problem:** build PRF for just 1-bit inputs using PRG

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  - For general case: Apply the "double and choose" idea repeatedly!

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- Construction for n-bit inputs $x = x_1 x_2 \dots x_n$

$$F_k(x) = G_{x_n}\big(G_{x_{n-1}}\big(\dots\big(G_{x_1}(k)\big)..\big)$$

PRF from PRG (contd.)

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- At level ℓ, 2^ℓ nodes, one for each path, denoted by $k_{x_1 \ldots x_\ell}$