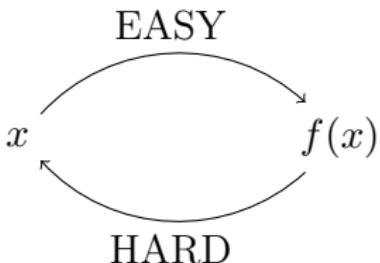


One-Way Functions (II)

601.642/442: Modern Cryptography

Fall 2022

Recap: One Way Functions



- A function is one-way if it “easy to compute,” but “hard to invert”
- Necessary for the existence of most cryptographic primitives (e.g., multi-message encryption, digital signatures)
- Also sufficient for some cryptographic primitives (e.g., pseudorandom generators, secret-key encryption, digital signatures).

Recap: One Way Functions (Definition)

Definition (One Way Function)

A function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ is a one-way function (OWF) if it satisfies the following two conditions:

- **Easy to compute:** there is a polynomial-time algorithm \mathcal{C} s.t.

$$\forall x \in \{0, 1\}^*,$$

$$\Pr[\mathcal{C}(x) = f(x)] = 1.$$

- **Hard to invert:** there exists a negligible function $\nu : \mathbb{N} \rightarrow \mathbb{R}$ s.t. for every non-uniform PPT adversary \mathcal{A} and $\forall n \in \mathbb{N}$:

$$\Pr[x \stackrel{\$}{\leftarrow} \{0, 1\}^n, x' \leftarrow \mathcal{A}(1^n, f(x)) : f(x') = f(x)] \leq \nu(n).$$

- The above definition is also called **strong** one-way functions.

Recap: Factoring Problem

- Consider the **multiplication** function $f_{\times} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$:

$$f_{\times}(x, y) = \begin{cases} \perp & \text{if } x = 1 \vee y = 1 \\ x \cdot y & \text{otherwise} \end{cases}$$

- The first condition helps exclude the trivial factor 1.
- Is f_{\times} a OWF?
- Clearly not!** With prob. 1/2, a random number (of any fixed size) is even. I.e., xy is even w/ prob. $\frac{3}{4}$ for random (x, y) .
- Inversion: given number z , output $(2, z/2)$ if z is even and $(0, 0)$ otherwise! (succeeds 75% time)

Factoring Problem (continued)

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- This is unlikely to have small trivial factors.

Assumption (Factoring Assumption)

For every (non-uniform PPT) adversary \mathcal{A} , there exists a negligible function ν such that

$$\Pr \left[p \xleftarrow{\$} \Pi_n; q \xleftarrow{\$} \Pi_n; N = pq : \mathcal{A}(N) \in \{p, q\} \right] \leq \nu(n).$$

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- **Note:** Factoring can be solved in polynomial time with a quantum computer!
- Can we construct OWFs from the Factoring Assumption?

Multiplication Function

- Going back to the multiplication function $f_{\times} : \mathbb{N}^2 \rightarrow \mathbb{N}$.

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- Now suppose that ε is a **noticeable function** (say e.g. an inverse polynomial, i.e., $\frac{1}{p(\cdot)}$)
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⇒ **every** \mathcal{A} must fail to invert f_{\times} with **noticeable** probability.
- **This is already useful!**
- Usually called a **weak OWF**.

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Let us start by formally defining noticeable functions. These are functions that are **at most polynomially small**.

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Note that a non-negligible function is not necessarily a noticeable function. Example:

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2^{-n} & \text{if } n \text{ is odd} \end{cases}.$$

This function is non-negligible, but not noticeable. **Why?**

Weak One Way Functions

Definition (Weak One Way Function)

A function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ is a *weak one-way function* if it satisfies the following two conditions:

- **Easy to compute:** there is a polynomial-time algorithm \mathcal{C} s.t.

$$\forall x \in \{0, 1\}^*,$$

$$\Pr [\mathcal{C}(x) = f(x)] = 1.$$

- **Somewhat hard to invert:** there is a noticeable function $\varepsilon : \mathbb{N} \rightarrow \mathbb{R}$ s.t. for every non-uniform PPT \mathcal{A} and $\forall n \in \mathbb{N}$:

$$\Pr [x \leftarrow \{0, 1\}^n, x' \leftarrow \mathcal{A}(1^n, f(x)) : f(x') \neq f(x)] \geq \varepsilon(n).$$

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Assuming the factoring assumption, function f_x is a weak OWF.

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- **Chebyshev's theorem:** An n bit number is a prime with probability $\frac{1}{2n}$

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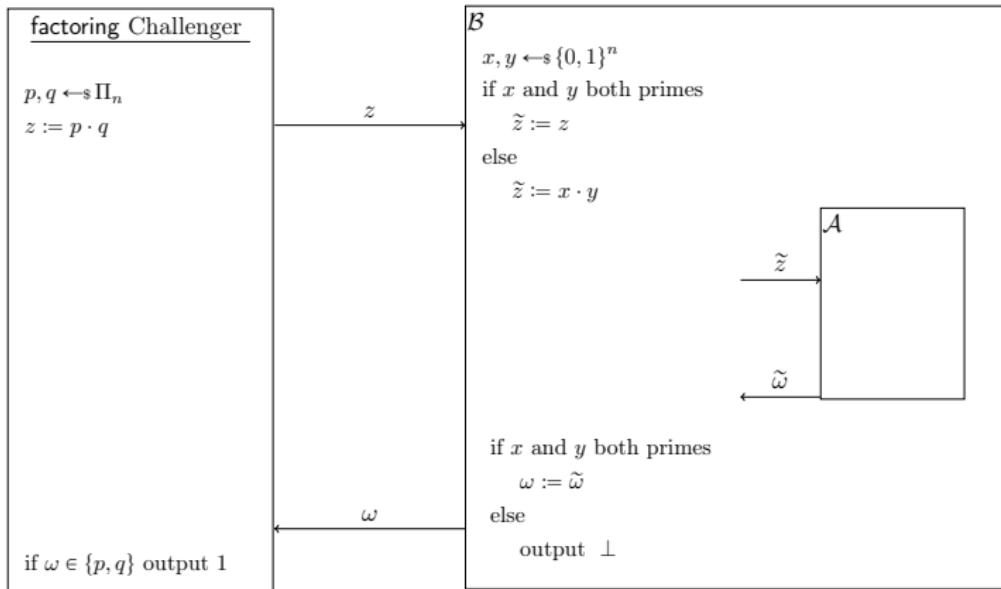
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- But if $\Pr[(x, y) \in \text{GOOD}]$ is noticeable, then overall, the adversary can only invert with some bounded noticeable probability.
- Formally: Let $q(n) = 8n^2$. Will show that no non-uniform PPT adversary can invert f_x with probability greater than $1 - \frac{1}{q(n)}$

Proof via Reduction

Goal: Given an adversary \mathcal{A} that breaks weak one-wayness of f_x with probability *at least* $1 - \frac{1}{q(n)}$, we will construct an adversary \mathcal{B} that breaks the factoring assumption with noticeable probability

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The input of \mathcal{B} is a product of two random n -bit **primes** while that of \mathcal{A} is the product of two random n -bit **numbers**. Passing the input directly to \mathcal{A} would not emulate the distribution of the inputs given to \mathcal{A} .

Analysis of \mathcal{B}

- Since \mathcal{A} is non-uniform PPT, so is \mathcal{B} (using polynomial-time primality testing)

$$\begin{aligned}\Pr[\mathcal{B} \text{ fails}] &= \Pr[\mathcal{B} \text{ passes input to } \mathcal{A}] \cdot \Pr[\mathcal{A} \text{ fails to invert } f_x] \\ &\quad + \Pr[\mathcal{B} \text{ fails to pass input to } \mathcal{A}] \\ &\leq \Pr[\mathcal{A} \text{ fails to invert } f_x] + \Pr[\mathcal{B} \text{ fails to pass input to } \mathcal{A}] \\ &\leq \frac{1}{8n^2} + \left(1 - \frac{1}{4n^2}\right) \leq \left(1 - \frac{1}{8n^2}\right)\end{aligned}$$

- \mathcal{B} succeeds with probability at least $\frac{1}{8n^2}$: **Contradiction to factoring assumption!**

Back to Strong OWFs

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- **Yao's Hardness Amplification:** YES!

Weak to Strong OWFs

Theorem (Yao)

Strong OWFs exist if and only weak OWFs exist

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Theorem (Yao)

Strong OWFs exist if and only weak OWFs exist

- This is called **hardness amplification**: convert a somewhat hard problem into a really hard problem
- Intuition: Use the weak OWF *many* times
- Think: Is $f(f(\dots f(x)))$ a good idea?

Weak to Strong OWFs

Theorem

For any weak one-way function $f : \{0, 1\}^n \rightarrow \{0, 1\}^n$, there exists a polynomial $N(\cdot)$ s.t. the function $F : \{0, 1\}^{n \cdot N(n)} \rightarrow \{0, 1\}^{n \cdot N(n)}$ defined as

$$F(x_1, \dots, x_N(n)) = (f(x_1), \dots, f(x_N(n)))$$

is strongly one-way.

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- If N is sufficiently large and the inputs are chosen independently at random, then the probability of inverting all of them should be small

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- Nevertheless, it can be shown via a non-trivial proof that hardness does amplify for one-way functions (albeit not all the way to exponentially small inversion probability – there are counterexamples to this!)
- In fact, hardness amplification is not a general phenomenon; for other cases such as interactive arguments (we will study later), hardness does not amplify in general

Weak to Strong OWFs: Example

- We will show that Yao's hardness amplification works for f_{\times}
- The general case requires a different and careful proof; see lecture notes for details

Hardness Amplification for f_{\times}

Theorem

Assume the factoring assumption and let $m = 4n^3$. Then,

$\mathcal{F} : (\{0, 1\}^{2n})^m \rightarrow (\{0, 1\}^{2n})^m$ is a strong OWF:

$$\mathcal{F}((x_1, y_1), \dots, (x_m, y_m)) = (f_{\times}(x_1, y_1), \dots, f_{\times}(x_m, y_m)).$$

- **Intuition:** Recall that by Chebyshev's Thm, a pair of random n -bit numbers are both primes with prob $\frac{1}{4n^2}$

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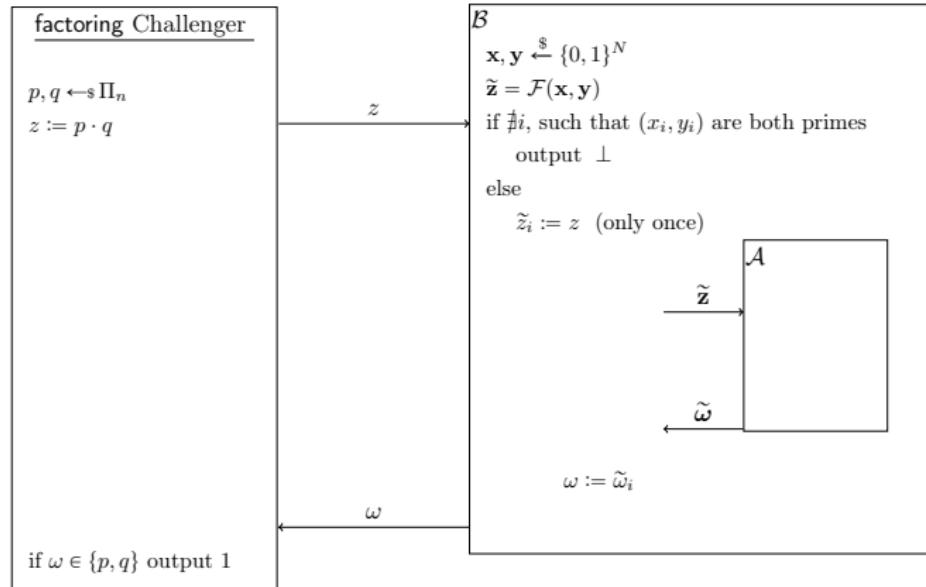
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- **Intuition:** Recall that by Chebyshev's Thm, a pair of random n -bit numbers are both primes with prob $\frac{1}{4n^2}$
- When we choose $m = 4n^3$ pairs, then the prob that no pair consists of primes is at most e^{-n} , which is negligible

Hardness Amplification for f_x : Proof Details

- Let $N = 2n \cdot 4n^3 = 8n^4$. Let $(\mathbf{x}, \mathbf{y}) = (x_1, y_1), \dots, (x_m, y_m)$
- Suppose \mathcal{F} is not a strong OWF. Then, \exists a non-uniform PPT adversary \mathcal{A} that inverts \mathcal{F} with prob at least $\varepsilon(2n)$ for some non-negligible function $\varepsilon(\cdot)$
- We will use \mathcal{A} to construct a non-uniform PPT adversary \mathcal{B} that breaks the factoring assumption

Hardness Amplification for f_x : Reduction



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- Overall, \mathcal{B} fails with prob at most $(1 - \varepsilon(2n)) + e^{-n} < (1 - \frac{\varepsilon(2n)}{2})$
- Thus, \mathcal{B} succeeds with prob at least $\frac{\varepsilon(2n)}{2}$, which is a contradiction to the factoring assumption.