Pseudorandomness (V)
&
Key Exchange

601.642/442: Modern Cryptography

Fall 2022

PRF from PRG

Theorem (Goldreich-Goldwasser-Micali (GGM))

If pseudorandom generators exist then pseudorandom functions exist

- **Notation:** define G_0 and G_1 as

$$G(s) = G_0(s) \| G_1(s)$$

  i.e., G_0 chooses left half of G and G_1 chooses right half

- Construction for n-bit inputs $x = x_1 x_2 \dots x_n$

$$F_k(x) = G_{x_n}\big(G_{x_{n-1}}\big(\dots\big(G_{x_1}(k)\big)..\big)$$

PRF from PRG (contd.)

$$F_k(x) = G_{x_n}\big(G_{x_{n-1}}\big(\ldots\big(G_{x_1}(k)\big)..\big)$$

- We can represent F_k as a binary tree of size 2^n
- The root corresponds to k
- Left and right child on level 1 and 2 are:

$$k_0 = G_0(k) \text{ and } k_1 = G_1(k)$$

- Second level children:

$$k_{00} = G_0(k_0),\ k_{01} = G_1(k_0),\ k_{10} = G_0(k_1),\ k_{11} = G_1(k_1)$$

- At level ℓ, 2^ℓ nodes, one for each path, denoted by $k_{x_1 \ldots x_\ell}$

Proof Strategy

- Let's use Hybrid Arguments!
- Problem: If we replace each node in the tree one-by-one with random, then exponentially many hybrids. Hybrid lemma doesn't apply!
- **Observation:** Efficient adversary can only make polynomial queries
- Thus, only need to change polynomial number of nodes in the tree

Proof Strategy (contd.)

Two layers of hybrids:

- First, define hybrids over the n levels in the tree. For every i, H_i is such that the nodes up to level i are random, but the nodes below are pseudorandom.

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- Use PRG security to argue indistinguishability

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  1. Adversary only makes polynomial number of queries
  2. A random function can be efficiently implemented (using second method) if the number of queries are polynomial
- Think: Formal proof?

Concluding Remarks

- **Efficient PRFs from concrete assumptions:** [Naor-Reingold97], [Banerjee-Peikert-Rosen12]
- **Constrained PRFs:** PRFs with "punctured" keys that are disabled on certain inputs [Boneh-Waters13, Kiayias-Papadopoulos-Triandopoulos-Zacharias13, Boyle-Goldwasser-Ivan14, Sahai-Waters14]
- **Related-key Security:** Evaluation of $F_s(x)$ does not help in predicting $F_{s'}(x)$ [Bellare-Cash10]
- **Key-homomorphic PRFs:** Given $f_s(x)$ and $f_{s'}(x)$, compute $f_{g(s,s')}(x)$ [Boneh-Lewi-Montgomery-Raghunathan13]

Key Exchange

Groups

- A group G is defined by a set of elements and an operation which maps two elements in the set to a third element
- $(G, \bullet)$ is a group if it satisfies the following conditions:
  - Closure: For all $a, b \in G$, we have $a \bullet b \in G$
  - Associativity: For all $a, b, c \in G$, we have $(a \bullet b) \bullet c = a \bullet (b \bullet c)$
  - Identity: There exists an element e such that for all $a \in G$, we have $e \bullet a = a$
  - Inverse: For every $a \in G$, there exists $b \in G$ such that $a \bullet b = e$
- Abelian Groups: $a \bullet b$ equal to $b \bullet a$
- Example: $(\mathbb{Z}, +)$

Cyclic Groups

- A group $(G, \cdot)$ is a cyclic group if it is generated by a single element
- That is: $G = \left\{1 = e = g^0, g^1, \ldots, g^{n-1}\right\}$, where $|G| = n$. (Here we are implicitly considering multiplicative groups.)
- Written as: $G = \langle g \rangle$
- Order of G: n

Discrete Logarithm Problem

- Let $(G, \cdot)$ be a cyclic group of order p with generator g, where p is an n-bit "safe prime" number (i.e., $p = 2q + 1$ for some large prime q).
- Given $(g, b = g^a)$, where $a \xleftarrow{\$} \{0, \ldots, p-1\}$, it is hard to predict a

Discrete Logarithm Problem: Definition

Definition (Discrete Logarithm Problem)

Let $(G, \cdot)$ be a cyclic group of order p (where p is a safe prime) with generator g, then for every non-uniform PPT adversary $\mathcal{A}$, there exists a negligible function ε such that

$$\Pr[a \xleftarrow{\$} \{0, \ldots, p-1\}, a' \leftarrow \mathcal{A}(G, p, g, g^a) : a = a'] \leqslant \varepsilon$$

Computational Diffie-Hellman Assumption

- Let G be a cyclic group $(G, \cdot)$ of order p with generator g, where p is an n-bit safe prime number.
- Give (g, g^a, g^b) to the adversary
- Hard to find g^{ab}

Computational Diffie-Hellman Assumption: Definition

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Decisional Diffie-Hellman Assumption

- Let $(G, \cdot)$ be a cyclic group of order p with generator g, where p is an n-bit safe prime number.
- Pick $b \xleftarrow{\$} \{0, 1\}$
- If $b = 0$, send (g, g^a, g^b, g^{ab}), where $a, b \xleftarrow{\$} \{0, \ldots, p-1\}$
- If $b = 1$, send (g, g^a, g^b, g^r), where $a, b, r \xleftarrow{\$} \{0, \ldots, p-1\}$
- Adversary has to guess b
- Effectively: $(g, g^a, g^b, g^{ab}) \approx (g, g^a, g^b, g^r)$, for $a, b, r \xleftarrow{\$} \{0, \ldots, p-1\}$ and any g

Decisional Diffie-Hellman Assumption: Definition

Definition (Decisional Diffie-Hellman Assumption)

Let $(G, \cdot)$ be a cyclic group of order p (where p is a safe prime) with generator g, then the following two distributions are computationally indistinguishable:

- $\{a, b \xleftarrow{\$} \{0, \ldots, p-1\} : (G, p, g, g^a, g^b, g^{ab})\}$
- $\{a, b, r \xleftarrow{\$} \{0, \ldots, p-1\} : (G, p, g, g^a, g^b, g^r)\}$

Relationship

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$$\text{DDH} \implies \text{CDH} \implies \text{DL}$$

Key Agreement

- Alice and Bob want to share a key.
- They want to establish a shared by by sending each other messages over a channel.
- However, there is an adversary (Eavesdropper) that is eavesdropping on this channel and sees the messages that are sent over it.
- How to securely establish a shared key while keeping it hidden from the eavesdropper?

Key Agreement: Definition

- Alice picks a local randomness r_A
- Bob picks a local randomness r_B
- Alice and Bob engage in a protocol and generate the transcript τ
- Alice's view $V_A = (r_A, \tau)$ and Bob's view $V_B = (r_B, \tau)$
- Eavesdropper's view $V_E = \tau$
- Alice outputs k_A as a function of V_A and Bob outputs k_B as a function of V_B
- Correctness: $\Pr_{r_A, r_B}[k_A = k_B] \approx 1$
- Security: $(k_A, V_E) \equiv (k_B, V_E) \approx (r, \tau)$

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- Adversary sees: (g^a, g^b)
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- Think: Is this scheme still secure if the adversary is allowed to modify the messages?