Hard Core Predicates

601.642/442: Modern Cryptography

Fall 2022

Last Time

- Proof via Reduction: $f_\times$ is a weak OWF
- Amplification: From weak to strong OWFs

Today

- Hard Core Predicate
- 1-bit stretch PRGs from hard core predicate.

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Is there any non-trivial (non-identity) function of x, even 1 bit, that OWFs hide?

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- In other words, learning the hardcore bit of x, even given $f(x)$, is "as hard as" inverting f itself.
- Think: What does "hard to compute" mean for a single bit?
  - you can always guess the bit with probability 1/2.

Hard Core Predicate: Definition

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Definition (Hard Core Predicate)

A predicate $h : \{0,1\}^* \to \{0,1\}$ is a hard-core predicate for $f(\cdot)$ if h is efficiently computable given x and there exists a negligible function ν s.t. for every non-uniform PPT adversary $\mathcal{A}$ and $\forall n \in \mathbb{N}$:

$$\Pr\left[x \leftarrow \{0,1\}^n : \mathcal{A}(1^n, f(x)) = h(x)\right] \leqslant \frac{1}{2} + \nu(n).$$

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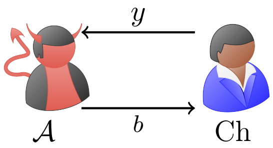

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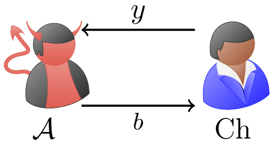

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We want that for all n.u. PPT adversary $\mathcal{A}$, the adversary wins with probability only at most negligible more than 1/2.

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Theorem (Goldreich-Levin)

Let f be a OWF. Define function

$$g(x, r) = (f(x), r)$$

where $|x| = |r|$. Then g is a OWF and

$$h(x, r) = \langle x, r \rangle$$

is a hard-core predicate for f

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- **Main challenge**: Adversary $\mathcal{A}$ for h only outputs 1 bit. Need to build an inverter $\mathcal{B}$ for f that outputs n bits.

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  - Yes! Levin gives us a method!