Pseudorandomness (III) & One-Way Functions

601.642/442: Modern Cryptography

Fall 2022

Recap: One-bit stretch PRG $\implies$ Poly-bit stretch PRG

- A PRG with **one-bit stretch** can be used to construct a PRG with arbitrary **polynomial-bit stretch**.

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Construction of $G_{poly} : \{0,1\}^n \rightarrow \{0,1\}^{\ell(n)}$

Let $G : \{0,1\}^n \rightarrow \{0,1\}^{n+1}$ be a one-bit stretch PRG.

$$\begin{aligned} s &= x_0 \\ G(x_0) &= x_1 \| b_1 \\ &\vdots \\ G(x_{\ell(n)-1}) &= x_{\ell(n)} \| b_{\ell(n)} \end{aligned}$$

$G_{poly}(s) := b_1 \ldots b_{\ell(n)}$

Recap: Pseudorandomnes of G_{poly}

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$s = x_0$	$s = x_0$	$s = X_0$
$G(x_0) = x_1 \| b_1$	$s_1 \| u_1 = x_1 \| u_1$	$s_1 \| u_1 = x_1 \| u_1$
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$\ldots$	$\ldots$	$\ldots$
$\ldots$	$\ldots$	$\ldots$
$G(x_{\ell(n)-1}) = x_{\ell(n)} \| b_{\ell(n)}$	$G(X_{\ell(n)-1}) = x_{\ell(n)} \| b_{\ell(n)}$	$s_{\ell(n)} \| u_{\ell(n)} = x_{\ell(n)} \| u_{\ell(n)}$
$\boxed{\text{Output } G(s) := b_1 b_2 \ldots b_{\ell(n)}}$	$\boxed{\text{Output } G(s) := u_1 b_2 \ldots b_{\ell(n)}}$	$\boxed{\text{Output } G(s) := u_1 u_2 \ldots u_{\ell(n)}}$

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- And established that $\forall i \in [\ell(n) - 1]$, $\mathcal{H}_i \approx_c \mathcal{H}_{i+1}$.

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- If G_{poly} is not a PRG, then there exists a n.u. PPT adversary $\mathcal{A}$ who can distinguish between its output on a random input and a uniformly sampled string with some **non-negligible** advantage μ.

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Can we somehow use $\mathcal{A}$ to also break security of G?

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Lemma (Alternate way to state Hybrid Lemma)

Let $X^1, \ldots, X^m$ be distribution ensembles for $m = \mathsf{poly}(n)$. Suppose there exists a distinguisher/adversary $\mathcal{A}$ that distinguishes between X^1 and X^m with probability μ. Then $\exists i \in [m-1]$, such that $\mathcal{A}$ distinguishes between X^i and X^{i+1} with advantage at least μ/m.

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- We will now use $\mathcal{A}$ that has non-negligible advantage in distinguishing between $\mathcal{H}_i$ and $\mathcal{H}_{i+1}$, to construct another adversary $\mathcal{B}$ to break security of G.
- However, since G is a secure PRG, no such n.u. PPT $\mathcal{A}$ should exist. This will give us a contradiction and imply that our assumption was incorrect. G_{poly} is in fact secure.

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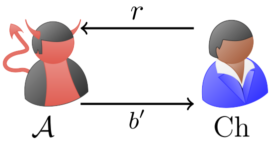

$b \xleftarrow{\$} \{0,1\}$

if $b = 1$: $s \xleftarrow{\$} \{0,1\}^n$

$r := G(s)$

if $b = 0$: $r \xleftarrow{\$} \{0,1\}^{\ell(n)}$

$\boxed{\mathcal{A} \text{ wins if } b' = b}$

$$\left| \Pr[b' = 1 | b = 1] - \Pr[b' = 1 | b = 0] \right| \leqslant \nu(n)$$

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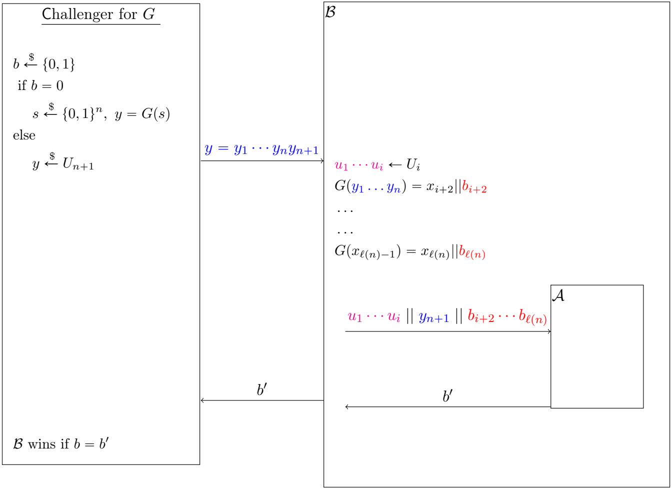

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- Moreover, since $\mathcal{A}$ is n.u. PPT, so is $\mathcal{B}$. This is a contradiction!
- Hence, $G_{\mathsf{poly}(n)}$ is a PRG.

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  3. **Output Mapping:** How do we map the output that $\mathcal{A}$ provides to an output for $\mathcal{B}$?
  4. **Probability:** When we assume existence of $\mathcal{A}$, we also assume that $\mathcal{A}$ wins with non-negligible advantage. What is the probability/advantage that $\mathcal{B}$ wins, given the mappings above?

One-Way Functions

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How to define one-way functions?

Defining One Way Functions: Attempt 1

Attempt 1: A function $f : \{0,1\}^* \to \{0,1\}^*$ is a one-way function (OWF) if it satisfies the following two conditions:

- **Easy to compute:** there is a polynomial-time algorithm $\mathcal{C}$ s.t. $\forall x \in \{0,1\}^*$,

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Probability of Inversion is Negligible

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$$\Pr\left[\mathcal{A} \text{ inverts } f(x) \text{ for random } x\right] \leqslant \textit{negligible}.$$

This is called **average-case** hardness.

One Way Functions: Definition

Definition (One Way Function)

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- **Hard to invert:** there exists a negligible function $\nu : \mathbb{N} \to \mathbb{R}$ s.t. for every non-uniform PPT adversary $\mathcal{A}$ and $\forall n \in \mathbb{N}$:
$$\Pr\left[x \xleftarrow{\$} \{0,1\}^n, x' \leftarrow \mathcal{A}(1^n, f(x)) : f(x') = f(x)\right] \leqslant \nu(n).$$

- The above definition is also called **strong** one-way functions.

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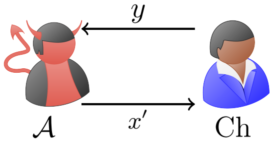

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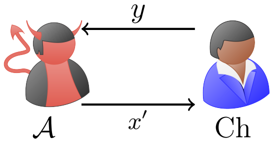

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We say that $f : \{0,1\}^* \to \{0,1\}^*$ is a one-way function if there exists a negligible function $\nu : \mathbb{N} \to \mathbb{R}$ s.t. for every n.u. PPT adversary $\mathcal{A}$ and $\forall n \in \mathbb{N}$:

$$\Pr[\mathcal{A} \text{ wins}] \leqslant \nu(n).$$

Injective OWFs and One Way Permutations (OWP)

- **Injective or 1-1 OWFs**: each image has a unique pre-image:

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

- **One Way Permutations (OWP)**: 1-1 OWF with the additional conditional that "each image has a pre-image"

  (Equivalently: domain and range are of same size.)

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- Such constructions are sometimes called "candidates" because they are based on an assumption or a conjecture.

Factoring Problem

- Consider the **multiplication** function $f_\times : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$:

$$f_\times(x, y) = \begin{cases} \perp & \text{if } x = 1 \vee y = 1 \\ x \cdot y & \text{otherwise} \end{cases}$$

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- **Clearly not!** With prob. 1/2, a random number (of any fixed size) is <u>even</u>. I.e., xy is even w/ prob. $\frac{3}{4}$ for random (x, y).
- Inversion: given number z, output $(2, z/2)$ if z is even and $(0, 0)$ otherwise! (succeeds 75% time)

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- This is unlikely to have small trivial factors.
  [Factoring Assumption] For every (non-uniform PPT) adversary $\mathcal{A}$, there exists a negligible function ν such that

$$\Pr\left[p \xleftarrow{\$} \Pi_n; q \xleftarrow{\$} \Pi_n; N = pq : \mathcal{A}(N) \in \{p, q\}\right] \leqslant \nu(n).$$

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\begin{array}{ll}
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2^{O\left(\sqrt[3]{n \log^2 n}\right)} & \text{(heuristic)}
\end{array}
$$

- Can we construct OWFs from the Factoring Assumption?

Multiplication Function

- Going back to the multiplication function $f_\times : \mathbb{N}^2 \to \mathbb{N}$.

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- Usually called a **weak** OWF.

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Note that a non-negligible function is not necessarily a noticeable function. Example:

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2^{-n} & \text{if } n \text{ is odd} \end{cases}.$$

This function is non-negligible, but not noticeable. Why?