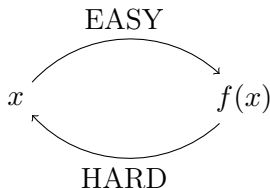


# One-Way Functions (II)

601.642/442: Modern Cryptography

Fall 2022

## Recap: One Way Functions



- A function is one-way if it “easy to compute,” but “hard to invert”
- Necessary for the existence of most cryptographic primitives (e.g., multi-message encryption, digital signatures)
- Also sufficient for some cryptographic primitives (e.g., pseudorandom generators, secret-key encryption, digital signatures).

# Recap: One Way Functions (Definition)

## Definition (One Way Function)

A function  $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$  is a one-way function (OWF) if it satisfies the following two conditions:

- **Easy to compute:** there is a polynomial-time algorithm  $\mathcal{C}$  s.t.  
 $\forall x \in \{0, 1\}^*,$

$$\Pr [\mathcal{C}(x) = f(x)] = 1.$$

- **Hard to invert:** there exists a negligible function  $\nu : \mathbb{N} \rightarrow \mathbb{R}$  s.t.  
for every non-uniform PPT adversary  $\mathcal{A}$  and  $\forall n \in \mathbb{N}$ :

$$\Pr \left[ x \stackrel{\$}{\leftarrow} \{0, 1\}^n, x' \leftarrow \mathcal{A}(1^n, f(x)) : f(x') = f(x) \right] \leq \nu(n).$$

- The above definition is also called **strong** one-way functions.

## Recap: Factoring Problem

- Consider the **multiplication** function  $f_{\times} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ :

$$f_{\times}(x, y) = \begin{cases} \perp & \text{if } x = 1 \vee y = 1 \\ x \cdot y & \text{otherwise} \end{cases}$$

- The first condition helps exclude the trivial factor 1.
- Is  $f_{\times}$  a OWF?
- Clearly not!** With prob.  $1/2$ , a random number (of any fixed size) is even. I.e.,  $xy$  is even w/ prob.  $\frac{3}{4}$  for random  $(x, y)$ .
- Inversion: given number  $z$ , output  $(2, z/2)$  if  $z$  is even and  $(0, 0)$  otherwise! (succeeds 75% time)

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## Assumption (Factoring Assumption)

*For every (non-uniform PPT) adversary  $\mathcal{A}$ , there exists a negligible function  $\nu$  such that*

$$\Pr \left[ p \xleftarrow{\$} \Pi_n; q \xleftarrow{\$} \Pi_n; N = pq : \mathcal{A}(N) \in \{p, q\} \right] \leq \nu(n).$$

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- **Note:** Factoring can be solved in polynomial time with a quantum computer!
- Can we construct OWFs from the Factoring Assumption?

# Multiplication Function

- Going back to the multiplication function  $f_{\times} : \mathbb{N}^2 \rightarrow \mathbb{N}$ .

$$f_{\times}(x, y) = \begin{cases} \perp & \text{if } x = 1 \vee y = 1 \\ x \cdot y & \text{otherwise} \end{cases}$$

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- This is already useful!
- Usually called a **weak** OWF.

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Note that a non-negligible function is not necessarily a noticeable function. Example:

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2^{-n} & \text{if } n \text{ is odd} \end{cases}.$$

This function is non-negligible, but not noticeable. **Why?**

# Weak One Way Functions

## Definition (Weak One Way Function)

A function  $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$  is a *weak one-way function* if it satisfies the following two conditions:

- **Easy to compute:** there is a polynomial-time algorithm  $\mathcal{C}$  s.t.  
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$$\Pr [\mathcal{C}(x) = f(x)] = 1.$$

- **Somewhat hard to invert:** there is a noticeable function  
 $\varepsilon : \mathbb{N} \rightarrow \mathbb{R}$  s.t. for every non-uniform PPT  $\mathcal{A}$  and  $\forall n \in \mathbb{N}$ :

$$\Pr \left[ x \leftarrow \{0, 1\}^n, x' \leftarrow \mathcal{A}(1^n, f(x)) : f(x') \neq f(x) \right] \geq \varepsilon(n).$$



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- Proof Idea: The fraction of prime numbers between 1 and  $2^n$  is noticeable!
- **Chebyshev's theorem:** An  $n$  bit number is a prime with probability  $\frac{1}{2^n}$

# Proof Idea

- Let GOOD be the set of inputs  $(x, y)$  to  $f_x$  s.t. both  $x$  and  $y$  are prime numbers

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- Let **GOOD** be the set of inputs  $(x, y)$  to  $f_{\times}$  s.t. both  $x$  and  $y$  are prime numbers
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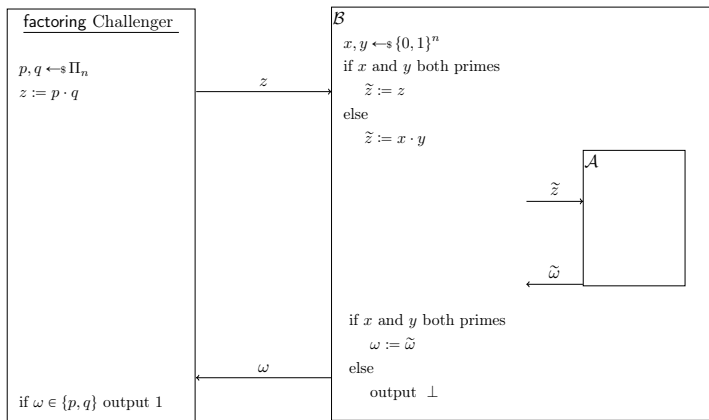
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- But if  $\Pr[(x, y) \in \text{GOOD}]$  is noticeable, then overall, the adversary can only invert with some bounded noticeable probability.
- Formally: Let  $q(n) = 8n^2$ . Will show that no non-uniform PPT adversary can invert  $f_{\times}$  with probability greater than  $1 - \frac{1}{q(n)}$

# Proof via Reduction

**Goal:** Given an adversary  $\mathcal{A}$  that breaks weak one-wayness of  $f_{\times}$  with probability *at least*  $1 - \frac{1}{q(n)}$ , we will construct an adversary  $\mathcal{B}$  that breaks the factoring assumption with noticeable probability

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The input of  $\mathcal{B}$  is a product of two random  $n$ -bit **primes** while that of  $\mathcal{A}$  is the product of two random  $n$ -bit **numbers**. Passing the input directly to  $\mathcal{A}$  would not emulate the distribution of the inputs given to  $\mathcal{A}$ .

# Analysis of $\mathcal{B}$

- Since  $\mathcal{A}$  is non-uniform PPT, so is  $\mathcal{B}$  (using polynomial-time primality testing)

$$\begin{aligned}\Pr[\mathcal{B} \text{ fails}] &= \Pr[\mathcal{B} \text{ passes input to } \mathcal{A}] \cdot \Pr[\mathcal{A} \text{ fails to invert } f_{\times}] \\ &\quad + \Pr[\mathcal{B} \text{ fails to pass input to } \mathcal{A}] \\ &\leq \Pr[\mathcal{A} \text{ fails to invert } f_{\times}] + \Pr[\mathcal{B} \text{ fails to pass input to } \mathcal{A}] \\ &\leq \frac{1}{8n^2} + \left(1 - \frac{1}{4n^2}\right) \leq \left(1 - \frac{1}{8n^2}\right)\end{aligned}$$

- $\mathcal{B}$  succeeds with probability at least  $\frac{1}{8n^2}$ : **Contradiction to factoring assumption!**



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- **Yao's Hardness Amplification: YES!**

# Weak to Strong OWFs

## Theorem (Yao)

*Strong OWFs exist if and only weak OWFs exist*

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## Theorem (Yao)

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- Intuition: Use the weak OWF *many* times
- Think: Is  $f(f(\dots f(x)))$  a good idea?

# Weak to Strong OWFs

## Theorem

*For any weak one-way function  $f : \{0, 1\}^n \rightarrow \{0, 1\}^n$ , there exists a polynomial  $N(\cdot)$  s.t. the function  $F : \{0, 1\}^{n \cdot N(n)} \rightarrow \{0, 1\}^{n \cdot N(n)}$  defined as*

$$F(x_1, \dots, x_{N(n)}) = (f(x_1), \dots, f(x_{N(n)}))$$

*is strongly one-way.*



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- If  $N$  is sufficiently large and the inputs are chosen independently at random, then the probability of inverting all of them should be small

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- The above intuition does not quite work as you expect because even though the instances are chosen independently, adversary gets to see them all together and does not have to invert them independently.



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- Nevertheless, it can be shown via a non-trivial proof that hardness does amplify for one-way functions (albeit not all the way to exponentially small inversion probability – there are counterexamples to this!)
- In fact, hardness amplification is not a general phenomenon; for other cases such as interactive arguments (we will study later), hardness does not amplify in general

# Weak to Strong OWFs: Example

- We will show that Yao's hardness amplification works for  $f_{\times}$
- The general case requires a different and careful proof; see lecture notes for details

# Hardness Amplification for $f_{\times}$

## Theorem

Assume the factoring assumption and let  $m = 4n^3$ . Then,  $\mathcal{F} : (\{0, 1\}^{2n})^m \rightarrow (\{0, 1\}^{2n})^m$  is a strong OWF:

$$\mathcal{F}((x_1, y_1), \dots, (x_m, y_m)) = (f_{\times}(x_1, y_1), \dots, f_{\times}(x_m, y_m)).$$

- **Intuition:** Recall that by Chebyshev's Thm, a pair of random  $n$ -bit numbers are both primes with prob  $\frac{1}{4n^2}$

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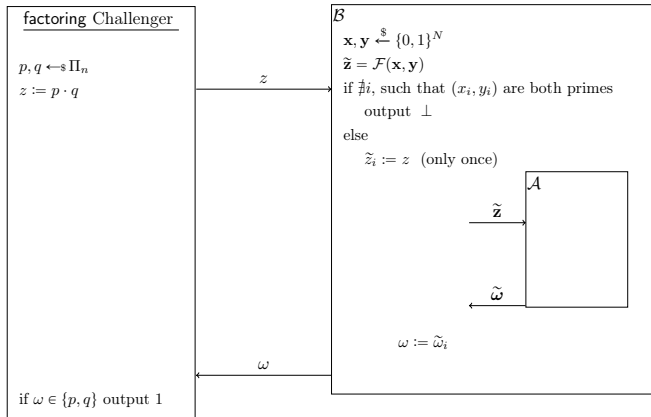
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- **Intuition:** Recall that by Chebyshev's Thm, a pair of random  $n$ -bit numbers are both primes with prob  $\frac{1}{4n^2}$
- When we choose  $m = 4n^3$  pairs, then the prob that no pair consists of primes is at most  $e^{-n}$ , which is negligible

# Hardness Amplification for $f_{\times}$ : Proof Details

- Let  $N = 2n \cdot 4n^3 = 8n^4$ . Let  $(\mathbf{x}, \mathbf{y}) = (x_1, y_1), \dots, (x_m, y_m)$
- Suppose  $\mathcal{F}$  is not a strong OWF. Then,  $\exists$  a non-uniform PPT adversary  $\mathcal{A}$  that inverts  $\mathcal{F}$  with prob at least  $\varepsilon(2n)$  for some non-negligible function  $\varepsilon(\cdot)$
- We will use  $\mathcal{A}$  to construct a non-uniform PPT adversary  $\mathcal{B}$  that breaks the factoring assumption

# Hardness Amplification for $f_{\times}$ : Reduction



# Analysis of $\mathcal{B}$

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- Overall,  $\mathcal{B}$  fails with prob at most  $(1 - \varepsilon(2n)) + e^{-n} < (1 - \frac{\varepsilon(2n)}{2})$
- Thus,  $\mathcal{B}$  succeeds with prob at least  $\frac{\varepsilon(2n)}{2}$ , which is a contradiction to the factoring assumption.