Computational Intractability (II) & Pseudorandomness (I)

601.642/442: Modern Cryptography

Fall 2022

Asymptotic Cost of an Attack

- As we saw in the last class, exponential-time (e.g., $\Theta(2^n)$) brute-force attacks do not scale well as we increase the length of the key or the security parameter. We view them as *infeasible*.
- To rule out only *feasible* attacks, we will focus on polynomial-time attacks (e.g., n^2, n^5, etc) that scale reasonably well (especially if exponent is small). Recall that polynomial-time algorithms are also referred to as *efficient* algorithms.
- **Our goal:** Ensure that no polynomial-time attack can successfully break security.
- A useful property of polynomial-time algorithms is **closure**: *repeating a poly-time algorithm polynomial times is still polynomial time!*

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- It might also use a *different* algorithm for each input size, each of which might be efficient. This still counts as efficient since he is using polynomial-time resources!
- We call this a **non-uniform** adversary since the algorithm is not uniform across all input sizes.

Non-Uniform PPT

Definition (Non-Uniform PPT)

A <u>non-uniform probabilistic polynomial time Turing machine</u> A is a sequence of probabilistic machines $A = \{A_1, A_2, \ldots\}$ for which there exists a polynomial $p(\cdot)$ such that for every $A_i \in A$, the description size $|A_i|$ and the running time of A_i are at most $p(i)$. We write $A(x)$ to denote the distribution obtained by running $A_{|x|}(x)$.

- Our adversary will usually be a non-uniform PPT Turing machine. (most general)

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- Essentially, for security, we don't need to worry about the following:
  - Attacks that are as expensive as a brute-force attack.
  - Attacks whose success probability is as low as a blind-guess attack.
- While an attack with success probability 2^{-128} should not really count as an attack, one with success probability $1/2$ should. Where should we draw the line?

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Probability	Equivalent
2^{-10}	full house in 5-card poker
2^{-20}	royal flush in 5-card poker
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- While these examples are good to get some intuition, we need more concrete way to draw the line between *reasonable* and *unreasonable*.
- Just like running time, we will use an asymptotic approach to capture **low success probability** so that it can be tweaked as desired by changing the security parameter.

Negligible Functions

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Definition (Negligible Function)

A function $\nu(\cdot)$ is negligible if for every polynomial $p(\cdot)$, we have $\lim_{n\to\infty} p(n)\nu(n) = 0$

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- Since $f(\cdot)$ and $g(\cdot)$ are both negligible functions, we know that $\exists n_f, n_g$ corresponding to $c+1$, such that $\forall n > n_f$, $f(n) \leqslant \frac{1}{n^{c+1}}$ and $\forall n > n_g$, $g(n) \leqslant \frac{1}{n^{c+1}}$.

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For a given c, let $n_0 = \mathsf{max}(n_f, n_g, 2)$. $\forall n > n_0$:

$$\begin{aligned} f(n) + g(n) &\leqslant \frac{1}{n^{c+1}} + \frac{1}{n^{c+1}} \\ &\leqslant \frac{2}{n^{c+1}} \\ &\leqslant \frac{n}{n^{c+1}} \text{ (Since } n \geqslant n_0 \geqslant 2) \\ &\leqslant \frac{1}{n^c} \end{aligned}$$

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For a given c, let $n_0 = \mathsf{max}(n_\nu, n_p)$. $\forall n > n_0$:

$$\begin{aligned}
\nu(n) \cdot p(n) &\leqslant \frac{1}{n^{c+c_p}} \cdot n^{c_p} \\
&\leqslant \frac{1}{n^{c+c_p-c_p}} \\
&\leqslant \frac{1}{n^c}
\end{aligned}$$

Distributions & Ensembles

- Recall: X is a distribution over sample space $\mathcal{S}$ if it assigns probability p_s to the element $s \in \mathcal{S}$ s.t. $\sum_s p_s = 1$

Ensemble

A sequence $\{X_n\}_{n\in\mathbb{N}}$ is called an ensemble if for each $n \in \mathbb{N}$, X_n is a probability distribution over $\{0,1\}^*$.

- Generally, X_n will be a distribution over the sample space $\{0,1\}^{\ell(n)}$ (where $\ell(\cdot)$ is a polynomial)

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How do we formalize this?

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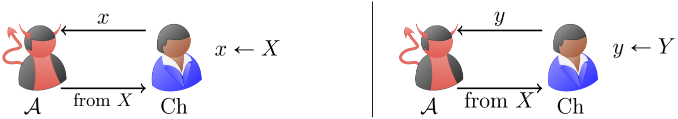

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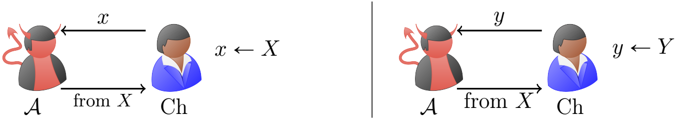

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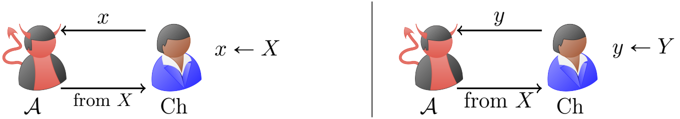

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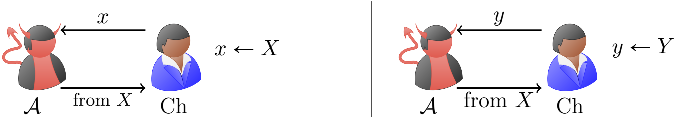

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  $\Pr\left[x \leftarrow X; \mathcal{A}(1^n, x) = 1\right] \approx \Pr\left[y \leftarrow Y; \mathcal{A}(1^n, y) = 1\right] \implies$

  $$\left|\Pr\left[x \leftarrow X; \mathcal{A}(1^n, x) = 1\right] - \Pr\left[y \leftarrow Y; \mathcal{A}(1^n, y) = 1\right]\right| \leqslant \nu(n).$$

Computationally Indistinguishability: Definition

Definition (Computationally Indistinguishability)

Two ensembles of probability distributions $X = \{X_n\}_{n\in\mathbb{N}}$ and $Y = \{Y_n\}_{n\in\mathbb{N}}$ are said to be **computationally indistinguishable** if for every non-uniform PPT $\mathcal{A}$ there exists a negligible function $\nu(\cdot)$ s.t.:

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- Therefore, X and Y are computationally indistinguishable if all non-uniform PPT $\mathcal{A}$ have negligible advantage in distinguishing them.

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  Proof Idea: If not, $\mathcal{A}$ can use M to tell them apart!
- **Transitivity:** If X, Y are computationally indistinguishable, and Y, Z are computationally indistinguishable; then X, Z are also computationally indistinguishable.

Generalizing Transitivity: Hybrid Lemma

Lemma (Hybrid Lemma)

Let $X^1, \ldots, X^m$ be distribution ensembles for $m = \mathsf{poly}(n)$. If for every $i \in [m-1]$, X^i and X^{i+1} are computationally indistinguishable, then X^1 and X^m are computationally indistinguishable.

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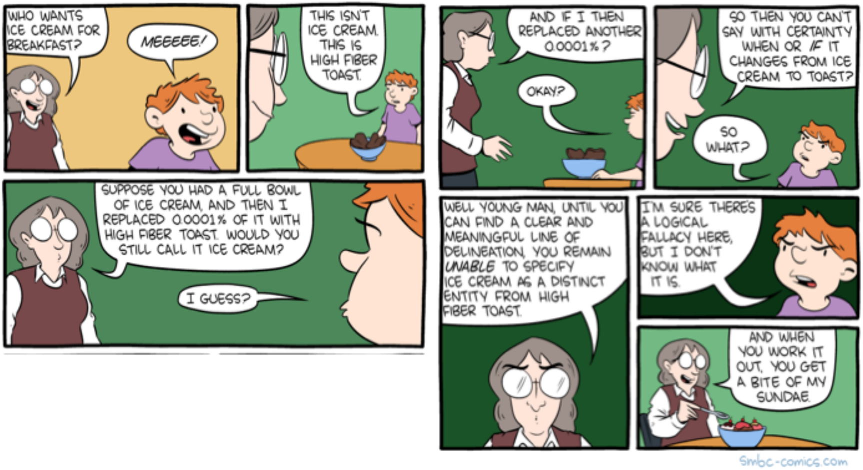
Used in most crypto proofs!

Hybrid Lemma



Looking Back at OTP

One-Time Pad

- $\mathsf{KeyGen}(1^n) := k \xleftarrow{\$} \{0,1\}^n$
- $\mathsf{Enc}(k, m) := c = k \oplus m$
- $\mathsf{Dec}(k, c) := m = k \oplus c$

Uniform ciphertext security and perfect security of OTP crucially depends on the following features:

1. A key is as long as a message.
2. A key is sampled uniformly at random.

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These features of OTP are also its key drawbacks! Why?

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Can we **expand** few random bits into many random “looking” bits?

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- $G : \{0,1\}^n \to \{0,1\}^{n+1}$ is called a **pseudorandom generator** (PRG) and its output is a **pseudorandom** string.

Pseudorandom Generators (PRG)

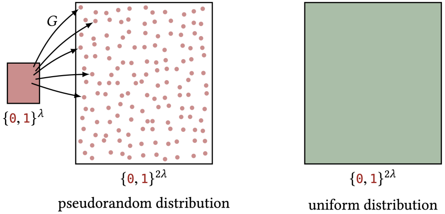
Definition (Pseudorandom Generator)

A deterministic algorithm G is called a pseudorandom generator (PRG) if:

- G can be computed in polynomial time
- $|G(x)| > |x|$
- $\left\{x \xleftarrow{\$} \{0,1\}^n; G(x)\right\} \approx_c \left\{U_{\ell(n)}\right\}$ where $\ell(n) = |G(0^n)|$

The **stretch** of G is defined as: $|G(x)| - |x|$

Illustrating PRG $G : \{0,1\}^\lambda \to \{0,1\}^{2\lambda}$



- From a relative perspective, the PRG's output distribution is tiny. Out of the $2^{2\lambda}$ strings in $\{0,1\}^{2\lambda}$, only 2^λ are possible outputs of G. These strings make up $2^\lambda/2^{2\lambda} = 1/2^\lambda$ fraction of $\{0,1\}^{2\lambda}$ — a **negligible fraction!**

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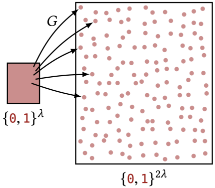

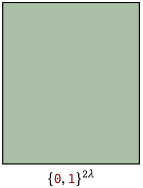

- From a relative perspective, the PRG's output distribution is tiny. Out of the $2^{2\lambda}$ strings in $\{0,1\}^{2\lambda}$, only 2^{λ} are possible outputs of G. These strings make up $2^{\lambda}/2^{2\lambda} = 1/2^{\lambda}$ fraction of $\{0,1\}^{2\lambda}$ — a **negligible fraction!**
- From an absolute perspective, the PRG's output distribution is huge. There are 2^{λ} possible outputs of G, which is an **exponential amount!** This is large enough that an efficient adversary cannot distinguish it from the set $\{0,1\}^{2\lambda}$.

Pseudorandom OTP

- We can now use a PRG to modify one-time pads as follows.

Pseudorandom One-Time Pad

Let n be the security parameter and $\ell(\cdot)$ be a polynomial. Let $G : \{0,1\}^n \to \{0,1\}^{\ell(n)}$ be a PRG, and let the message space and ciphertext space be $\{0,1\}^{\ell(n)}$.

- $\mathsf{KeyGen}(1^n) := k \xleftarrow{\$} \{0,1\}^n$
- $\mathsf{Enc}(k, m) := c = G(k) \oplus m$
- $\mathsf{Dec}(k, c) := m = G(k) \oplus c$

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Does this encryption scheme satisfy our original definitions of one-time uniform ciphertext security or one-time perfect security?