Zero Knowledge Proofs

601.642/442: Modern Cryptography

Fall 2022

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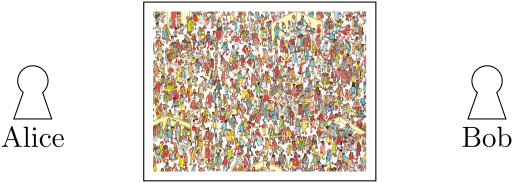
- Shafi and Silvio won the 2012 Turing Award for work on encryption and proof systems.

Scenario: Where's Waldo?



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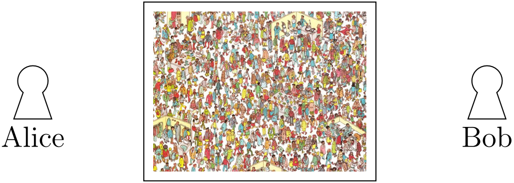
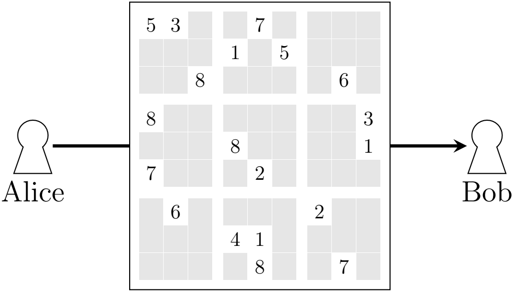


A "Hey Bob, I found Waldo!"

B "That was way too fast, I don't believe you."

Scenario: Sudoku



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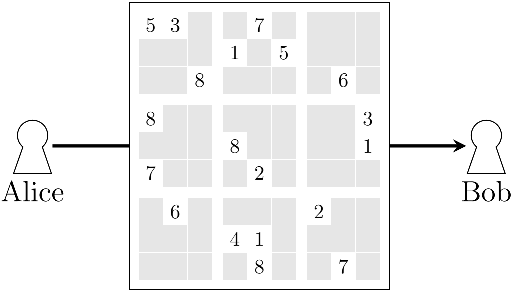

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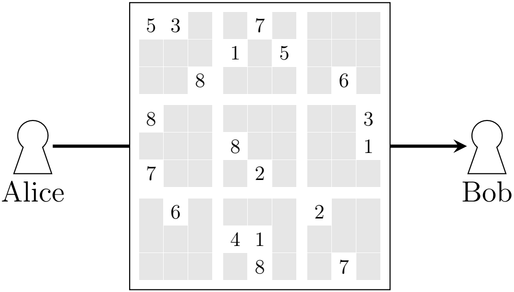

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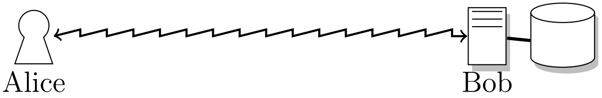
A "This one has a solution, trust me."

Scenario: Authentication



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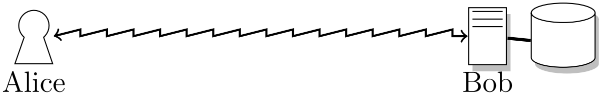

A "Can I have access to the database? It's me, Alice."

B "OK, send me your password so I know it's you."

Scenario: Nuclear Disarmament

Alice
(USA)

Bob
(Russia)

A "Hey Bob, As per our treaty, I have dismantled my nuclear warheads."

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A “Hey Bob, As per our treaty, I have dismantled my nuclear warheads.”

B “What if you dismantled fake or obsolete warheads and are still keeping high quality fissile material? I don’t believe you.”

A Problem of Trust and Information

Alice wants to convince Bob of something

- Waldo is in the picture
- Sudoku puzzle has a solution
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What might a possible solution look like?

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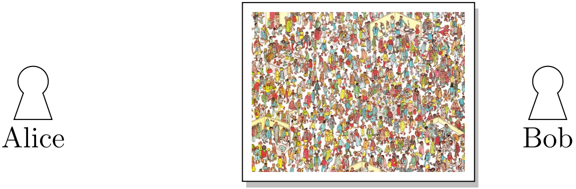

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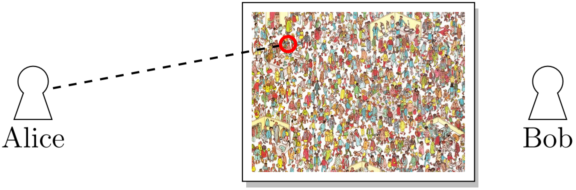

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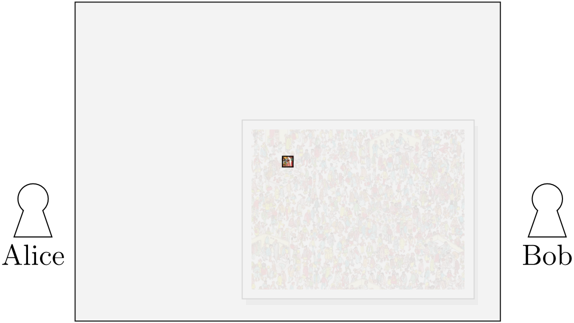

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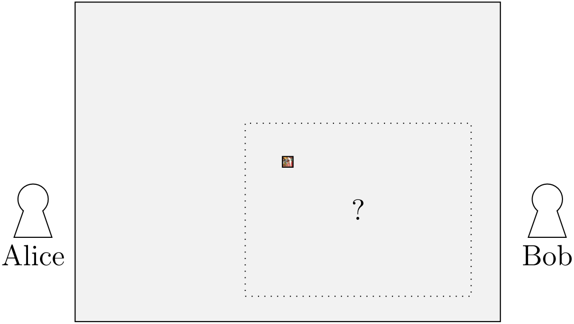

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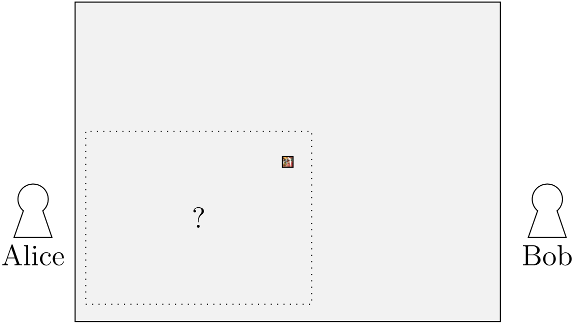

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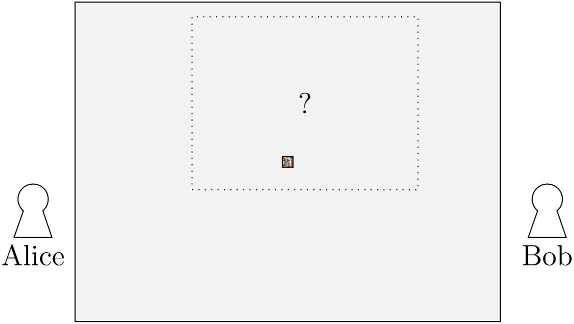

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- E.g., Proof that there are infinitely many primes should not simply be a list of all the primes. Not only would it take forever to generate that proof, it would also take forever to verify it

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2. Question 2: Must a proof be non-interactive?
   - Or can a proof be a conversation? (i.e., interactive)

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- $\mathsf{View}_{M_i}(e)$: View of M_i in an execution e consists of its input, random tape, auxiliary input and all the protocol messages it sees.

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<u>Remark</u>: In the above definition, prover is not required to be efficient. Later, we will also consider efficient provers.

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So why use interactive proofs after all?

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2. Achieving privacy guarantee for prover
    - Zero knowledge: Verifier learns nothing from the proof beyond the validity of the statement!

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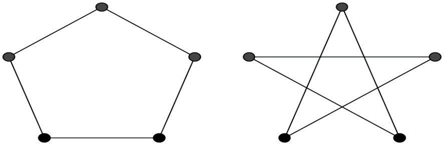

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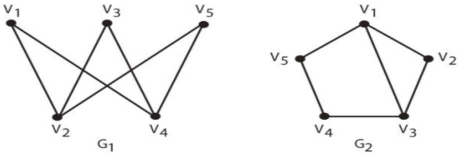

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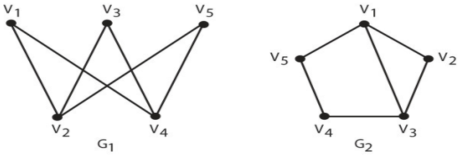

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- How to design an efficiently verifiable interactive proof?

Interactive Proof for Graph Non-Isomorphism

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Protocol (P, V)**:** Repeat the following procedure n times using fresh randomness

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$P \to V$: P computes b' s.t. H and $G_{b'}$ are isomorphic and sends b' to V

Interactive Proof for Graph Non-Isomorphism

Common Input: $x = (G_0, G_1)$

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$V(x, b, b')$: V outputs 1 if $b' = b$ and 0 otherwise

(P, V) is an Interactive Proof

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- **Soundness:** If G_0 and G_1 are isomorphic, then H is isomorphic to both G_0 and G_1! Therefore, in one iteration, any (unbounded) prover can correctly guess b with probability at most $\frac{1}{2}$. Since each iteration is independent, prover can succeed in all iterations with probability at most 2^{-n}.

Interactive Proofs with Efficient Provers

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- **Main Goal:** <u>Zero Knowledge</u>, i.e., ensuring that verifier does not gain any knowledge from its interaction with prover beyond learning the validity of the statement x (e.g., P's witness w remains private from V)

Towards Zero Knowledge

- Q. 1: How to formalize "does not gain any knowledge?"

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- Q. 2: What is knowledge?

Towards Zero Knowledge (contd.)

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That is, by learning the result of a random process or result of a polynomial time computation, we gain no knowledge

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We can generate 100-bit random string for free by flipping a coin, and we can also multiply on our own for free. But an exponential-time computation is hard to perform on our own, since we are PPT. So we should reject first and second offers, but seriously consider the third one!

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- Formalized via notion of Simulator, as in definition of semantic security for encryption

Zero Knowledge: Definition I

Definition (Honest Verifier Zero Knowledge)

An interactive proof (P, V) for a language L with witness relation R is said to be honest verifier zero knowledge if there exists a PPT simulator S s.t. for every non-uniform PPT distinguisher D, there exists a negligible function $\nu(\cdot)$ s.t. for every $x \in L$, $w \in R(x)$, $z \in \{0,1\}^*$, D distinguishes between the following distributions with probability at most $\nu(n)$:

- $\left\{\mathsf{View}_V[P(x,w) \leftrightarrow V(x,z)]\right\}$
- $\left\{S(1^n, x, z)\right\}$

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- What if V is malicious and deviates from the honest strategy?
- Want: Existence of a simulator S for every, possibly malicious (efficient) verifier strategy V^*

Zero Knowledge: Definition II

Definition (Zero Knowledge)

An interactive proof (P, V) for a language L with witness relation R is said to be zero knowledge if for every non-uniform PPT adversary V^*, there exists an (expected) PPT simulator S s.t. for every non-uniform PPT distinguisher D, there exists a negligible function $\nu(\cdot)$ s.t. for every $x \in L$, $w \in R(x)$, $z \in \{0,1\}^*$, D distinguishes between the following distributions with probability at most $\nu(n)$:

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- If the distributions are identical, then we call it perfect zero knowledge

Reflections on Zero Knowledge

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- Eve: "Oh really?"
- Bob: "Yes, you can see this accepting transcript"
- Eve: "That doesn't mean anything. Anyone can come up with such a transcript without knowing a witness for x!"
- Bob: "But I computed this transcript by talking to Alice who answered my challenge correctly every time!"

Reflections on Zero Knowledge (contd.)

Moral of the story:

- Bob participated in a "live" conversation with Alice, and was convinced by how the transcript was generated

Reflections on Zero Knowledge (contd.)

Moral of the story:

- Bob participated in a "live" conversation with Alice, and was convinced by <u>how</u> the transcript was generated
- But to Eve, who did not see the live conversation, there is no way to tell whether the transcript is from real execution or produced by simulator