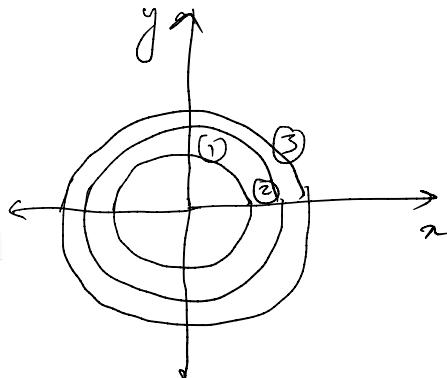
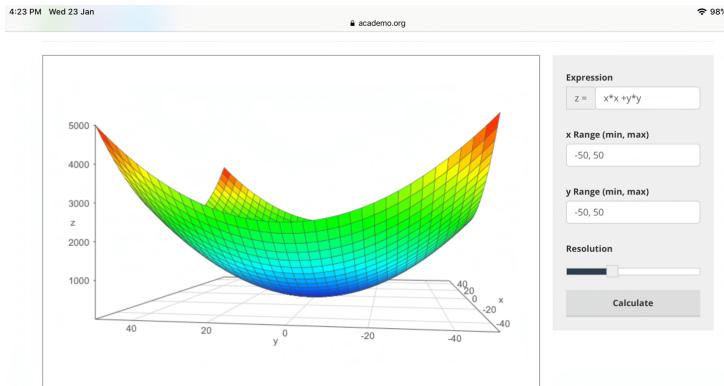


Mathematics for Machine learning - II

CONTOUR PLOT

$$f(x, y) = x^2 + y^2$$

Plot $f(x, y) = K$ for varying K

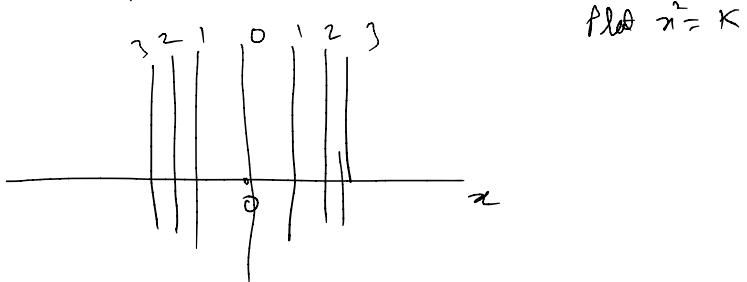


This demo allows you to enter a mathematical expression in terms of x and y . When you hit the calculate button, the demo will calculate the value of the expression over the x and y ranges provided and then plot the result as a surface. The graph can be zoomed in by scrolling with your mouse, and rotated by dragging around. Clicking on the graph will reveal the x , y and z values at that particular

You might also be interested in

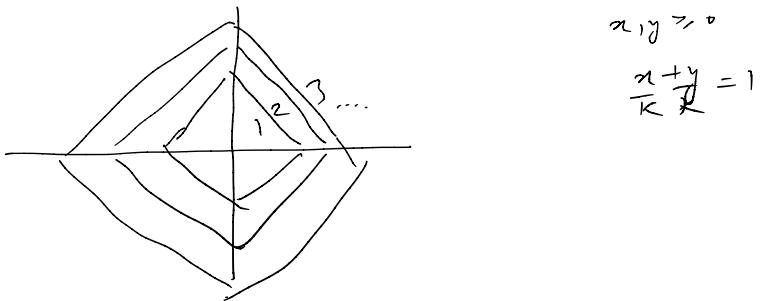
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①) Draw contour plot for $f(x) = x^2$



$$\text{plot } x^2 = K$$

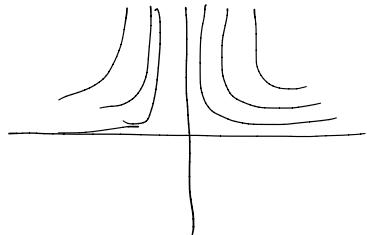
②) Draw contour plot for $f(x, y) = |x| + |y|$



$$x, y \geq 0$$

$$\frac{x+y}{K} = 1$$

① Draw contour for $f(x,y) = x^2y$



$$x^2y = K$$

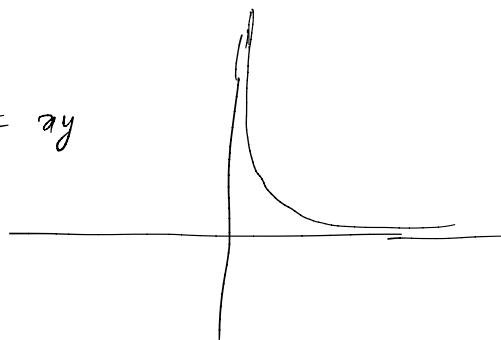
$$x^2 = \frac{K}{y}$$

$$xy = \frac{K}{x^2}$$

$$(K=1)$$

$$y = \frac{1}{x^2}$$

② $f(x,y) = xy$



$$xy = K \Rightarrow x = \frac{K}{y}$$

$$K=1$$

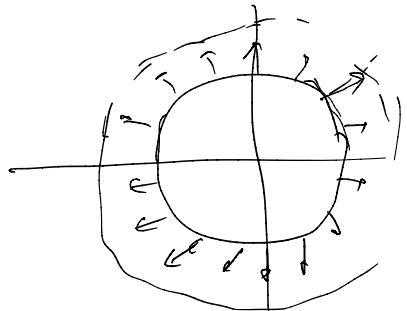
$$x = \frac{1}{y}$$

$$y = \frac{1}{x}$$

Contour plots and (gradients)

↳ steepest change...

$$f(x,y) = x^2 + y^2$$



* All points on
contour have
same $f(x,y)$

* which direction
more to
increase $f(x,y)$
then

⇒ I go from
point on the
curve

Gradient of: $\nabla f(x,y)$

$$= \begin{bmatrix} \frac{\partial}{\partial x} f(x,y) \\ \frac{\partial}{\partial y} f(x,y) \end{bmatrix}$$

$$= \begin{bmatrix} 2x \\ 2y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

Constrained
Optimisation

Extrema
(max or
minima)

$$\begin{aligned} f(x, y) &= x^2 + y^2 \\ \text{s.t. } & xy = K \end{aligned}$$

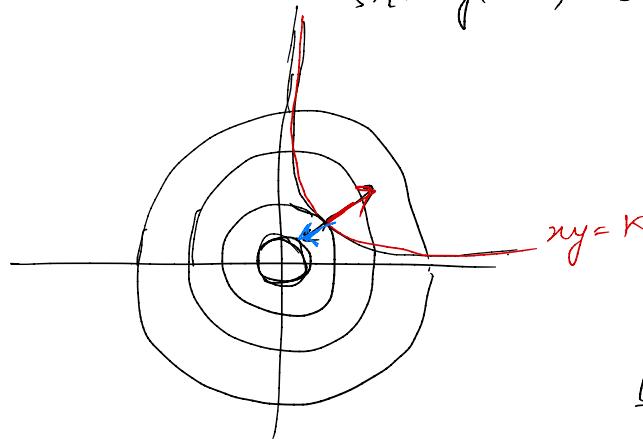
Q. Extrema is minima?
maxima?

More generally, we want extrema $f(x, \dots)$

$$\text{s.t. } g(x, \dots) = 0$$

$\because xy = 1$
 $x \neq 0$
 $y \neq 0$

- Gradient $\nabla f(x, y)$
- Gradient of $g(x, y)$



$$\boxed{\nabla f(x, y) = \lambda (\nabla g(x, y))}$$

$$\nabla f(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \Rightarrow \nabla g(x,y) \Rightarrow \begin{bmatrix} y \\ x \end{bmatrix}$$

$$\begin{array}{l} 2x = \lambda y \\ 2y = \lambda x \end{array} \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad \begin{array}{l} 3 \text{ variables} \\ 2 \text{ eqns} \end{array} \quad \begin{array}{l} \dots \textcircled{1} \\ \dots \textcircled{2} \end{array}$$

$$xy = K \quad \dots \textcircled{3}$$

① and ③

$$x = \frac{\lambda y}{2} \quad & xy = K \Rightarrow \frac{\lambda y}{2} \cdot y = K \Rightarrow \lambda y^2 = 2K$$

① and ②

$$4xy = \lambda^2 xy \Rightarrow \lambda = 2$$

$$2x = 2y \Rightarrow x = y \quad \Rightarrow \boxed{x = y = \sqrt{K}} \\ \lambda = 2$$

Constrained
Optimisation

Extrema
(max or
minima)

$$f(x, y) = x^2 + y^2$$

s.t.

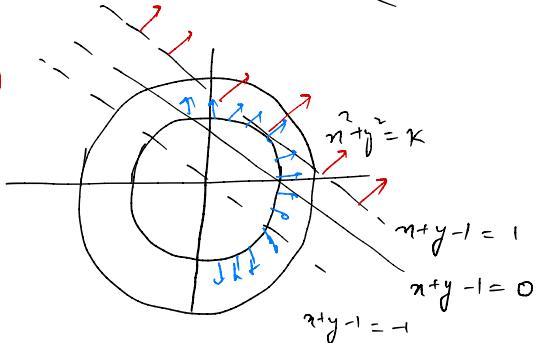
$$x+y=1$$

Q. Extrema is minima?
or maxima?

More generally, we want { Extrema $f(x, \dots)$ }
s.t. $g(x, \dots) = 0$

$x+y=1$
 x w.l.o.g.
 y may be

- Gradient of $f(x, y)$
- Gradient of $g(x, y)$



All points on contour have

$$\nabla f(x, y) = \lambda (\nabla g(x, y))$$

$$\nabla f(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix} = \lambda \nabla g(x,y) = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{array}{l} 2x = \lambda \\ 2y = \lambda \end{array} \quad \left. \begin{array}{l} 2 \text{ eqns} \\ 2 \text{ variables} \end{array} \right\}$$

But we had

$$x+y-1=0 \quad \dots \quad \text{2nd eqn}$$

$$\begin{aligned} \therefore 2x = \lambda &\Rightarrow x = y = \frac{\lambda}{2} \\ 2y = \lambda & \\ x+y=1 &\Rightarrow \lambda = 1 \Rightarrow x=y=\frac{\lambda}{2} \end{aligned}$$

λ -Lagrangian multiplier

Lagrangian $L(\eta, y, \lambda)$

$$\nabla L = 0$$

$$\frac{\partial L}{\partial \eta} = 0$$

$$\frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial \lambda} = 0$$

a) Find extreme for $f(x, y) = x^2y$
 s.t. $g(x, y) = x^2 + y^2 - 1 = 0$

$$L = x^2y + \lambda(x^2 + y^2 - 1)$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow 2xy + \lambda(2x) = 0$$

$$\frac{\partial L}{\partial y} \Rightarrow x^2 + \lambda(2y) = 0$$

$$\frac{\partial L}{\partial x} \Rightarrow x^2 + y^2 - 1 = 0$$

$$\Rightarrow 2x(y + \lambda) = 0$$

$$x^2 + 2\lambda y = 0$$

$$x^2 + y^2 = 1$$

Case I

$$x = 0$$

$$\Rightarrow y^2 = 1 \Rightarrow y = \pm 1$$

$$f(x, y) = 0$$

$$\lambda = 0$$

Case II

$$\lambda \neq 0$$

$$\Rightarrow y = -\lambda$$

$$\Rightarrow x^2 + 2\lambda(-\lambda) = 0$$

$$\Rightarrow x^2 = 2\lambda^2 \quad \Rightarrow \lambda = \sqrt{\frac{1}{2}}$$

$$\Rightarrow 2\lambda^2 = 1$$

$$\Rightarrow x^2 + \lambda^2 = 1$$

$$x^2 = 2x^2 = \frac{2}{3}$$

$$y = \pm \sqrt{\frac{1}{3}}$$

$$\text{max } \frac{2}{3}y = \frac{2}{3}\sqrt{\frac{1}{3}}$$

KKT conditions

Minimize $f(x)$

$$\text{st. } h_i(x) = 0 \quad \forall i = 1, \dots, m$$

$$g_i(x) \leq 0 \quad \forall j = 1, \dots, n$$

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^m \lambda_i h_i(x) + \sum_{i=1}^n \mu_i g_i(x)$$

Now, if $g_i(x^*) < 0$, then μ_i can be set to zero (constraint doesn't participate)

else $g_i(x^*) = 0$

$$\therefore \boxed{\lambda_i g_i(x^*) = 0}$$

$$\mu_i > 0$$

Stationarity

$$\nabla_x f(x) + \sum_{i=1}^m \nabla_x \lambda_i h_i(x) + \sum_{i=1}^m \nabla_x \mu_i g_i(x) = 0$$

Equality

$$\nabla_x f(x) + \sum_{i=1}^m \nabla_x \lambda_i h_i(x) + \sum_{i=1}^m \nabla_x \mu_i g_i(x) = 0$$

Inequality / complementary slackness

$$\begin{aligned} \mu_i g_i(x) &= 0 \\ \mu_i &\geq 0 \end{aligned}$$