Weiss Variation of AdS/BCFT

Rohan, Minguye

January 13, 2026

1 Checklist

- Consider boundary conditions for BCFT eg. time slice, gauge conditions..etc
- Compare Weiss Variation of BCFT and Gravity side using Witten Ansatz

2 Weiss Variation of BCFT

Recall a Weiss variation is one where we vary the field and the region itself. Under a transformation of the region including the boundary $x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$The matter field and metric transform as $\varphi' = \varphi + \delta\varphi$ and $g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$. We also know under a conformal transformation the metric must transform as $g'_{\mu\nu} = \Omega(x) g_{\mu\nu}$.

2.1 Neother Method

Recall from Feng's Weiss Variation

$$\delta S = \int_W d^d x \left(\frac{\partial \mathcal{L}}{\partial \varphi} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \partial_\mu \varphi \right) \delta\varphi + \int_{\partial W} d^{d-1} x \left(\mathcal{L} \delta x^{\mu} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta\varphi \right) \tag{1}$$

Recall for a CFT the constraint on the coordinate diffeomorphis: $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon) g_{\mu\nu}$ We define a primary field, that transforms under a conformal transformation as follows.

$$\phi'(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\Delta/d} \phi(x) = (1 + \partial_\mu \epsilon^{\mu})^{-\Delta/d} \phi(x) \tag{2}$$

Where we are expanding in small ϵ to first order. To further expand the last term in equation 1. we want to find the form variation $\delta\varphi(x) = \varphi'(x) - \varphi(x)$. We can do so using the total variation $\Delta\varphi(x) = \varphi'(x') - \varphi(x)$.

$$\begin{aligned}
\phi'(x') &= (1 + \partial_\mu \epsilon^{\mu})^{-\Delta/d} \phi(x) \\
&= \left(1 - \frac{\Delta}{d} \partial_\mu \epsilon^{\mu} \right) \phi(x' - \epsilon) \\
&= \left(1 - \frac{\Delta}{d} \partial_\mu \epsilon^{\mu} \right) \left(\phi(x') - \epsilon^{\mu} \partial_\mu \phi(x') \right) \\
&= \phi(x') - \epsilon^{\mu} \partial_\mu \phi(x') - \frac{\Delta}{d} \partial_\mu \epsilon^{\mu} \phi(x')
\end{aligned}$$

Set $x' = x$

$$\delta\varphi(x) = \varphi'(x) - \varphi(x) = -\epsilon^{\mu} \partial_\mu \varphi(x) - \frac{\Delta}{d} \varphi(x) \partial_\mu \epsilon^{\mu}$$

The last term in equation 1. becomes:

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi = -\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\left(\epsilon^\mu \partial_\mu \phi + \frac{\Delta}{d}\phi \partial_\mu \epsilon^\mu\right)$$

The boundary integral now becomes:

$$\int_{\partial W} d^{d-1}x \left(-\delta^{\mu\nu}\mathcal{L}\epsilon_\nu - \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\epsilon_\nu \partial^\nu \phi - \frac{\Delta}{d}\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\phi \partial^\nu \epsilon_\nu\right) n_\mu \tag{3}$$

Now, integrate by parts the term:

$$-\frac{\Delta}{d}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\phi \partial^\nu \epsilon_\nu = -\frac{\Delta}{d}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\nu(\phi\epsilon_\nu) + \frac{\Delta}{d}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\nu \phi \epsilon_\nu$$

Now integrate by parts again on the first term on the RHS:

$$\begin{aligned} &= -\frac{\Delta}{d}\partial^\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\phi\epsilon_\nu\right) + \frac{\Delta}{d}\partial^\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\phi\epsilon^\nu \\ &\quad + \frac{\Delta}{d}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\nu \phi \epsilon_\nu \end{aligned}$$

The first term is a total derivative and as there is no boundary of the boundary, it vanishes.

$$-\frac{\Delta}{d}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\phi \partial_\nu \epsilon^\nu = \frac{\Delta}{d}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\nu \phi + \partial^\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\phi\right)\epsilon_\nu$$

So the boundary integral becomes:

$$\int_{\partial W} d^{d-1}x \left(\delta^{\mu\nu}\mathcal{L} - P^\mu \partial^\nu \phi + \frac{\Delta}{d}\partial^\nu\left(P^\mu \phi\right)\right)\epsilon_\nu$$

$P^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}$. Define $\tilde{T}^{\mu\nu} = -T^{\mu\nu} + R^{\mu\nu}$, where $R^{\mu\nu} = \frac{\Delta}{d}\partial^\nu(P^\mu \phi)$. The variation of the action due to local Weiss variation becomes

$$\delta S = \int_W d^d x T^{\mu\nu}\partial_\mu \epsilon_\nu + \int_{\partial W} d^{d-1}x \tilde{T}^{\mu\nu} n_\mu \epsilon_\nu \tag{4}$$

Where n_μ is the unit normal to the boundary. We can check if it's traceless. In the following we use the on-shell condition $\partial_\mu \partial^\mu \phi = 0$

$$\begin{aligned} \delta_{\mu\nu}\tilde{T}^{\mu\nu} &= -\left(\frac{d}{2}+1\right)\partial_\mu \phi \partial^\mu \phi + \frac{(d-2)}{2d}\partial_\mu \phi \partial^\mu \phi + \frac{\Delta}{d}\phi \partial_\mu \partial^\mu \phi \\ &= \frac{d-2}{2}\left(1-\frac{1}{d}\right) \quad \text{which is zero for } d = 1, 2. \end{aligned}$$

and otherwise not traceless. As conformal invariance required a tracless energy momentum tensor (this can be seen by using the conformal killing equation on $\partial_\mu \epsilon_\nu$)), we must turn to the Rosenfeld-Hilbert prescription.

2.2 Rosenfeld Method

We start with the action for a CFT in a general curved space, do the Weiss variation and then take the flat space limit. As our action is constructed to be conformally invariant from the get-go, it shall give us a traceless and symmetric energy momentum tensor.

$$S = \int_M d^d x \sqrt{-g}\left(\mathcal{L}_s + \xi R \varphi^2\right) \tag{5}$$

Where $\mathcal{L}_s = \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$ is the standard kinetic Lagrangian for a scalar field. The term with the Ricci curvature is needed to manifest the action conformally invariant. To begin with the Weiss variation we split the variation of the field and variation of the boundary.

$$\Delta S = \int_M d^dx\,\delta(\sqrt{-g}\mathcal{L}_s) + \delta(\sqrt{-g}\xi R\varphi^2) + \int_{\partial M} d^{d-1}x\sqrt{-h}\left(\mathcal{L}_s + \xi R\varphi^2\right)\delta x^\mu \tag{6}$$

Where the boundary integral comes from change in the jacobian from the coordinate/boundary transformation. We now need find the variation of individual terms.

1. Variation of the standard part, $\delta(\sqrt{-g}\mathcal{L}_s)$. First doing integration by parts and using that $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$.

$$\delta(\sqrt{-g}\mathcal{L}_s) = \left(\left(\frac{\partial\mathcal{L}_s}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}_s}{\partial(\partial_\mu\varphi)}\right)\delta\varphi + \partial_\mu\left(\frac{\partial\mathcal{L}_s}{\partial(\partial_\mu\varphi)}\delta\varphi\right) + \frac{\partial\mathcal{L}_s}{\partial g_{\mu\nu}}\delta g_{\mu\nu}\right)\sqrt{-g} + \mathcal{L}_s\left(\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\right) \tag{7}$$

2. Variation of the Ricci Scalar term , $\delta(\sqrt{g}\xi\varphi^2 R)$:
For clarity we split up the field variation into the variation of the metric of δ_g and the matter field $\delta\varphi$

$$\delta_g(\sqrt{-g}\xi R\varphi^2) = \delta_g(\sqrt{-g})\xi R\varphi^2 + \sqrt{-g}\delta_g(\xi R\varphi^2) \tag{8}$$

$$\delta_g R\xi\varphi^2 = \xi(g_{\mu\nu}\nabla_\mu\nabla^\mu - \nabla_\mu\nabla_\nu)\delta g_{\mu\nu}\varphi^2 + \xi R_{\mu\nu}\delta g^{\mu\nu}\varphi^2 \qquad \delta_\varphi(\sqrt{-g}\xi R\varphi^2) = 2\sqrt{-g}\,\xi R\varphi\delta\varphi \tag{9}$$

Let's do integration by parts on individual terms of $\xi\varphi^2 g^{\mu\nu}(\nabla_\mu\nabla^\mu\varphi - \nabla_\mu\nabla_\nu)\delta g^{\mu\nu}$.

$$\begin{aligned}
-\varphi^2\nabla_\mu\nabla_\nu\delta g^{\mu\nu} &= -\nabla_\mu(\varphi^2\nabla_\nu\delta g^{\mu\nu}) + \nabla_\nu\varphi^2\nabla_\mu\delta g^{\mu\nu}\\
&= -\nabla_\mu(\varphi^2\nabla_\nu\delta g^{\mu\nu}) + \nabla_\nu(\nabla_\mu\varphi^2\delta g^{\mu\nu}) - \nabla_\nu\nabla_\mu\varphi^2\delta g^{\mu\nu}
\end{aligned}$$

Next,

$$\begin{aligned}
\varphi^2 g^{\mu\nu}\nabla_\alpha\nabla^\alpha\delta g^{\mu\nu} &= \nabla_\alpha(\varphi^2 g^{\mu\nu}\nabla^\alpha\delta g^{\mu\nu}) - \nabla_\alpha(\varphi^2 g^{\mu\nu})\nabla^\alpha\delta g^{\mu\nu}\\
&= \nabla_\alpha(\varphi^2 g^{\mu\nu}\nabla^\alpha\delta g^{\mu\nu}) - \nabla^\alpha(\nabla_\alpha\varphi^2 g^{\mu\nu}\delta g^{\mu\nu}) + \nabla^\alpha\nabla_\alpha\varphi^2 g^{\mu\nu}\delta g^{\mu\nu}
\end{aligned}$$

By Stokes, we know that total derivatives will move to the boundary integral. The remaining bulk terms are the following

$$(\nabla_\alpha\nabla^\alpha\varphi^2 g^{\mu\nu} - \nabla_\nu\nabla_\mu\varphi^2)\delta g^{\mu\nu} \tag{10}$$

The terms that become the integrand of the boundary integral:

$$-n_\mu\varphi^2\nabla_\nu\delta g^{\mu\nu} + n_\nu\nabla_\mu\varphi^2\delta g^{\mu\nu} + \varphi^2 g^{\mu\nu}\nabla_\alpha\delta g^{\mu\nu}n^\alpha - \nabla_\alpha\varphi^2 g^{\mu\nu}\delta g^{\mu\nu}n^\alpha \tag{11}$$

Now more integration by parts on the boundary terms so we can have everything $(...)\delta g^{\mu\nu}$.

$$\begin{aligned}
-n_\mu\varphi^2\nabla_\nu\delta g^{\mu\nu} &= \nabla_\nu(n_\mu\varphi^2)\delta g^{\mu\nu}\\
&= (\nabla_\nu n_\mu\varphi^2 + n_\mu\nabla_\nu\varphi^2)\delta g^{\mu\nu}
\end{aligned}$$

Where we got rid of the total derivative term as there is no boundary of the boundary.

$$\begin{aligned}
\varphi^2 g^{\mu\nu}n_\alpha\nabla^\alpha\delta g^{\mu\nu} &= -\nabla^\alpha(\varphi^2 n_\alpha g^{\mu\nu})\delta g^{\mu\nu}\\
&= -(\nabla^\alpha\varphi^2 n_\alpha g^{\mu\nu} + \varphi^2\nabla^\alpha n_\alpha g^{\mu\nu})\delta g^{\mu\nu}
\end{aligned}$$

$$\nabla^\alpha n_\alpha = K \tag{12}$$

We can choose a gauge condition such that

$$\nabla_\mu n_\nu = K_{\mu\nu} \tag{13}$$

The contribution to the boundary integral from equation 10 is

$$\Xi_{\mu\nu} = K_{\mu\nu}\varphi^2 + n_\mu \nabla_\nu \varphi^2 - \nabla^\alpha \varphi^2 n_\alpha g^{\mu\nu} + \varphi^2 K g^{\mu\nu} + \nabla_\mu \varphi^2 n_\nu - \nabla_\alpha \varphi^2 n^\alpha g_{\mu\nu} \tag{14}$$

Since φ is a scalar field $\nabla_\mu \varphi = \partial_\mu \varphi$. We then have:

$$\begin{aligned}\int_M \delta_g(\sqrt{-g}\xi R\varphi^2) = \int_M \sqrt{-g}\left(\xi\left(R_{\mu\nu}\varphi^2 - \frac{1}{2}Rg_{\mu\nu}\varphi^2 + g_{\mu\nu}\partial_\alpha\partial^\alpha\varphi^2 - \partial_\mu\partial_\nu\varphi^2\right)\delta g^{\mu\nu} + 2\,\xi R\varphi\delta\varphi\right) \\ + \int_{\partial M}\sqrt{-h}\xi\left(\nabla_\nu n_\mu \varphi^2 + n_\mu\partial_\nu\varphi^2 - 2\partial^\alpha\varphi^2 n_\alpha g^{\mu\nu} + \varphi^2 K g^{\mu\nu} + \partial_\mu\varphi^2 n_\nu\right)\delta g^{\mu\nu}\end{aligned} \tag{15}$$

3. Let's now write out the full Weiss Variation for a scalar field CFT

$$\begin{aligned}\Delta S = \int_M d^d x\sqrt{-g}\left(EL + 2\xi\varphi^2 R\right)\delta\varphi + \Theta'_{\mu\nu}\delta g_{\mu\nu} \\ + \int_{\partial M} d^{d-1}x\sqrt{h}\left(\frac{\partial \mathcal{L}_s}{\partial(\partial_\mu\varphi)}\delta\varphi - \Xi_{\mu\nu}\delta g^{\mu\nu} + (\mathcal{L}_s + \xi R\varphi^2)\delta x^\mu\right)\end{aligned} \tag{16}$$

$$\Theta'_{\mu\nu} = \frac{\partial \mathcal{L}_s}{\partial g_{\mu\nu}} + \frac{1}{2}g_{\mu\nu}\mathcal{L}_s + \xi R_{\mu\nu}\varphi^2 - \frac{1}{2}\xi R g_{\mu\nu}\varphi^2 + \xi g_{\mu\nu}\partial_\alpha\partial^\alpha\varphi^2 - \xi\partial_\mu\partial_\nu\varphi^2 \tag{17}$$

Where EL is the Euler Lagrange equation for $\mathcal{L}_s$. Since we are dealing with a scalar field in flat space, we can set $R_{\mu\nu} \to 0 \quad \text{and} \quad g^{\mu\nu} \to \eta^{\mu\nu}$.

$$\begin{aligned}\Delta S = \int_M d^d x\sqrt{-g}\left(-T^{\mu\nu}\partial_\mu\epsilon_\nu + \Theta_{\mu\nu}\delta g^{\mu\nu}\right) \\ + \int_{\partial M} d^{d-1}x\sqrt{-h}\left(\frac{\partial \mathcal{L}_s}{\partial(\partial_\mu\varphi)}\delta\varphi n_\mu - \mathcal{L}_s\delta x^\mu n_\mu + \Xi_{\mu\nu}\delta g^{\mu\nu}\right)\end{aligned} \tag{18}$$

Where $T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\partial^\nu\varphi - \eta^{\mu\nu}\mathcal{L}$. To get $T^{\mu\nu}$ we used that for a scalar field $\Delta\varphi = 0$, and therefore $\delta\varphi = -\epsilon_\nu\partial^\nu\varphi$. Using $g'^{\mu\nu} = \frac{\partial x'_\mu}{\partial x_\alpha}\frac{\partial x'_\nu}{\partial x_\beta}g^{\alpha\beta}$ we get the following:

$$\delta g^{\mu\nu} = \partial^\mu\epsilon^\nu + \partial^\nu\epsilon^\mu \quad \text{when} \quad \delta x^\mu = \epsilon^\mu(x) \tag{19}$$

Since $\Theta_{\mu\nu}$ is symmetric in $\mu\nu$. We can write the bulk integral as

$$\int_M d^d x\left(-T^{\mu\nu} + 2\Theta^{\mu\nu}\right)\partial_\mu\epsilon_\nu \tag{20}$$

And the boundary integral as:

$$\int_{\partial M} d^{d-1}x\sqrt{-h}(T^{\mu\nu}\epsilon_\nu n_\mu + \Xi_{\mu\nu}\delta g^{\mu\nu}) \tag{21}$$

3 Dynamical Boundary and Symmetry Breaking?

If we are considering a dynamical boundary than it's shape can carry some energy, in which case it gets its own action.

$$S_b = \int_{\partial W} d^{d-1} y \sqrt{h} \tau \tag{22}$$

Where y are the boundary coordinates, and τ is the energy density of the boundary, which we take to be constant. We can think of this tension opposing the pressure from the bulk on the boundary. We consider an infinitesimal deformation of the boundary location $\delta x^\mu = \epsilon n^\mu$, where n is the unit normal to the boundary. The variation of the boundary action with respect to the boundary shape is derived by varying the intrinsic boundary volume element $\sqrt{h}$: $\delta S_b = \int_\Sigma d^{d-1}y\, \delta\sqrt{h}\, \tau$

3.0.1 Derivation of the Boundary Term Variation (δS_Σ)

We want to calculate $\delta S_\Sigma = \int_\Sigma d^{d-1}y\, \tau\, \delta(\sqrt{h})$. From Fen'g paper we know that $\delta(\sqrt{h}) = \frac{1}{2\sqrt{h}}\delta h = \frac{1}{2}\sqrt{h} h^{ab} \delta h_{ab}$. So, we need to find δh_{ab} due to the normal displacement $\delta x^\mu = \epsilon n^\mu$. The induced metric h_{ab} on the boundary ∂W is defined by $h_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial y^a} \frac{\partial x^\nu}{\partial y^b}$, where $g_{\mu\nu}$ is the metric of the bulk W. The variation of h_{ab} is:

$$\delta h_{ab} = \delta \left(g_{\mu\nu} \frac{\partial x^\mu}{\partial y^a} \frac{\partial x^\nu}{\partial y^b} \right) = g_{\mu\nu} \left(\frac{\partial(\delta x^\mu)}{\partial y^a} \frac{\partial x^\nu}{\partial y^b} + \frac{\partial x^\mu}{\partial y^a} \frac{\partial(\delta x^\nu)}{\partial y^b} \right) \quad \delta g_{\mu\nu} = 0$$

Substitute $\delta x^\mu = \epsilon n^\mu$:

$$\begin{aligned} \delta h_{ab} &= g_{\mu\nu} \left(\frac{\partial(\epsilon n^\mu)}{\partial y^a} \frac{\partial x^\nu}{\partial y^b} + \frac{\partial x^\mu}{\partial y^a} \frac{\partial(\epsilon n^\nu)}{\partial y^b} \right) \\ &= g_{\mu\nu} \left(\left(\frac{\partial \epsilon}{\partial y^a} n^\mu + \epsilon \frac{\partial n^\mu}{\partial y^a} \right) \frac{\partial x^\nu}{\partial y^b} + \frac{\partial x^\mu}{\partial y^a} \left(\frac{\partial \epsilon}{\partial y^b} n^\nu + \epsilon \frac{\partial n^\nu}{\partial y^b} \right) \right) \end{aligned}$$

Since n^μ is the unit normal vector, it is orthogonal to any tangent vector of the boundary. Therefore, terms containing $\frac{\partial \epsilon}{\partial y^a} n^\mu \frac{\partial x^\nu}{\partial y^b}$ or $\frac{\partial x^\mu}{\partial y^a} \frac{\partial \epsilon}{\partial y^b} n^\nu$ vanish. So we have,

$$\delta h_{ab} = \epsilon\, g_{\mu\nu} \left(\frac{\partial n^\mu}{\partial y^a} \frac{\partial x^\nu}{\partial y^b} + \frac{\partial x^\mu}{\partial y^a} \frac{\partial n^\nu}{\partial y^b} \right) \tag{23}$$

We can connect this to the extrinsic curvature K_{ab}. We can use the chain rule: $\frac{\partial n^\mu}{\partial y^a} = \frac{\partial x^\rho}{\partial y^a} \frac{\partial n^\mu}{\partial x^\rho} = e^\rho_a \partial_\rho n^\mu$. Substituting this into Equation 6.

$$\begin{aligned} \delta h_{ab} &= \epsilon \left(\eta_{\mu\nu} (e^\rho_a \partial_\rho n^\mu) e^\nu_b + \eta_{\mu\nu} e^\mu_a (e^\sigma_b \partial_\sigma n^\nu) \right) \\ &= \epsilon \left(e^\rho_a e^\nu_b \eta_{\mu\nu} \partial_\rho n^\mu + e^\mu_a e^\sigma_b \eta_{\mu\nu} \partial_\sigma n^\nu \right) \end{aligned}$$

Using $\eta_{\mu\nu} \partial_\rho n^\mu = \partial_\rho n_\nu$:

$$\delta h_{ab} = \epsilon \left(e^\rho_a e^\nu_b \partial_\rho n_\nu + e^\mu_a e^\sigma_b \partial_\sigma n_\mu \right) \tag{24}$$

$$K_{ab} \equiv e^\rho_a e^\nu_b \partial_\rho n_\nu \tag{25}$$

Therefore we have,

$$\delta h_{ab} = 2\epsilon K_{ab} \tag{26}$$

Substituting this into the expression for $\delta(\sqrt{h})$:

$$\begin{aligned}\delta(\sqrt{h}) &= \frac{1}{2}\sqrt{h}h^{ab}(2\epsilon K_{ab}) \\ &= \sqrt{h}\epsilon h^{ab}K_{ab} \\ &= \sqrt{h}\epsilon\,\mathrm{tr}(K)\end{aligned}$$

where $K = \mathrm{tr}(K) = h^{ab}K_{ab}$ is the mean curvature of ∂W

Finally, substituting this back into δS_b:

$$\delta S_b = \int_{\Sigma} d^{d-1}y\,\sqrt{h}\,\tau\,K\,\epsilon \tag{27}$$

$$\delta S = \int_{\partial M} d^{d-1}x\sqrt{h}\left(\tilde{T}^{\mu\nu}n_{\mu}n_{\nu} + K\right)\epsilon \tag{28}$$

$$\delta S = 0 \Longrightarrow \tau\,h^{\alpha\beta}K_{\alpha\beta} = \widetilde{T}_{nn}$$

Boundary Conditions:

- Dirichlet: $\varphi|_{\partial M} = 0$
- Neumann: $\partial_{\mu}\varphi\,n^{\mu} = 0$

Dirichlet boundary condition:

$$\tilde{T}_{nn} = 0 \Longrightarrow K = 0$$

Zero curvature means the boundary is a straight line.

3.1 Questions

- If the boundary solution $K = 0$ has less symmetry than the action, then there is spontaneous symmetry breaking at the boundary.
- Can we find a Goldstone boson?
- Can we use Neumann boundary conditions? We will have to consider the subtlety of a scalar field not being primary, as the boundary condition must also satisfy the primary transformation definition? For the Dirichlet B.C I don't think it matters since the field is zero anyways.

4 Weiss Variation of AdS/BCFT

Consider the AdS/BCFT framework as proposed by Takayanagi, where the $(d+1)$-dimensional bulk AdS space N is bounded both by the asymptotic AdS boundary W (where the BCFT lives), and by a co-dimension hypersurface Q known as the end-of-the-world brane. The full gravitational action is given by :

$$S = S_{\mathrm{bulk}} + S_Q + S_{\mathrm{GHY}} \tag{29}$$

$$S = \frac{1}{16\pi G_N}\int_N \mathrm{d}^{d+1}x\,\sqrt{\gamma}(R - 2\Lambda) + \frac{1}{8\pi G_N}\int_Q \mathrm{d}^d y\sqrt{h}(K - T) + \frac{1}{8\pi G_N}\int_M \mathrm{d}^d z\sqrt{g}K \tag{30}$$

Where constant T is interpreted as the tension of the boundary Q. (Aspects of Ads/BCFT 2.11). Without the addition of the Brane we have the following from Feng's paper for the Weiss variation of the gravitational action:

$$\delta S_{\mathrm{GR}} = \frac{1}{2\kappa}\int_N ...+\frac{\varepsilon}{2\kappa}\int_M (K_{\mu\nu}-Kg_{\mu\nu})\,\delta g^{\mu\nu}\,\sqrt{g}\,d^3y+\frac{\varepsilon}{2\kappa}\int_M \delta x^\mu\,n_\mu\left({}^3R+\varepsilon(K^2-K_{\alpha\beta}K^{\alpha\beta})\right)\sqrt{g}\,d^3y \tag{31}$$

The second term is the Brown-York tensor dual to the CFT energy-momentum tensor.

Would have there been any terms with $\int_{\partial W}...$ that the authors got rid of that we would have to keep?

4.1 Rederive the weiss variation of AdS/BCFT

$$S = S_{\mathrm{bulk}} + S_Q + S_{\mathrm{GHY}} \tag{32}$$

$$\Rightarrow \quad \frac{1}{2\kappa}\int_{\mathcal{N}} d^{d+1}x\,\sqrt{|\gamma|}\,(R-2\Lambda)+\frac{1}{\kappa}\int_Q d^dy\,\sqrt{|h|}\,(K-T)+\frac{1}{\kappa}\int_M d^{d-1}z\,\sqrt{|g|}\,K$$

Separate the Einstein-Hilbert action into 2 parts:

$$\frac{1}{2\kappa}\int_{\mathcal{N}} d^{d+1}x(R-2\Lambda) = \frac{1}{2\kappa}\int_{\mathcal{M}} d^{d+1}x\,\sqrt{|\gamma|}\,R - \frac{1}{2\kappa}\int_{\mathcal{M}} d^{d+1}x\,\sqrt{|\gamma|}\,2\Lambda$$

The variation of the first part, follows Feng's paper, including both Q and M boundary

$$\begin{aligned}\delta\left(\frac{1}{2\kappa}\int_N d^{d+1}x\,\sqrt{|\gamma|}R\right) &= \frac{1}{2\kappa}\int_N G_{\mu\nu}\delta\gamma^{\mu\nu}\sqrt{|\gamma|}d^{d+1}x+\frac{\varepsilon}{2\kappa}\int_Q(\delta h^{\alpha\beta}k_{\alpha\beta}-2\delta K+R\delta x^\mu n_\mu)\sqrt{|h|}d^dy\\ &+\frac{\varepsilon}{2\kappa}\int_M(\delta g^{\alpha\beta}k_{\alpha\beta}-2\delta K+R\delta x^\mu n_\mu)\sqrt{|g|}d^dz\end{aligned}$$

The Variation of the second part

$$\delta\left(\frac{1}{2\kappa}\int_N d^{d+1}x\,\sqrt{|\gamma|}2\Lambda\right) = \frac{1}{2\kappa}\int_N \delta\sqrt{|g|}2\Lambda\,d^{d+1}x+\frac{1}{2\kappa}\int_Q 2\Lambda\delta x^\mu d^d\bar{\Sigma}_\mu+\frac{1}{2\kappa}\int_M 2\Lambda\delta x^\mu d^d\bar{\Sigma}_\mu$$

where the covariant directed surface element $d^d\bar{\Sigma}_\mu$ is given by the following expression:

$$d^d\bar{\Sigma}_\mu = \varepsilon n_\mu\sqrt{|h|}d^dy$$

According to

$$\delta\sqrt{|\gamma|} = -\frac{1}{2}\sqrt{|\gamma|}\gamma_{\mu\nu}\delta\gamma^{\mu\nu}$$

The variation of the second part is:

$$\begin{aligned}\delta\left(\frac{1}{2\kappa}\int_{\mathcal{N}} d^{d+1}x\,\sqrt{|\gamma|}2\Lambda\right) &= -\frac{1}{2\kappa}\int_{\mathcal{N}} d^{d+1}x\,\sqrt{|\gamma|}\gamma^{\mu\nu}\delta\gamma^{\mu\nu}\Lambda\\ &+\frac{\varepsilon}{2\kappa}\int_Q 2\Lambda\delta x^\mu n_\mu\sqrt{|h|}\,d^dy\\ &+\frac{\varepsilon}{2\kappa}\int_M 2\Lambda\delta x^\mu n_\mu\sqrt{|g|}\,d^dz\end{aligned}$$

The variation of S_Q term:

$$\delta\left(\frac{1}{\kappa}\int_Q d^dy\,\sqrt{|h|}\,(K-T)\right) = \frac{\varepsilon}{2\kappa}\int_Q \left(2\delta K-(K-T)h_{\mu\nu}\delta h^{\mu\nu}\right)\sqrt{|h|}\,d^dy$$

$$+\ \delta_{\delta x}\left(\frac{1}{\kappa}\int_Q d^dy\,\sqrt{|h|}\,K-\frac{1}{\kappa}\int_Q d^dy\,\sqrt{|h|}\,T\right)$$

$$=\frac{\varepsilon}{2\kappa}\int_Q \left(2\delta K-(K-T)h_{\mu\nu}\delta h^{\mu\nu}\right)\sqrt{|h|}\,d^dy$$

$$+\ \frac{\varepsilon}{2\kappa}\int_Q \delta x^\mu n_\mu\left[{}^dR+\varepsilon\left(K^2-K_{\alpha\beta}K^{\alpha\beta}\right)-R\right]\sqrt{|h|}\,d^dy$$

$$+\,\varepsilon\int_{\partial Q} K\,\lambda^\mu_\alpha\delta X^\alpha\,\sqrt{|\lambda|}\,\bar{r}_i d^{d-1}y$$

$$-\ \delta_{\delta x}\left(\frac{1}{\kappa}\int_Q d^dy\,\sqrt{|h|}\,T\right)$$

Where $\bar{r}_i$ is the normal for ∂Q

The variation for the S_M term,

$$\delta\left(\frac{1}{\kappa}\int_{\mathcal{M}} d^dz\,\sqrt{|g|}\,K\right)=\frac{1}{\kappa}\int_{\mathcal{M}} d^dz\left[\delta(\sqrt{|g|})\,K+\delta K\,\sqrt{|g|}\right]+\delta_{\delta x}\left(\frac{1}{\kappa}\int_{\mathcal{M}} d^dz\,\sqrt{|g|}\,k\right)$$

$$=\frac{\varepsilon}{2\kappa}\int_{\mathcal{M}} \left(2\delta K-K\,g_{\mu\nu}\delta g^{\mu\nu}\right)\sqrt{|g|}\,d^dz$$

$$+\ \frac{\varepsilon}{2\kappa}\int_{\mathcal{M}} \delta x^\mu n_\mu\left[{}^dR+\varepsilon\left(K^2-K_{\alpha\beta}K^{\alpha\beta}\right)-R\right]\sqrt{|g|}\,d^dz$$

$$+\,\varepsilon\int_{\partial\mathcal{M}} K\,\sigma^\mu_\alpha\,\delta X^\alpha\,r_i\,\sqrt{|\sigma|}\,d^{d-1}z$$

Where r_i is normal to ∂M

Combine everything together:

$$= \frac{1}{2\kappa}\int_N G_{\mu\nu}\delta g^{\mu\nu}\sqrt{|\gamma|}d^{d+1}x + \frac{\varepsilon}{2\kappa}\int_Q (\delta h^{\alpha\beta}K_{\alpha\beta} - 2\delta K + R\delta x^\mu n_\mu)\sqrt{|h|}d^d y$$
$$+ \frac{\varepsilon}{2\kappa}\int_M (\delta g^{\alpha\beta}k_{\alpha\beta} - 2\delta K + R\delta x^\mu n_\mu)\sqrt{|g|}d^d z$$
$$- \frac{1}{2\kappa}\int_N d^{d+1}x\,\sqrt{|\gamma|}\gamma^{\mu\nu}\delta\gamma^{\mu\nu}\Lambda$$
$$+ \frac{\varepsilon}{2\kappa}\int_Q 2\Lambda\delta x^\mu n_\mu \sqrt{|h|}\,d^d y$$
$$+ \frac{\varepsilon}{2\kappa}\int_M 2\Lambda\delta x^\mu n_\mu \sqrt{|g|}\,d^d z$$
$$+ \frac{\varepsilon}{2\kappa}\int_Q (2\delta K - (K-T)h_{\mu\nu}\delta h^{\mu\nu})\,\sqrt{|h|}\,d^d y$$

$$+ \frac{\varepsilon}{2\kappa}\int_Q \delta x^\mu n_\mu \left[{}^d R + \varepsilon\left(K^2 - K_{\alpha\beta}K^{\alpha\beta}\right) - R\right]\sqrt{|h|}\,d^d y$$

$$+ \varepsilon\int_{\partial Q} K\,\lambda^\mu_\alpha \delta x^\alpha\,\sqrt{|\lambda|}\,\bar{r}_i d^{d-1}y$$
$$- \delta_{\delta x}\left(\frac{1}{\kappa}\int_Q d^d y\,\sqrt{|h|}\,T\right)$$
$$+ \frac{\varepsilon}{2\kappa}\int_{\mathcal{M}} (2\delta K - K\,g_{\mu\nu}\delta g^{\mu\nu})\,\sqrt{|g|}\,d^d z$$

$$+ \frac{\varepsilon}{2\kappa}\int_{\mathcal{M}} \delta x^\mu n_\mu \left[{}^d R + \varepsilon\left(K^2 - K_{\alpha\beta}K^{\alpha\beta}\right) - R\right]\sqrt{|g|}\,d^d z$$

$$+ \varepsilon\int_{\partial\mathcal{M}} K\,\sigma^\mu_\alpha\,\delta x^\alpha\,r_i\,\sqrt{|\sigma|}\,d^{d-1}z$$

Grouping terms:

$$=> \frac{1}{2\kappa}\int_N (G_{\mu\nu} - \Lambda\gamma^{\mu\nu})\delta\gamma^{\mu\nu}\sqrt{|\gamma|}d^{d+1}x$$
$$+ \frac{\varepsilon}{2\kappa}\int_Q (\delta h^{\alpha\beta}K_{\alpha\beta} + (R + 2\Lambda + ({}^d R + \varepsilon(K^2 - K_{\alpha\beta}K^{\alpha\beta}) - R)\delta x^\mu n_\mu - (K-T)h_{\mu\nu}\delta h^{\mu\nu})\sqrt{|h|}d^d y$$
$$+ \frac{\varepsilon}{2\kappa}\int_M (\delta g^{\alpha\beta}k_{\alpha\beta} + (R + 2\Lambda + ({}^d R + \varepsilon(K^2 - K_{\alpha\beta}K^{\alpha\beta}) - R)\delta x^\mu n_\mu - K g_{\mu\nu}\delta g^{\mu\nu})\sqrt{|g|}d^d z$$
$$+ \varepsilon\int_{\partial Q} K\,\lambda^\mu_\alpha \delta x^\alpha\,\bar{r}_i,\sqrt{|\lambda|}\;d^{d-1}z$$
$$+ \varepsilon\int_{\partial\mathcal{M}} K\,\sigma^\mu_\alpha\,\delta x^\alpha\,r_i\,\sqrt{|\sigma|}\,d^{d-1}z$$
$$- \delta_{\delta x}\left(\frac{1}{\kappa}\int_Q d^d y\,\sqrt{|h|}\,T\right)$$

Focusing on the Corner Terms

$$S_{corner} = \varepsilon\int_{\partial Q} K\,\lambda^\mu_\alpha \delta x^\alpha\,\bar{r}_i,\sqrt{|\lambda|}\;d^{d-1}z + \varepsilon\int_{\partial\mathcal{M}} K\,\sigma^\mu_\alpha\,\delta x^\alpha\,r_i\,\sqrt{|\sigma|}\,d^{d-1}z - \delta_{\delta x}\left(\frac{1}{\kappa}\int_Q d^d y\,\sqrt{|h|}\,T\right) \tag{33}$$

4.2 Question

- What is $\delta_{\delta x}(\frac{1}{\kappa}\int_Q d^d y\,\sqrt{|h|}\,T)$?

4.3 Weiss Variation of the Brane

To find the Weiss variation of the brane, we need to find the Weiss variation of the brane GHY term.

$$\Delta S_{brane} = \int_Q d^d x \delta\left(\sqrt{-h}(K-T)\right) + \int_{\partial Q} d^{d-1}x\sqrt{-h}(K-T)\delta x^\mu \tag{34}$$

Again, the boundary integral comes from the diffeomorphism of the coordinates. Now we must find the variation due to the field or metric.As such we need:

$$\delta(\sqrt{-h}K) = (\delta\sqrt{-h})K + \sqrt{-h}(\delta K) \tag{35}$$

$$\delta\sqrt{-h} = \frac{1}{2}\sqrt{-h}\,h^{\mu\nu}\delta h_{\mu\nu} \tag{36}$$

We know that $K = \nabla_\mu n^\mu$,

$$\delta K = \delta(\nabla_\mu n^\mu) = \delta(g^{\mu\nu}\nabla_\mu n_\nu) = (\delta g^{\mu\nu})\nabla_\mu n_\nu + g^{\mu\nu}\delta(\nabla_\mu n_\nu) \tag{37}$$

and,

$$\delta(\nabla_\mu n_\nu) = \nabla_\mu(\delta n_\nu) - \delta\Gamma^\lambda_{\mu\nu}n_\lambda \tag{38}$$

We can split up the variation of δn_μ into components perpendicular and parallel to the normal vector.

$$\delta n_\mu = \frac{1}{2}n_\mu n^\lambda n^\nu \delta g_{\nu\lambda} = \frac{1}{2}\delta g_{\mu\nu}n^\nu + c_\mu,$$

where c_μ is orthogonal to n^μ, and defined as:

$$c_\mu = -\frac{1}{2}h^\lambda_\mu \delta g_{\nu\lambda}n^\nu.$$

Using the variation of the connection

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}(\nabla_\mu\delta g_{\nu\lambda} + \nabla_\nu\delta g_{\mu\lambda} - \nabla_\lambda\delta g_{\mu\nu}) \tag{39}$$

We now have:

$$\delta K = -\frac{1}{2}K^{\mu\nu}\delta g_{\mu\nu} - \frac{1}{2}n^\mu\left(\nabla^\nu\delta g_{\mu\nu} - g^{\nu\lambda}\nabla_\mu\delta g_{\nu\lambda}\right) + \nabla^\gamma_\mu c^\mu. \tag{40}$$

∇^γ_μ is the covariant derivative compatible with the induced boundary metric $h_{\mu\nu}$. The others are bulk covariant derivatives. Finally, the variation of the Gibbons–Hawking–York boundary term is:

$$\begin{aligned}\delta S_{\text{brane}} = \frac{1}{2\kappa}\int_Q d^d x\sqrt{-h}\bigg(\frac{1}{2}h^{\mu\nu}\delta g_{\mu\nu}(K+T) + \delta K - \nabla^\gamma_\mu c^\mu\bigg)\\ - \frac{1}{2\kappa}\int_{\partial Q} d^{d-1}x\sqrt{-\gamma}\,(c^\mu + (K-T)\delta x^\mu)\,n_\mu\end{aligned} \tag{41}$$

Where we moved the total derivative into the boundary integral. Combining with the EH action

$$
\begin{aligned}
\delta S = &\frac{1}{2\kappa}\int_N \sqrt{g}\cdots \\
&+\frac{1}{2\kappa}\int_Q d^dx\sqrt{h}\,(K_{\mu\nu}-Kh_{\mu\nu}+Th_{\mu\nu})\,\delta h^{\mu\nu}\,d^dy \\
&+\frac{1}{2\kappa}\int_Q d^dx\sqrt{h}\,\delta x^\mu\,n_\mu\left({}^3R+(K^2-K_{\alpha\beta}K^{\alpha\beta})\right) \\
&+\frac{1}{2\kappa}\int_{\partial Q} d^{d-1}x\sqrt{-\gamma}n_\mu\left(\frac{1}{2}h^\lambda_\mu\delta g_{\nu\lambda}n^\nu+(K-T)\delta x^\mu\right)
\end{aligned}
\tag{42}
$$

4.4 Weiss Variation of AdS

We can now start with the full action

$$
S = S_{\text{bulk}} + S_Q + S_{\text{GHY}} \tag{43}
$$

Focusing on the corner terms we get:

$$
\delta S = \frac{1}{\kappa}\int_{\partial Q} d^{d-1}x\sqrt{-\gamma}\left(\frac{1}{2}h^\lambda_\mu\delta g_{\nu\lambda}n^\nu+(K-T)\delta x^\mu\right)+\frac{1}{\kappa}\int_{\partial M} d^{d-1}x\sqrt{-\gamma}\left(\frac{1}{2}\sigma^\lambda_\mu\delta g_{\nu\lambda}r^\nu+K\delta x^\mu\right) \tag{44}
$$

Where σ is the induced metric on asymptotic AdS, M, and r is the unit normal to M.

Quoting from corner integral from Feng's paper " Weiss Variation of Gravitational Action" equation 4.30, we should get:

$$
\delta S = \frac{1}{\kappa}\int_{\partial M} d^{d-1}x\sqrt{-\gamma}K_M\sigma^\mu_\alpha\delta x^\alpha r_\mu+\frac{1}{\kappa}\int_{\partial Q} d^{d-1}x\sqrt{-\gamma}(K_Q-T)h^\mu_\alpha\delta x'^\alpha n_\mu \tag{45}
$$

Where r_μ, n_μ are the unit normal vectors for the surfaces M and Q, respectively?

4.5 Questions

- Why is there an extra term that Feng does not have?
- Add something on the Brane or boundary so Stress energy tensor on BCFT does reflect perfectly.

$$\int_{\partial M} d^{d-1}x\sqrt{h}\left(-\xi\left(\partial_\mu\varphi^2 n_\nu+\partial_\nu\varphi^2 n_\mu\right)\left(\partial^\mu\epsilon^\nu+\partial^\nu\epsilon^\mu\right)+\left(\frac{\partial\mathcal{L}_s}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi-\mathcal{L}_s\eta^{\mu\nu}\right)\epsilon_\mu n_\nu\right) \tag{46}$$

$$\left(\partial_\mu\varphi^2 n_\nu+\partial_\nu\varphi^2 n_\mu\right)\left(\partial^\mu\epsilon^\nu+\partial^\nu\epsilon^\mu\right)=\partial_\mu\varphi^2 n_\nu\partial^\mu\epsilon^\nu+\partial_\nu\varphi^2 n_\mu\partial^\mu\epsilon^\nu+\partial_\mu\varphi^2 n_\nu\partial^\nu\epsilon^\mu+\partial_\nu\varphi^2 n_\mu\partial^\nu\epsilon^\mu \tag{47}$$

$$\partial_\mu\varphi^2 n_\nu\partial^\mu\epsilon^\nu=\partial^\mu(\partial_\mu\varphi^2 n_\nu\epsilon^\nu)-\partial_\mu\partial^\mu\varphi^2 n_\nu\epsilon^\nu \tag{48}$$

Where the first term vanishes, being a total derivative

$$\left(\partial_\mu\varphi^2 n_\nu+\partial_\nu\varphi^2 n_\mu\right)\left(\partial^\mu\epsilon^\nu+\partial^\nu\epsilon^\mu\right)=-(\partial_\mu\partial^\mu\varphi^2 n_\nu\epsilon^\nu+\partial^\mu\partial_\nu\varphi^2 n_\mu\epsilon^\nu+\partial^\nu\partial_\mu\varphi^2 n_\nu\epsilon^\mu+\partial_\nu\partial^\nu\varphi^2 n_\mu\epsilon^\mu) \tag{49}$$

If we take the deformation ϵ to be normal to to the boundary then we have.

$$\left(\partial_\mu\varphi^2 n_\nu+\partial_\nu\varphi^2 n_\mu\right)\left(\partial^\mu\epsilon^\nu+\partial^\nu\epsilon^\mu\right)=-2(\partial_\mu\partial^\mu\varphi^2 n_\nu\epsilon^\nu+\partial^\mu\partial_\nu\varphi^2 n_\mu\epsilon^\nu)=-2(\partial_\mu\partial^\mu\varphi^2\eta_{\mu\nu}+\partial^\nu\partial^\mu\varphi^2)n_\nu\epsilon_\mu \tag{50}$$

The boundary integral becomes

$$\int_{\partial M} d^{d-1}x\sqrt{h}\left(T^{\mu\nu}-2(\partial_\mu\partial^\mu\varphi^2\eta_{\mu\nu}+\partial^\nu\partial^\mu\varphi^2)\right)\epsilon_\mu n_\nu \tag{51}$$

$$V(x,\dot{x},\theta,\dot{\theta})=\frac{1}{2}k_1\,\dot{x}^2\ +\ \frac{1}{2}k_2 l^2\dot{\theta}^2\ +\ k_3 gl\big(1-\cos\theta\big)\ +\ \frac{1}{2}kx^2.$$