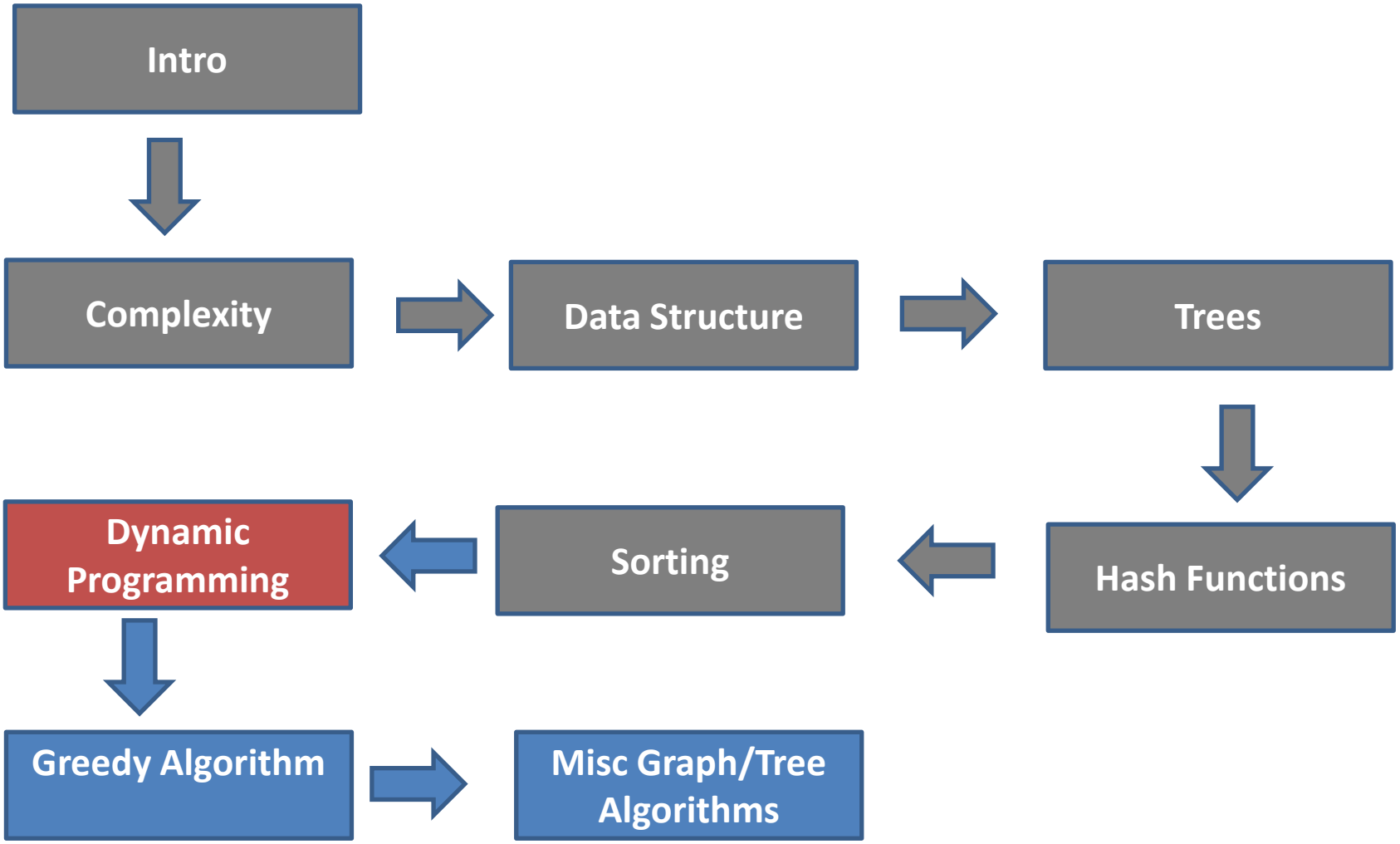
An Introduction to Algorithms

By

Hossein Rahmani

h_rahmani@iust.ac.ir

http://webpages.iust.ac.ir/h_rahmani/


Intro
Complexity
Data Structure
Trees
Hash Functions
Sorting
Dynamic Programming
Greedy Algorithm
Misc Graph/Tree Algorithms

Languages

Languages

A language is a set of strings

String: A sequence of letters

Examples: “cat”, “dog”, “house”, …

Defined over an alphabet:

$$\Sigma = \{a, b, c, \ldots, z\}$$

Alphabets and Strings

We will use small alphabets: $\Sigma = \{a, b\}$

Strings a

ab $\quad u = ab$

$abba$ $\quad v = bbbaaa$

$baba$ $\quad w = abba$

$aaabbbaabab$

Grammars

Grammars

Grammars express languages

Example: the English language

$$\langle sentence \rangle \rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle$$

$$\langle noun\_phrase \rangle \rightarrow \langle article \rangle \ \langle noun \rangle$$

$$\langle article \rangle \rightarrow a$$

$$\langle article \rangle \rightarrow the$$

$$\langle noun \rangle \rightarrow boy$$

$$\langle noun \rangle \rightarrow dog$$

$$\langle verb \rangle \rightarrow runs$$

$$\langle verb \rangle \rightarrow walks$$

A derivation of "the boy walks":

$$\langle sentence \rangle \Rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle$$
$$\Rightarrow \langle noun\_phrase \rangle \ \langle verb \rangle$$
$$\Rightarrow \langle article \rangle \ \langle noun \rangle \ \langle verb \rangle$$
$$\Rightarrow the \ \langle noun \rangle \ \langle verb \rangle$$
$$\Rightarrow the \ boy \ \langle verb \rangle$$
$$\Rightarrow the \ boy \ walks$$

A derivation of "a dog runs":

$$\langle sentence \rangle \Rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle$$
$$\Rightarrow \langle noun\_phrase \rangle \ \langle verb \rangle$$
$$\Rightarrow \langle article \rangle \ \langle noun \rangle \ \langle verb \rangle$$
$$\Rightarrow a \ \langle noun \rangle \ \langle verb \rangle$$
$$\Rightarrow a \ dog \ \langle verb \rangle$$
$$\Rightarrow a \ dog \ runs$$

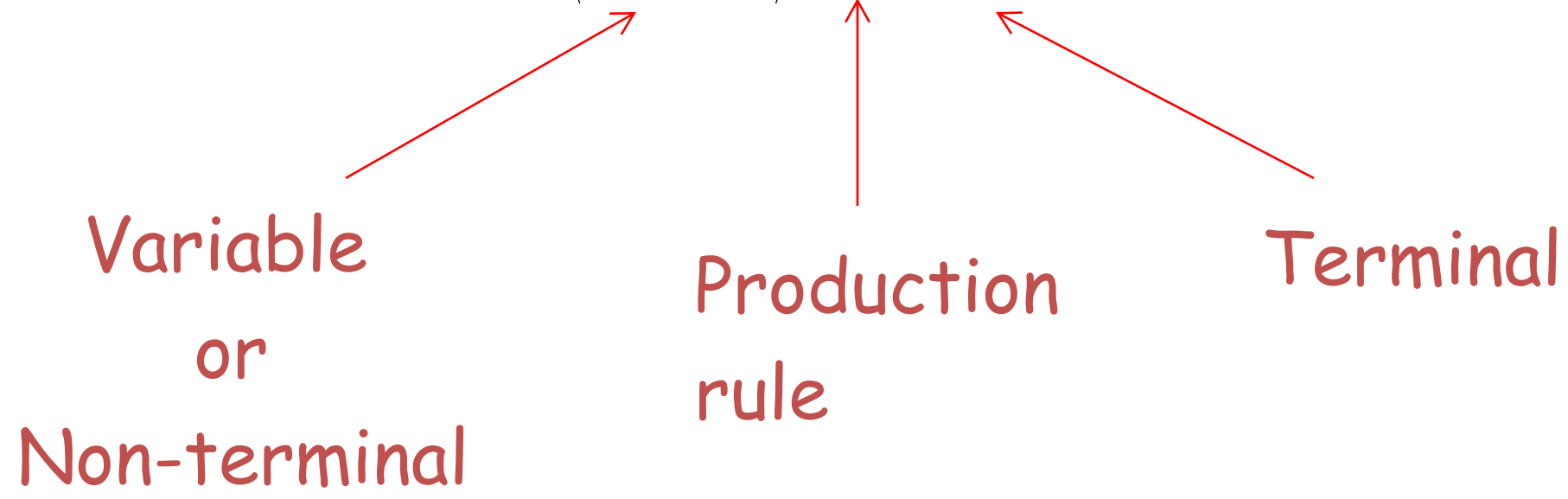
Language of the grammar:

L = { "a boy runs",
"a boy walks",
"the boy runs",
"the boy walks",
"a dog runs",
"a dog walks",
"the dog runs",
"the dog walks" }

Notation

$$\langle noun \rangle \rightarrow boy$$

$$\langle noun \rangle \rightarrow dog$$



Another Example

$$S \to aSb$$

Grammar: $S \to \lambda$

Derivation of sentence ab:

$$S \Rightarrow aSb \Rightarrow ab$$

$S \to aSb$ $\qquad$ $S \to \lambda$

Grammar: $S \to aSb$

$S \to \lambda$

Derivation of sentence $aabb$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

$S \to aSb$ $\quad$ $S \to \lambda$

Other derivations:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb$$
$$\Rightarrow aaaaSbbbb \Rightarrow aaaabbbb$$

Language of the grammar

$$S \to aSb$$

$$S \to \lambda$$

$$L = \{a^n b^n : n \geq 0\}$$

Example

For grammar G:

$$S \to Ab$$

$$A \to aAb$$

$$A \to \lambda$$

$$L(G) = \{???\}$$

Example

For grammar G:

$$S \to Ab$$

$$A \to aAb$$

$$A \to \lambda$$

$$L(G) = \{a^n b^n b : \quad n \geq 0\}$$

Since: $S \overset{*}{\Rightarrow} a^n b^n b$

A grammar

$$G: \quad S \to aSb$$
$$S \to SS$$
$$S \to \lambda$$

A derivation:

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow abaSb \Rightarrow abab$$

$$S \to aSb$$

$$S \to SS$$

$$S \to \lambda$$

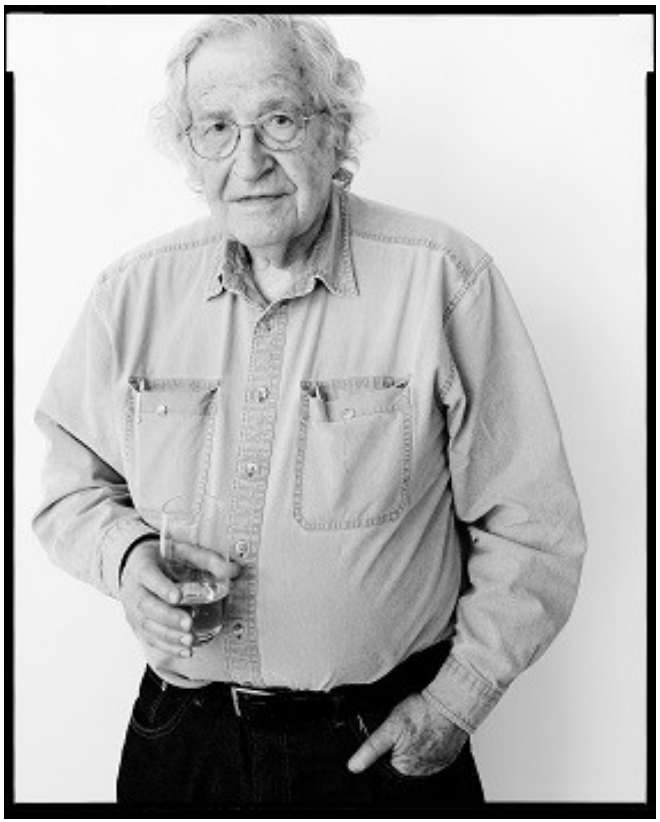
$$L(G) = \{w \ : n_a(w) = n_b(w),$$

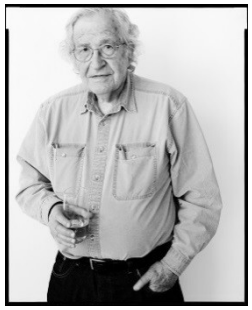
$$\text{and } n_a(v) \geq n_b(v)$$

$$\text{in any prefix } v\}$$

() ((( ))) (( ))

Chomsky Normal Form





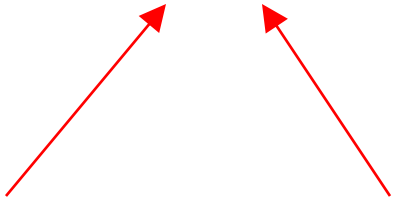
Avram Noam Chomsky (/ˈnoʊm ˈtʃɒmski/; born December 7, 1928) is an American linguist, philosopher, cognitive scientist, historian, logician, social critic, and political activist. Sometimes described as "the father of modern linguistics," Chomsky is also a major figure in analytic philosophy, and one of the founders of the field of cognitive science. He has spent more than half a century at the Massachusetts Institute of Technology (MIT), where he is Institute Professor Emeritus, and is the author of over 100 books on topics such as linguistics, war, politics, and mass media. Ideologically, he aligns with anarcho-syndicalism and libertarian socialism.

Noam Chomsky: The United States, Not Iran, Poses Greatest Threat to World Peace

Chomsky Normal Form

All productions have form:

$A \to BC$ and $A \to a$



variable variable terminal

Examples:

$$S \to AS$$
$$S \to a$$
$$A \to SA$$
$$A \to b$$

Chomsky Normal Form

$$S \to AS$$
$$S \to AAS$$
$$A \to SA$$
$$A \to aa$$

Not Chomsky Normal Form

The CYK Membership Algorithm

(J. Cocke, D. H. Younger, and T. Kasami)

Input:

- Grammar G in Chomsky Normal Form
- String w

Output:

find if $w \in L(G)$

Construction: Dynamic Programming

$$w = a_1 a_2 \dots \ a_n$$

$$w[i,j] = a_i a_{i+1} \dots \ a_j$$

$$V[i,j] = \{A \in V : A \Rightarrow_G^* w[i,j]\}$$

$$A \Rightarrow_G^* w[i,j]$$

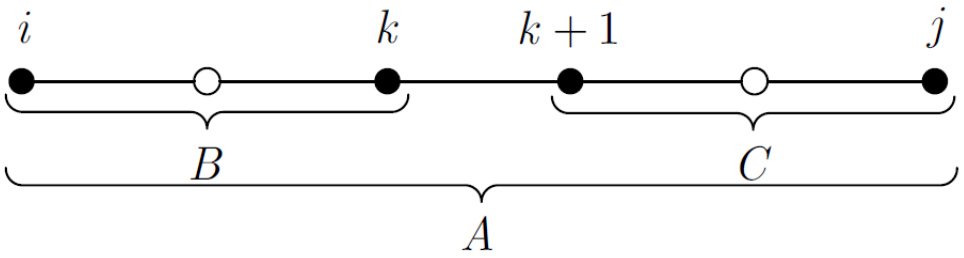
$A \to BC$ $\qquad B \Rightarrow^* w[i,k] \qquad C \Rightarrow^* w[k+1,j]$

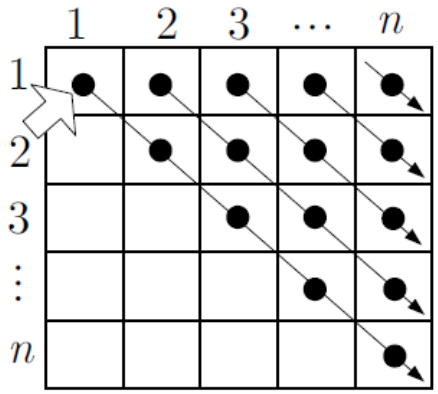
$$i \leq k < j$$

Construction: Dynamic Programming

$$V[i,i] = \{A \in V : A \to a_i\}$$

$$V[i,j] = \bigcup_{i \le k < j} \{A : A \to BC, B \in V[i,k] \wedge C \in V[k+1,j]\}$$





$$w \in L(G) \Leftrightarrow S \in V[1,n]$$

The Algorithm

Input example:

- Grammar G:

$$S \to AB$$

$$A \to BB$$

$$A \to a$$

$$B \to AB$$

$$B \to b$$

- String w : $aabbb$

aabbb

$$V[i,i] = \{A \in V : A \to a_i\}$$

a	a	b	b	b
aa	ab	bb	bb	
aab	abb	bbb		
aabb	abbb			
aabbb				

$S \to AB$

$A \to BB$

$A \to a$

$B \to AB$

$B \to b$

$$V[i,i] = \{A \in V : A \to a_i\}$$

a	a	b	b	b
A	A	B	B	B

aa ab bb bb

aab abb bbb

aabb abbb

aabbb

$S \rightarrow AB$

$A \rightarrow BB$

$A \rightarrow a$

$B \rightarrow AB$

$B \rightarrow b$

$$V[i,j] = \bigcup_{i \leq k < j} \{A : A \rightarrow BC, B \in V[i,k] \wedge C \in V[k+1,j]\}$$

a	a	b	b	b
A	A	B	B	B
aa	ab	bb	bb	
	S,B	A	A	
aab	abb	bbb		
aabb	abbb			
aabbb				

$S \to AB$

$A \to BB$

$A \to a$

$B \to AB$

$B \to b$

a	a	b	b	b
A	A	B	B	B
aa	ab	bb	bb	
	S,B	A	A	
aab	abb	bbb		
S,B	A	S,B		
aabb	abbb			
A	S,B			
aabbb				
S,B				

Therefore: $aabbb \in L(G)$

Time Complexity: $|w|^3$

Observation: The CYK algorithm can be easily converted to a parser

Quiz 1

- Grammar G:
  - S → AB | BC
  - A → BA | a
  - B → CC | b
  - C → AB | a

Question Is **ababa** in L(G)? Use dynamic programming.

Quiz 2

Productions;

- $S \rightarrow AA|AS|b$
- $A \rightarrow SA|AS|a$

Input string; abaab(n=5)

- Is input string generated by Grammar G?