Graph theory project

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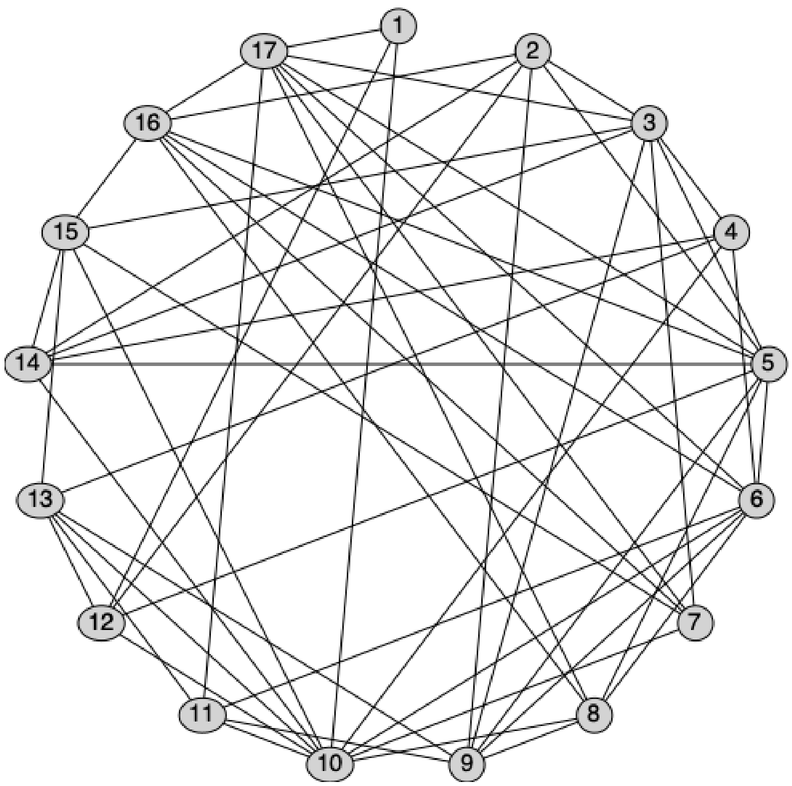


Contents

1 GB

In this section, I will discuss the numerical invariants and graph properties of the randomly generated graph on the first page, which I have named The Gay Bar because there is a gay bar in Southampton right next to my house called The Edge and every time I walked past it reminded me that I still hadn't chosen a name for my graph. I haven't called it the edge because that would make things extremely confusing. I will shorten it for GB.

All references to definitions, lemmas, etc. in this section are from the Lecture notes for MATH3033 unless stated otherwise.

1.1 Adjacency and its related invariants

GB has a vertex set $V(GB)$ and an edge set $E(GB)$, we start by considering its ***order***, which is, according to definition 1.1.0.6, the number of vertices in GB, $|V(GB)|$. Since the vertices are labelled 1 to 17 it is easy to see that $order(GB) = 17$. The ***size***, which is the number of edges $|E(GB)|$, can be found by considering the degree of each vertex, where the degree of a vertex is the number of neighbours it has, and a neighbour is an adjacent vertex. So, we write down for each vertex all the neighbours it has and its degree.

Vertex: v	Neighbours of v: $N(v)$	Degree of v: $deg(v)$
1	10,12,17	3
2	3,5,9,12,14,16	6
3	2,4,5,7,9,14,15,17	8
4	3, 6,10,13, 14,	5
5	2,3,6,8,9,12,14,16,17	9
6	4, 5, 8,9,10,11,16,17	8
7	3,10,15,16,17	5
8	5,6,9,10,16,17	6
9	2,3,5,6,8,11,13	7
10	1,4,6,7,8, 11,12,13,14,15	10
11	6,9,10, 13,17	5
12	1,2,5,10,13	5
13	4,9,10,11,12,15	6
14	2,3,4,5,10,15	6
15	,3,7,10,13,14, 16	6
16	2,5,6,7,8,15, 17	7
17	1,3,5,6,7,8,11,16	8

By the Handshaking lemma, the sum of all the degrees gives twice the number of edges in GB since each edge is adjacent to two vertices, so we are double-counting each edge. Hence, the size is the sum of all the degrees divided by two, $Size(GB) = 110/2 = 55$.

By definition 3.1.0.4 we see that the ***maximum degree*** of GB is $\Delta(GB) = 10$, since vertex 10 has degree 10 and all other vertices have degrees smaller than 10. Similarly, by definition 3.1.0.2, the ***minimum degree*** is $\delta(GB) = 3$, since vertex 1 has degree 3 and all other vertices have degrees greater than 3. According to definition 3.1.0.5, GB is not ***regular***, since 10=$\Delta(GB) \neq \delta(GB)$=3, so not all vertices have the same degree. By lemma 7.1.0.4 it cannot be ***transitive*** either, since GB is not regular an automorphism could not send, for example, vertex 1 to vertex 2 as their degrees are different. There is no ***dominating vertex***, since $\Delta(GB) = 10$, which is smaller than $Order(GB) - 1 = 17 - 1 = 16$, so no vertex is adjacent to all other vertices.

From definition 1.1.0.2 we know that GB is ***finite*** since the vertex set $V(GB)$ is finite, it has 17 elements, and the edge set E(GB) is finite, it has 55 elements. GB is ***simple***, it satisfies definition 1.1.0.3, every edge has two distinct end vertices and there isn't more than one edge between every pair of vertices in GB. Definition 2.1.0.8 states that a graph is ***connected*** if there is a walk between every pair of vertices. To find if GB is connected we try to find a walk from all other vertices to one vertex. If we can find them then there will be a walk from any vertex to any other vertex. We can guess that it will be easier to find walks from all other vertices to vertex 10 since vertex 10 has the maximum degree, and so it will be adjacent to more vertices. The following walks:

1·10
2·12·10
3·4·10
4·10
5·12·10
6·10
7·10
8·10
9·13·10
11·10
12·10
13·12·10
14·10
15·10
16·8·10
17·8·10

show that there is in fact a walk from all vertices in GB to 10 and so, from 10 to every vertex in GB. Hence we can find a walk from every vertex in GB to every other vertex in $V(GB)$, namely a walk going from any vertex in GB to 10 and from 10 to any other vertex in GB. So, GB is connected. Hence, GB is simple, connected and finite, i.e. GB is a ***SCaF*** graph, by Definition 2.1.0.11.

Since GB contains at least one cycle, the ***girth*** and ***circumference*** of GB are the lengths of the shortest and longest cycles respectively, according to definitions 4.1.0.1 and 4.1.0.2. The shortest cycle possible in any SCaF is a cycle of length 3, so, since GB is SCaF, if we can find a cycle in GB of length 3 then we know it will be of minimum length. We see that vertex 10 is adjacent to 1 and 12 and 1 is adjacent to 12 so we have a cycle of length 3 and $girth(GB) = 3$.

The same vertex cannot appear twice in a cycle, so, the circumference of GB will be at most the order of GB, 17, as given in page 49 of the notes. We now look for a cycle of length 17 starting with vertex 10, since it has maximum degree and it will be easier to close the walk. We find edges from one vertex to another one with lowest degree available. If there is more than one we choose the closest one. If there is still more than one we choose arbitrarily.

Edge	
10·1	since 1 is the vertex of lowest degree adjacent to 10
1·12	since 12 is the vertex of lowest degree adjacent to 1
12·13	since 13 is the closest available vertex of lowest degree adjacent to 12
13·11	since 11 is the closest available vertex of lowest degree adjacent to 13 (so is 15, but we choose 11 arbitrarily)
11·9	since 9 is the closest available vertex of lowest degree adjacent to 11
9·8	since 8 is the closest available vertex of lowest degree adjacent to 9
8·16	since 16 is the vertex of lowest degree available adjacent to 13
16·7	since 7 is the vertex of lowest degree adjacent to 16
7·15	since 15 is the vertex of lowest degree adjacent to 7
15·14	since 14 is the vertex of lowest degree available adjacent to 15
14·4	since 14 is the vertex of lowest degree adjacent to 15
4·3	since 2 is the closest available vertex of lowest degree adjacent to 3
3·2	since 3 is the closest available vertex of lowest degree adjacent to 5
2·5	since 5 is the only available vertex adjacent to 2

At this point we see there are only 2 vertices left, 17 and 6, both of which are adjacent to 5. We are looking for a walk from 5 to 10 going through 17 and 6. However 17 is not adjacent to 10, which is the vertex we started with and should end with. So, if we want to have a closed walk, we should have 5·17·6·10. Putting the edges together we get the cycle 10·1·12·13·11·9·8·16·7·15·14·4·3·2·5·17·6·10. Since we found a cycle of length 17 we know that the circumference of GB is circum(GB)=17.

The ***arithmetic mean*** of the degrees of GB is obtained by dividing the sum of the degrees of all vertices, 110, by the order of GB, 17, which gives arith(GB)=6.47. The ***geometric mean*** of the degrees of GB can be obtained by multiplying all the degrees and raising the product to the power of 1/order(GB)= 1/17. This gives, geom$(GB) = 32920473600000^{1/17} = 6.239$. Since the order of GB is odd, the ***median*** of GB is the middle value of the values of the degrees written in ascending order. The list $[3, 5, 5, 5, 5, 6, 6, 6, 6, 6, 7, 7, 8, 8, 8, 9, 10]$ has all the degrees of GB in ascending order, which gives that the median is median(GB)=6. The value of the degrees that appears most often is 6, so, the ***mode*** of GB is mode(GB)=6.

By definition 4.1.0.3 the ***kissing number*** is the number of cycles of length girth(GB)=3. It can be found by considering all edges and the common neighbours of its two adjacent vertices. Each common neighbour together with the other two vertices will be pairwise adjacent creating a cycle of length 3. So, we write down all the edges and the common neighbours of the adjacent vertices.

To avoid writing down the same edges or cycles more than once we write down only the edges of the form $a{\cdot}b$ where $a < b$ and the common neighbours of v and w when $N(v) \cap N(w) > v$ and $N(v) \cap N(w) > w$. So, we only write cycles of the form $a{\cdot}b{\cdot}c$ where $a < b < c$, for $a, b, c \in V(GB)$. This will avoid repetitions since the edge $a{\cdot}b$ is the same as the edge b·a and the cycles $a{\cdot}b{\cdot}c{\cdot}a$, $b{\cdot}c{\cdot}a{\cdot}b$, $c{\cdot}a{\cdot}b{\cdot}c$, $c{\cdot}b{\cdot}a{\cdot}c$, $b{\cdot}a{\cdot}c{\cdot}b$, $a{\cdot}c{\cdot}b{\cdot}a$ are all the same, these are just different ways of expressing the same cycle. By making $a < b < c$ we make the notation unique:

Edge $v{\cdot}w$	Common neighbours of v and w: $N(v) \cap N(w)$	Cycles of length 3
1·10	12	{1·10·12·1}
1·12	10	
1·17	empty	
2·3	5, 9, 14	{2·3·5·2}, {2·3·9·2}, {2·3·14·2}
2·5	3, 9, 12, 14, 16	{2·5·9·2}, {2·5·12·2}, {2·5·14·2}, {2·5·16}
2·9	3, 5	
2·12	5	
2·14	3, 5	
2·16	5	
3·4	14	{3·4·14·3}
3·5	2, 9, 14, 17	{3·5·9·3}, {3·5·14·3}, {3·5·17·3}
3·7	15, 17	{3·7·15·3}, {3·7·17·3}
3·9	2, 5	
3·14	5, 15	{3·14·15·3}
3·15	7, 14	
3·17	5, 7	
4·6	10	{4·6·10·4}
4·10	6, 13, 14	{4·10·13·4}, {4·10·14·4}
4·13	10	
4·14	3, 10	
5·6	8, 9,16, 17	{5·6·8·5}, {5·6·9·5}, {5·6·16·5}, {5·6·17·5}
5·8	6, 9, 16, 17	{5·8·9·5}, {5·8·16·5}, {5·8·17·5}
5·9	2, 3, 6, 8	
5·12	2	

5·14	2, 3	
5·16	2, 6, 8, 17	{5·16·17·5}
5·17	3, 6, 8, 16	
6·8	9, 10, 16, 17	{6·8·9·6}, {6·8·10·6}, {6·8·16·6}, {6·8·17·6}
6·9	5, 8, 11	{6·9·11·6}
6·10	4, 8, 11	{6·10·11·6}
6·11	9, 10, 17	{6·11·17·6}
6·16	5, 8, 17	{6·16·17·6}
6·17	5, 8, 11, 16	
7·10	15	{7·10·15·7}
7·15	3, 10, 16	{7·15·16·7}
7·16	15, 17	{7·16·17·7}
7·17	3, 16	
8·9	5, 6	
8·10	empty	
8·16	5, 17	{8·16·17·8}
8·17	5, 6, 16	
9·11	6, 13	{9·11·13·9}
9·13	11	
10·11	6, 13	{10·11·13·10}
10·12	1, 13	{10·12·13·10}
10·13	4, 11, 12, 15	{10·13·15·10}
10·14	4, 15	{10·14·15·10}
10·15	7, 13, 14	
11·13	9, 10,	
11·17	empty	
12·13	10	
13·15	10	
14·15	3, 10	
15·16	7	
16·17	5, 6, 7, 8	

Counting the number of 3·cycles we get 43, and since they are all distinct the kissing number is $kiss(GB) = 43$.

One way to find the ***cycles of length 4*** $\Sigma_4(GB)$ would be to consider for each pair of adjacent vertices v, w the intersection of the neighbours of the neighbours of v with the neighbours of w.

But since we cannot have a repeated vertex in a cycle and it might be that the neighbours of the neighbours of w contain w we actually want to consider the neighbours of the neighbours of w without w, we will denote this $N(N(w)) - \{w\} = N^2(w) - \{w\}$.

Since w is in the set by assumption we are not leaving out any cycles by deleting w from $N^2(w)$.

If the intersection of $N^2(w) - \{w\}$ with $N(w)$ is not empty then there is some cycle of length 4 containing the edge between v and w.

For $k \geq 5$ we cannot use the same line of argument to find $\Sigma_k(GB)$, as $N^p(v)$ can intersect non-trivially with $N^j(w)$ and also with $N^i(v)$ for some p, i, j so this can give paths with repeated vertices, so not necessarily cycles.

GB is not ***Eulerian*** in the sense of definition 9.1.0.1, since it contains vertices of odd degree, namely, vertex 1, so Lemma 9.1.0.2 tells us it cannot be Eulerian. Clearly, it would be impossible to have a walk going through all 3 edges adjacent to 1 without crossing them more than once or starting and finishing the walk at different vertices, which would give a non-closed walk.

GB doesn't contain exactly 2 vertices of odd degree, so it is not ***semi-Eulerian*** either by Lemma 9.1.0.4. Vertices 1, 4, 5, 7, 9, 11, 12 and 16 have odd degree, that is, 8 vertices have odd degree in GB, so the best we could (possibly) do is find 4 walks such that their disjoint union contained all edges in GB, in such a way that each odd degree vertex is the starting or finishing vertex, but not both, of one of the four walks.

GB satisfies definition 9.2.0.1, so GB is ***Hamiltonian***. We have already found a cycle that contains every vertex exactly once in GB when we considered its circumference, that is, 10·1·12·13·11·9·8·16·7·15·14·4·3·2·5·17·6·10.

The complement from definition 15.1.0.1 of GB is given by:

Vertex name	**Degree in $\overline{GB}$**	**Neighbours in $\overline{GB}$**
1	13	2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15, 16
2	10	1, 4, 6, 7, 8, 10, 11, 13, 15, 17
3	8	1, 6, 8, 10, 11, 12, 13, 16
4	11	1, 2, 5, 7, 8, 9, 11, 12, 15, 16, 17
5	7	1, 4, 7, 10, 11, 13, 15
6	8	1, 2, 3, 7, 12, 13, 14, 15
7	11	1, 2, 4, 5, 6, 8, 9, 11, 12, 13, 14
8	10	1, 2, 3, 4, 7, 11, 12, 13, 14, 15
9	9	1, 4, 7, 10, 12, 14, 15, 16, 17
10	6	2, 3, 5, 9, 16, 17
11	11	1, 2, 3, 4, 5, 7, 8, 12, 14, 15, 16
12	11	3, 4, 6, 7, 8, 9, 11, 14, 15, 16, 17
13	10	1, 2, 3, 5, 6, 7, 8,14, 16, 17
14	10	1, 6, 7, 8, 9, 11, 12, 13, 16, 17
15	10	1, 2, 4, 5, 6, 8, 9, 11, 12, 17
16	9	1, 3, 4, 9, 10, 11, 12, 13, 14,
17	8	2, 4, 9, 10, 12, 13, 14, 15,

From page 181 from the notes we have that $size(\overline{GB}) = \binom{order(GB)}{2} \cdot size(GB) = \binom{17}{2} \cdot 17 = 119$

1.2 Edge adjecency and its related invariants

We can extrapolate the definition of neighbour and degree of a vertex for edges. We can modify definition 3.1.0.1 and the concept of neighbour from the notes to get:

Definition 1. *The set of edge neighbors $N'(e)$ of e is the set of edges adjacent to e.*

Definition 2. *Let G be a finite simple graph. For each edge $e \in E(G)$ the edge degree of e $deg'(e)$ is the number of edges of G adjacent to e.*

We note that the edge neighbours of an edge $v{\cdot}w$ are the edges adjacent to its end vertices without $v{\cdot}w$ itself, and since every vertex is adjacent to the same number of vertices than edges we have that the edge degree of an edge is equal to the sum of the degrees of its end-vertices without including the end-vertices themselves. That is, $deg'(e) = N(v) + N(w) - 2$, where substracting 2 comes from the fact that $v \in N(w)$ and $w \in N(v)$ since they are both end vertices of e.

For finding the edge degrees it is useful to write down all the edges each vertex is adjacent to:

Vertex: v	Edges adjacent to v
1	1·10,1·12,1·17
2	2·3,2·5,2·9,2·12,2·14,2·14,2·16
3	3·2,3·4,3·5,3·7,3·9,3·14,3·15,3·17
4	4·3,4·6,4·10,4·13,4·14
5	5·2,5·3,5·6,5·8,5·9,5·12,5·14,5·16,5·17
6	6·4,6·5,6·8,6·9,6·10,6·11,6·16,6·17
7	7·3,7·10,7·15,7·16,7·17
8	8·5,8·6,8·9,8·10,8·16,8·17
9	9·2,9·3,9·5,9·6,9·8,9·11,9·13
10	10·1,10·4,10·6,10·7,10·8,10·11,10·12,10·13,10·14,10·15
11	11·6,11·9,11·10,11·13,11·17
12	12·1,12·2,12·5,12·10,12·13
13	13·4,13·9,13·10,13·11,13·12,13·15
14	14·2,14·3,14·4,14·5,14·10,14·15
15	15·3,15·7,15·10,15·13,15·14,15·16
16	16·2,16·5,16·6,16·7,16·8,16·15,16·17
17	17·1,17·3,17·5,17·6,17·7,17·8,17·11,17·16

We can now compute the edge degree of each edge $v{\cdot}w$ in GB and its edge neighbours by writing the edges adjacent to its end vertices v and w but without the edge $v{\cdot}w$ itself.

Edge: $v{\cdot}w$	Edge neighbours: $N'(v{\cdot}w)$	Edge degree: $deg'(v{\cdot}w)$
1·10	1·12,1·17,10·4,10·6,10·7,10·8,10·11,10·12,10·13,10·14,10·15	11
1·12	1·10,1·17,12·2,12·5,12·10,12·13	6
1·17	1·10,1·12,17·3,17·5,17·6,17·7,17·8,17·11,17·16	9
2·3	2·5,2·9,2·12,2·14,2·14,2·16,3·4,3·5,3·7,3·9,3·14,3·15,3·17	12
2·5	2·3,2·9,2·12,2·14,2·14,2·16,5·3,5·6,5·8,5·9,5·12,5·14,5·16,5·17	13
2·9	2·3,2·5,2·12,2·14,2·14,2·16,9·3,9·5,9·6,9·8,9·11,9·13	11
2·12	2·3,2·5,2·9,2·14,2·14,2·16,12·1,12·5,12·10,12·13	9
2·14	2·3,2·5,2·9,2·12,2·14,2·16,14·3,14·4,14·5,14·10,14·15	10
2·16	2·3,2·5,2·9,2·12,2·14,2·14,16·5,16·6,16·7,16·8,16·15,16·17	11
3·4	3·2,3·5,3·7,3·9,3·14,3·15,3·17,4·6,4·10,4·13,4·14	11
3·5	3·2,3·4,3·7,3·9,3·14,3·15,3·17,5·2,5·6,5·8,5·9,5·12,5·14,5·16,5·17	15
3·7	3·2,3·4,3·5,3·9,3·14,3·15,3·17,7·10,7·15,7·16,7·17	11
3·9	3·2,3·4,3·5,3·7,3·14,3·15,3·17,9·2,9·5,9·6,9·8,9·11,9·13	13
3·14	3·2,3·4,3·5,3·7,3·9,3·15,3·17,14·2,14·4,14·5,14·10,14·15	12
3·15	3·2,3·4,3·5,3·7,3·9,3·14,3·17,15·7,15·10,15·13,15·14,15·16	12
3·17	3·2,3·4,3·5,3·7,3·9,3·14,3·15,17·1,17·5,17·6,17·7,17·8,17·11,17·16	14
4·6	4·3,4·10,4·13,4·14,6·5,6·8,6·9,6·10,6·11,6·16,6·17	11
4·10	4·3,4·6,4·13,4·14,10·1,10·6,10·7,10·8,10·11,10·12,10·13,10·14,10·15	13
4·13	4·3,4·6,4·10,4·14,13·9,13·10,13·11,13·12,13·15	9
4·14	4·3,4·6,4·10,4·13,14·2,14·3,14·5,14·10,14·15	9
5·6	5·2,5·3,5·8,5·9,5·12,5·14,5·16,5·17,6·4,6·8,6·9,6·10,6·11,6·16,6·17	15
5·8	5·2,5·3,5·6,5·9,5·12,5·14,5·16,5·17,8·6,8·9,8·10,8·16,8·17	10
5·9	5·2,5·3,5·6,5·8,5·12,5·14,5·16,5·17,9·2,9·3,9·6,9·8,9·11,9·13	14
5·12	5·2,5·3,5·6,5·8,5·9,5·14,5·16,5·17,12·1,12·2,12·10,12·13	12
5·14	5·2,5·3,5·6,5·8,5·9,5·12,5·16,5·17,14·2,14·3,14·4,14·10,14·15	13
5·16	5·2,5·3,5·6,5·8,5·9,5·12,5·14,5·17,16·2,16·6,16·7,16·8,16·15,16·17	14
5·17	5·2,5·3,5·6,5·8,5·9,5·12,5·14,5·16,17·1,17·3,17·6,17·7,17·8,17·11,17·16	15
6·8	6·4,6·5,6·9,6·10,6·11,6·16,6·17,8·5,8·9,8·10,8·16,8·17	12
6·9	6·4,6·5,6·8,6·10,6·11,6·16,6·17,9·2,9·3,9·5,9·8,9·11,9·13	13
6·10	6·4,6·5,6·8,6·9,6·11,6·16,6·17,10·1,10·4,10·7,10·8,10·11,10·12,10·13,10·14,10·15	16
6·11	6·4,6·5,6·8,6·9,6·10,6·16,6·17,11·9,11·10,11·13,11·17	11
6·16	6·4,6·5,6·8,6·9,6·10,6·11,6·17,16·2,16·5,16·7,16·8,16·15,16·17	13
6·17	6·4,6·5,6·8,6·9,6·10,6·11,6·16,17·1,17·3,17·5,17·7,17·8,17·11,17·16	14
7·10	7·3,7·15,7·16,7·17,10·1,10·4,10·6,10·8,10·11,10·12,10·13,10·14,10·15	13
7·15	7·3,7·10,7·16,7·17,15·3,15·10,15·13,15·14,15·16	9
7·16	7·3,7·10,7·15,7·17,16·2,16·5,16·6,16·8,16·15,16·17	10
7·17	7·3,7·10,7·15,7·16,17·1,17·3,17·5,17·6,17·8,17·11,17·16	11
8·9	8·5,8·6,8·10,8·16,8·17,9·2,9·3,9·5,9·6,9·11,9·13	11
8·10	8·5,8·6,8·9,8·16,8·17,10·1,10·4,10·6,10·7,10·11,10·12,10·13,10·14,10·15	14
8·16	8·5,8·6,8·9,8·10,8·17,16·2,16·5,16·6,16·7,16·15,16·17	11
8·17	8·5,8·6,8·9,8·10,8·16,17·1,17·3,17·5,17·6,17·7,17·11,17·16	12
9·11	9·2,9·3,9·5,9·6,9·8,9·13,11·6,11·10,11·13,11·17	10
9·13	9·2,9·3,9·5,9·6,9·8,9·11,13·4,13·10,13·11,13·12,13·15	11
10·11	10·1,10·4,10·6,10·7,10·8,10·12,10·13,10·14,10·15,11·6,11·9,11·13,11·17	13
10·12	10·1,10·4,10·6,10·7,10·8,10·11,10·13,10·14,10·15,12·1,12·2,12·5,12·13	13
10·13	10·1,10·4,10·6,10·7,10·8,10·11,10·12,10·14,10·15,13·4,13·9,13·11,13·12,13·15	14
10·14	10·1,10·4,10·6,10·7,10·8,10·11,10·12,10·13,10·15,14·2,14·3,14·4,14·5,14·15	14
10·15	10·1,10·4,10·6,10·7,10·8,10·11,10·12,10·13,10·14,15·3,15·7,15·13,15·14,15·16	14
11·13	11·6,11·9,11·10,11·17,13·4,13·9,13·10,13·12,13·15	9
11·17	11·6,11·9,11·10,11·13,17·1,17·3,17·5,17·6,17·7,17·8,17·11	11
12·13	12·1,12·2,12·5,12·10,13·4,13·9,13·10,13·11,13·15	9
13·15	13·4,13·9,13·10,13·11,13·12,15·3,15·7,15·10,15·14,15·16	10
14·15	14·2,14·3,14·4,14·5,14·10,15·3,15·7,15·10,15·13,15·16	10
15·16	15·3,15·7,15·10,15·13,15·14,16·2,16·5,16·6,16·7,16·8,16·17	12
16·17	16·2,16·5,16·6,16·7,16·8,16·15,17·1,17·3,17·5,17·6,17·7,17·8,17·11	13

From the definition of edge degree we can now define the maximum and minimum edge degree

Definition 3. *Let G be a finite simple graph. The* ***maximum edge degree*** $\Delta'(G)$ *of G is defined by*

$\Delta'(G) = max\{deg'(e)|x \in V(G)\}$.

Definition 4. *Let G be a finite simple graph. The **minimum degree** $\delta'(G)$ of G is defined by $\delta'(G) = min\{deg'(x)|x \in V(G)\}$.*

By inspection we see that $\Delta'(G) = deg'\{6{\cdot}10\} = 16$ and $\delta'(G) = deg'\{1{\cdot}12\}$

We can also define the following:

The ***arithmetic mean*** of the edge degrees is

$$arith'(G) = [\sum_{e \in E(G)} deg'(e)]/size(G),$$

The ***geometric mean*** of the edge degrees is

$$geom'(G) = \prod_{e \in E(G)} deg'(e)^{1/size(G)},$$

the ***median*** $median'(G)$ of the edge degrees, the middle value (if the size is odd) or the average of the two middle values (if the size is even) of the list of edge degrees in ascending order and the ***mode*** $mode'(G)$ of the edge degrees, the value of the edge degree that appears most often.

For GB $arith'(GB) = 648/55 = 11,78181818$, $geom'(GB) = 648/55 = 11,60218164$, $median'(GB) = 12$ and $mode'(GB) = 11$

1.3 Automorphisms

An ***automorphism*** is, as given in Chapter 7, a bijection $\phi : V(GB) \to V(GB)$ so that two vertices $v, w \in V(GB)$ are adjacent if and only if $\phi(v)$, $\phi(w) \in V(GB)$ are adjacent. We know that any automorphism $\phi \in aut(GB)$, will fix vertices 1, 5 and 10 since for each of them there is no other vertex in GB with the same degree, and so, any function that would take, for example, vertex 1 to something other than 1 would imply that deg(1) is not equal to $deg(\phi(1))$, contradicting part 2 of Lemma 7.1.0.2 . We have $\phi(1) = 1$, $\phi(5) = 5$ and $\phi(10) = 10$,

Vertices 17, 6 and 3 will either be fixed or will be permuted among themselves since they are the only ones with degree 8. We require $\phi(N(17)) = N(\phi(17))$. $1 \in N(17)$ and $\phi(1) = 1$ but $1 \notin N(3)$, $1 \notin N(6)$. 17 can only be sent to either 3 or 6 thus $\phi(17) = 17$. For the same reason, 6 cannot be sent to 3; $10 \in N(6)$ and 10 is fixed but $10 \notin N(3)$. So, 3 and 6 are fixed.

The only two vertices with degree 7 are vertices 9 and 16, so an automorphism will either fix both 9 and 16 or swap them. $17 \in N(16)$ and 17 is fixed but $17 \notin N(9)$. Hence, any automorphism in GB fixes 9 and 16.

4, 7, 11 and 12 have degree 5 and no other vertex has degree 5. We now consider the intersection of the neighbours of each vertex with the set {1,3,5,6,9,10,16,17} of vertices that we know are fixed so far. We see that $3, 6, 10 \in N(4)$, all of which are fixed. Neither 7 nor 11 nor 12 is adjacent to all three vertices. So, 4 is fixed. $16 \in N(7)$ and 16 is fixed but neither 11 nor 12 are adjacent to 16, hence, 7 is fixed. $6 \in N(11)$ and 6 is fixed but $6 \notin N(17)$, hence, 11 and 12 are fixed, since there are no other vertices of degree 5. We add them to the fixed vertices set {1,3,4,5,6,7,9,10,11,12,16,17}.

Vertices 2, 8, 13, 14 and 15 are the only vertices of degree 6. 15 is the only vertex out of these that is adjacent to 7, which is fixed. Hence, vertex 15 is fixed. Vertex 2 is the only vertex out of the ones of degree 6 that is adjacent to 3, 5 and 16, which are fixed. Hence, 2 is fixed. Vertex 8 is the only vertex out of the ones of degree 6 that is adjacent to 5, 6, 9 and 10, which are fixed. Hence, 8 is fixed. Vertex 13 is the only vertex out of the ones of degree 6 that is adjacent to 4, 9 and 10, which are fixed. Hence, vertices 13 and 14 are fixed, since there are no other vertices of degree 6.

So, any automorphism will fix all vertices; $\phi(v) = v$ for all $v \in V(GB)$ and $\phi \in$ Aut(G). That is, any automorphism from GB to itself is trivial.

1.4 Eccentricities and its related invariants

To find the ***eccentricity***, $\epsilon(v)$, as given in definition 8.1.0.3, of each vertex in GB we look for a path of greatest length from each vertex to any other vertex.

We first consider all paths from each vertex to every other vertex. To make the process a bit easier to execute I will only consider paths from a vertex a to a vertex b if $a < b$. We can do this since a path from a to b exists if and only if a path from b to a exists, namely the path consisting of the edges in the path from a to b reversing their order and the order of the vertices of each edge. Also, these two paths have the same length.

Vertex 1	Vertex 2	Vertex 3	Vertex 4	Vertex 5	Vertex 6	Vertex 7	Vertex 8
1.12.2							
1.17.3	2.3						
1.10.4	2.3.4	3.4					
1.12.5	2.5	3.5	4.6.5				
1.17.6	2.5.6	3.5.6	4.6	5.6			
1.17.7	2.3.7	3.7	4.10.7	5.17.7	6.17.7		
1.17.8	2.5.8	3.5.8	4.6.8	5.8	6.8	7.10.8	
1.12.5.9	2.9	3.9	4.6.9	5.9	6.9	7.3.9	8.9
1.10	2.14.10	3.7.10	4.13.10	5.8.10	6.10	7.10	8.10
1.10.11	2.9.11	3.17.11	4.13.11	5.6.11	6.11	7.10.11	8.10.11
1.12	2.12	3.5.12	4.13.12	5.12	6.10.12	7.10.12	8.10.12
1.12.13	2.9.13	3.15.13	4.13	5.12.13	6.10.13	7.10.13	8.10.13
1.10.14	2.14	3.14	4.14	5.14	6.10.14	7.15.14	8.10.14
1.10.15	2.16.15	3.15	4.14.15	5.14.15	6.10.15	7.15	8.16.15
1.17.16	2.16	3.17.16	4.16	5.16	6.16	7.16	8.16
1.17	2.16.17	3.17	4.6.17	5.17	6.17	7.17	8.17
$\epsilon(1) = 3$	$\epsilon(2) \geq 2$	$\epsilon(3) \geq 2$	$\epsilon(4) \geq 2$	$\epsilon(5) \geq 2$	$\epsilon(6) \geq 2$	$\epsilon(7) \geq 2$	$\epsilon(8) \geq 2$

Vertex 9	Vertex 10	Vertex 11	Vertex 12	Vertex 13	Vertex 14	Vertex 15	Vertex 16
9.8.10							
9.11	10.11						
9.13.12	10.12	11.13.12					
9.13	10.13	11.13	12.13				
9.2.14	10.14	11.10.14	12.5.14	13.15.14			
9.3.15	10.15	11.10.15	12.13.15	13.15	14.15		
9.2.16	10.15.16	11.6.16	12.2.16	13.15.16	14.15.16	15.16	
9.3.17	10.1.17	11.17	12.1.17	13.11.17	14.3.17	15.16.17	16.17
$\epsilon(9) \geq 2$	$\epsilon(10) \geq 2$	$\epsilon(11) \geq 2$	$\epsilon(12) \geq 2$	$\epsilon(13) \geq 2$	$\epsilon(14) \geq 2$	$\epsilon(15) \geq 2$	$\epsilon(16) \geq 1$

All paths except one of them have lengths 1 or 2. We know that these are the shortest walks since they are either adjacent giving a path of length 1 or they are not adjacent. If they aren't adjacent the shortest walk possible is of length 2. So, if we find a walk of length 2 we know it's of minimum length for non-adjacent vertices, and so, it's a path.

The only case where we are not able to find a path of length 1 or 2 is when finding a path from 1 to 9. 1 is only adjacent to 10, 12 and 17 but 9 is not adjacent to any of those, so any walk from 1 to 9 will have length at least 3. Since we found a walk of length 3 from vertex 1 to 9 we know that it is a path. And so, the eccentricities of 1 and 9 are $\epsilon(1) = \epsilon(9) = 3$. This is the only path of length greater than 2 in the table, so we know that all vertices other than 1 and 9 will have eccentricity smaller than or equal to 2. A vertex of eccentricity 1 would be a dominating vertex, but we know that GB doesn't have any dominating vertices, so the eccentricity of all vertices other than 1 and 9 is 2.

From definitions 8.1.0.6, 8.1.0.7 and 8.1.0.9 we now conclude that the ***radius*** of GB is rad(GB)=2. The ***centre*** of GB is the set $[2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17]$, that is $v \in V(GB)$ such that $\epsilon(v) = 2 = rad(GB)$ for all $v \in G - \{1, 9\}$. The diameter is diam(GB)=3, since we have $\epsilon(1) = 3 \geq \epsilon(v)$ for all $v \in G$.

1.5 Clique number and its variants

Definition 5.1.0.5 states that the ***clique number*** is the size of the largest complete subgraph. For the clique number to be n we need n vertices such that they are all pairwise adjacent. For this to be a possibility we need n vertices with degree at least $n-1$. So, the clique number cannot be 8 or more since there are only seven vertices with degree at least 7. We cannot use the same argument for $n \leq 7$ since there are twelve vertices with degree greater than or equal to 6. We now consider the possibilities for the value of the clique number starting from 7 in decreasing order.

For each clique number n we consider only the vertices that have degree at least $n-1$, and delete the ones with degree $< n-1$ from the graph GB to get an induced subgraph GB' with vertex set $V' = V(GB) -$ [vertices with degree$[V] < n-1$]. We then consider the degrees of the vertices in GB', which we will denote degree[V']. Once again we delete the ones with degree[V'] smaller than $n-1$ and consider the induced graph GB" with vertex set $V'' = V' -$ [vertices with $degree[V'] < n-1$]. We keep doing this until we get a vertex set with $n-1$ vertices or until all vertices in the set are pairwise adjacent, and hence form an n-clique.

For all vertex sets V' in each of the induced subgraphs we also consider whether the deg$[V'](v) = n-1$ for some vertex v in an induced subgraph GB'. When this is the case, we have that v cannot be in an n-clique unless all the neighbours of v in GB' are pairwise adjacent. So, we check if this is the case for each induced subgraph and if it's not then we can delete v from GB' and get once again another induced subgraph.

We first consider $\omega(GB) = 7$: The vertices of degree lower than six are 1, 4, 7, 11 and 12, we delete them from the vertex set $V(GB)$ and denote $V' = [2, 3, 5, 6, 8, 9, 10, 13, 14, 15, 16, 17]$. We consider the degrees of the vertices in the set in their induced subgraph, degree[V'].

Vertex	Neighbours in V'	Degree[V']
2	3,5,9,14,16	5
3	2,5,9,14,15,17	6
5	2,3,6,8,9,14,16,17	8
6	5, 8,9,10,16,17	6
8	5,6,9,10,16,17	6
9	2,3,5,6,8,13	6
10	6,8,13,14,15	5
13	9,10,15	3
14	2,3,5,10,15	5
15	3,10,13,14,16	5
16	2,5,6,8,15,17	6
17	3,5,6,8,16	5

So, the only 5 vertices with degree[V"] at least 7 are 3, 5, 6, 8, 9 and 16 which is not enough since we need 6 of them. Hence the clique number cannot be 7.

Now we consider $\omega(GB) = 6$. The only vertex with degree smaller than five is vertex 1. As discussed before, for each vertex v of degree 5 if v is in the clique then all of its neighbours are in the clique too, and so, they are all pairwise adjacent. We check if this is the case for any vertex of degree 5.

Vertex v	Neighbours of v: $N(v)$	
4	3, 6,10,13, 14,	3 is not adjacent to 6
7	3,10,15,16,17	3 is not adjacent to 10
11	6,9,10, 13,17	6 is not adjacent to 9
12	1,2,5,10,13	5 is not adjacent to 10

It is not the case, so we delete vertices 1, 4, 7, 11 and 12 from $V(GB)$ and denote $V' = [2, 3, 5, 6, 8, 9, 10, 13, 14, 15, 16, 17]$, which is the same vertex set that we had when considering $\omega(GB) = 7$, so we now consider the vertices v with degree[V'](v)=5 and check if their neighbours are pairwise adjacent.

Vertex v	Neighbours of v in V': $N(v) \cap V'$	
2	3,5,9,14,16	3 is not adjacent to 16
10	6,8,13,14,15	6 is not adjacent to 13
17	3,5,6,8,16	3 is not adjacent to 6

So, we have that vertices 2, 10 and 17 cannot be in the 6-clique, and so we delete them from V' and denote $V'' = [3, 5, 6, 8, 9, 13, 14, 15, 16]$. We now consider the $degree[V'']$ of the vertices in V":

Vertex: v	Neighbours of v in V'': $N(v) \cap V''$	$Degree[V'']$ of v
3	5,7,9,14,15	5
5	3,6,8,9,12,14,16	7
6	5,8,9,16	4
8	5,6,9,16	4
9	3,5,6,8,13	5
13	9,15	2
14	3,5,15	3
15	,3,13,14, 16	4
16	,5,6,8,15,	4

We see that only 3 vertices have $degree[V'']$ at least 5, which is not enough since we need at least 6 for them to be a 6-clique set. Hence, the clique cannot be 6.

Now we consider $\omega(GB) = 5$. The only vertex with degree smaller than five is vertex 1, so we delete it from V(GB) and denote $W := [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]$. Now we consider the vertices with the lowest degree.

- Vertex 12 has degree[W](12)=4. Vertex 12 is adjacent to 2 and 10 but 2 is not adjacent to 10, so we delete 12 from W and denote $W' := [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17]$.

- Vertex 13 has $degree[W'](13) = 5$. For a clique with vertex 13 in it we need 4 of its 5 neighbours to be pairwise adjacent. 11 and 15 are adjacent to 13 but aren't adjacent to one another. So, they cannot both be in the clique. 9 and 10 are adjacent to 13 but aren't adjacent to one another. So, they cannot both be in the clique. Hence, there can only be 3 vertices in the clique if vertex 13 is in it. So, vertex 13 cannot be in the clique and we delete it from W' and denote $W'' := [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 15, 16, 17]$.

We write down the neighbours of each vertex in this set and their degree[W"]:

Vertex: V	Neighbours of v in W'': $N(v) \cap W''$	$Degree[W'']$ of v
2	3,5,9,14,16	5
3	2,4,5,7,9,14,15,17	8
4	3, 6,10,14,	4
5	2,3,6,8,9,14,16,17	8
6	4, 5, 8,9,10,11,16,17	8
7	3,10,15,16,17	5
8	5,6,9,10,16,17	6
9	2,3,5,6,8,11	6
10	4,6,7,8, 11,14,15	8
11	6,9,10,17	4
14	2,3,4,5,10,15	6
15	,3,7,10,14, 16	6
16	2,5,6,7,8,15, 17	7
17	3,5,6,7,8,11,16	8

We check if the neighbours of vertices with $degree[W'']$ 5 are pairwise adjacent.

Vertex: v	Neighbours in W''	
4	3, 6,10,14,	3 is not adjacent to 6
11	6,9,10,17	9 is not adjacent to 10

They aren't, so we delete vertices 4 and 11 from V(GB), and denote $W''' = [2, 3, 5, 6, 7, 8, 9, 10, 14, 15, 16, 17]$. Vertex 2 has degree$[W'''](2) = 5$. Hence, we need 4 of the 5 vertices to be pairwise adjacent. 12 and 14 are adjacent to 2 but aren't adjacent to one another. So, they cannot both be in the clique. 3 and 16 are adjacent to 2 but aren't adjacent to one another. So, they cannot both be in the clique. Hence, there can only be 3 vertices in the clique if vertex 2 is in it. So, we delete vertex 2 from W''' and denote $Y = [3, 5, 6, 7, 8, 9, 10, 14, 15, 16, 17]$.

Vertex	**Neighbours in Y**	**Degree[Y]**
3	5,7,9,14,15,17	6
5	3,6,8,9,14,16,17	7
6	5, 8,9,10,16,17	6
7	3,10,15,16,17	5
8	5,6,9,10,16,17	6
9	3,5,6,8	4
10	6,7,8,14,15	6
14	3,5,10,15	4
15	,3,7,10,14, 16	6
16	5,6,7,8,15, 17	6
17	3,5,6,7,8,16	7

We now check if vertices with degree[Y] 4 have neighbours all pairwise adjacent:

Vertex	**Neighbours in Y**	
9	3,5,6,8	3 is not adjacent to 8
14	3,5,10,15	3 is not adjacent to 10

We delete 9 and 14 from Y, and denote $Y' = [3, 5, 6, 7, 8, 10, 15, 16, 17]$

Vertex	**Neighbours in Y**	$Degree[Y']$
3	5,7,15,17	4
5	3,6,8,16,17	5
6	5, 8,10,16,17	5
7	3,10,15,16,17	5
8	5,6,10,16,17	5
10	6,7,8,15	4
15	,3,7,10,16	4
16	5,6,7,8,15, 17	6
17	3,5,6,7,8,16	7

Now we consider vertices with $degree[Y'] = 4$

Vertex	**Neighbours in** Y'	
3	5,7,15,17	5 is not adjacent to 7
10	6,7,8,15	6 is not adjacent to 7
15	,3,7,10,16	3 is not adjacent to 10

We delete vertices 2, 3, 10 and 15 from Y' and denote $Y'' = [5, 6, 7, 8, 16, 17]$

Vertex	**Neighbours in Y''**	$Degree[Y'']$
5	6,8,16,17	4
6	5,8,16,17	4
7	16,17	2
8	5,6,16,17	4
16	5,6,7,8,17	5
17	5,6,7,8,16	5

We see that vertex 7 cannot be in the 5- clique since it has $degree[Y''](7) = 2$. So, the only vertices that could be in the set are vertices 5, 6, 8, 16 and 17. We check if they are pairwise adjacent:

5 is adjacent to 6, 8, 16 and 17
6 is adjacent to 8, 16 and 17
8 is adjacent to 16 and 17
16 is adjacent to 17

Thus, 5, 6, 8, 16 and 17 are pairwise adjacent and since we discarded all other vertices they form the largest clique and so the clique number is $\omega(GB) = 5$. Also, this is the only clique of order 5.

The ***edge clique number*** is simply the value of the maximum degree, which is the value of the degree of vertex 10, $\Delta(GB) = 10 = \omega'(GB)$. This is because all these ten vertices are pairwise adjacent but couldn't all be adjacent to any other vertex that wasn't adjacent to vertex 10 in the first place.

For the total clique number we make use of Question of the week 2, in which we showed that in most cases $\omega''(GB) = \Delta(GB) + 1$, in our case $\Delta(GB) + 1 = 11$. Indeed, if the total clique set were to contain all edges it could also contain one more vertex, the one all 10 edges in the edge clique set are adjacent to. Hence, the total clique number is at least $10 + 1 = 11$.

All other combinations of adjacent vertices and edges would be smaller than 11: If the total clique set contains two vertices then there could be at most one edge in the total clique set, since two edges can only share one vertex, giving a total of $3 < 11$. If there are three or more vertices in the total clique set then there cannot be any edges in the set, but the maximum number of adjacent vertices in the set is $\omega(GB) = 5 < 11$. If we have more than one edge then there can only be one vertex and this is maximized by $\Delta(GB)+1$. Hence, the total clique number is $\omega''(GB) = 11$. The largest total clique set is the set of all edges adjacent to vertex 10 together with vertex 10.

1.6 Independence number and its variants

The ***independence number*** $\alpha(GB)$ is, according to definition 5.1.0.4, the number of vertices that are in the largest set of pairwise non-adjacent vertices. In order to calculate it it is useful to write down all possible complete subgraphs. To do this we make use of the table in the kissing number section, where we computed all possible cycles. Since 3-cycles are the same as complete graphs with 3 vertices now we write down those cycles and write the common neighbours of the vertices in the cycles. Again, we only consider the common neighbours greater than all vertices in the complete graphs. To do this, it is easier to check the intersection of common neighbours of any two pairs in each K_3. We then write down the K_4 subgraphs that each K_3 subgraph together with each common neighbour create.

$K_3 : \{u, v, w\}$	$N(u) \cap N(v) \cap N(w)$	K_4
1,10,12	empty	
2,3,5	9, 14	{2,3,5,9}, {2,3,5,14}
2,3,9	empty	
2,3,14	empty	
2,5,9,	empty	
2,5,12	empty	
2,5,14	empty	
2,5,16	empty	
3,4,14	empty	
3,5,9	empty	
3,5,14,	empty	
3,5,17	empty	
3,7,15,	empty	
3,7,17	empty	
3,14,15	empty	
4,6,10	empty	
4,10,13	empty	
4,10,14	empty	
5,6,8,	9, 16, 17	{5,6,8,9}, {5,6,8,16}, {5,6,8,17}
5,6,9,	empty	
5,6,16,	17	{5,6,16,17}
5,6,17	empty	
5,8,9	empty	
5,8,16,	17	{5,8,16,17}
5,8,17	empty	
6,8,9	empty	
6,8,10	empty	
6,8,16	17	{6,8,16,17}
6,8,17	empty	
6,9,11	empty	
6,10,11	empty	
6,11,17	empty	
6,16,17	empty	
7,10,15	empty	
7,15,16	empty	
7,16,17	empty	
8,16,17	empty	
9,11,13	empty	
10,11,13	empty	
10,12,13	empty	
10,13,15	empty	
10,14,15	empty	

Again, we now write down all K_4 subgraphs and consider the common neighbours of the vertices in each subgraph and the K_5 subgraphs they create.

$K_4 : \{u, v, w, z\}$	$N(u) \cap N(v) \cap N(w) \cap N(z)$	K_5
2,3,5,9	empty	
2,3,5,14	empty	
5,6,8,9	empty	
5,6,8,16	17	{5,6,8,16,17}
5,6,8,17	empty	
5,6,16,17	empty	
5,8,16,17	empty	
6,8,16,17	empty	

We see that there is only one K_5 subgraph so there cannot be a K_6 subgraph. We already obtained this result when finding the clique number, and the largest clique we found is precisely {5,6,8,16,17}. Using this method it is easier to answer how many cliques of each order there are in GB. There is only one clique of order 5, there are four cliques of order 4, 43 cliques of order 3 (the kissing number), 55 cliques of order 2 (the number of edges) and 17 cliques of order 1 (the order).

In order to find the independence number we can notice that for all vertices in an induced subgraph to be non adjacent they cannot be in the same complete subgraph. That means that, for example, in an independent set there can only be one vertex out of the 5 from the complete induced subgraph with vertices $\{5, 6, 8, 16, 17\}$. If there was another one they would be adjacent, contradicting the fact that the set is independent. In order to find a set with the minimum number of complete subgraphs such that the set contains all vertices in GB we choose disjoint complete subgraphs. The following list of vertices are each complete induced subgraphs, K_i:

$\{5, 6, 8, 16, 17\}$
$\{1, 10, 12\}$
$\{3, 4, 14\}$ (*)
$\{9, 11, 13\}$
$\{7, 15\}$
$\{2\}$

The union of all sets contains all the vertices in GB. We can only take at most one vertex from each complete subgraph, meaning that the independent set will contain at most 6 vertices. If it contained 7 or more vertices, two of them would have to be in the same complete subgraph. So, we conclude that 6 is an upper bound for the $\alpha(GB)$.

However, this doesn't mean that there will be an independent set with 6 vertices. To prove that there is one we guess with vertices with lowest degree and discard those that lie in the same complete subgraph. We get, for example, [1,2,4,7,8,11]. Hence, $\alpha(G) = 6$.

To compute the ***number of independent sets of size 6*** we note that in any independent set of size 6 we can only have one vertex from each complete subgraph K_i in the list (*), but if we want to have 6 vertices, because we cannot have two from the same K_i we must have vertex 2, since 2 is the only vertex in its complete subgraph. This yields that the neighbours of 2 cannot be in the independent set, so we delete all $N(2)$ from the list of sets:

$\{6, 8, 17\}$
$\{1, 10\}$
$\{4\}$
$\{11, 13\}$
$\{7, 15\}$
$\{2\}$

These are still complete induced subgraphs, hence we can still only choose one from each set. Again, we see that, since vertex 4 is the only one in its set it must be in the independent set. So, we delete its neighbours from the sets:

$\{8, 17\}$
$\{1\}$
$\{4\}$
$\{11\}$
$\{7, 15\}$
$\{2\}$

Vertices 11 and 1 must be in the independent set since they are the only vertices in their respective set. We delete their neighbours from the sets:

$\{8\}$
$\{1\}$
$\{4\}$
$\{11\}$
$\{7, 15\}$
$\{2\}$

Vertex 8 must be in the set. All its neighbours have been deleted already. At this point all sets are independent from one another since we deleted the adjacent vertices of 5 of the 6 sets, thus we see there are two independent sets with 6 vertices, $\{8, 1, 4, 11, 7, 2\}$ and $\{8, 1, 4, 11, 15, 2\}$.

The ***edge independence number***, $\alpha'(GB)$, described in definition 5.1.0.7 as the largest number of independent edges in GB, is, according to Lemma 5.1.0.8, $\alpha'(GB) \leq 1/2\ order(GB) = 8.5$. This gives $\alpha'(GB) \leq 8$ because $\alpha'(GB)$ is an integer.

In order to find an edge independent set we can take a cycle of maximum length in GB. We can note that, since the cycle contains every vertex only once, we can take alternating edges in the cycle and they will not be adjacent. If they were adjacent then a vertex would appear twice in the cycle, which cannot happen. When we calculated the circumference of GB we found a cycle of length 17, $\{10{\cdot}1{\cdot}12{\cdot}13{\cdot}11{\cdot}9{\cdot}8{\cdot}16{\cdot}7{\cdot}15{\cdot}14{\cdot}4{\cdot}3{\cdot}2{\cdot}5{\cdot}17{\cdot}6{\cdot}10\}$. If we delete every other edge we get the set of independent edges, $\{10{\cdot}1, 12{\cdot}13, 11{\cdot}9, 8{\cdot}18, 7{\cdot}15, 14{\cdot}4, 3{\cdot}2, 5{\cdot}17\}$. These are clearly non-adjacent, since no two edges share an end vertex. The set has 8 elements, and we know that $\alpha'(GB) \leq 8$, which yields $\alpha'(GB) = 8$.

Definition 5.1.0.11 states that a total independent set is a set of vertices and edges where no two elements are adjacent. The ***total independence number***, $\alpha''(GB)$ is the greatest number of elements in a total independent set.

We try to find a total independent set with 11 elements. Having 7 edges in the set would force the set to have $11 - 7 = 4$ vertices giving that GB has $7 \times 2 + 4 = 18$ vertices, which is not the case. Having more than 7 edges in the set would increase the order of GB, that is, $order(GB) > 18$, which is not the case. So, the possibilities for the number of edges and vertices in the total independent set are:

6 edges and 5 vertices
5 edges and 6 vertices
4 edges and 7 vertices
3 edges and 8 vertices
2 edge and 9 vertices
1 edges and 10 vertices
0 edge and 11 vertices

The last 5 options give a contradiction since we would need an independent set with more than 6 vertices for them to be the case. So, the only options available are:

6 edges and 5 vertices
5 edges and 6 vertices

We consider the second option. The only 6 independent vertices in GB are the ones in the independent set, there is only one independent set in GB, $V := \{1, 2, 4, 7, 8, 11\}$; we take it to be in the total independent set. From the vertices remaining we cannot choose any other vertex, since the independent set V is maximal. We can, however, add edges to the set. We try to join with edges 10 of the 11 remaining vertices $\{3, 5, 6, 9, 10, 12, 13, 14, 15, 16, 17\}$ in a way that the edges don't share end vertices with each other and such that their end vertices are not in V. We get, for example, the set of edges $E := \{5{\cdot}6, 9{\cdot}13, 10{\cdot}12, 14{\cdot}15, 16{\cdot}17\}$. Each edge in E is not adjacent to any other edge in E nor to any vertex in V. So, the union $\{1, 2, 4, 7, 8, 11\} \cup \{5{\cdot}6, 9{\cdot}13, 10{\cdot}12, 14{\cdot}15, 16{\cdot}17\}$ is a total independent set and the total independence number is $\alpha''(GB) \geq 11$.

Suppose the total independence number was 12. The 12 elements will be a union of a set of edges E and a set of vertices V, so $|E| + |V| = 12$. V cannot have more than 6 vertices, since $\alpha(GB) = 6$ hence $|V| \leq 6$ and $|E| \geq 6$. Now, each edge will be adjacent to 2 different vertices and no two edges share an end vertex, as that would make them adjacent, hence the at least 6 edges are adjacent to at least 12 distinct vertices. Also, none of the vertices in the set are end vertices of the edges, as that would make them adjacent. So, for $|V| = 6$ and $|E| = 6$ we have $6 + 2 \times 6 = 18$ distinct vertices in GB, which is not the case. Having more edges and fewer vertices would imply having more than 18 vertices in GB, so we reach a contradiction and $\alpha''(GB) = 11$.

1.7 Chromatic number and its variants

For finding out the ***chromatic number*** $\chi(GB)$, described in definition 10.1.0.1 as the smallest number of colors in a proper colouring of GB, we make use of the complete subgraphs in GB. We know that in a complete graph the chromatic number is the order, since if we choose the same colour for any two vertices they will be adjacent and the coloring won't be proper. So, we consider the largest complete graph in GB, {5,6,8,16,17}, and we assign different colours to each of the vertices in it since they are all adjacent to one another. So far we have used 5 colours, and by Lemma 10.1.0.4, we know the chromatic number is monotone, hence $\chi(GB) \geq 5$.

We use the following algorithm to find a proper 5-colouring: We start assigning 5 different colours c_1, c_2, c_3, c_4, c_5 to the vertices in $5, 6, 8, 16, 17$, respectively. Then we order the vertices from highest degree to lowest and assign the first colour in c_1, c_2, c_3, c_4, c_5 that hasn't been assigned to any of their neighbours. If some vertices have the same degree we use their labeling as the order.

Vertex	Colour	
10	c_1	since none of its neighbours are coloured c_1
3	c_2	3 is adjacent to 5 which has colour c_1, so
		we use c_2, since none of its neighbours are coloured c_2
6	c_2	6 is adjacent to 5 which has colour c_1, so we
		use c_2, since none of its neighbours are coloured c_2
9	c_4	9 is adjacent to 5, 6 and 8 which have colours c_1, c_2 and
		c_3, so we use c_4, since none of its neighbours are coloured c_4
2	c_3	2 is adjacent to 5 and 6 which have colours c_1 and c_2,
		so we use c_3 since none of its neighbours are colour c_3
13	c_2	13 i adjacent to 10 which has colour c_1, so
		we use c_2 since none of its neighbours are coloured c_2
14	c_4	14 is adjacent to 5, 2 and 6 which have colour c_1, c_2
		and c_3 so we use c_4, since none of its neighbours are coloured c_4
15	c_3	15 is adjacent to 5 and 3 which have colours c_1 and c_2
		so we use c_3, since none of its neighbours are coloured c_3
4	c_3	4 is adjacent to 10 and 3 which have colours c_1 and
		c_2, so we use c_3, since none of its neighbours are coloured c_3
7		7 is adjacent to 10, 3, 15, 16 and 17 which have colours
		c_1, c_2, c_3, c_4 and c_5, so there is no available colour for 7.

The algorithm fails so we try something different. We notice that the vertices in the largest independent set $\{1, 2, 4, 7, 8, 11\}$ can all have the same colour, since none of them are adjacent to any other vertex in the set. So we colour them with c_3. We won't use this color again since all other vertices will be adjacent to some vertex in the set $\{1, 2, 4, 7, 8, 11\}$, (otherwise the $\alpha(GB)$ would be greater than 6). As before we colour each vertex in the clique $\{5, 6, 8, 16, 17\}$ with different colours $\{c_1, c_2, c_3, c_4, c_5\}$, we order the vertices by degree and we use the first colour in $\{c_1, c_2, c_4, c_5\}$ not used by their neighbours.

Vertex	Colour	
3	c_2	3 is adjacent to 5 which has colour c_1, so we
		use c_2, since none of its neighbours are coloured c_2
9	c_4	9 is adjacent to 5, 6 and 8 which have colours c_1, c_2 and
		c_3, so we use c_4, since none of its neighbours are coloured c_4
10	c_1	we use c_1, since none of its neighbours are coloured c_1
12	c_2	12 is adjacent to 5 which has colour c_1, so we
		use c_2, since none of its neighbours are coloured c_2
13	c_4	13 is adjacent to 10, 12, 11 and 9 which have colours c_1, c_2,
		c_3 and c_4, so we use c_5, since none of its neighbours are coloured c_5
14	c_4	14 is adjacent to 5, 3 and 4 which have colours c_1, c_2 and
		c_3, so we use c_4, since none of its neighbours are coloured c_4
15	c_5	15 is adjacent to 10, 3, 7 and 13 which have colours c_1, c_2,
		c_3 and c_4, so we use c_5, since none of its neighbours are coloured c_5

This time the algorithm works and we get the following 5-coloring:

$colour1 = \{5, 15\}$
$colour2 = \{6, 12, 14\}$
$colour3 = \{1, 2, 4, 7, 8, 11\}$
$colour4 = \{3, 13, 16\}$
$colour5 = \{9, 10, 17\}$

The union of all vertices in each colour set has 17 vertices, and they are all distinct, hence all vertices in GB have been coloured. Each colour set is an independent set since the vertices in each colour set are pairwise non adjacent. Hence, we found a proper 5-colouring, and since the chromatic number has to be at least 5 (for the reason given above) we have found a minimal chromatic set, and the chromatic number is $\chi(GB) = 5$.

Alternatively, to find the chromatic number we could realise that finding the chromatic number is finding the smallest number of disjoint independent sets such that we use all vertices.

We take list (*) from page 16 in the independence number section and notice that we can use the same colour for all vertices in $\{1, 2, 4, 7, 8, 11\}$ since this is an independent set, and so we can delete these vertices from the list to find other disjoint independent sets.

$\{5, 6, 16, 17\}$
$\{10, 12\}$
$\{3, 14\}$
$\{9, 13\}$
$\{15\}$

We now choose non-adjacent vertices from different sets to create independent sets. We get {6,12,3}, {15,9,17}, {13,14,16} and {5,10}. These sets together with $\{1, 2, 4, 7, 8, 11\}$ contain all vertices in GB and are independent sets, so we can colour each of them with a different colour and this will be a proper 5-colouring:

$colour1 = \{6, 12, 3\}$
$colour2 = \{15, 9, 17\}$
$colour3 = \{13, 14, 16\}$
$colour4 = \{5, 10\}$
$colour5 = \{1, 2, 4, 7, 8, 11\}$

We now consider the ***chromatic index*** $\chi'(GB)$ as given in definition 10.2.0.1. The maximum degree of GB is 10 and Lemma 10.2.0.2 tells us that the number of colours in any proper edge colouring will have at least 10 colours. This is because there are at least 10 pairwise adjacent edges. So, we try to show that there is a proper 10-edge colouring. We start by colouring each edge adjacent to vertex 10, $\{10{\cdot}1, 10{\cdot}4, 10{\cdot}6, 10{\cdot}7, 10{\cdot}8, 10{\cdot}11, 10{\cdot}12, 10{\cdot}13, 10{\cdot}14, 10{\cdot}15\}$ with a different colour from $\{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}\}$ in order. We then list all vertices in order from greatest to lowest degree and then for each of them we list all the vertices they are adjacent to in order of labeling. We then assign a colour to the edge between the two vertices using the first colour available from $\{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}\}$.

End vertex 1	**End vertex 2**		
5	2	c_1	
5	3	c_2	
5	6	c_4	since we used c_3 for 10·6
5	8	c_3	
5	9	c_5	
5	12	c_5	
5	14	c_7	
5	16	c_8	
5	17	c_9	
3	2	c_3	since we used c_1 and c_2 for 5·2 and 5·3
3	4	c_4	
3	7	c_5	
3	9	c_5	
3	14	c_8	since 5·14 is c_7
3	15	c_7	
3	17	c_{10}	since 5·17 is c_9
6	4	c_1	
6	8	c_2	
6	9	c_7	since we used c_3, c_4, c_5 and 6 for 10·6, 5·6, 5·9 and 3·9
6	11	c_5	since we used c_3 for 10·6 and c_4 for 5·6
6	16	c_5	
6	17	c_8	
17	1	c_2	since we used c_1 for 10·1
17	7	c_1	
17	8	c_4	since we used c_3 for 5·8
17	11	c_3	
17	16	c_5	
9	2	c_2	since we used c_1 for 5·2
9	8	c_1	
9	11	c_4	since we used c_3 for 17·11
9	13	c_3	
16	2	c_4	since we used c_1, c_2 and c_3 for 3·2, 5·2 and 9·2
16	7	c_2	since we used c_1 for 17·7
16	8	c_7	since we used c_1, c_3, c_5 and c_6 for 9·8, 5·8, 10·8 and 6·16
16	15	c_1	
2	12	c_5	since we used c_1, c_2, c_3 and c_4 for 5·2, 9·2, 3·2 and 16·2
2	14	c_6	since we used c_1, c_2, c_3 and c_4 for 5·2, 9·2, 3·2 and 16·2
13	4	c_5	since we used c_1, c_2, c_3 and c_4 for 6·4, 10·4, 9·13 and 3·4
13	11	c_1	
13	12	c_2	
13	15	c_4	since we used c_3 for 9·13
14	4	c_3	since we used c_1 and c_2 for 6·4 and 10·4
14	15	c_1	
15	7	c_3	since we used c_1 and c_2 for 14·15 and 16·7
15	16	c_9	since we used c_1, c2, c_4, c_5, c_5, c_7 and c_8 for
			15·14, 7·16, 7·15, 13·15, 17·16, 6·16, 8·16 and 5·16
12	1	c_3	since we used c_1 and c_2 for 10·1 and 17·1

We get the following edge 10-colouring:

c_1	2·5	4·6	7·17	8·9	15·16	11·13	14·15	1·10
c_2	3·5	6·8	12·13	1·17	2·9	7·16	4·10	
c_3	4·14	1·12	7·15	11·17	9·13	2·3	5·8	6·10
c_4	3·4	9·11	8·17	2·16	13·15	5·6	7·10	
c_5	4·13	2·12	16·17	6·11	3·7	5·9	8·10	
c_6	2·14	3·9	5·12	6·16	10·11			
c_7	8·16	5·14	3·15	6·9	10·12			
c_8	5·16	3·14	6·17	10·13				
c_9	5·17	10·14						
c_{10}	3·17	10·15						

There are 55 edges in the coloring and no edge appears more than once, so all edges have been coloured. In each coloring no two edges share an end vertex, so they are not adjacent. Hence this is a proper edge 10-coloring and the chromatic index is $\chi'(GB) = 10$.

Finally, we consider the ***total chromatic number***, $\chi''(GB)$, from definition 10.3.0.1. Lemma 10.3.0.5 tells us that $\chi''(GB) \geq 1 + \Delta(G)$. We see that vertex 10, which is the vertex with maximum degree, will be adjacent to 10 edges such that all of them are adjacent to one another, since they all share the end vertex 10. So, each edge adjacent to 10 will have a different colour and vertex 10 will have to be a new different colour giving a total of 11 colours.

To find the total colouring of GB with 11 colours we work from the edge colouring of GB and assign a new colour c_{11} to vertex 10. We list all the vertices in order of labelling and assign to each vertex the first colour available from the list of colours $[c_{11}, c_{10}, c_9, c_8, c_7, c_6, c_5, c_4, c_3, c_2, c_1]$, where we reversed the order of the colours with respect to the edge coloring since the most used colours were the ones at the beginning of the list.

Vertex	**Colour**	
1	c_{10}	since 1 is adjacent to 10 which has colour c_{11}
2	c_{11}	
3	c_9	since we used c_{11} and c_{10} for vertex 2 (adjacent to 3) and edge 17·3
4	c_{10}	since 4 is adjacent to 10 which has colour c_{11}
5	c_{10}	since 5 is adjacent to 2 which has colour c_{11}
6	c_9	since 6 is adjacent to 10 and 5 which have colours c_{11} and c_{10}
7	c_{10}	since 7 is adjacent to 10 which has colour c_{11}
8	c_8	since 8 is adjacent to vertex 10, 5 and 6 which have colours c_{11}, c_{10} and c_9
9	c_{11}	
10	c_{11}	
11	c_{10}	since 11 is adjacent to 10 which has colour c_{11}
12	c_9	since 12 is adjacent to 1 and 2 which have colours c_{11} and c_{10}
13	c_7	since 13 is adjacent to vertices 10, 4 and 12 and edge 10·13 which have colours c_{11}, c_{10}, c_9 and c_8
14	c_5	since 14 is adjacent to vertices 10, 5 and 3 and edges 3·14, 5·14 and 2·14
		which have colours c_{11}, c_{10}, c_9, c_8, c_7 and c_6
15	c_8	since 15 is adjacent to 10, 7 and 3 which have colours c_{11}, c_{10} and c_9
16	c_{11}	
17	c_7	since 17 is adjacent to 1, 5, 6 and 8 which have colours c_{11}, c_{10}, c_9 and c_8

We get the following total coloring:

c_1	2·5	4·6	7·17	8·9	15·16	11·13	14·15	1·10
c_2	3·5	6·8	12·13	1·17	2·9	7·16	4·10	
c_3	4·14	1·12	7·15	11·17	9·13	2·3	5·8	6·10
c_4	3·4	9·11	8·17	2·16	13·15	5·6	7·10	
c_5	4·13	2·12	16·17	6·11	3·7	5·9	8·10	14
c_6	2·14	3·9	5·12	6·16	10·11			
c_7	8·16	5·14	3·15	6·9	10·12	13	17	
c_8	5·16	3·14	6·17	10·13	8	15		
c_9	5·17	10·14	3	6	12			
c_{10}	3·17	10·15	2	4	5	7	11	
c_{11}	1	9	10	16				

There are 17 vertices in the coloring and no vertex appears twice so all vertices have been coloured. Since we worked from the edge coloring we already know that every edge has been coloured and no two edges are adjacent. It is not the case in any of the colour sets that a vertex is in the set and is also an end vertex of an edge in the set, so no edge is adjacent to any vertex. Also, no two vertices in a set are adjacent, so each colour set is an independent set and hence this is a proper total 11-coloring and the total chromatic number is $\chi''(GB) = 11$.

1.8 Domination number and its variants

By lemma 11.1.0.4 we know that the ***domination number***, given in definition 11.1.0.1, is smaller than or equal to $\alpha(GB)$. Indeed, the independent set {1,2,4,7,8,11} is the largest one in GB, hence all other vertices are adjacent to some vertex in {1,2,4,7,8,11}. If this weren't the case then there would be some vertex v such that it is not adjacent to any of the vertices in {1,2,4,7,8,11}, and so, the set {1,2,4,7,8,11,v} would be an independent set larger than {1,2,4,7,8,11}, contradicting the maximality of the independence number. So, all vertices in GB are adjacent to some vertex in {1,2,4,7,8,11}, and so {1,2,4,7,8,11} is a dominating set.

However, we can do better. There is no dominating vertex, hence the dominating number is at least 2. The most promising vertices to be in the smallest dominating set are the ones with the highest degree. Vertex 10 has maximum degree 10. Vertex 5 has degree 9, which is the next biggest degree after vertex 10. So, we try with vertices 5 and 10 to see if they form a dominating set. $N(10) = \{1, 4, 6, 7, 8, 11, 12, 13, 14, 15\}$ and $N(5) = \{6, 8, 9, 12, 14, 16, 17, 2, 3\}$, the union of their neighbours together with vertices 5 and 10 contain the whole vertex set, hence, the set {5, 10} is a dominating set and has two vertices, so it is a minimal dominating set and the dominating number is $\gamma(GB) = 2$.

Now we consider what the ***number of dominating sets of two vertices*** in GB is. We note that for two vertices to be a dominating set their degrees must sum up at least 15 since the order of GB is 17 and the set has 2 vertices in it. However this could be the case for two vertices which are not a dominating set if they share too many vertices. So, we need the sum of their degrees minus the number of neighbours they share to be 15, or 17 if the two vertices are adjacent. We list all pairs of vertices $\{v, w\}$ such that $deg(v) + deg(w)$ is at least 15. We then compute $N(v, w) := \{deg(v) + deg(w) - |\{N(v) \cup v\} \cap \{N(w) \cup w\}\}|$.

$\{v,w\}$	$deg(v)$	$deg(w)$	$N(v,w) := \{\{N(v) \cup v\} \cap \{N(w) \cup w\}\}$	$\|N(v,w)\|$	$deg(v) + deg(w) - \|N(v,w)\|$
{10,2}	10	6	12,14	2	14
{10,3}	10	8	4,7,14,15	4	14
{10,4}	10	5	6,13,14,10,3	5	10
{10,5}	10	9	6,8,12,14	4	15
{10,6}	10	8	4,8,11,6,10	5	13
{10,7}	10	5	15,10,7	3	12
{10,8}	10	6	6,8,10	3	13
{10,9}	10	7	6,8,11,13	4	13
{10,11}	10	5	6,13,10,7	4	11
{10,12}	10	5	1,13,10,12	4	11
{10,13}	10	6	4,11,12,15,10,13	6	10
{10,14}	10	6	4,15,10,14	4	12
{10,15}	10	6	7,13,14,10,15	5	11
{10,16}	10	7	6,7,8,15	4	13
{10,17}	10	8	1,6,7,8,11	5	13
{5,2}	9	6	3,9,12,14,16,5,2	7	8
{5,3}	9	8	2, 9, 14, 17,5,3	6	11
{5,6}	9	8	8,9,16,17,5,6	6	11
{5,8}	9	6	6, 9, 16, 17,5,8	6	9
{5,9}	9	7	2, 3, 6, 8,5,9	6	10
{5,13}	9	6	9,12	2	13
{5,14}	9	6	2, 3,5,14	4	11
{5,15}	9	6	3,14,16	3	12
{5,16}	9	7	2,6,8,17,5,16	6	10
{5,17}	9	8	3,6,8,16,5,17	6	11
{3,6}	8	8	4,5,9	3	13
{3,9}	8	7	2,5,3,9	4	11
{3,16}	8	7	2,5,7,15,17	5	10
{3,17}	8	8	5,7,3,17	4	12
{6,9}	8	7	5,8,11,6,9	5	10
{6,16}	8	7	5,8,17,6,16	5	10
{6,17}	8	8	5,8,11,16,6,17	6	10
{9,17}	7	8	2,5,6,8	4	11
{16,17}	7	8	5,6,7,8,16,17	6	9

The only pair that satisfies $deg(v) + deg(w) - |\{\{N(v) \cup v\} \cap \{N(w) \cup w\}\}| = 15$ is $[10, 5]$, so this is the only dominating set of two vertices in GB.

The dominating set $[5, 10]$ is also an independent set, since 5 and 10 are not adjacent, hence it's an independent dominating set, and so, the ***independent domination number*** is $\iota(GB) = 2$ since $\iota(GB) \geq \gamma(GB)$. $[5, 10]$ is also the only independent dominating set.

By Lemma 11.1.0.7 the ***total domination number*** $\gamma_t(GB)$, as described in definition 11.1.0.6, cannot be 1 since there are no vertices adjacent to themselves. For the total domination number to be two we would need to find a total dominating set with two vertices. For these two vertices to be adjacent to all other vertices their degrees would have to sum at least $Order(GB) = 17$. The only pairs of vertices that satisfy this are

$$\{10, 5\}, \{10, 3\}, \{10, 6\}, \{10, 17\}, \{10, 9\}, \{10, 16\}, \{5, 3\}, \{5, 6\}, \{5, 17\}$$

However, the sum of the degrees of a non-dominating pair can be 17 if they have too many neighbours in common. So, to have a dominating set of two vertices we need for them to be adjacent to 17 distinct vertices. That is, the sum of their degrees minus the number of common neighbours should be 17. So, we proceed by listing all the pairs, the sum of their degrees and the number of common neighbours.

Pair {v,w}	$deg(v) + deg(w)$	$N(v) \cap N(w)$	$\lvert N(v) \cap N(w)\rvert$	$deg(v) + deg(w) - \lvert N(v) \cap N(w)\rvert$
{10, 5}	19	6,8,12,14	4	15
{10,3}	18	4,7,14,15	4	14
{10,6}	18	4,8,11	3	15
{10,17}	18	1,6,7,8,11	5	13
{10,9}	17	6,8,11,13	4	13
{10,16}	17	6,7,8,15	4	13
{5,3}	17	2,9,14,17	4	13
{5,6}	17	8,9,16,17	4	13
{5,17}	17	3,6,8,16	4	13

There is no pair such that the sum of their degrees minus the number of common neighbours they have is 17, hence, it is not possible that any pair in GB will be a total dominating set.

However, we see that the pairs {10, 5} and {10, 6} are the ones adjacent to the highest number of vertices, so the vertices in them are the most promising ones to be in a dominating set of three vertices. Since 5 and 10 are adjacent to all vertices but themselves and vertex 6 is adjacent to both 5 and 10, the set {5,6,10} is a total dominating set, and the total domination number is $\gamma_t(GB) = 3$.

We now consider the ***edge domination number*** as given in Definition 11.1.0.10. From question of the week 4 we have that $order(GB) \leq \alpha(GB) + 2\gamma'(GB)$. Order(GB) = 17 and $\alpha(GB) = 6$, so $11/2 \leq \gamma'(GB)$, that is, $6 \leq \gamma'(GB)$.

An independent set, although not the largest one, is $\{2, 4, 7, 8, 11\}$, we take all vertices that aren't in this set, that is, $\{1, 3, 5, 6, 9, 10, 12, 13, 14, 15, 16, 17\}$. We find 6 non adjacent edges with end vertices in $\{1, 3, 5, 6, 9, 10, 12, 13, 14, 15, 16, 17\}$. We start with vertex 1, we find the next vertex adjacent to it in the set, and we take the edge joining them. We continue in order without using the same vertex twice. We get the set $\{1{\cdot}10, 3{\cdot}5, 6{\cdot}9, 12{\cdot}13, 14{\cdot}15, 16{\cdot}17\}$.

If an edge shares an end vertex with the edges in $\{1{\cdot}10, 3{\cdot}5, 6{\cdot}9, 12{\cdot}13, 14{\cdot}15, 16{\cdot}17\}$ then they are adjacent, but the only vertices that aren't end vertices of these edges are $\{2, 4, 7, 8, 11\}$ which create an independent set, so any edge adjacent to them must have its other end vertex in $V(GB) - \{2, 4, 7, 8, 11\} = \{1, 3, 5, 6, 9, 10, 12, 13, 14, 15, 16, 17\}$, which are all end vertices of edges in $\{1{\cdot}10, 3{\cdot}5, 6{\cdot}9, 12{\cdot}13, 14{\cdot}15, 16{\cdot}17\}$. Hence all edges in E(GB) have end vertices in $\{1{\cdot}10, 3{\cdot}5, 6{\cdot}9, 12{\cdot}13, 14{\cdot}15, 16{\cdot}17\}$ and hence all edges are adjacent to it and it is an edge dominating set. Since $6 \leq \gamma'(GB)$ we conclude that $\gamma'(GB) = 6$

The ***independent edge domination number***, the minimum number of edges in an edge dominating set such that it is an independent set, is 6 since the set $\{1{\cdot}10, 3{\cdot}5, 6{\cdot}9, 12{\cdot}13, 14{\cdot}15, 16{\cdot}17\}$ is as well as an edge dominating set, an edge independent set.

For the ***total edge domination number***, from definition 11.1.0.11, we note that a total edge dominating set will have no isolated edges, that is, each edge in the total edge dominating set will be adjacent to some other edge in the set. This is because, in contrast to the edge dominating sets, the total edge dominating set does not allow an edge to not be adjacent to anything in the set if that edge is itself in the set. In practice this implies that we should be looking for sets of size at least 2 of connected edges.

To construct an 8-total edge dominating set fairly easily we can use a similar argument to the one for the edge dominating set. We take a largest independent set, {1,2,4,7,8,11} and we try to join the remaining vertices $V(GB) - \{1, 2, 4, 7, 8, 11\}$ with a cycle. The remaining 11 vertices are {3,5,6,9,10,12,13,14,15,16,17}, and a cycle joining them is {6·9·5·16·17·3·14·15·13·12·10·6}.

Now we can take two edges out of every three in order, making sure that no edge is isolated, for example $S := \{6{\cdot}9, 9{\cdot}5, 16{\cdot}17, 17{\cdot}3, 14{\cdot}15, 15{\cdot}13, 12{\cdot}10, 10{\cdot}6\}$. Every vertex in the set $\{3, 5, 6, 9, 10, 12, 13, 14, 15, 16, 17\}$ is an end vertex of some edge in S. Since $\{1, 2, 4, 7, 8, 11\}$ is an independent set, all the edges adjacent to them will be adjacent to some vertex in the set $\{3, 5, 6, 9, 10, 12, 13, 14, 15, 16, 17\}$ and thereby to the edges in S. Also, every edge in S is adjacent to some other edge in S since they were chosen in connected sets. So the set of edges $\{6{\cdot}9, 9{\cdot}5, 16{\cdot}17, 17{\cdot}3, 14{\cdot}15, 15{\cdot}13, 12{\cdot}10, 10{\cdot}6\}$ is a total edge dominating set, and $\gamma_t' \leq 8$.

We now consider whether the total edge domination number could be 7. The 7 edges cannot be isolated so they must come in connected sets. The possible configurations of the sizes of the connected sets of edges are $[2, 2, 3], [2, 5], [4, 3]$ and $[7]$, where each number denotes the size of a connected set of edges.

Now, for any SCaF graph G we have that $Size(G) \geq Order(G) - 1$. So, we can compute the number of vertices that each configuration of edges is adjacent to. For any set of edges with configuration [2,2,3] the two sets of two edges are adjacent to at most 3 vertices each and the 3 edges to at most 4 vertices, hence a total of 10 vertices. We do the same for the other configurations:

Edges configuration	Number of vertices
[2,2,3]	10
[2,5]	9
[4,3]	9
[7]	8

So, a total edge dominating set will be adajcent to at most 10 vertices. Now, there are 17 vertices in GB, so there will be at least 7 remaining vertices in GB not adjacent to the total edge dominating set of 6 edges but such that all their adjacent edges are adjacent to the total edge dominating set. By Question of the week 2 we know that these 7 vertices will create an independent set. This contradicts the fact that $\alpha(GB) = 6$. So, it cannot be the case that there are only 7 edges in a total edge dominating set, and so, $\gamma_t(GB) = 8$.

We now consider the ***complete domination number*** $\gamma''(GB)$ from definition 11.1.0.13. The set $\{1{\cdot}10, 3{\cdot}5, 6{\cdot}9, 12{\cdot}13, 14{\cdot}15, 16{\cdot}17\}$ is an edge dominating set and the set $\{5{,}10\}$ is a dominating set, so their union $\{1{\cdot}10, 3{\cdot}5, 6{\cdot}9, 12{\cdot}13, 14{\cdot}15, 16{\cdot}17\} \cup \{5, 10\}$ is a complete dominating set since every edge is either in $\{5{,}10\}$ or adjacent to some vertex in $\{5, 10\}$ and every edge is either in $\{1{\cdot}10, 3{\cdot}5, 6{\cdot}9, 12{\cdot}13, 14{\cdot}15, 16{\cdot}17\}$ or adjacent to some edge in $\{1{\cdot}10, 3{\cdot}5, 6{\cdot}9, 12{\cdot}13, 14{\cdot}15, 16{\cdot}17\}$. The set $\{1{\cdot}10, 3{\cdot}5, 6{\cdot}9, 12{\cdot}13, 14{\cdot}15, 16{\cdot}17\} \cup \{5, 10\}$ has 8 elements, so $\gamma''(GB) \leq 8$.

Since 5 and 10 is a dominating set and all edges with end vertices 5 or 10 are adjacent to either 5 or 10 we try to find a set of edges adjacent to all other edges except those adjacent to 5 or 10. As before, we take all vertices except those in the independent set {1,2,4,7,8,11}, that is, {3,6,9,12,13,14,15,16,17}. A set of edges with end-vertices in {3,6,9,12,13,14,15,16,17} is $\{6{\cdot}9,\ 12{\cdot}13,\ 15{\cdot}16,\ 17{\cdot}3,\ 3{\cdot}14\}$. The set $\{6{\cdot}9, 12{\cdot}13, 15{\cdot}16, 17{\cdot}3, 3{\cdot}14\} \cup \{5, 10\}$ is a complete dominating set and $\gamma''(GB) \leq 7$.

We now argue that there isn't a complete dominating set of 6 elements. Such a set is a union of edges and vertices, we now go through the possible configurations:

Having 6 edges would not work since at most 12 vertices are adjacent to them, so not all 17 vertices are adjacent to them.

Having 5 edges and 1 vertex $\{e_1, e_2, e_3, e_4, e_5\} \cup \{v\}$ would yield that the 5 edges must be adjacent to all edges in $E(GB)$ other than those with end-vertex v. Since the 5 edges are adjacent to at most 10 vertices, the vertices that are not adjacent to the 5 edges must be an independent set of 6 vertices. We know that there are only 2 such sets: {1,2,4,7,8,11} and {1,2,4,8,11,15}. Since $\{e_1, e_2, e_3, e_4, e_5\}$ is not adjacent to {1,2,4,7,8,11} or {1,2,4,8,11,15} we must have that v is adjacent to {1,2,4,7,8,11} or {1,2,4,8,11,15}, but we see by inspection that no single vertex is adjacent to all vertices in {1,2,4,7,8,11} nor {1,2,4,8,11,15}.

Now we consider a total dominating set of 4 edges and 2 vertices $\{e_1, e_2, e_3, e_4\} \cup \{v_1, v_2\}$. Vertices v_1 and v_2 are only adjacent to the edges with end-vertices v_1 or v_2. Hence $\{e_1, e_2, e_3, e_4\}$ must be adjacent to all edges other than those with end vertices v_1 or v_2. So, the vertices which aren't end vertices of $\{e_1, e_2, e_3, e_4\}$ and aren't v_1 or v_2 must be an independent set. However, $\{e_1, e_2, e_3, e_4\}$ is adjacent to at most 8 edges, which implies that there is an independent set of 7 vertices, which is not the case.

Having fewer edges or an empty set of edges would yield having a larger independent set, so there cannot be a complete dominating set with 6 elements and $\gamma''(GB) \geq 7$, which yields $\gamma''(GB) = 7$.

We now consider the ***total complete domination number*** $\gamma_t''(GB)$ from definition 11.1.0.14. The set {5,6,10} is a total dominating set and $\{6{\cdot}9, 9{\cdot}5, 16{\cdot}17, 17{\cdot}3, 14{\cdot}15, 15{\cdot}13, 12{\cdot}10, 10{\cdot}6\}$ is a total edge dominating set. Hence their union is a total complete dominating set, since every vertex is adjacent to {5,6,10} and every edge is adjacent to $\{6{\cdot}9, 9{\cdot}5, 16{\cdot}17, 17{\cdot}3, 14{\cdot}15, 15{\cdot}13, 12{\cdot}10, 10{\cdot}6\}$. The set $\{6{\cdot}9, 9{\cdot}5, 16{\cdot}17{,}17{\cdot}3, 14{\cdot}15, 15{\cdot}13, 12{\cdot}10, 10{\cdot}6\} \cup \{5, 6, 10\}$ has 11 elements so $\gamma_t''(GB) \leq 11$.

We note that the total edge dominating set is adjacent to all edges and all vertices except those in the independent set $\{1, 2, 4, 7, 8, 11\}$, hence a set adjacent to $\{1, 2, 4, 7, 8, 11\}$ together with $\{6{\cdot}9, 9{\cdot}5, 16{\cdot}17, 17{\cdot}3, 14{\cdot}15, 15{\cdot}13, 12{\cdot}10, 10{\cdot}6\}$ is a complete total dominating set. We see that all vertices in $\{1, 2, 4, 7, 8, 11\}$ are adjacent to either 3 or 17. Hence $\{3, 17\} \cup \{6{\cdot}9, 9{\cdot}5, 16{\cdot}17, 17{\cdot}3, 14{\cdot}15, 15{\cdot}13, 12{\cdot}10, 10{\cdot}6\}$ is a complete total dominating set.

However, since 3 is adjacent to 17 and all edges with end-vertices 3 or 17 are adjacent to $\{3, 17\}$ we look for a connected set adjacent to all edges except those with end vertices 3 or 17. As we argued before, we take all vertices in $V(GB)$ except those in the independent set $\{1, 2, 4, 7, 8, 11\}$, that is, $\{5, 6, 9, 10, 12, 13, 14, 15, 16\}$ and we find non-isolated edges with end-vertices in $\{5, 6, 9, 10, 12, 13, 14, 15, 16\}$. We get, for example, $\{5{\cdot}6, 6{\cdot}9, 10{\cdot}12, 12{\cdot}13, 14{\cdot}15, 15{\cdot}16\}$. Hence, the union $\{5{\cdot}6, 6{\cdot}9, 10{\cdot}12, 12{\cdot}13, 14{\cdot}15, 15{\cdot}16\} \cup \{3, 17\}$ is a complete dominating set and $\gamma_t''(GB) \leq 8$.

We now argue that $\gamma_t''(GB) \geq 8$ considering the possibilities:

For a set with 7 edges $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ there are only 14 vertices adjacent to them, so this isn't a total complete dominating set.

For a set with 6 edges and one vertex $\{e_1, e_2, e_3, e_4, e_5, e_6\} \cup \{v\}$ the single vertex v cannot be isolated so it must be an end vertex of some edge in $\{e_1, e_2, e_3, e_4, e_5, e_6\}$. The edges in $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ cannot be isolated so the possible configurations are $[2, 2, 2], [2, 4], [3, 3][6]$, which are adjacent to at most $9, 8, 8$ and 7 vertices respectively.

So, $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ will be adjacent to at most 9 vertices and 8 vertices are not adjacent to any edge in $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ and so the edges joining the 8 vertices won't be adjacent to $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ and neither to v. So there cannot be any edge joining these 8 vertices and they must be an independent set, which contradicts $\alpha(GB) = 6$.

For a set with 5 edges and 2 vertices $\{e_1, e_2, e_3, e_4, e_5\} \cup \{v_1, v_2\}$ this time v_1 and v_2 could be end-vertices of $\{e_1, e_2, e_3, e_4, e_5\}$ or not. The possible configuration of the connected sets of edges in $\{e_1, e_2, e_3, e_4, e_5\}$ are $[2, 3], [5]$ from which we see that the edges are adjacent to at most 7 distinct vertices. So, there are 10 vertices not adjacent to $\{e_1, e_2, e_3, e_4, e_5\}$.

If v_1 or v_2 or both are end-vertices of $\{e_1, e_2, e_3, e_4, e_5\}$ then there cannot be any edge joining the 9 or 10 remaining vertices, which contradicts $\alpha(GB) = 6$. If v_1 and v_2 are not end-vertices of $\{e_1, e_2, e_3, e_4, e_5\}$ then there are 8 vertices which cannot have edges joining them, which again contradicts $\alpha(GB) = 6$.

Having fewer edges or an empty edge set would yield having a larger independent set, thus there cannot be a total complete dominating set with 6 elements and $\gamma_t''(GB) = 8$

1.9 Connectivity and edge connectivity

We know by Lemma 12.1.0.5 that the ***connectivity*** of GB is $\kappa(GB) \leq \delta(GB) = 3$, since the order of GB is greater than 3. Indeed, if we delete vertices 10, 12 and 17 then we have isolated vertex 1 and so, GB is not connected any more.

To show that $\kappa(GB) = 3$ we find 3 disjoint walks from every vertex to every other vertex, if we can, then after deleting any two vertices from GB there will be at least one walk from every vertex to every other vertex, so GB will be connected and $\kappa(GB) > 2$.

We already found a walk from every vertex to every other vertex when considering the connectedness of GB, but we used too many vertices which makes it more difficult to find other disjoint walks. We look for walks using as few vertices as possible.

We notice that a total dominating set is adjacent to every other vertex, hence there are walks going through them joining every pair of vertices. E.g. the total dominating set {5,6,10} joins every pair of vertices since each vertex is adjacent to some vertex in the set {5,6,10} and the set is connected. So, there are walks from every vertex to every other vertex going only through vertices in {5,6,10} (not necessarilly all of them). If we can find 3 disjoint dominating sets then we will have 3 disjoint walks from every vertex to every other vertex.

The sets

$$S_1 = \{6, 9, 10\}, S_2 = \{5, 12, 13, 15\}, S_3 = \{2, 3, 4, 17\}$$

are total dominating sets; all vertices in GB (including the vertices in the set themselves) are adjacent to some vertex in each of the sets. We found them by taking adjacent pairs of vertices with high number at the last column to the right from the table for the domination number in page 25 and then adding some vertex adjacent to those not adjacent to the pair.

So, there are three walks joining every pair of vertices x and y. Namely, for x adjacent to $v_i \in S_i$ and y adjacent to $w_i \in S_i$ the walk $x{\cdot}v_i{\cdot}(...){\cdot}w_i{\cdot}y$ where (...) denotes a walk from v_i to w_i, which exists since S_i is connected.

S_1, S_2, S_3 are disjoint, which implies that, for each pair of vertices $x, y \in GB$, deleting any two other vertices from GB will destroy at most two of the walks from x to y but there will still remain at least one walk from x to y. Hence, after deleting any two vertices, GB will still be connected and $\kappa(GB) \geq 3$.

Notice that we cannot find a forth disjoint total dominating set since vertex 1 is only adjacent to $10, 12$ and 17 so, any total dominating set must include one of these but these are already in S_1, S_2 and S_3, so no total dominating set can be disjoint from all of them.

More explicitly, the following table shows that there are disjoint walks from every vertex $v \in GB$ to 10, 5 and 17, hence there are three disjoint walks from every vertex to every other vertex.

Walks from v to 10 through {6,9,10}	Walks from v to 5 through {5,12,13,15}	Walks from v to 17 through {2,3,4,17}
1·10	1·12·5	1·17
2·9·6·10	2·5	2·3·17
3·9·6·10	3·5	3·17
4·10	4·13·12·5	4·3·17
5·6·10	6·5	5·17
6·10	7·15·13·12·5	6·17
7·10	8·5	7·17
8·10	9·5	8·17
9·6·10	10·12·5	9·3·17
11·10	11·13·12·5	10·4·3·17
12·10	12·5	11·17
13·10	13·12·5	12·2·3·17
14·10	14·5	13·4·3·17
15·10	15·13·12·5	14·3·17
16·6·10	16·5	15·3·17
17·6·10	17·5	16·17

The ***edge connectivity*** $\kappa'(GB)$ as given in definition 12.1.0.14 is, by lemma 12.1.0.16, at most $3 = \delta(GB)$ and at least $3 = \kappa(GB)$, hence $\kappa'(GB) = 3$. Indeed, deleting edges 1·10, 1·12 and 1·17 would result in GB being disconnected.

1.10 Domatic number

The ***domatic number*** of GB is the largest number of disjoint dominating sets such that their union is the whole of $V(GB)$.

We see from the argument for connectivity that GB has 3 disjoint total dominating sets $S_1 = \{6, 9, 10\}, S_2 = \{5, 12, 13, 15\}, S_3 = \{2, 3, 4, 17\}$ which are also disjoint dominating sets.

However, for a dominating set we don't have the restriction that a vertex in a dominating set must also be adjacent to some other vertex in the set. So, since vertex 1 is only adjacent to 10,12 and 17, which are all already in S_1, S_2 or S_3 we can try to find another dominating set with vertex 1 in it.

We try with all the remaining vertices not in the sets, that is $V(GB) - \{S_1 \cup S_2 \cup S_3\} = \{1, 7, 8, 11, 14, 16\} =: S_4$, which we see by inspection is in fact a dominating set.

We have found a partition of $V(GB)$ into four disjoint dominating sets $S_1 = \{6, 9, 10\}, S_2 = \{5, 12, 13, 15\}, S_3 = \{2, 3, 4, 17\}, S_4 = \{1, 7, 8, 11, 14, 16\}$, and the domatic number is at least 4.

We argue the domatic number cannot be 5. Any dominating set must be adjacent to vertex 1 or have vertex 1 in the set, vertex 1 only has 3 vertices adjacent to it, so there are only 4 possible disjoint dominating sets. Therefore, the domatic number is 4.

We could modify the definition of domatic number and define the ***total domatic number*** as the largest number of disjoint total dominating sets in GB such that their union is a partition of GB. We have already argued in the connectivity section that there cannot be more than 3 disjoint total dominating sets in GB and we already found three sets $S_1 = \{6, 9, 10\}, S_2 = \{5, 12, 13, 15\}, S_3 = \{3, 4, 13, 17\}$.

Their union is not a partition of GB but since after adding vertices to a dominating set the set is still a dominating set, we can simply add all the remaining vertices to one of the sets to get $S_1^* = \{1, 7, 8, 11, 14, 16, 6, 9, 10\}, S_2 = \{5, 12, 13, 15\}, S_3 = \{2, 3, 4, 17\}$. Hence, the total domatic number is 3.

1.11 Planarity

There is a K_5 subgraph in GB, hence, by Theorem 14.1.0.11 (Kuratowski's Theorem), GB is not ***planar***.

1.12 The join

Suppose GB is a ***join*** of two induced subgraphs of GB, H and G, with disjoint vertex sets, so that all vertices in H are adjacent to all vertices in G.

Without loss of generality assume that $|G| \leq |H|$. G cannot consist of a single vertex v because if it did then v would be adjacent to every vertex in H, that is, every vertex in GB other than v, implying that v is a dominating vertex, but there is no dominating vertex in GB.

For $|G| \in [1, 6]$, H has at least 11 vertices and all vertices in G are adjacent to all vertices in H, contradicting the fact that the maximum degree in GB is 10.

For $|G| = 7$, we would need 7 vertices of degree at least 10 in GB, which is not the case. For $|G| = 7$ we would need 8 vertices of degree at least 9 in GB, which is not the case. $|G| \geq 9$ would force $|H| \leq 8 < |G|$, contradicting our assumption that $|G| \leq |H|$. We have discarded all possibilities, so there are no subgraphs H and G in GB such that their join is the graph GB.

1.13 Graph homology

We can consider graphs as a special case of the topological objects called simplicial complexes, which are unions of n-simplices with a few conditions, and n-simplices are the smallest convex sets containing $n + 1$ points (vertices) in general position.

We can define a graph as a simplicial complex that only has 0-simplices (vertices) and 1-simplices (edges).

The n-homology group of a topological space is a formalisation of the number of n-dimensonal "holes" the space has. The ***homology*** of a graph describes the homology groups of a graph.

The 0-homology group describes the number of connected pieces in GB, which we already know is 1, so the 0-homology group of GB is $H_0(GB) = \mathbb{Z}$, where $\mathbb{Z}$ denotes the integers.

The first homology group of a connected graph is $H_1(G) = \mathbb{Z}^{Size(G)-Order(G)+1}$. For GB $H_1(GB) = \mathbb{Z}^{Size(GB)-Order(GB)+1} = \mathbb{Z}^{55-17+1} = \mathbb{Z}^{39}$.

The higher dimensional homology groups are 0 since there is no simplex in GB of dimension greater than 1.

2 Numerical invariant: The distinguishing number

I came across the distinguishing number when doing research on different numerical invariants. I was looking for something that would be related to some other subject that I like and since the distinguishing number is about breaking the symmetries of a graph it seemed to me that it would be related to group theory, which I am interested in. It was also clear to me that it was related to the automorphism group of a graph, which I found interesting and with potential when we covered it in lectures, and so the topic wasn't completely unfamiliar.

2.1 Definition and properties of the distinguishing number

As given in the Wikipedia page for Distinguishing coloring [3], a puzzle by Frank Rubin asks how many colours you need to colour the keys in a ring of keys so that you are able to distinguish them. The keys can be moved around the ring but this will not change which key is next to which. We can take the ring of keys to be a graph where the keys are vertices and two keys are adjacent if they are next to each other. Albertson & Collins first introduced the distinguishing number to solve this puzzle.

Definition 5. *A distinguishing colouring for a graph G is an assignment of colors to the vertices in G such that no automorphism other than the trivial one will preserve the colouring of the vertices.*
The ***distinguishing number*** *of a graph G, D(G), is the smallest number such that there exists a distinguishing colouring with that number of colours.*

Frank Rubin's puzzle can be solved by considering the ring of keys as a cycle, then the puzzle is asking what the distinguishing number of a cycle is.

From the definition of distinguishing number some trivial consequences follow:

Lemma 1. *The distinguishing number is at most the order of the graph $D(G) \leq order(G)$.*

Proof. We can always colour each vertex with a different colour implying that any automorphism preserving colouring will fix every vertex, so only the trivial automorphism will preserve the colouring. We might be able to do better, but this colouring will work for any graph □

Lemma 2. *The distinguishing number of a graph will be 1 if and only if it has no non-trivial automorphisms.* $D(G) = 1 \iff Aut(G) = 1$

Proof. If the only automorphism of a graph is the trivial one then we can colour every vertex the same colour and we will have that the only automorphism preserving the colouring is the trivial one, since the coloring didn't change the automorphism group, so the distinguishing number is one. Conversely, if the distinguishing number is 1 then there is no non-trivial automorphism preserving the colouring, but since the 1-colouring does not impose any restriction on where the vertices can get mapped to, we have that there is no automorphism other than the trivial one.

□

2.2 The distinguishing number and the automorphism group

Lemma 3. *Let X be the set of all vertices in G that are fixed by all automorphisms in G, i.e.* $X = \{x \in V(G) | f(x) = x \text{ for all } f \in Aut(G)\}$*. Then,* $D(G) \leq Order(G) - |X| = |V(G) - X|$*. Where* $V(G) - X$ *denotes the set* $V(G)$ *without* X*.*

Proof. A coloring of the vertices in $V(G) - X$ with $|V(G) - X|$ colors will colour each vertex with a different color, so any automorphism preserving coloring will fix all vertices in $V(G) - X$. Also, vertices in X are fixed by all automorphisms by assumption. Hence, all vertices in V(G) are fixed by all automorphisms preserving coloring. Hence, an $|V(G) - X|$-coloring is distinguishable for all graphs. This is an upper bound for the distinguishing number although there might be a distinguishable coloring with fewer colours.

□

For not so trivial properties we can consider the orbits of the action of an element of aut(G) on the vertices, that is, the list of all vertices each vertex gets mapped to by some action of the automorphism group of a graph:

Definition 6. *The orbit of v is the set $O = \{x|f(v) = x, f \in A\}$. Where A is some action in Aut(G).*

Remark 1. *Note that orbits are either disjoint or coincident, that is, two vertices either have the exact same orbit or they lie in different and disjoint orbits.*

We can take the action to be the whole group Aut(G). Then the orbit in Aut(G) of a vertex is the set of all vertices it gets mapped to by all automorphisms in Aut(G).

Example 1. *Consider a graph S with 4 vertices $\{v_1, v_2, v_3, v_4\}$ where v_1 is adjacent to v_2, v_3 and v_4 and no other vertices are adjacent. Then the orbit of v_1 is $O_1 = \{v_1\}$ and the orbit of v_2, v_3 and v_4 is $O_2 = \{v_2, v_3, v_4\}$, since any $\phi \in Aut(G)$ fixes v_1 and possibly permutes v_2, v_3 and v_4.*

Definition 7. *The distinguishing coloring of an orbit O in a graph G by the action of Aut(G), denoted $D(O)$, is a coloring of the vertices in O where the only automorphism in Aut(G) preserving the coloring of the vertices in O is the trivial one.*
The distinguishing number of an orbit O is the smallest number d such that there exists a distinguishing coloring of O with d colours.

Example 2. *Going back to example 1, the distinguishing number of the different orbits are $D(O_1) = 1$ and $D(O_2) = 3$.*

Corollary 1. *The distinguishing number of an orbit O by $Aut(G)$ is at most the size of the orbit, $D(O) \leq |O|$.*

Proof. This yields from the proof of lemma 1. □

Definition 8. *A $max\{a_1, a_2, \cdots, a_n\}$-distinguishing coloring is a coloring with a_i colours for some $i \in [1, n]$ where $a_i \geq a_j$ for all $j \in [1, n]$.*

Lemma 4. *For all orbits $O_1, ...O_n$ of a graph G by the action of Aut(G) $D(G) \leq max\{|O_1|, ..., |O_n|\}$.*

Proof. We can colour each element in an orbit O_{max} of greatest length with a different colour. The same colours are enough for colouring the other orbits O_i with one colour for each element in an orbit since $|O_i| \leq |O_{max}|$ for all i. For any two vertices with the same colour they lie in different orbits, so no automorphism maps one to the other. So, any automorphism preserving this colouring fixes every element. □

The paper *Distinguishing number for graphs and groups* [1] generalises the notion of the distinguishing number for graph to a group action acting on a set X. We now use one of the results in the paper, Lemma 2.3, which gives:

Lemma 5. *For a graph G with an orbit O in Aut(G) with n-distinguishing colouring and such that $G - O$ has an m-distinguishing colouring, G has a $max[n, m]$-distinguishing colouring.*

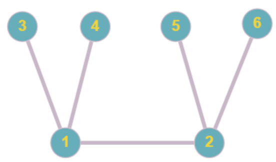
Proof. For $n \leq m$ we can colour O with the n-distinguishing colouring and $G - O$ with the m-distinguishing colouring relabeling them so that we use all the colours used in $O \in \{G - O\}$ as well, which gives a total of m colours used. For all vertices v in O, v cannot be sent to a vertex $v' \in O$ if $v \neq v'$ since we have a distinguishable coloring for O, it cannot be sent to a vertex outside of its orbit either, by definition 6. Similarly, for all vertices $w \in \{G - O\}$, w cannot be sent to a vertex $w' \in \{G - O\}$ if $w \neq w'$ since we have a distinguishable coloring for $G - O$, it cannot be sent to a vertex in O either. Hence, all vertices are fixed by a colour preserving automorphism. The same argument shows that for $m \leq n$ there is an n-distinguishable colouring of G. □

We can use the same argument to generalise lemma 5:

Corollary 2. *For all the orbits $O_1, \cdots, O_n$ of a graph G by the action of Aut(G) we let d_i be the distinguishing number of the orbit O_i. Then the distinguishing number of G is $D(G) \leq max\{d_1, \cdots, d_n\}$*

Note that we cannot yield that $D(G) = max\{d_1, \cdots, d_n\}$. As a counter example:

Example 3. *Consider the graph in the image below and denote it H. The orbits of H are* $O_1 = \{1, 2\}$ *and* $O_2 = \{3, 4, 5, 6\}$*. Hence, by corollary 2,* $D(H) \leq 4$*. However the colouring* $c_1 = [1, 3, 5]; c_2 = [2, 4, 6]$ *is a distinguishable colouring with two colours. Since Aut(H) is non-trivial,* $D(H) \neq 1$ *by lemma 1. Hence* $D(H) = 2$*.*



The paper *Symmetry Breaking in Graphs* [2] gives a discussion on various proofs of theorems relating group properties and the distinguishing number. To be able to introduce some results given in this paper we introduce the notion of a stabilizer and the Orbit-Stabilizer Theorem:

Definition 9. *The stabilizer of a vertex v is the set of elements g in Aut(G) that fix v, that is* $Stab(v) = \{g \in Aut(G) | g(v) = v\}$*.*

Theorem 1. *The Orbit-Stabilizer Theorem states that for a vertex v with orbit O and stabiliser Stab(v) in a graph G we have that:* $|O| = \frac{|Aut(G)|}{|Stab(v)|}$ *.*

Now, Corollary 4.1 in Symmetry Breaking in Graphs [2] gives the following:

Lemma 6. *If the size of the automorphism group of a group G has the size of some orbit of Aut(G) then the distinguishing number is at most 2.*

Proof. By the Orbit-Stabilizer Theorem, we have that, if the size of the orbit is $|aut(G)|$, then the stabilizer of all vertices in the orbit is 1, that is, trivial. So, any non-trivial automorphism in $aut(G)$ will send every vertex in the orbit to some other vertex in the orbit. By colouring one of the vertices w in the orbit with c_1 and all the rest with c_2 we have that any non-trivial automorphism will send w to a vertex with colour c_2 , so it won't preserve the colouring. □

Theorem 5 in Symmetry breaking in graphs [2] states the following result:

Theorem 2. *Consider the orbits of Aut(G) and take one vertex* x_i *from each orbit. If the intersection of all the stabilizers of* x_i *is trivial, then, the distinguishing number of G is 2.*

Proof. We can colour one vertex in each orbit with colour c_1 and all the rest with c_2 . Since the intersection of the stabilizers is trivial, no automorphism will fix all x_i, that is for all automorphisms there is an x_i that will not be fixed. Since x_i is not fixed it will be sent to some distinct vertex in its orbit, but x_i is the only vertex with colour c_1, so the coloring won't be preserved. □

Note that although the automorphism group of a graph is linked to the distinguishing number, it is not the case that the automorphism group determines the distinguishing number. Two different graphs can have the same automorphism group but have different distinguishing numbers.

Example 4. *To see this consider the complete graph* K_3*, which has automorphism group* S_3 *and distinguishing number 3 (a proof of this is in the examples section). Now consider the graph obtained from* K_3 *by joining a new vertex to each vertex in* K_3 *by an edge, as shown in figure 1 below, and call it* K_3*'. The automorphism group of* K_3*' is also* S_3 *but this time the distinguishing number is 2.*

Using the labeling in the image below we can colour the vertices of K_3*' like so:* $c_1 = \{1, 4, 5\}; c_2 = \{2, 3, 5\}$*. This gives a 2-distinguishable colouring of a graph with automorphism group* S_3*. An automorphism can only send vertices 2 and 4 to either 1 and 4 or 3 and 6 respectively. But neither of those pairs have the same colours as vertices 2 and 4. Hence, 2 and 4 are fixed. Vertices 1 and 4 can only be mapped to 3 and 6 respectively, but they don't have the same colouring, so they are all fixed too. The only map preserving adjacency and colouring is the identity. Hence, we have found two graphs with the same automorphism group but different distinguishing numbers.*

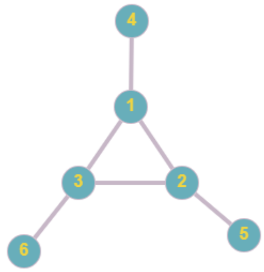


Figure 1

2.3 The distinguishing number and transitivity

We now have a way of relating the distinguishing number and transitivity as given in Definition 7.1.0.5.

Theorem 3. *For a graph G, if we have that $D(G) = Order(G)$ then G is transitive.*

Proof. We prove the contrapositive, if G is not transitive then $D(G) \neq Order(G)$. Let G be a non-transitive graph. Then, there are two vertices v and w such that there is no automorphism sending one to the other. Let O_v and O_w be the orbits of v and w under the action of $Aut(G)$. Then we have that, $w \notin O_v$ and $v \notin O_w$, which yields $|O_v| \leq Order(G) - 1$ and $|O_w| \leq Order(G) - 1$. Hence, by corollary 1, $D(O_v) \leq Order(G) - 1$ and $D(O_w) \leq Order(G) - 1$. Corollary 2 tells us $D(G) \leq max\{D(O_w), D(O_v)\} \leq Order(G) - 1$. Since G was arbitrary we have that if a graph is not transitive then $D(G) \leq Order(G) - 1$, which is an equivalent statement to theorem 3: If it is not the case that $D(G) \leq Order(G) - 1$ then $D(G) = Order(G)$ since $D(G) \leq Order(G)$ by lemma 1. Putting everything together we get that if $D(G) = Order(G)$ then G is transitive. □

Note that the converse is not true since a graph can be transitive but have distinguishing number not equal to its order.

Example 5. *As a counterexample consider the cycle C_6. C_6 is transitive since every vertex can be mapped to every other vertex. In the example section we prove that $D(C_6) = 2$, so $D(C_6) = 2 \neq 6 = Order(C_n)$*

The only assertion we can make about the distinguishing number of transitive graphs is that its distinguishing number is at most 2 since their automorphism group is non-trivial.

2.4 Monotonicity and anti-monotonicity of the distinguishing number

By Definition 6.1.0.1 from the notes, a numerical invariant $\mu(G)$ is ***monotone*** if $\mu(H) \leq \mu(G)$ for all SCaF subgraphs H of G. The distinguishing number is not monotone. As a counterexample, consider again the complete graph K_3 as a subgraph of the graph in figure 1, K_3'. We have that D(K_3)=3 while D(K_3') = 2, clearly $D(K_3') < D(K_3)$ and $K_3 \subseteq K_3'$, thus $D(G)$ is not monotone.

By Definition 6.1.0.2 from the notes we know that a numerical invariant is anti-monotone if $\mu(G) \leq \mu(H)$ for all SCaF subgraphs H of G. The distinguishing number is not anti-monotone either. As a counterexample consider the complete graphs K_4 and K_5. K_4 and K_5 have distinguishing numbers 3 and 4 respectively, a proof of this is in the examples section. K_4 is a subgraph of K_5 but $D(K_4) = 4 < 5 = D(K_5)$, so the distinguishing number is not anti-monotone.

2.5 Distinguishing colourings of cycles

We now focus our attention on cycles. We want to find a condition that will allow us to determine if a coloring of a cycle is distinguishable, i.e., a condition satisfied by all distinguishable colourings of cycles. In the examples section we prove that for n at least 6, a cycle with n vertices C_n has D(C_n) = 2, so we will only examine cycles of length at least 6. We start by introducing some notation. We partition a coloured cycle into its connected colour sets, S_i, that is, the sets of paths of vertices with the same colour in G. Since there are only two colours in G we can express G in terms of the lengths of their colour sets in clockwise order without having to specify the colour of each colour set.

Definition 10. *For a graph G with a given coloring we define its colour structure to be:* $s_1 \sim s_2 \sim s_3 \sim ... \sim s_n$*, where* s_i *is the length of a colour set* s_i*.*

Remark 2. *Note that, if we know the colour of any* s_i *then we know the colour of all colour sets, since every two adjacent sets are different colours and there are only two colours.*

Example 6. *A 6-cycle with coloring* $c_1 = [v_1, v_2, v_4]$,$c_2 = [v_3, v_5, v_6]$ *would have colour structure* $2 \sim 1 \sim 1 \sim 2$.

With this notation we can try to spot some symmetry and hence determine if a coloring is distinguishable, i.e. if we can find a pattern.

For a more systematic approach we can consider the colour structures for all vertices in a graph and compare them. We want to be able to tell exactly where a given vertex v lies in $s_1 \sim s_2 \sim s_3 \sim ... \sim s_n$ but without specifying where every vertex is. So, we take v to partition a colour set if it lies in the middle of it and we do not include v in our counting of the size of the colour set it lies in it. If the vertex before or after v (in clockwise order) is a different colour than v then we write ~ 0 or $0 \sim$ at the end or beginning of the colour structure respectively.

Example 7. *Consider the following colouring of the 10-cycle:* $c_1 = \{v_1, v_2, v_3, v_5, v_6, v_8\}$;$c_2 = \{v_4, v_7, v_9, v_{10}\}$, *where* v_i *is adjacent to* v_{i-1} *and* v_{i+1} *and* v_1 *is adjacent to* v_{10}*. The colour structure of* v_1 *in* C_{10} *with this colouring is* $2 \sim 1 \sim 2 \sim 1 \sim 1 \sim 2 \sim 0$.

Placing the vertices of the n-cycle on an n-gon we see that:

Remark 3. *The automorphism group of* C_n *is isomorphic to the dihedral group* D_n*.*

From which we get the following result:

Lemma 7. *For a cycle* C_n *with a given coloring we have:*

i) If two vertices in C_n *have the same colour and have the same colour structure then there is a rotation that sends one to the other and preserves the colouring.*

ii) If two vertices have the same colour and the same colour structure but reversed then there is a reflection swapping them that preserves the colouring.

Where rotations and reflections are elements of the dihedral group

Proof. The map can send the neighbours of v_1 to the neighbours of v_2 and viceversa and since they have the same colour structure this will be possible. The same can be done for the neighbours of the neighbours and adjacency and coloring will be preserved. If the structures are the same (as opposed to reversed) then the neighbours will be mapped preserving their position relative to v. If the colour structures are reversed then the map will send the neighbours in reverse position with respect to v.

Conversely, if a coloring is distinguishable at least two vertices will have the same colour structure or reversed. An automorphism f preserving adjacency will send the neighbors of a vertex v to the neighbors of $f(v)$ and will preserve their coloring. This will also be true for the neighbors, i.e. $f(v)$ will have the same colour structure as v. $\square$

This gives us a way of telling if a colouring is distinguishable or not, that is:

Corollary 3. *A coloring of a cycle is distinguishable if and only if no two vertices have the same colour and colour structure nor reversed.*

Example 8. *Consider the example of C_9 with the colouring $c_1 = \{v_1, v_3, v_4, v_6, v_7, v_9\}, c_2 = \{v_2, v_5, v_8\}$. Vertices v_1 and v_4 are the same colour and have the same colour structure: $0 \sim 1 \sim 2 \sim 1 \sim 2 \sim 1 \sim 1$. By Lemma 7 i) there is a rotation of the vertices preserving the coloring, an automorphism sending v_i to v_{i+4}. Adjacency will be preserved since v_i is adjacent to v_{i-1} and v_{i+1} and they get mapped to v_{i+3} and v_{i+5} which are neighbours of v_{i+4}.*

2.6 The distinguishing number for some examples

I will first consider the distinguishing number for simple cases, starting with ***cycles***. For cycles C_3, C_4, C_5 the distinguishing number is 3 but for greater cycles it is 2:

For C_3, with $V(C_3) = [v_1, v_2, v_3]$. The automorphism group of C_3 is non-trivial, e.g. $\phi(v_1) = v_2, \phi(v_2) = v_1, \phi(v_3) = v_3$ preserves adjacency and is not the identity. Hence, by lemma 2 the $D(C_3) \neq 1$. If we have two colours, without loss of generality, the coloring will be $c_1 = [v_i]$ and $c_2 = [v_j, v_k]$ where $i, j, k \in [1, 2, 3]$. Now, v_j and v_k have colour structures $1 \sim 1 \sim 0$ and $0 \sim 1 \sim 1$, which are the same but reversed, so by lemma 7 there is a reflection swapping them that preserves the colouring. Since there are no more colourings with two colours $D(C_3) \geq 3$. By lemma 1 ii) we also have that $D(C_3) \leq 3 = order(C_3)$, which yields $D(C_3) = 3$.

For C_4, if we try to use 2 colours to destroy its symmetries we run into problems. The only two options are having two pairs of vertices with the same colour each or having 3 vertices with one colour and one vertex a different colour. This gives 3 options in total, which are, without loss of generality:

$c_1 = [v_1, v_2]$; $c_2 = [v_3, v_4]$
$c_1 = [v_1, v_3]$; $c_2 = [v2, v_4]$
$c_1 = [v_1]$; $c_2 = [v_2, v_3, v_4]$

Where v_i is adjacent to v_{i+1}. Any other colouring with two three colours will be a relabelling of one of these three. All three options have symmetries in them: For the first option v_1 and v_2 have colour structures $1 \sim 2 \sim 0$ and $0 \sim 2 \sim 1$, which are the same but reversed. For the second option v_1 and v_3 have the same colour structure $0 \sim 1 \sim 1 \sim 1 \sim 0$. For the third option v_2 and v_4 have colour structures $2 \sim 1 \sim 0$ and $0 \sim 1 \sim 2$ which are the same but revered. So, since each of these pairs have the same colour, for each colouring there is a non-trivial automorphism preserving the colouring by lemma 7.There is no other colouring of C_4 with 2 colours, thus there is no distinguishing colouring of C_4 with 2 colours.

The 3-colouring of C_4, $c_1 = [v_1, v_2]$; $c_2 = [v_3]$; $c_3 = [v_4]$, has no non-trivial automorphisms that preserve the colourings. Suppose there is such an automorphism ϕ, then ϕ fixes v_3 and v_4, since they are the only ones in their respective colour set. For ϕ to be non-trivial we need $\phi(v_1) = v_2$ and that forces $\phi(v_2) = v_1$. v_1 is adjacent to v_4, so we need its image to be as well, but $\phi(v_1) = v_2$ is not adjacent to v_4. Hence, ϕ is not an automorphism, and so, there is no non-trivial automorphism preserving the colouring of the vertices. Hence, $c_1 = [v_1, v_2]$; $c_2 = [v_3]$; $c_3 = [v_4]$, is a distinguishing colouring and the distinguishing number of C_4 is 3.

C_5 has 3 possible colourings using two colours up to relabelling. These are:

$c_1 = [v_4]; c_2 = [v_1, v_2, v_3, v_5]$
$c_1 = [v_1, v_2]; c_2 = [v_3, v_4, v_5]$
$c_1 = [v_1, v_2, v_4]; c_2 = [v_3, v_5]$

Where v_i is adjacent to v_{i+1}. As for C_4 all options have symmetries in them. It can be checked that v_1 and v_2 have the same colour structure but reversed for all three options. So we have that $D(C_n) \geq 3$.

The colouring $c_1 = [v_1]; c_2 = [v_2, v_4]; c_3 = [v_3, v_5]$ is distinguishable. A colour preserving automorphism ϕ will fix v_1 since it is the only vertex in its colour set. ϕ can swap v_2, v_4 or v_3, v_5 or both. v_2 cannot be sent to v_4 since v_4 is not adjacent to $\phi(v_1) = v_1$, so v_2 and v_4 are fixed. Similarly, $v3$ cannot be sent to v_5 since

v_5 is not adjacent to $\phi(v_2) = v_2$, so v_3 and v_5 are fixed and ϕ is the identity. Since ϕ was arbitrary, we have that the colouring is distinguishable and $D(C_5) = 3$.

For cycles of length greater than or equal to 6 the distinguishing number is 2. For a cycle of length 6 we can have the colouring $c_1 = [v_1, v_3, v_4]$: $c_2 = [v2, v5, v6]$ where v_i is adjacent to vi+1 and v_6 is adjacent to v_1. We can compute the colour structures of each vertex, which are:

$v_1 : 0 \sim 1 \sim 2 \sim 2 \sim 0$
$v_2 : 0 \sim 2 \sim 2 \sim 1 \sim 0$
$v_3 : 1 \sim 2 \sim 1 \sim 1 \sim 0$
$v_4 : 0 \sim 2 \sim 1 \sim 1 \sim 1$
$v_5 : 1 \sim 1 \sim 1 \sim 2 \sim 0$
$v_6 : 0 \sim 1 \sim 1 \sim 2 \sim 1$

At first glance it seems like v_1 and v_2 have the same structure but reversed, but since v_1 and v_2 are different colours their structures have the colours swapped, so no automorphism swapping them will preserve colouring. The same works for v_4 and v_5, they have different colours. v_3 and v_6 also have different colours. So, the colour structure of each vertex is unique, and so, by corollary 3 the colouring is distinguishable. Hence, $D(C_6) = 2$.

Now we discuss the distinguishing number for cycles of length n at least 7. First we note that an n-cycle will have automorphism group isomorphic to the dihedral group D_n, as given in the first remark in chapter 2.4 . We chose a coloring that will destroy the symmetries of D_n, that is, its rotations and reflections. To do that we chose to have only one colour set with a single vertex v_1, so that it has to be fixed, and another colour set with the same colour with two vertices not opposite to v1, so that a reflection won't swap them. We get, for example, $c_1 = [a_2, a5, \cdots, a_n]$; $c_2 = [a_1, a_3, a_4]$. Note that a_3 and a_4 are not opposite to a_1 since n is at least 7. The colour structures of the vertices with the colour of the vertex given in brackets are:

$a_1(c_2) : 0 \sim 1 \sim 2 \sim n-4$
$a_2(c_1) : 0 \sim 2 \sim n-4 \sim 1 \sim 0$
$a_3(c_2) : 1 \sim n-4 \sim 1 \sim 1 \sim 0$
$a_4(c_2) : 0 \sim n-4 \sim 1 \sim 1 \sim 1$
$a_5(c_1) : n-5 \sim 1 \sim 1 \sim 2 \sim 0$
$a_i(c_1) : n-i \sim 1 \sim 1 \sim 2 \sim i-5$ for $i \in [6, n-1]$
$a_n(c_1) : 0 \sim 1 \sim 1 \sim 2 \sim n-5$

For the colour structures of a_i and a_5 to be the same we would have that $n-5 = n-i$ giving that $i = 5$ which is not in the range specified $[6, n-1]$. For the colour structures of a_n and a_3 to be the same we would have that $n-5 = 0$ giving that $n = 5$ contradicting that n is at least 7. For the colour structures of an and a_i to be the same we would have that $n-i = 0$ giving that $i = n$, which is not in the range $[6, n-1]$. By inspection we see that no two other colour structures of same coloured vertices are the same nor the same but reversed, so, the 2-colouring is distinguishable and $D(C_n) = 2$ for n at least 6.

Then, the answer to the puzzle by Frank Rubin is that if we have 3, 4 or 5 keys in the ring then we need three colors to distinguish them, if we have 2, 6 or more than 6 keys then we only need two colors and if we have one key then obviously one colour is enough.

The ***complete graph*** K_n as given in definition 1.2.0.1 from the notes has automorphism group isomorphic to Sn, since every vertex can be sent to every other vertex by some automorphism f because $f(v)$ is adjacent to $f(v')$ for all v and all v' adjacent to v. For a colouring with $n-1$ colours, we would have 2 vertices with the same colour, and so there would exist an automorphism that swapped them and left every other vertex fixed, which would preserve the coloring. So, we need n colours so that each vertex has a unique colour and there is no non-trivial automorphism preserving the colourings. Hence, the distinguishing number of K_n is n.

Consider the ***complete bipartite graph*** $K_{m,n}$ as given in definition 1.3.0.3 from the notes. Assume, without loss of generality, that $m \leq n$. Let $[a_1, \cdots, a_m]$ be the m independent vertices in $K_{m,n}$ and $[b_1, \cdots, b_n]$ be the n independent vertices in $K_{m,n}$ such that each vertex a_i in $[a_1, \cdots, a_m]$ is adjacent to every vertex b_i in $[b_1, \cdots, b_n]$. We consider two cases:

- When $m \neq n$ an automorphism cannot send a_i to b_i since $deg(a_i) = n$ and $deg(b_j) = m$ but $m \neq n$. An automorphism can send any a_i to any a_j and any b_i to any b_i since adjacency is preserved because all a_i and all b_i are adjacent to the same vertices respectively. So, the orbits of $K_{m,n}$ are $O_a = [a_1, ..., a_m]$ and $O_b = [b_1, ..., b_n]$ which have sizes m and n respectively. Hence, $D(O_a) \leq m$ and $D(O_b) \leq n$ by corollary 1. A colouring of O_a with $m-1$ colours would have two vertices with the same colour and an automorphism could swap them, so the colouring wouldn't be distinguishable, so $D(O_a) = m$. Using the same argument we have that $D(O_b) = n$. By corollary 2 $D(Km, n = max\{D(O_b), D(O_a)\} = max\{m, n\} = n$ since $m \geq n$ by assumption.
- In the case where $m = n$ we still need at least n colours, one for each pair a_i, b_i, but this time there is an automorphism sending all a_i to b_i and all b_i to a_i, which would preserve the colouring. Thus, n colours are not enough, we need $n+1$ colours. We can colour each a_i with a different colour and b_i with the same colour as a_i for all $i \in [1, n-1]$ and a_n and b_n with two other different colours, giving a total of $n+1$ colours. This colouring is distinguishable. A non-trivial automorphism, f, preserving colouring would have to swap some a_j and b_j for some $j \in [1, n-1]$, since they are the only vertices with the same colour, but in order to preserve adjacency all a_i have to be swapped with b_i for all $i \in [1, n-1]$. Concretely, $f(a_n) = b_n$, but a_n and b_n have different colours, so the colouring is not preserved by f. Since f was arbitrary, the only automorphism preserving the colouring is the identity. Therefore, the colouring we defined is distinguishable and the distinguishing number of $K_{n,n}$ is $n+1$.

To calculate the distinguishing number of some of the ***co-prime graphs*** N_n, as given in definition 1.4.0.1 from the notes, we first consider its automorphisms. We start with the simple cases:

For $n = 1$ the automorphism is trivial so, $D(N_1) = 1$.

For $n = 2$ there is an automorphism swapping the two vertices, so $D(N_2) = 2$.

For $n = 3$ there is an automorphism swapping 2 and 3, but 1 has to be fixed, so $D(N_3) = 2$.

For $n = 4$ 2 and 4 can be swapped by an automorphism and 1 and 3 have to be fixed, so $D(N_4) = 2$.

For $n = 5$ 3 and 5 are leafs adjacent to 1, so an automorphism can swap them. We also have that 2 and 4 can be swapped. Colouring 2 and 4 different colours and 3 and 5 different colours would make all vertices distinguishable. We can use the same colour for 2 and 3 and for 4 and 5, since they lie in different orbits. So, $D(N_5) = 2$.

For n=6 there is only one leaf, 3, so it is fixed. 2 and 6 are both adjacent to 3 and 1, so they can be swapped. So, the automorphism group is not trivial and we need at least two colours. Colouring all vertices except 6 with the same colour will make all vertices distinguishable, so $D(N_6) = 2$.

More generally, we observe that if there is a prime smaller than n but such that $2p$ is larger than n, then p will only be adjacent to v_1. If we can find two different prime numbers p_1 and p_2 between $n/2$ and n then there exists an automorphism sending p_1 to p_2 and fixing everything else, the two primes will be leafs of 1 so they can be swapped. If that is the case the automorphism group is non-trivial, so we need at least two colours.

5 and 7 are primes, so for n between 7 and 9 $D(N_n)$is at least 2.
7 and 11 are primes, so for n between 11 and 13 $D(N_n)$ is at least 2.
11 and 13 are primes, so for n between 13 and 21, $D(N_n)$ is at least 2.
17 and 19 are prime, so for n between 19 and 33 $D(N_n)$ is at least 2.
29 and 31 are primes, so for n between 31 and 57 $D(N_n)$ is at least 2.

It looks like, for most n there will be at least 2 leafs adjacent to 1, making the automorphism group non-trivial and the distinguishing number at least 2. For $n = 10$ this is not the case. N_{10} has no non-trivial automorphisms, so $D(N_{10}) = 1$. We do not get a general formula, but combining the results gives:

For $n = 1, 10; D(N_n) = 1$
For $n = 2, 3, 4, 5, 6; D(N_n) = 2$
For $n \in (7, 57)$; $D(N_n) \geq 2$

For the ***Kneser graph*** KG_n the only boundary we have is the order of the graph, $D(KG_n) \leq size(KG_n) = n$. For $n = 5, 6$ $D(KG_n)$ is at least two since the automorphism group is non-trivial.

The calculation of the distinguishing number for ***GB*** is straightforward. We know, from the section 1.3 that GB has no non-trivial automorphisms, hence, by lemma 2 $D(GB) = 1$. For any colouring at all, the only automorphism preserving colouring will be the identity. This gives that the smallest number of colours in a distinguishing colouring of GB is one.

3 Reflective section

Quite frankly, the most difficult part of this project has been using LaTeX. I started using Google Documents but changed to LaTeX for aesthetic reasons, which meant inserting thousands of dollar signs and underscores. Other than that I have really enjoyed working through the assessment.

The properties from section 1.1, section 1.2, the automorphisms, the eccentricities, the domination number, the edge connectivity, the planarity, the homology, etc. were all quite straightforward to determine. The domination number was quite easy to calculate since I found a dominating set with two vertices, so it was easy to show that it was the smallest one. It was also an independent set, so it was easy to show that $\iota(GB) = 2$. The eccentricities for all vertices except two of them was 2, so it was easy to argue that the paths I found were of minimal length.

My mathematics project is on simplicial complexes and homology so I was initially gonna choose graph homology as my invariant but ended up changing to the distinguishing number since I felt there was more to say about it. Even so, I found it appropriate to include a short discussion of graph homology at the end.

I enjoyed coming up with arguments for the number of independent sets of 6 elements, the total independence number, the chromatic number (using the last method), the number of dominating sets of two elements, the connectivity, the domatic number and for showing that GB is not a join. Finding the connectivity of GB was a bit tricky since $\delta(GB) = 3$ so I had to show that $\kappa(GB) \geq 3$. Having worked through the total domination number made this a lot easier.

When working through the connectivity I used disjoint total dominating sets so I wondered how many of them there could be in GB, after doing a bit of research I found the domatic number is a very similar numerical invariant, so I considered it for GB too. That's why I had the domatic number in a rather strange place, instead of having it in the domination number section.

I found the determination of the clique number, the independence number and the chromatic index to be the most time-consuming. To find the clique number I used my own argument, which was different from the one in the notes, and took less time to find the answer but more time to write down. When finding the chromatic number and the circumference, I had to try a few algorithms before I was able to find the ones that worked.

The total edge domination number and the (total) complete domination number were more challenging and working on them was amusing but the arguments were difficult to structure nicely.

Writing the discussion of the distinguishing number was my first time writing mathematics formally. The literature I found was at most times quite technical and used concepts we hadn't covered in lectures but some parts I was able to use for my discussion. It was gratifying to apply the group theory concepts I know to the discussion of the distinguishing number.

References

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[2] Albertson, M. and Collins, K., 1996. Symmetry Breaking in Graphs. The Electronic Journal of Combinatorics, 3.