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Hongdou Li

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A **time series** is a **stochastic process** indexed by time. Specially we have a sequence of random variables.

$\{Y_t : t \in N\} = \{Y_0, Y_1, Y_2...\} = \{Y_t\}$

lower case y are observed random variables. the sequence $\{y_t\} = \{y_1, y_2, y_3...\}$ represent an observed time series ehat y_t is a realization of the random variable Y_i (stochastic process: sequence of random variables)

The idea is to understand the probabilistic behavior of $\{Y_t\}$ in order to model/forecast $\{y_t\}$

A time series model is the specification of the joint distribution of the random variables in $\{Y_t\}$:

$$P(Y_1 \leq y_1, Y_2 \leq y_2, ..., Y_n \leq y_n) \text{ where } y_1, y_2, y_3... \in R$$

(however, it is not practical in practice. because when data set increases there will be too much parameters.()

However, specifying such a joint distribution is typically infeasible in practice since there typically won't be enough information to estimate all of the parameters required.

But: Most of a distributions characteristics can be described with 1^{st} and 2^{nd} moments:

- $E[Y_t], t = 1, 2, ...$ (means)
- $E[Y_t, Y_{t+h}], t = 1, 2, ...h >= 0$ (variance covariance)
- special case : $\{\bar{Y_t}\} \sim MVN(\bar{\mu}, \Sigma)$

MVN :multivariate normal

$\mu_i = E[Y_i]$, $\Sigma_{ii} = Var(Y_i)$, $\Sigma_{ij} = Cov[Y_i, Y_j]$

Take away: we don't need the whole joint distribution of $\{Y_t\}$, we can rely on knowing the 1^{st} and 2^{nd} moments there will be a loss of information, but it won't be substantial.

Zero-Mean Models

• IID Noise:

$\{\epsilon_t\} \sim IID(0, \sigma^2)$

- $E[\epsilon_t] = 0$
- $Var[\epsilon_t] = \sigma^2$
- ϵ_t and ϵ_{t+h} (for h > 0) are independent
- ϵ_t all follows the same distribution.

• White Noise:

$\{\epsilon_t\} \sim WN(0, \sigma^2)$

- $E[\epsilon_t] = 0$
- $Var[\epsilon_t] = \sigma^2$
- $Cov(\epsilon_t, \epsilon_{t+h}) = 0$ (for h > 0)
- IID is a special case of WN

• Random Walk:

$\epsilon_t = \begin{cases} +1 & with\ prob\ 1/2 \\ -1 & with\ prob\ 1/2 \end{cases}$ independently

$\{S_t\}$ where $S_t = \sum_{k=1}^{t} \epsilon_k$ and $S_0 = 0$

This is zero -mean because $E[S_t] = \sum_{k=1}^{t} E[\epsilon_t] = \sum_{k=1}^{t} [1/2 + (-1)1/2] = 0$

Stationary

A time series $\{Y_t\}$ is said to **strictly stationary** if the joint distribution of $Y_{t1}, Y_{t2}, Y_{t3}, ..., Y_{tn}$ is the same as that of $Y_{t1+h}, Y_{t2+h}, Y_{t3+h}, ..., Y_{tn+h}$ for $t1, t2, t3...tn, h \in N$. In other words $\{Y_t^2\}$ is strictly stationary if **all** of its statistical properties are preserved under time shifts.

Verifying the strict stationary conditions is difficult in practice and so we will use a weaker definition of stationary defined in terms of the 1^{st} and 2^{nd} moments.

• Define first:

- Mean function:

$\mu(t) = E[Y_t]$

- Covariance Function:

$$\gamma(r, s) = Cov(Y_r, Y_s) = E(Y_r, Y_s) - E[Y_r]E[Y_s]$$

A time series $\{Y_t\}$ is said to weakly stationary if :

(i) $\mu(t) = E[Y_t]$ must be independent of t.

(ii) $\gamma(t, t+h) = Cov(Y_t, Y_{t+h})$ must be independent of t for all h

- covariance only depends on the distance h , not t.

This for a weakly stationary time series $\mu(t) \equiv \mu$ and $\gamma(t, t+h) \equiv \gamma(h)$

• Example: Is IID$(0, \sigma^2)$ (weakly) stationary?

(i) $\mu(t) = E[\epsilon_t] = 0$ (independent of t)

(ii) $\gamma(t, t+h) = Cov(\epsilon_t, \epsilon_{t+h}) = \begin{cases} \sigma^2 & h = 0 \\ 0 & h > 0 \end{cases}$ (independent of t)

thus, IID is weakly stationary.

more: By similar arguments $WN(0, \sigma^2)$ is weakly stationary.

• Eaxmple: Is $\{S_t\}$ (the random walk) with $S_t = \sum_{k=1}^{t} \epsilon_k$, $\{\epsilon_k\} \sim IID(0, \sigma^2)$ weakly stationary?

(i) $\mu(t) = E[S_t] = 0$

(ii) $\gamma(t, t+h) = Cov(S_t, S_{t+h})$

$$= Cov(S_t, S_t + \epsilon_{t_1} + \epsilon_{t_2} + \epsilon_{t+3} ... + \epsilon_{t+h})$$

$$= Cov(S_t, S_t) + Cov(S_t, \epsilon_{t+1}) ... + Cov(S_t, \epsilon_{t+h})$$

$$= var(S_t)$$

$$= E[S_t^2] - E[S_t]^2$$

$$= E[(\textstyle\sum_{k=1}^{t} \epsilon_k)^2]$$

$$= E[\textstyle\sum_{k=1}^{t} [\epsilon_k^2]] + 2 \sum_{i<j} E[\epsilon_i \epsilon_j]$$

$=\sum_{k=1}^{t} Var[\epsilon_k] + 2\sum_{i<j} Cov(\epsilon_i, \epsilon_j)$

$=\sum_{k=1}^{t} \sigma^2 = t\sigma^2$

Thus, the random walk is not stationary.

• Example: First order moving average: $\{Y_t\} \sim MA(1)$

$Y_t = \epsilon_t + \theta\epsilon_{t-1}$

Where $\theta \in R$ and $\{\epsilon_t\} \sim wn(0, \sigma^2)$. Show that $\{Y_t\}$ IS weakly stationary.

(i) $\mu(t) = E[Y_t] = E[\epsilon_t + \theta\epsilon_{t-1}] = E[\epsilon_t] + \theta E[\epsilon_{t-1}] = 0$

(ii)

$\gamma(t, t+h) = cov(Y_t, Y_{t+h})$

$=Cov(\epsilon_t + \theta\epsilon_{t-1}, \epsilon_{t+h} + \theta\epsilon_{t+h-1})$

$=cov(\epsilon_t, \epsilon_{t+h}) + \theta Cov(\epsilon_t, \epsilon_{t+h-1}) + \theta Cov(\epsilon_{t-1}, \epsilon_{t+h}) + \theta^2 Cov(\epsilon_{t-1}, \epsilon_{t+h-1})$

$$=\begin{cases} \sigma^2(1+\theta^2) & h = 0 \\ \theta\sigma^2 & h = 1 \ \text{\textbackslash end\{center\}} \\ 0 & h > 1 \end{cases}$$