Nov1 Note

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Recap:

- $\{Y_t\} \sim AR(p)$ if

  $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \epsilon_t$

  $\phi(B)Y_t = \epsilon_t$

  where $\phi(Z) = 1 - \phi_1 Z - \phi_2 Z^2 - ... - \phi_p Z^p$
- $\{Y_t\}$ ~ $\sim MA(q)$ if

  $Y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + ... + \theta_q \epsilon_{t-q}$

  $Y_t = \theta(B)\epsilon_t$

  where $\theta(Z) = 1 + \theta_1 Z + \theta_2 Z^2 + ... + \theta_q Z^q$
- $\phi(Z)$ and $\theta(Z)$ are **"generating functions"** AKA **"characteristic polynomials"**

Mathematical Prerequisites

A **power series** is an infinite sum representation of a function

$f(x) = \sum_{n=0}^{\infty} a_n x^n$

- EX1:

  $e^x = sum_{n=0}^{\infty} \frac{x^n}{n!}$ (exponential series)
- EX2:

  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ (geometric series) Converges if |x|<1
- EX3:

  $f(x) = \sum_{n=0}^{\infty} \frac{f^(n)(a)(x-a)^n}{n!}$ (Taylor Series)
- EX4:

  $\frac{1}{1-x+x^2-2x^3} = \frac{1}{1-(x-x^2+2x^3)} = \sum_{n=0}^{\infty}(x - x^2 + 2x^3)^n$ converage if $|x - x^2 + 2x^3| < 1$

Complex Numbers:

$\sqrt{-1} \equiv i$ (imaginery number)

A **complex number** can be generally written as:

$Z = a + bi \in C$ where $a, b \in R$, a is the real part and bi is the complex part.

(there is a graph)

- |Z| $>$ 1 implies that Z lies outside the unite circle
- |Z| $\leq$ 1 implies that Z lies on/insideside the unite circle

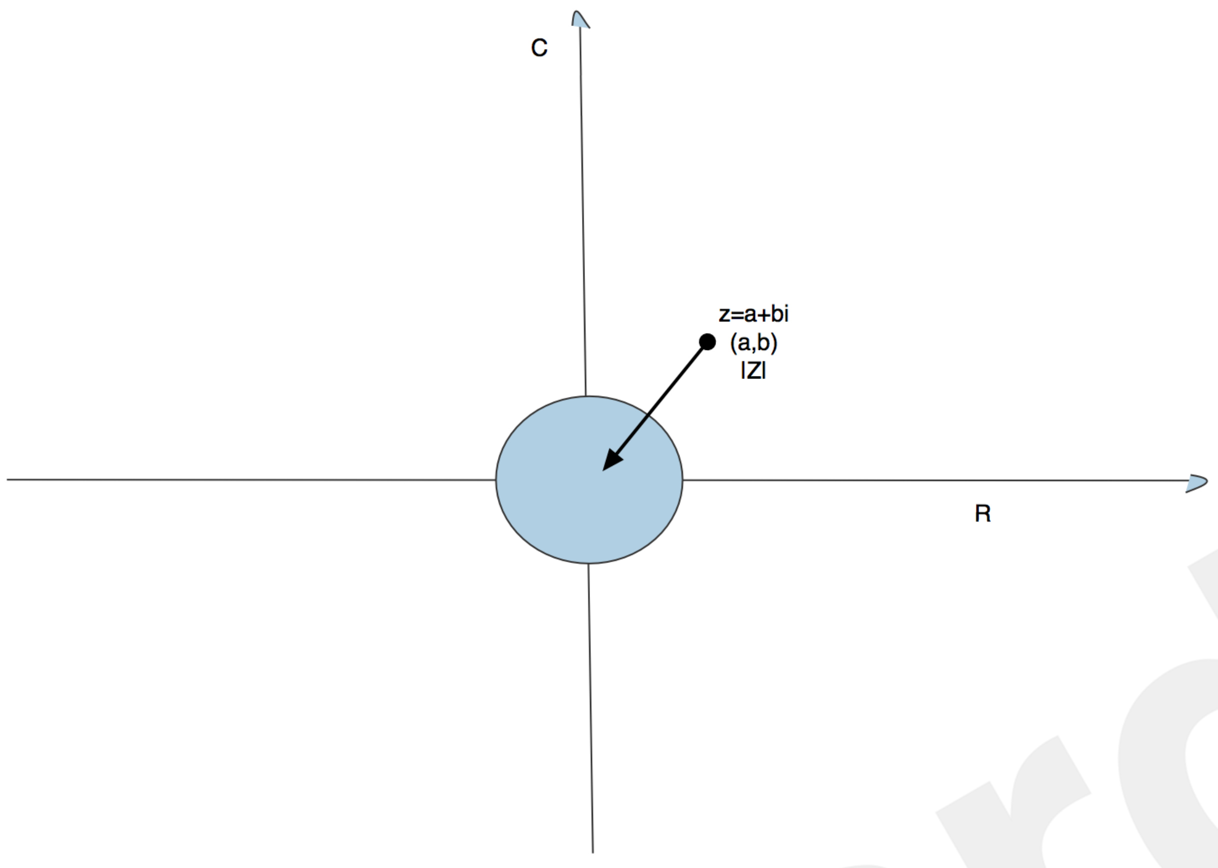

Figure 1: A caption

Remarks

- MA(q) is stationary for all q
- $AR(p) = MA(\infty)$ if "stationary conditions" hold
- $MA(q) = AR(\infty)$ if "invertibility conditions" hold

Stationarity Conditions

we know $\phi(B)Y_t = \epsilon_t$, $Y_t = \frac{1}{\phi(B)}\epsilon_t$ (1) (inverse the function)

because any function has a powerseries representation we know

$\frac{1}{\phi(B)} = \sum_{n=0}^{\infty} \psi_n B^n = \psi_0 + \psi_1 B + \psi_2 B^2 + ... \equiv \psi(B)$

let's substitute this into equation(1)

$$\begin{aligned} Y_t &= \psi(B)\epsilon_t \\ &= (\psi_0 + \psi_1 B + \psi_2 B^2 + ...)\epsilon_t \\ &= \psi_0\epsilon_t + \psi_1\epsilon_{t-1} + \psi_2\epsilon_{t-2} + \psi_3\epsilon_{t-3} + ... \end{aligned}$$

if $\psi_0 = 1$, this looks like $MA(\infty)$.

In order for this to be useful, we require that $\psi_0\epsilon_t + \psi_1\epsilon_{t-1} + \psi_2\epsilon_{t-2} + \psi_3\epsilon_{t-3} + ...$ converges. This converges if $\sum_{n=0}^{\infty} \psi_n Z^n$ converges which happens if and only if the zeros(roots) of $\psi(Z)$ lie outside the unit circle in the complex plane.

Thus, an AR(p) model can be written as an $MA(\infty)$ model is $\phi(Z)$, the AR generating function, has zeros outside the unit circle in the complex plane.

$$\phi(z) \neq 0 \text{ for any Z such that } |Z| \leq 1$$

or equivalently $\phi(Z) = 0$ only for Z such that $|Z| > 1$ (stationary condition)

Invertibility Conditions

We know $Y_t = \theta(B)\epsilon_t$ if $\{Y_t\} \sim MA(q)$

$\frac{1}{\theta(B)} Y_t = \epsilon_t$ (2)

Because any function has a power series representation we know

$\frac{1}{\theta(B)} = \sum_{n=0}^{\infty} \lambda_n B^n = \lambda_0 + \lambda_1 B + \lambda_2 B^2 = ... \equiv \lambda(B)$

substituting this into equation(2) yields:

$\lambda(B)Y_t = \epsilon_t$

$(\lambda_0 + \lambda_1 B + \lambda_2 B^2 + ...)Y_t = \epsilon_t$

$\lambda_0 Y_t + \lambda_1 Y_{t-1} + \lambda_2 Y_{t-2} + ... = \epsilon_t$

if $\lambda_0 = 1$, this looks like $AR(\infty)$

Inorder for this to be useful we require $\lambda_0 Y_t + \lambda_1 Y_{t-1} + \lambda_2 Y_{t-2} + ...$ to converge which happens if and only if the zeros of $\theta(Z)$ lie outside the unit circle in the complex plane.

Thus an MA(q) model is invertible iff:

$\theta(z) \neq 0$ for any Z such that $|Z| \leq 1$

or equivelently $\theta(Z) = 0$ only for z such that |Z| > 1