

Non-linear Dynamics

Fernando Freire

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Script 0.0.1 (python)

```
1 import warnings
2 warnings.filterwarnings("ignore")
3 import networkx as nx
4 import math as math
5 import pickle
6 import pandas as pd
7 import matplotlib.pyplot as plt
8 import matplotlib.lines as mlines
9 import matplotlib as mpl
10 from matplotlib import colors
11 %matplotlib inline
```

1 Bifurcations of equilibria: pitchfork bifurcation.

Consider the two dynamical systems:

$$\dot{x} = rx + x^3$$

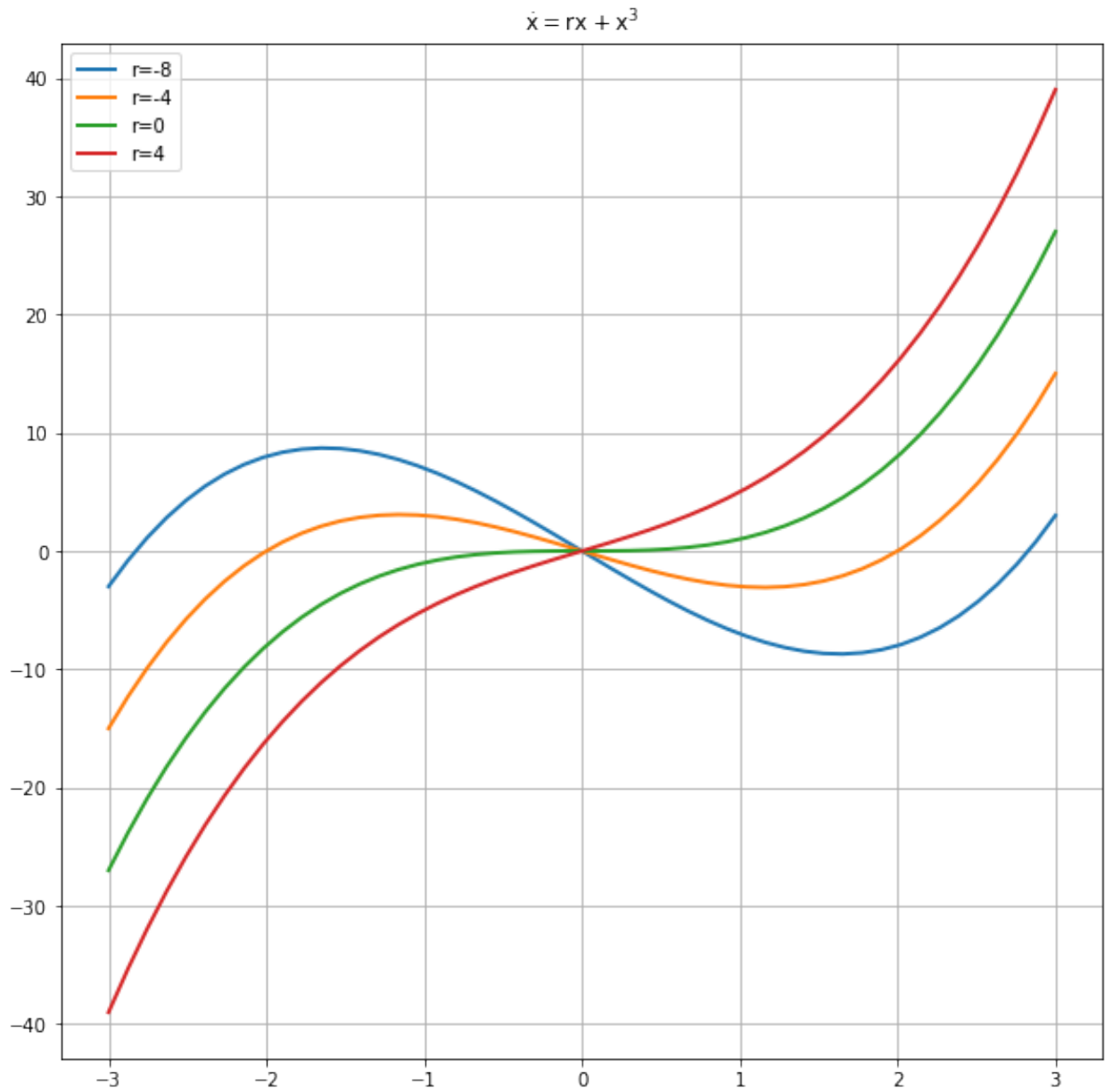
$$\dot{x} = rx - x^3$$

where r and x can take negative values. Calculate their equilibrium points and stability as function of parameter r and plot the bifurcation diagram

1.1 Equilibrium values of $\dot{x} = rx + x^3$

Script 1.1.1 (python)

```
1 r = -4
2 x = np.linspace(-3, 3, num = 50)
3 #print(x)
4 plt.figure(figsize=(10,10))
5 plt.grid()
6 for r in [-8, -4, 0, 4]:
7     f_x = r * x + x ** 3
8     plt.plot(x, f_x, linewidth=2.0, label="r=%d"%(r,))
9 leg = plt.legend(loc='best', ncol=1)
10 _ = plt.title("$ \dot{x} = rx + x^3 $")
```



$$f(x) = rx + x^3 = x(r + x^2) = 0$$

so, the roots are the solutions of:

$$\begin{aligned} x &= 0 \\ x^2 + r &= 0 \end{aligned}$$

therefore, we have three possible values of x that hold equilibrium points

$$\begin{aligned} x &= 0 \\ x &= \sqrt{-r} \\ x &= -\sqrt{-r} \end{aligned}$$

checking the sign of $f'(x) = 3x^2 + r$ in each point.

if $r < 0$, we have three real roots and so three equilibrium points:

$$x = 0, f'(0) = r < 0 \Rightarrow \text{stable}$$

$$x = \sqrt{-r}, f'(\sqrt{-r}) = -3r + r = -2r > 0 \Rightarrow \text{unstable}$$

$$x = -\sqrt{-r}, f'(-\sqrt{-r}) = -3r + r = -2r > 0 \Rightarrow \text{unstable}$$

if $r > 0$, we have only one real root:

$$x = 0, f'(0) = r > 0 \Rightarrow \text{unstable}$$

if $r = 0$, we have to take into account second order and third order term in Taylor development: $f''(x) = 6x$, also goes to 0 in 0, but $f'''(x) = 6 > 0$.

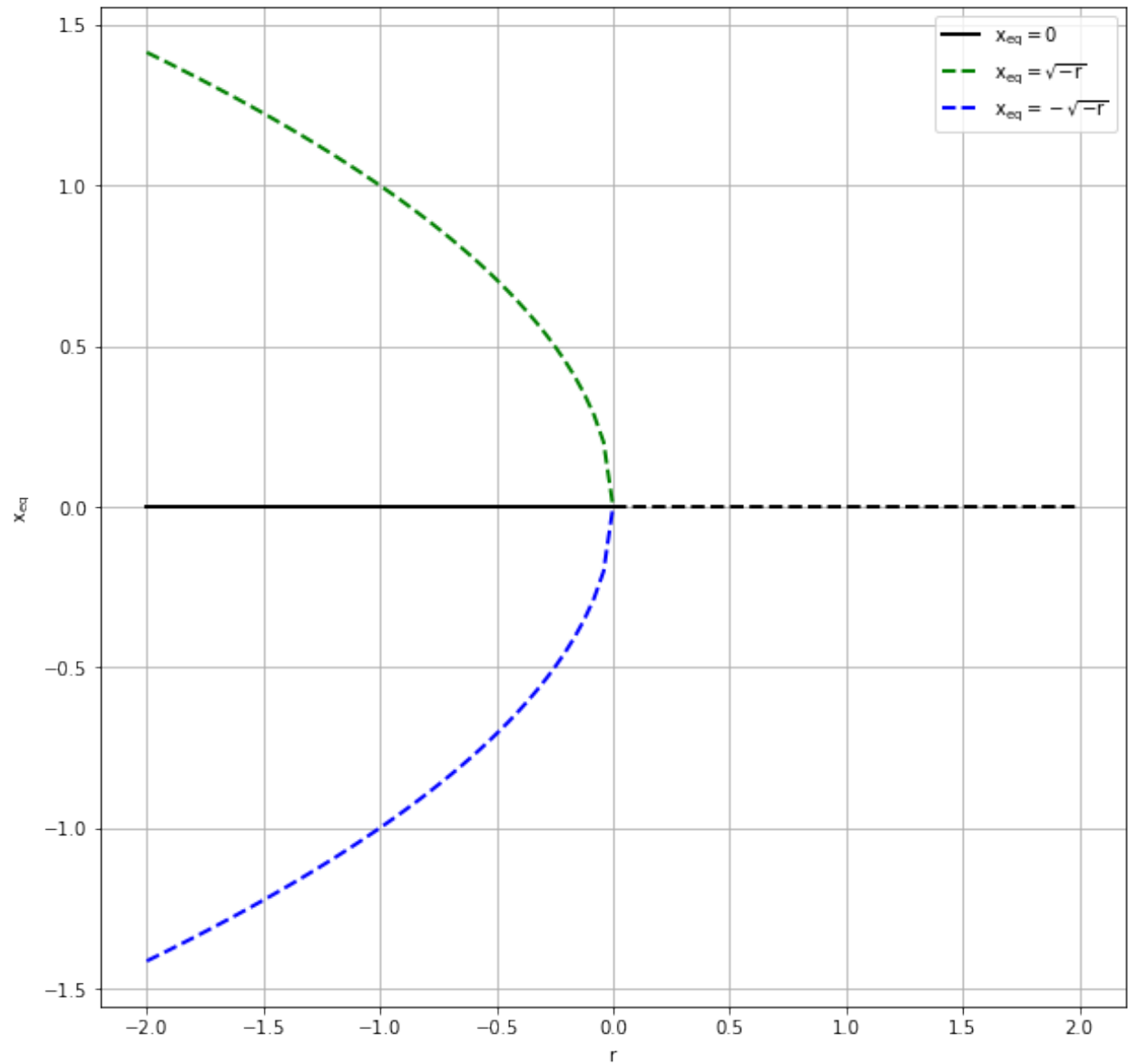
So, very near of 0: $f(x) = 6x^3/3!$ that is positive if $x > 0$ and negative if $x < 0$, so f is monotonically increasing around 0, and then x is an unstable equilibrium point.

This result is coherent with the observed in the graph (flux vectors).

1.2 Bifurcation diagram of $\dot{x} = rx + x^3$

Script 1.2.1 (python)

```
1 r = np.linspace(-2, 0, num = 50)
2 #print(x)
3 plt.figure(figsize=(10,10))
4 plt.grid()
5 plt.plot(r, np.zeros(50), linewidth=2.0, color='black', label="$ x_{eq} = 0$")
6 plt.plot(r, np.sqrt(-r), linewidth=2.0, linestyle = 'dashed',color='green', label="$ x_{eq} \to = \sqrt{-r}$")
7 plt.plot(r, -np.sqrt(-r), linewidth=2.0, linestyle = 'dashed',color='blue', label="$ x_{eq} \to = -\sqrt{-r}$")
8 r = np.linspace(0, 2, num = 50)
9 plt.plot(r, np.zeros(50), linewidth=2.0, color='black', linestyle = 'dashed')
10
11 leg = plt.legend(loc='best', ncol=1)
12 _ = plt.xlabel("$ r $")
13 _ = plt.ylabel("$ x_{eq} $")
```



1.3 Equilibrium values of $\dot{x} = rx - x^3$

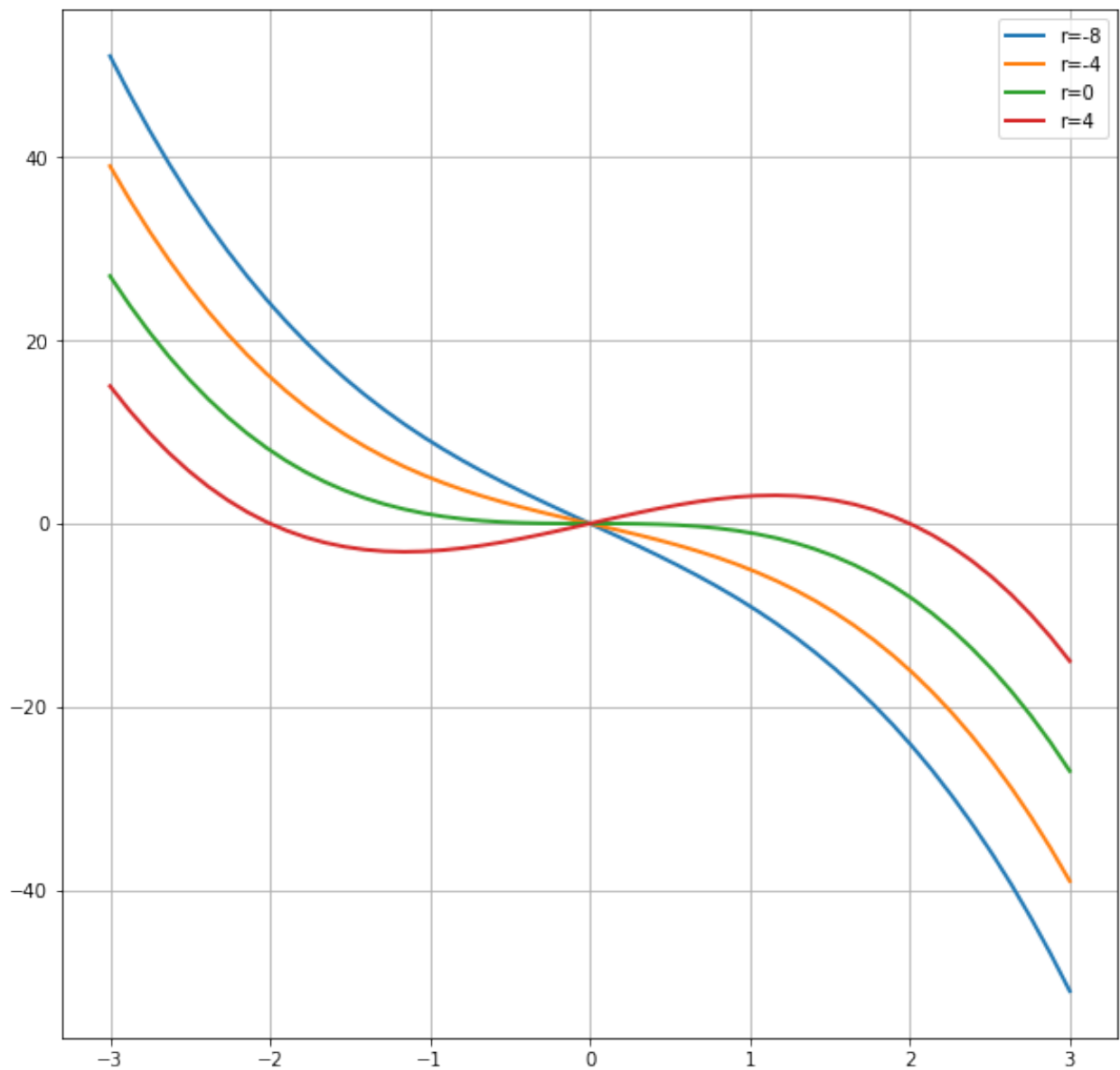
Script 1.3.1 (python)

```

1 x = np.linspace(-3, 3, num = 50)
2 #print(x)
3 plt.figure(figsize=(10,10))
4 plt.grid()
5 for r in [-8, -4, 0, 4]:
6     f_x = r * x - x ** 3
7     plt.plot(x , f_x, linewidth=2.0, label="r=%d"%(r,))

```

```
8 leg = plt.legend(loc='best', ncol=1)
```



$$f(x) = rx - x^3 = x(r - x^2) = 0$$

so, the roots are the solutions of:

$$\begin{aligned} x &= 0 \\ -x^2 + r &= 0 \end{aligned}$$

therefore, we have three possible values of x that hold equilibrium points

$$\begin{aligned}x &= 0 \\x &= \sqrt{r} \\x &= -\sqrt{r}\end{aligned}$$

checking the sign of $f'(x) = -3x^2 + r$ in each point.

if $r > 0$, we have three real roots and so three equilibrium points:

$$\begin{aligned}x = 0, f'(0) &= r > 0 \Rightarrow \text{unstable} \\x = \sqrt{r}, f'(\sqrt{r}) &= -3r + r = -2r < 0 \Rightarrow \text{stable} \\x = -\sqrt{r}, f'(-\sqrt{r}) &= -3r + r = -2r < 0 \Rightarrow \text{stable}\end{aligned}$$

if $r < 0$, we have only one real root:

$$x = 0, f'(0) = r < 0 \Rightarrow \text{stable}$$

if $r = 0$, we have to take into account second order and third order term in Taylor development: $f''(x) = -6x$, also goes to 0 in 0, but $f'''(x) = -6 < 0$.

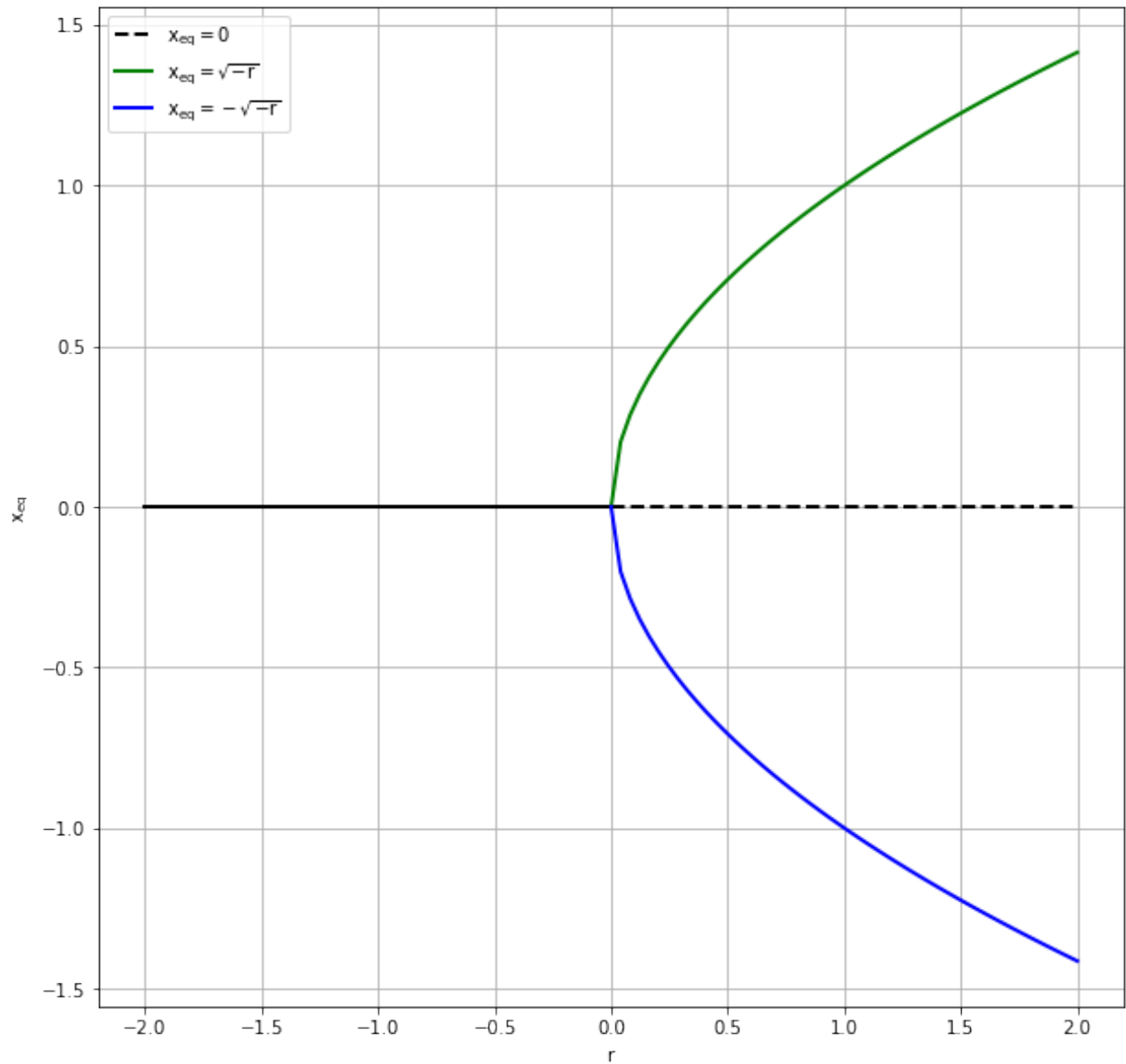
So, very near of 0: $f(x) = -6x^3/3!$ that is negative if $x > 0$ and positive if $x < 0$, so f is monotonically decreasing around 0, and then x is an stable equilibrium point.

This result is coherent with the observed in the graphs.

1.4 Bifurcation diagram of $\dot{x} = rx - x^3$

Script 1.4.1 (python)

```
1 plt.figure(figsize=(10,10))
2 plt.grid()
3 r = np.linspace(-2, 0, num = 50)
4 plt.plot(r, np.zeros(50), linewidth=2.0, color='black')
5
6 r = np.linspace(0, 2, num = 50)
7
8 plt.plot(r, np.zeros(50), linewidth=2.0, color='black', linestyle = 'dashed', label="$
  \hookrightarrow x_{eq} = 0$")
9 plt.plot(r, np.sqrt(r), linewidth=2.0, color='green', label="$ x_{eq} = \sqrt{-r}$")
10 plt.plot(r, -np.sqrt(r), linewidth=2.0, color='blue', label="$ x_{eq} = -\sqrt{-r}$")
11
12 leg = plt.legend(loc='best', ncol=1)
13 _ = plt.xlabel("$ r $")
14 _ = plt.ylabel("$ x_{eq} $")
```



2 Equilibria of two-variable systems

Calculate the equilibria and discuss their type and stability (based on the eigenvalues) of the simplified Lotka-Volterra equations:

$$\dot{u} = u(1 - v)$$

$$\dot{v} = \alpha v(u - 1)$$

Is there any bifurcation as a function of the parameter α ?

The equilibrium points (u, v) that make 0 the derivatives are clearly $(0, 0)$ and $(1, 1)$.

We will guess the qualitative patterns of behavior by means of linearization around equilibrium points. The Jacobian matrix:

$$\begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix}$$

is in this case:

$$\begin{bmatrix} 1-v & -1 \\ \alpha v & u-1 \end{bmatrix}$$

At equilibrium point (0,0)

$$\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

So, equating to zero the determinant for diagonalization:

$$\begin{vmatrix} 1-\lambda & -1 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

we obtain the characteristic equation:

$$(1-\lambda)(-1-\lambda) = 0$$

with solutions:

$$\begin{aligned} \lambda_+ &= 1 \\ \lambda_- &= -1 \end{aligned}$$

Applying this values to the linearized equations:

$$\begin{aligned} \delta y_+(t) &= \delta y(0) * e^t \\ \delta y_-(t) &= \delta y(0) * e^{-t} \end{aligned}$$

Therefore the first direction the derivative is increasing so unstable and the second and orthogonal direction the derivative is decreasing so stable. Therefore we have a **saddle point**.

This prevents that initial small populations of preys and predators doesn't evolve easily to extinction.

At equilibrium point (1,1)

$$\begin{bmatrix} 0 & -1 \\ \alpha & 0 \end{bmatrix}$$

So, equating to zero the determinant for diagonalization:

$$\begin{vmatrix} -\lambda & -1 \\ \alpha & -\lambda \end{vmatrix} = 0$$

we obtain the characteristic equation:

$$\lambda^2 + \alpha = 0$$

with solutions:

$$\begin{aligned} \lambda_+ &= \sqrt{-\alpha} \\ \lambda_- &= -\sqrt{-\alpha} \end{aligned}$$

If $\alpha < 0$ the eigenvalues are real and:

$$\begin{aligned}\lambda_+ &> 0 \\ \lambda_- &< 0\end{aligned}$$

thus, the first direction the derivative is increasing so unstable and the second and orthogonal direction the derivative is decreasing so stable. Therefore we have a **saddle point**. But the value of this parameter is always positive in this model.

If $\alpha > 0$ the eigenvalues are conjugate complex with no real part:

$$\begin{aligned}\lambda_+ &= i\sqrt{\alpha} \\ \lambda_- &= -i\sqrt{\alpha}\end{aligned}$$

So the point $(1, 1)$ **is a center** and the populations of prey and predator evolve synchronously following a periodic pattern.

For $\alpha = 0$ there is an abrupt change in behavior from center to saddle (the roots became real roots), but as I say before this parameter is always positive in the model definition (is equal to the product of increasing rates of prey and predator in absence or the other).