NP-Complete Problems

Prem Nair

Class *P and class NP*

Loosely speaking:

A problem belong to class ***P*** if it can be solved in polynomial time.

A problem belong to class ***NP*** if it can be solved in polynomial time using a non-deterministic algorithm.

Which is same as saying

A problem belong to class ***NP*** if it can be verified in polynomial time.

Solving vs Verifying

Solve the equation:

$7x^2 - 12x - 352 = 0$

Verify $x = 8$ is a solution to the equation

$7x^2 - 12x - 352 = 0$

Verification takes less time!

Example : A Non-deterministic Algorithm

```
IsPresent(A, x)
// A is an array of items. x is an item.
// Will return true if x is present and false otherwise.
        i <- guess(A, x)  // guess will return the correct index
                          // if x is present in the array A.
        if (A[i] == x)
                return true
        else
                return false
```

Note : If x is not present, then also guess will return a value.
Hence the need for verification in a nondeterministic algorithm

How Do We Know When a Problem Does Not Belong to ***P***?

- Hard to know for sure because even if there is no known polynomial time algorithm today, tomorrow someone may come up with one.

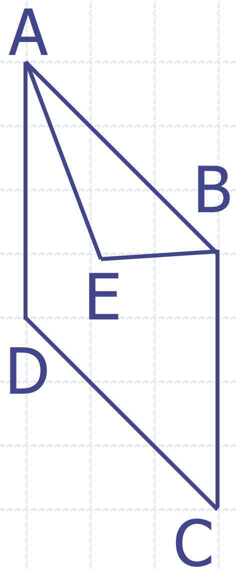
- Modern-day example: The **IsPrime** problem. Before 2002, all known deterministic algorithms to solve this problem ran in superpolynomial time.

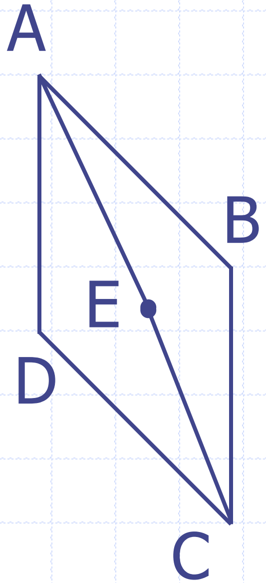
- **AKS Primality Test** was the first polynomial-time solution. Its fastest known implementation runs in
$$O(\mathit{length}(n)^6 * \log^k(\mathit{length}(n)))$$
for some k. (AKS stands for Agrawal–Kayal–Saxena)

$P \subseteq NP$

Is **P = NP** *we do not know.*

Hamiltonian Cycle And Vertex Covers





HC
Minimum VC = {A, C, E}

Not an HC
Minimum VC = {A, C}

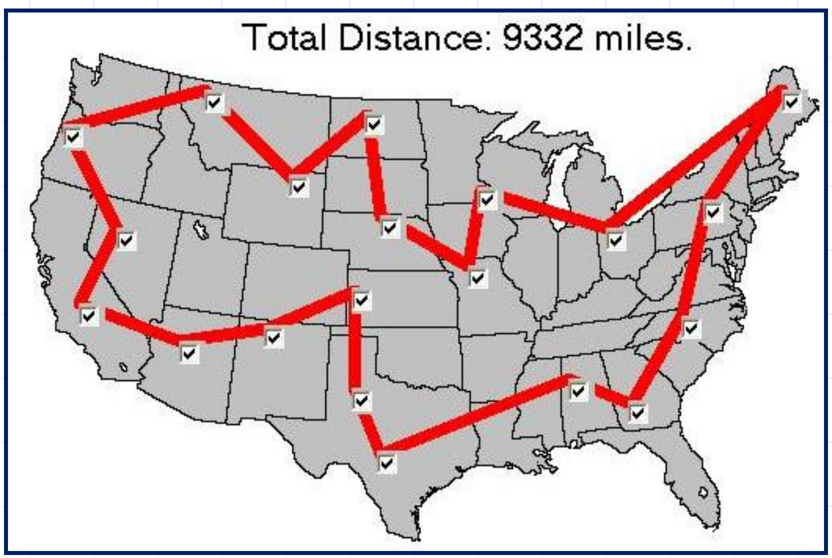
Hamiltonian Cycle And Vertex Cover

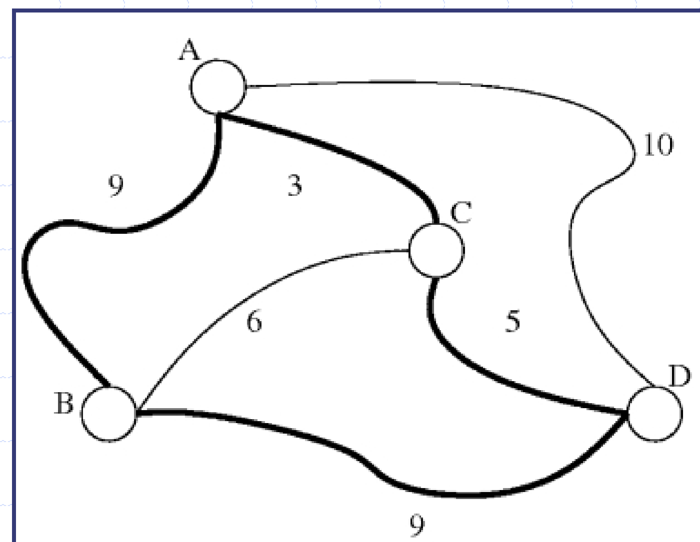
- A Hamiltonian cycle in a graph G is a simple cycle that contains every vertex of G. A graph is a Hamiltonian graph if it contains a Hamiltonian cycle.
- Examples. The Herschel graph is not a Hamiltonian graph.
- If G = (V,E) is a graph, a vertex cover for G is a set C $\subseteq$ V such that for every e $\in$ E, at least one end of e lies in C.
- Fact. The known algorithms for
  - determining whether a graph is Hamiltonian (HC), and
  - for computing the smallest size of a vertex cover (VC),

run in exponential time.

Traveling Salesperson Problem

- *Traveling Saleperson Problem* (TSP): Given a complete graph G with cost function c: E $\rightarrow$ N and a positive integer k, is there a Hamiltonian cycle C in G so that the sum of the costs of the edges in C is at most k? <u>Solution data</u>: a subset of E.





Problems in NP

HC

VC

TSP

are in ***NP.***

TSPmin (which computes minimum cost for TSP) is not in ***NP. Why? You cannot verify in polynomial time!***

Example: Not in ***NP***

- *PowerSet problem.* Given a set X of size n, a kind of optimization problem concerning the power set of X is to generate all subsets of X (“find the largest possible collection of subsets of X without duplicates”). Whatever method is used, just writing out the output requires at least 2^n steps. A corresponding decision problem is: Given a set X and a collection P, is P = P(X)? Any algorithm that solves this problem must check every element of P to see if it is a subset of X, so the algorithm requires at least 2^n steps in the worst case. So this problem does not belong to ***P***.

  Moreover, verifying correctness requires checking that each set that is output is a subset of X, and this has to be done 2^n times, so it doesn’t belong to ***NP*** either.

- We discusssed NxN chess in Lesson 1. We said it belongs to class EXP-complete. It is not known whether it belongs to ***NP***.

Reducibility (informal 1)

Let Q denote the problem of finding the area of a square.

Let R denote the problem of finding the area of rectangle.

Given an instance I_Q of Q, we can transform into an instance I_R of R.

I_Q has a solution iff I_R has a solution

- We write $Q \overset{\text{poly}}{\longrightarrow} R$
- Note that Q is not harder than R.

```
Algorithm areaSquare(double side)
    return (areaRectangle(side, side)
```

Reducibility (informal 2)

Let Q denote the problem computing the distance between two points in 2D.

Let R denote the problem computing the distance between two points in 3D.

Given an instance I_Q of Q, we can transform into an instance I_R of R.

I_Q has a solution iff I_R has a solution

- We write $Q \overset{\text{poly}}{\longrightarrow} R$
- Note that Q is not harder than R.

```
Algorithm distance2D((x1, y1), (x2, y2))
    return distance3D((x1, y1, 0), (x2, y2, 0))
```

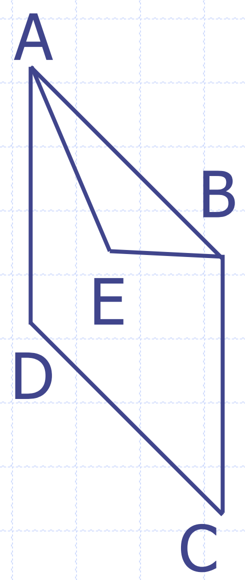
Hamiltonian Cycle $\xrightarrow{\text{poly}}$ TSP

We show HamiltonianCycle is reducible to TSP

- Given a graph G = (V,E) on n vertices (input for HamiltonianCycle) – notice G is a subgraph of K_n. Obtain an instance H, c, k of TSP as follows: Let H be the complete graph on n vertices (i.e. H is K_n), obtained by adding the missing edges to G. Let
  c(e) = 0 if e $\in$ E, else c(e) = 1. Let k = 0.
- Need to show: G has a Hamiltonian cycle if and only if H, c, k has a Hamiltonian cycle with edge cost $\leq$ k
- If G has Hamiltonian cycle C, C is Hamiltonian in H also. Since each edge e of C is in G, c(e) = 0. So cost sum $\leq$ k. Converse: A solution C for H,c,k implies all edges of C have weight 0; therefore, every edge of C also is an edge in G. Therefore C is an HC in G.

HamiltonianCycle $\xrightarrow{\text{poly}}$ TSP (Yes case)

G

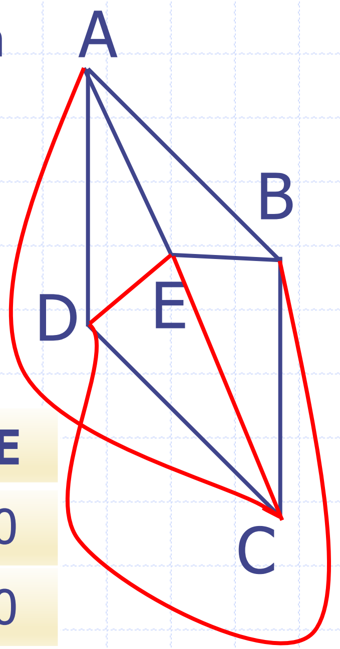


	A	**B**	**C**	**D**	**E**
A	0	1	0	1	1
B	1	0	1	0	1
C	0	1	0	1	0
D	1	0	1	0	0
E	1	1	0	0	0

Does G has a HC?

Can TSP visit with cost k = 0?

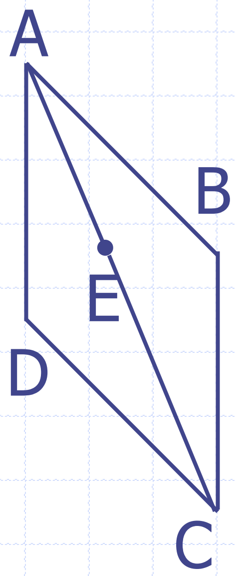
H



	A	**B**	**C**	**D**	**E**
A	0	0	1	0	0
B	0	0	0	1	0
C	1	0	0	0	1
D	0	1	0	0	1
E	0	0	1	1	0

HamiltonianCycle $\xrightarrow{\text{poly}}$ TSP (No case)

G

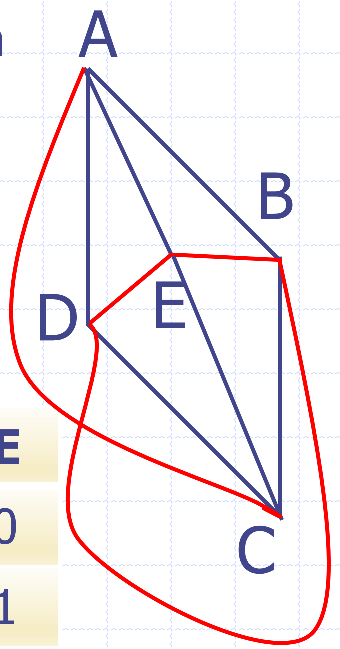


	A	**B**	**C**	**D**	**E**
A	0	1	0	1	1
B	1	0	1	0	0
C	0	1	0	1	1
D	1	0	1	0	0
E	1	0	1	0	0

Does G has a HC?

H

Can TSP visit with cost k = 0?



	A	**B**	**C**	**D**	**E**
A	0	0	1	0	0
B	0	0	0	1	1
C	1	0	0	0	0
D	0	1	0	0	1
E	0	1	0	1	0

$\text{HamiltonianCycle} \xrightarrow{\text{poly}} \text{TSP}$

Note: From a practical point of view, all we have to do is to change all non-diagonal values of 1's and 0's in the adjacency matrix of an instance of HC to 0's and 1's to obtain an instance of TSP.

Algorithm HCInstanceToTSPInstance(H)

Input : H an instance of HC as an adjacency matrix.

Output : T an instance of TSP as an adjacency matrix.

```
for i = 0 to n – 1 do
    for j = i + 1 to n – 2 do
        T[i, j ] <- (H[i, j] + 1) % 2
        T[j, i ] <- (H[i, j] + 1) % 2
```

//Arrays starts with index 0. T initialized with 0 during creation.

//Time complexity $O(n^2)$.

HamiltonianCycle $\xrightarrow{\text{poly}}$ TSP

Algorithm isHC(H)
Input : H an instance of HC as an adjacency matrix.
Output : **true** if H is Hamiltonian. **false** otherwise.

T <- HCInstanceToTSPInstance(H)
return isTSP(T, 0)

//isTSP(T, k)
//T an instance of TSP (adjacency matix)
//k a non-negative integer.
//isTSP(T, k) returns **true** if TSP can visit all cities at the cost of k and come back to home city. **false** otherwise.

NP-hard Problems

- A problem Q is ***NP-hard*** if for *every* problem R in ***NP***, R is polynomial reducible to Q.

$$R \xrightarrow{\text{poly}} Q$$

This means Q can never be less harder than any problem in ***NP***.

$$R \leq_{\text{poly}} Q.$$

NP-Complete Problems

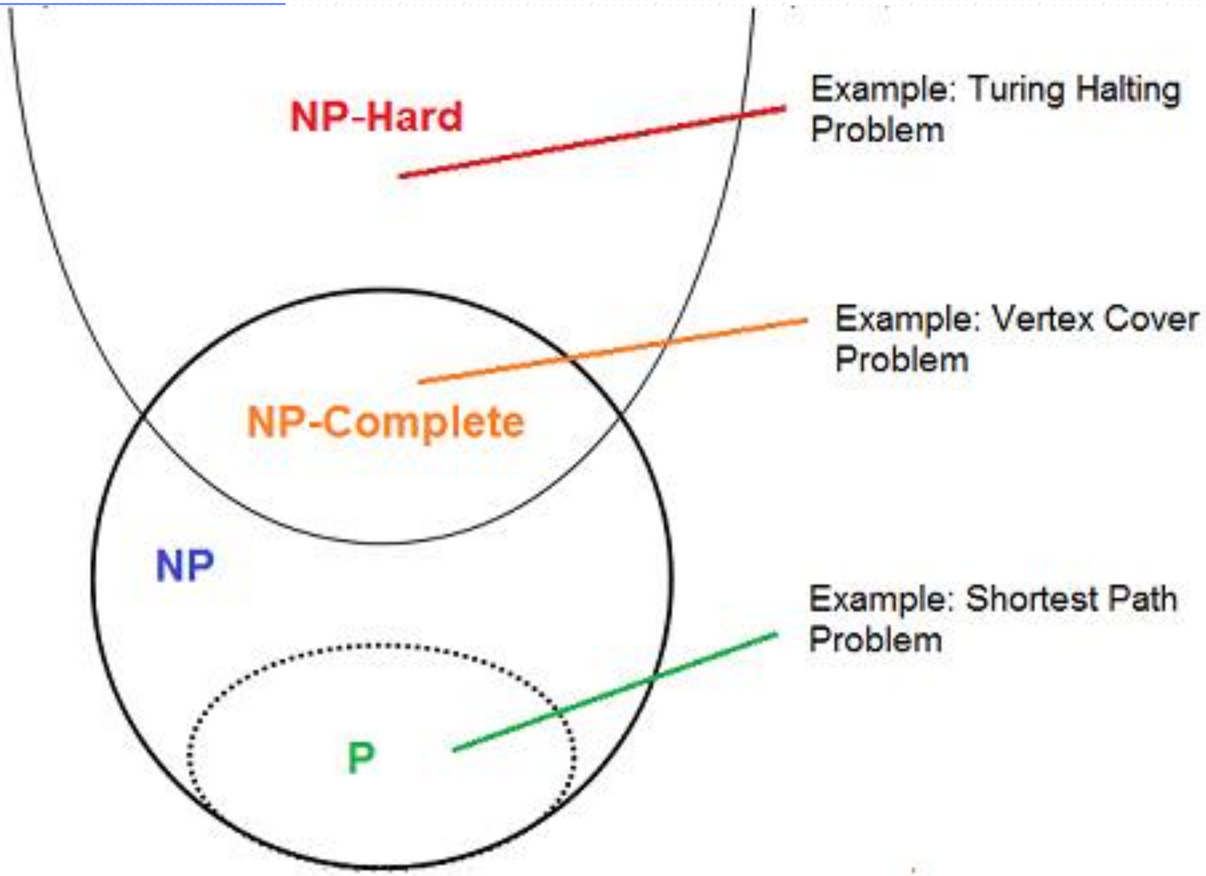
In 1971, Cook defined a class of ***NP*** problems called NP-complete that appeared to be fundamentally unsolvable in polynomial time. He proposed that if just one of the many ***NP-complete*** problems is in ***P***, then they all are – meaning ***P* = *NP***. But if just one of the ***NP-complete*** problems is not in ***P***, then none are.

NP-Complete Problems

A problem Q is ***NP-complete*** if

Q belongs to ***NP***,
and
Q is ***NP-hard***.

NP-Complete Problems



This diagram assumes that P != NP

HamiltonianCycle is NP-Complete

This is an outline of a proof that Hamiltonian Cycle(HC) is ***NP-Complete*** under the assumptions that Vertex Cover(VC) is ***NP-complete*** and VC is polynomial reducible to HC.

1. Show HC is in ***NP***.
2. Pick VC. A known ***NP-complete*** Problem. (Assumed)
3. Show VC is polynomial reducible to HC. (Assumed)

Note that by (2), VC is an ***NP-complete*** Problem. Hence *every* problem R in ***NP***, R is polynomial reducible to VC.

$$R \xrightarrow{\text{poly}} VC.$$

By (3), VC $\xrightarrow{\text{poly}}$ HC. Therefore, Hence *every* problem R in ***NP***, R is polynomial reducible to HC. Hence HC is ***NP-hard***.

Summary: To show Y is NP-Complete

- Show Y is in ***NP***.
- Show Y is in ***NP-Hard***.
  - Pick X. A known ***NP-complete*** Problem
  - Show X is polynomial reducible to Y.