

Study Slides 2, 7, 8, 9, 10, 15, 16, 17, 18, 19, 21, 22, 23.

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Optional Explanation by Prem for your better understanding. Following results you must know. No need memorize the proof. It will not be asked during the final exam.

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Property 1. (Slide 15) Sum of all vertex degrees =  $2m$ .

**Proof.** Every edge increases the degree of exactly two vertices by 1. Thus every edge increases the sum of all vertex degrees by 2. If there are  $m$  edges, sum of all vertex degrees =  $2m$ .

Property 2. (Slide 15)  $m \leq n(n-1) / 2 = \binom{n}{2}$

**Proof.** Given  $n$ , there are  $n(n-1)/2$  pairs possible. Each can be an edge. Hence the result.

Exercise on Slide 16.  $m = n(n-1) / 2 = \binom{n}{2}$  for  $K_n$ , the complete graph on  $n$  vertices.

**Proof.** Given  $n$ , there are  $n(n-1)/2$  edges possible. The complete graph has all possible edges. Thus  $m = n(n-1)/2$ .

Exercise on Slide 17. A graph is bipartite if and only if it has no odd (simple) cycle.

**Proof.** Case 1: There is no odd cycle. We can label adjacent vertices using labels 1 and 2 without any conflict. After labeling, let  $V_1$  denote the set of all vertices labeled as 1 and let  $V_2$  denote the set of all vertices labeled as 2. Clearly,  $V$  is partitioned into two subsets  $V_1$  and  $V_2$  such that all edges have one vertex in  $V_1$  and the other vertex  $V_2$ . Therefore, the graph is bipartite.

Case 2: There is an odd cycle. Consider an odd cycle on  $2n + 1$  vertices. Start labeling all 1 and 2 starting from any vertex in the cycle. This will result in 1 and  $2n + 1$  having label as 1. Since this is a cycle, there is an edge between 1 and  $2n + 1$ . Note that both 1 and  $2n + 1$  belongs to the same partition of  $V$ . From the definition of bipartite graph, we cannot have an edge between 1 and  $2n + 1$ . Therefore, graph is not bipartite.

Exercise on Slide 19. If  $m > \binom{n-1}{2}$ , then  $G$  is connected.

**Proof.** If there are  $n$  vertices, keep one vertex apart from all other  $n - 1$  vertices and try placing edges connecting those  $n - 1$  vertices. The maximum number of edges possible is  $\binom{n-1}{2}$ . Therefore if there is one more edge, graph will become connected.

Exercise on Slide 22. Show that every tree having 2 or more vertices has at least two vertices of degree one.

**Proof.** Consider a tree  $T$  having  $n > 1$  vertices. Since it is a tree,  $m = n - 1$ . Now by property 1, sum of all vertex degrees is  $2m = 2(n-1) = 2n - 2$ .

Since  $T$  is a tree, it is connected. Hence every vertex must have at least 1 vertex degree. Now you are left with only  $(2n - 2) - n = n - 2$  vertex degrees. You have  $n$  vertices and  $n - 2$  leftover vertex degrees. Thus there will be at least two vertices with vertex degree 1.

Q1. What is the minimum number of edges required to guarantee the graph is connected?

$$\binom{n-1}{2} + 1$$

Q2. If the graph is connected what is the minimum number of edges it has?

$$n - 1$$

Q3. What is the minimum number of edges required to guarantee a graph has a cycle?

$n$

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**Let  $C$  be a connected component with  $n$  vertices and  $m$  edges.**

(a) The minimum number of edges it has

$$n - 1$$

(a) If  $m = n - 1$  then  $C$  is tree.

(b) If  $m \geq n$ , then  $C$  has a cycle.

(c) If  $m = n$  and  $\deg(v) = 2$  for all vertices  $v$  in  $C$ , then  $C$  is a cycle.