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THEORETICAL NEUROSCIENCE

TD1: MODELS OF NEURONS I

All TD materials will be made available at https://github.com/helene-todd/TheoNeuro2425.

The goal of these first three tutorials is to understand the dynamics of a simple type of single neuron models: point neurons. In these models, a neuron's complex morphology is simplified to a single point, characterised by its membrane potential V, which is the difference in potential between inside/outside the neuron. In this tutorial, we will introduce the leaky integrate-and-fire model. In order to understand the different dynamics of this model, we will learn how to explicitly solve linear ODEs and implement them numerically.

1 Mathematical tools for ordinary differential equations (ODEs)

1.1 Analytical solution of linear ODEs

Consider the following differential equation for a variable x evolving with time t

$$\frac{dx(t)}{dt} = ax(t) + b(t), \tag{1}$$

with initial condition

$$x(t_0) = x_0. \tag{2}$$

1. **Homogeneous case** $b(t) = 0$.

   Show that equation (1) admits a solution given by

   $$x(t) = x_0 \exp\left(\int_{t_0}^{t} a ds\right) = x_0 \exp(at - at_0). \tag{3}$$

   Bonus: show that this solution is unique.

2. **Non-homogeneous autonomous case** $b(t) = b$.

   Using the solution to the homogeneous case, find a solution to equation (1).

3. **Bonus [general case]: Non-homogeneous non-autonomous case** $b(t)$.

   Using the variation of constants method, show that the solution to equation (1) with initial condition (2) is

   $$x(t) = \left(\int_{t_0}^{t} b(s) \exp(-as + at_0) ds + x_0\right) \exp(at - at_0). \tag{4}$$

   Hint: suppose the solution to equation (1) is of the form

   $$x(t) = K(t) \exp(at - at_0), \tag{5}$$

   and substitute into equation (1).

1.2 Numerical approximation of ODEs: Euler method

Any real function f of a variable t that is infinitely differentiable can be written as a Taylor series expansion in $t + \Delta t$

$$f(t + \Delta t) = f(t) + \sum_{n=0}^{+\infty} \frac{f^{(n)}(t)}{n!} (\Delta t)^n, \tag{6}$$

with $f^{(n)}(t)$ the n^{th} derivative of $f(t)$. For Δt small enough, the first order Taylor expansion is

$$f(t + \Delta t) = f(t) + f'(t)\Delta t + O(\Delta t^2). \tag{7}$$

4. Using equation (7), elaborate an algorithm that simulates a differential equation.

2 Leaky integrate-and-fire (LIF) neurons

In the following we will consider the parameters

C_m	g_l	E_l	V_{th}	V_{reset}
100 pF	10 nS	-70 mV	-50 mV	-80 mV

2.1 Modelling the leak term

Neuron membranes are permeable to ions, therefore differences in ion concentration and in electric potential between the interior and the exterior of the neuron result in a flow of ions across the membrane (through dedicated channels), according to

$$I = -g_l(V_m - E_l), \tag{8}$$

where g_l is the leak conductance and the leak potential E_l is the value of the membrane potential V_m for which there is no current across the membrane. This current results in the accumulation of charge Q close to the membrane of the neuron that acts as a capacitor, itself resulting in a membrane potential

$$Q = C_m V_m, \tag{9}$$

where C_m is the membrane capacitance.

5. Knowing the current I corresponds to the variation in time of charge Q in the neuron, obtain a differential equation governing the time course of the membrane potential V_m . Solve this equation for a given initial condition $V(t = 0) = V_0$. A characteristic relaxation time τ_m of the membrane should be introduced.

6. Depending on its initial value V_0, how does the membrane potential V_m behave in time?

2.2 The full model

The LIF model is a good starting point for simulating neurons. It reproduces some qualitative features of the membrane potential dynamics, and introduces a framework on which we can build more realistic models.

This model is described as a differential equation for the membrane potential of a neuron with a capacitance and a leak term combined with an additional tweak: when the membrane potential reaches a particular value, called the threshold, a spike is emitted and the membrane potential is returned to a reset value. The dynamical equation for the membrane potential is

$$C_m \frac{dV_m(t)}{dt} = g_l(E_l - V_m(t)) + I_{app}, \text{ if } V_m > V_{th} \text{ then } V_m = V_{reset}, \tag{10}$$

with I_{app} a constant applied current.

As we can observe on the simulations, there are different regimes depending on the injected current.

7. Find the condition on which the neuron is able to spike starting from a potential $V_0 < V_{th}$. Deduce the threshold current for which this condition is verified.

2.3 Firing rate as a function of current (fI curve)

When the applied current is held fixed, the time for the neuron membrane potential to increase from its reset value to the threshold can be calculated.

8. From the previous question, deduce the inter-spike interval $T_{ISI}(I)$ as well as the firing rate $f(I)$ of the neuron as a function of the input current.

2.4 Response to oscillating input current

We now inject a small oscillating current

$$I_{app}(t) = 2I_0 \cos(\omega t) = I_0(e^{i\omega t} + e^{-i\omega t}). \tag{11}$$

The membrane potential integrates this current, therefore when it reaches its steady-state solution, it oscillates at the same frequency with a certain time lag given by a phase ϕ. We can therefore write the membrane potential as

$$V_m(t) = E_l + 2A\cos(\omega t + \phi) = E_l + A(e^{i(\phi+\omega t)} + e^{-i(\phi+\omega t)}). \tag{12}$$

9. The goal is now to identify the time lag given by a phase ϕ. Show that

$$Ae^{i\phi} = \frac{I_0}{g_l + iC_m\omega}. \tag{13}$$

10. Compute the amplitude A and phase ϕ of the response. Provide an explanation of the limit behaviours at low ($\omega \ll g_l/C_m$) and high frequencies ($\omega \gg g_l/C_m$). Justify that the membrane behaves as a low-pass filter.