M2 Cogmaster 2024–2025 | October 17^{th}
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ENS | PSL

THEORETICAL NEUROSCIENCE

TD5: SYNAPSES & DENDRITES

All TD materials will be made available at https://github.com/helene-todd/TheoNeuro2425.

In previous tutorials, we have developed models of point neurons responding to injection of currents. Biologically, these currents correspond to synaptic inputs from other neurons, making the synapse a key element of the communication between neurons. Importantly, synapses are thought to be the substrate of learning, as the modulation of their strength, or plasticity, modifies the interaction between neurons. Plasticity can obey different rules depending on the structure of synapses. In this tutorial, we propose to model two phenomena related to plasticity at the level of the synapse: short-term plasticity (through STD and STF) and long-term plasticity (through STDP).

Note: we will use slightly different notations than those introduced in the lesson; A_{SE} instead of A, U_{SE} instead of U, U_{SE}^1 instead of u, τ_{facil} instead of τ_F and τ_{rec} instead of τ_D.

1 Short term plasticity (STP)

Short-term plasticity refers to the activity-dependent variation of synaptic efficacy on short time scales (milliseconds to seconds). Two main types of plasticity can coexist in the same synapse: facilitation (enhancement of synaptic efficacy with past activity and depression (reduction of synaptic efficacy with past activity).

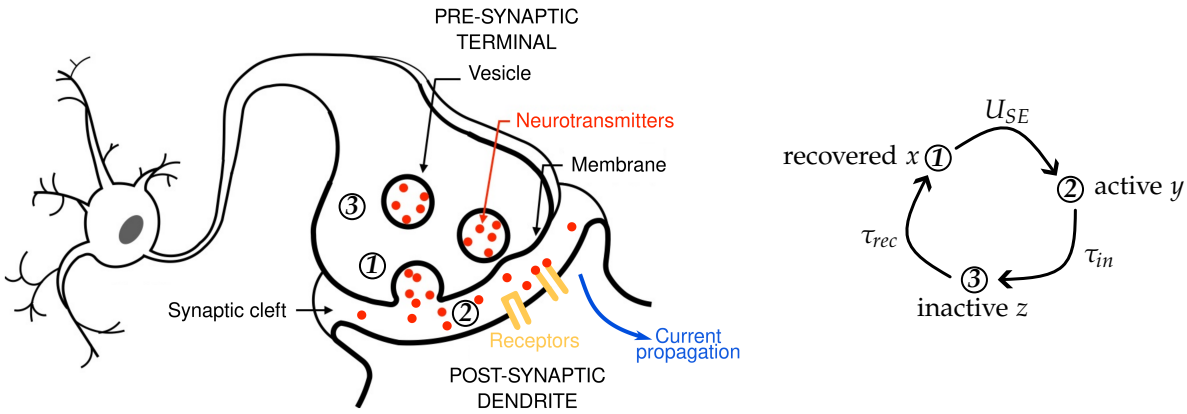
1.1 Vesicular pool dynamics

In this model, it is assumed that a synapse is characterised by a finite amount of resources (i.e. vesicles). Each vesicle is either in the recovered, active or inactive state. To model this, we denote the variables x, y and z as the fractions of resources in the recovered, active and inactive states respectively, such that $x + y + z = 1$.

- The variable U_{SE} stands for the release probability, which corresponds to the fraction of vesicles from the recovered pool released by a single spike arriving at the pre-synaptic axon terminal.
- Those vesicles immediately become active, and then quickly inactivate with a time constant τ_{in} (of the order of a few milliseconds).
- The pool of available resources is replenished by the recovery of inactivated vesicles with a time constant τ_{rec} (of the order of 1 sec).

On the post-synaptic side, the net input current is assumed to be directly proportional to the fraction of resources in the active state $y(t)$ by a factor A_{SE} (absolute synaptic efficacy)

$$I_{syn}(t) = A_{SE}y(t). \tag{1}$$



1.1.1 Short term depression (STD)

The short term depression (STD) effect is captured through the the fraction of resources available after synaptic vesicle depletion, i.e. the fraction of resources in the recovered state x.

1. Write down the equations governing the time evolution of variables x, y and z.

1.1.2 Short term facilitation (STF)

The short term facilitation (STF) effect is captured by the parameter U_{SE}, representing the fraction of available resources ready for use (i.e. the release probability). The parameter U_{SE} is not fixed, but rather increases by a certain amount at each pre-synaptic spike. Between spikes, U_{SE} decays back to 0 with time constant τ_{facil}.

The running value of U_{SE} will be referred to as U^1_{SE}, and $U_{SE} \in (0,1)$ will now denote a constant parameter that determines the increase in the value of U^1_{SE} due to each spike, or in other words, determines the contribution of facilitation. The following differential equation describes this dynamic

$$\frac{dU^1_{SE}(t)}{dt} = -\frac{U^1_{SE}(t)}{\tau_{facil}} + U_{SE}(1 - U^1_{SE}(t))\delta(t - t_{sp}). \tag{2}$$

2. The above equation can be transformed into an iterative expression for the value U_{SE} upon the arrival of the nth spike. Let t^+_{sp} and t^-_{sp} denote the times right before and after a spike t_{sp}. Determine the iterative expression that results from equation (2).

3. Suppose the pre-synaptic neuron emits a regular spiking train at a frequency $\delta t = 1/r$. Determine the steady-state solution of U^1_{SE}.

We have seen that this model captures both depressing and facilitating synaptic mechanisms. In what follows, we will consider only a depressing synapse - this is done by setting U^1_{SE} as a constant equal to U_{SE}.

1.2 A focus on STD

We now suppose that the synapse receives irregular spikes: this is captured in the model by replacing $\delta(t - t_{sp})$ by $\lambda(t)$, where $\lambda(t)$ is the rate of a Poisson process. Furthermore, we suppose that $\tau_{in} \ll \tau_{rec}$, i.e. the the dynamics of the active pool can be considered to be instantaneous.

4. Write down the new equations governing the time evolution of the variable x.

We now also consider the system at a slow time scale ($\gg \tau_{in}$).

5. Give the new expression of y and deduce a relationship between the post-synaptic current I_{syn} and the variable x.

We suppose the input firing rate is constant $\lambda(t) = \lambda$

6. What is the time course of x and I_{syn}?
7. What are the values x^* and I^*_{syn} at equilibrium? Contrast this to the case without STD.
8. **Bonus:** give the limits of x^* and I^*_{syn} for $\lambda \to 0$ and $\lambda \to \infty$.

The post-synaptic neuron then integrates this current according to

$$\tau_m \frac{dV(t)}{dt} = -V(t) + RI(t)$$

and emits a spike whenever the voltage reaches the threshold V_{th}.

9. **Bonus:** show that there is a critical threshold θ such that, whatever the input firing rate, the neuron eventually stops spiking.

2 Long term plasticity (LTP)

Spike-timing-dependent plasticity (STDP) is a class of long term modulation of synaptic efficacy, which depends on both pre-synaptic and post-synaptic mechanisms. In this sense, this is an associative process, which reinforces or weakens synaptic strengths between neurons based on their relative spike timings. This biological mechanism can be considered as an implementation of the Hebbian learning rule for adjusting the strength of connections between neurons. This computational rule has been proposed as a proxy for causality.

We consider a synapse connecting two neurons. The pre-synaptic neuron emits a train of n_{pre} pre-synaptic spikes $(t_{pre,1}, t_{pre,2}, \ldots, t_{pre,n_{pre}})$, while the post-synaptic neuron emits a train of n_{post} post-synaptic spikes $(t_{post,1}, t_{post,2}, \ldots, t_{post,n_{post}})$.

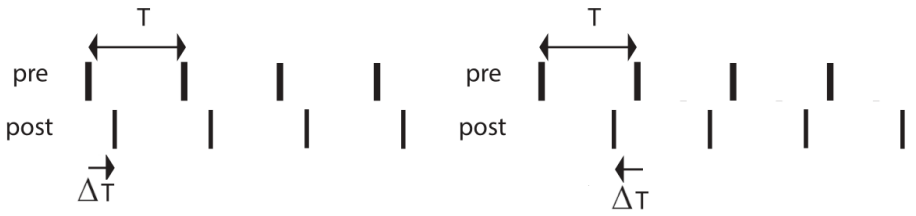
The synaptic weight w is modified according to the following rule: each pair of nearest-neighbour spikes $(t_{pre,i}, t_{post,j})$ induces a modification $\Delta w = f(t_{post,j} - t_{pre,i})$, with

$$f(s) = \begin{cases} A^+ e^{-s/\tau^+} \text{ if } s \geq 0, \\ -A^- e^{s/\tau^-} \text{ if } s < 0, \end{cases} \tag{3}$$

where A^+, A^-, τ^+ and τ^- are positive real parameters. Pairs of spikes are considered nearest-neighbour if and only if there is no other (pre-synaptic or post-synaptic) spike in the interval between the two spikes $]t_{pre,i}, t_{post,j}[$ (or $]t_{post,j}, t_{pre,i}[$). The synaptic modification is the sum of all modifications induced by the individual nearest-neighbour pairs.

2.1 Regular pre-synaptic train

Both neurons fire n spikes periodically with a frequency $F = 1/T$ (where T is the inter-spike interval), with a fixed interval ΔT between the spikes, $\Delta T = t_{post,i} - t_{pre,i}$ for all $i = 1, \ldots, n$.



10. Compute the total synaptic modification W induced by these spike trains for ΔT in $[-T/2, T/2]$.

11. Compute the synaptic modification W when the inter-spike interval T is much longer than the widths of the STDP windows τ^+ and τ^-, for both $\Delta T > 0$ and $\Delta T < 0$. Sketch how W depends on ΔT.

12. Compute the synaptic modification when the inter-spike interval T is much shorter than the widths of the STDP windows. Sketch again the dependence of W on ΔT.

2.2 Stochastic pre-synaptic train

We now consider spikes produced randomly. For the sake of simplification, spikes still alternate between pre-synpatic and post-synaptic neurons, but the duration between successive spikes is stochastic. In total, $2n$ spikes are generated, n per neuron. Spikes are produced following a Poisson process of mean frequency $R = 1/T$ and assigned to each neuron in alternation.

13. Recalling that the probability density of inter-spike intervals is given by $p_{ISI}(t) = Re^{-Rt}$, what is the average total synaptic modification W at the end of the spike trains, as a function of n, R, A^+, A^-, τ^+ and τ^-?

    Reminder: let X be a continuous random variable. For any function $f : \mathbb{R} \to \mathbb{R}$,

    $$\mathbb{E}(f(X)) \stackrel{\text{def}}{=\!=} \int_{\mathbb{R}} f(x) p_X(x) dx,$$

    where p_X is the probability density function of X.

14. What are the limits of W in the low ($R \to 0$) and high ($R \to \infty$) frequency limits?

15. In the BCM learning rule, the synaptic modification is negative at low frequencies, and positive at high frequencies. Determine the conditions on the parameters for implementing this rule.

16. Determine the frequency R_0 at which the total synaptic modification changes sign.