Ordinary Differential Equations

as an alternative to agent-based modelling

eX Modelo school

OpenMOLE

June 26, 2019

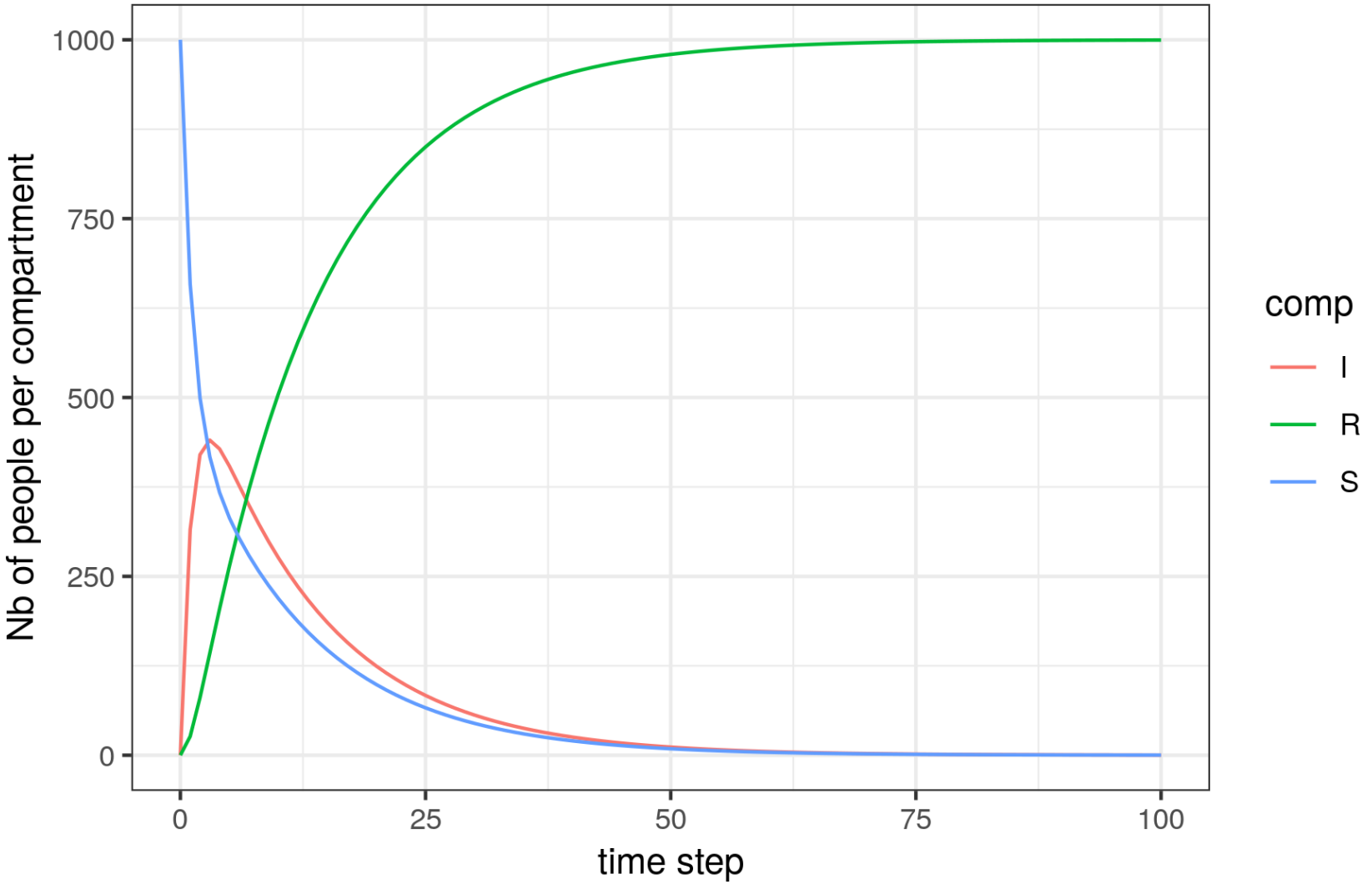
ODE systems

→ widely used to model transmission phenomena

- population split into compartments
- system of ordinary differential equations

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} &= -\beta S + \lambda I \\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \beta S - (\lambda + \gamma) I \\ \frac{\mathrm{d}R}{\mathrm{d}t} &= \gamma I \end{cases}$$

SIR model dynamics



ODE vs ABM

ODE	ABM
Equation-based	Individual-based
Generic mechanisms	Precise mechanisms
Population scale	Individual scale
Needs less resources	Computationally expensive

A Zombie situation

How could we model the Zombie invasion?

- Which mechanisms?
- Which parameters?

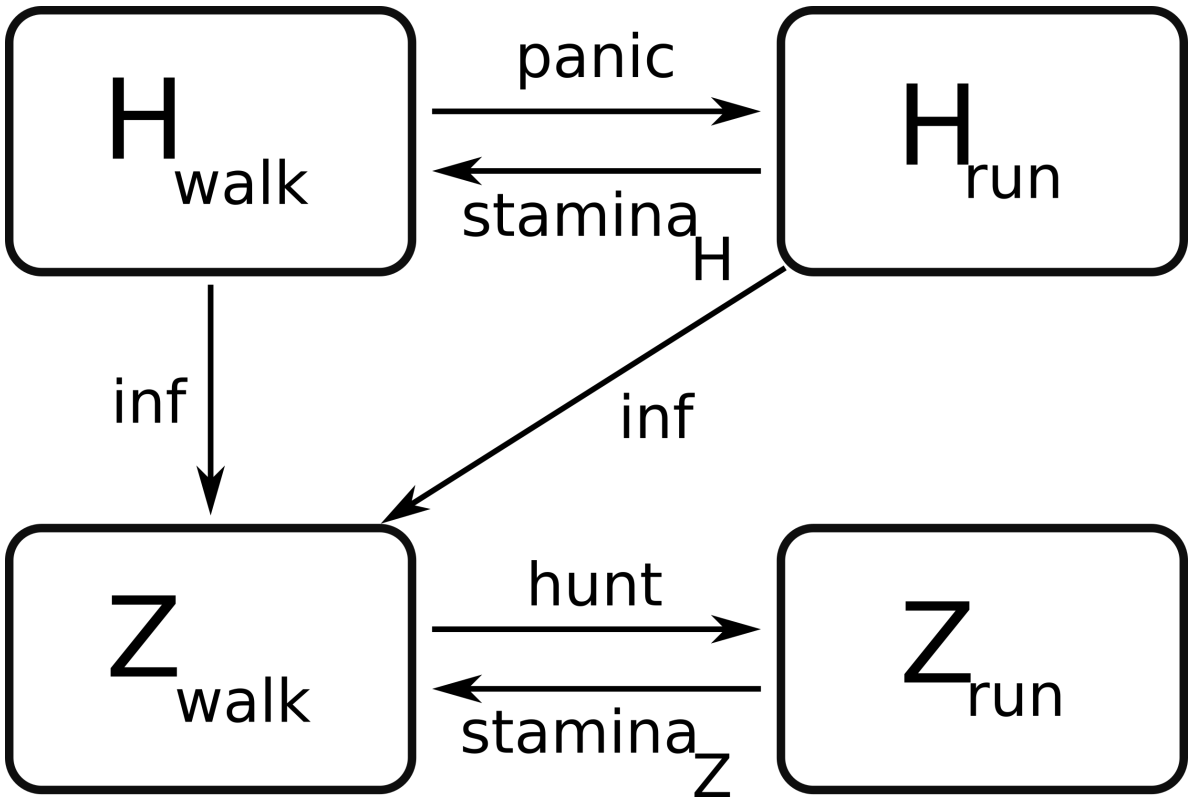
How could we model the Zombie invasion?

- Which mechanisms?
- Which parameters?

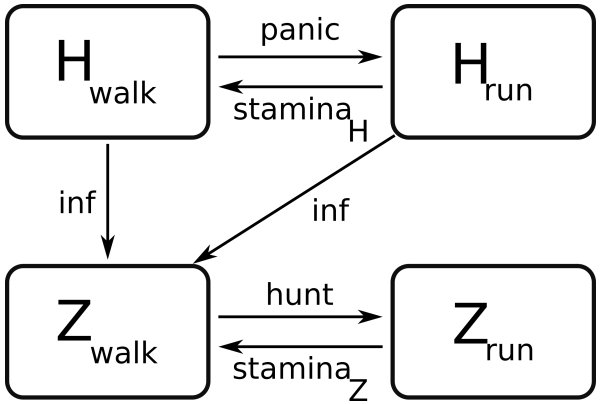
How can we assess our model's ability to reproduce the real data?

- Which metrics?
- Which fitness function?

A very simple ODE model



A very simple ODE model



$$
\begin{cases}
\dfrac{\mathrm{d}H_{walk}}{\mathrm{d}t} &= -(panic + inf) * H_{walk} + exhaustH * H_{run} \\
\dfrac{\mathrm{d}H_{run}}{\mathrm{d}t} &= panic * H_{walk} - (exhaustH + inf) * H_{run} \\
\dfrac{\mathrm{d}Z_{walk}}{\mathrm{d}t} &= inf * (H_{walk} + H_{run}) - hunt * Z_{walk} + exhaustZ * Z_{run} \\
\dfrac{\mathrm{d}Z_{run}}{\mathrm{d}t} &= hunt * Z_{walk} - exhaustZ * Z_{run}
\end{cases}
$$

$$\begin{cases} N & = & H_{walk} + H_{run} + Z_{walk} + Z_{run} \\ panic & = & panic_0 * (Z_{walk} + Z_{run})/N \\ hunt & = & hunt0 * (H_{walk} + H_{run})/N \end{cases}$$

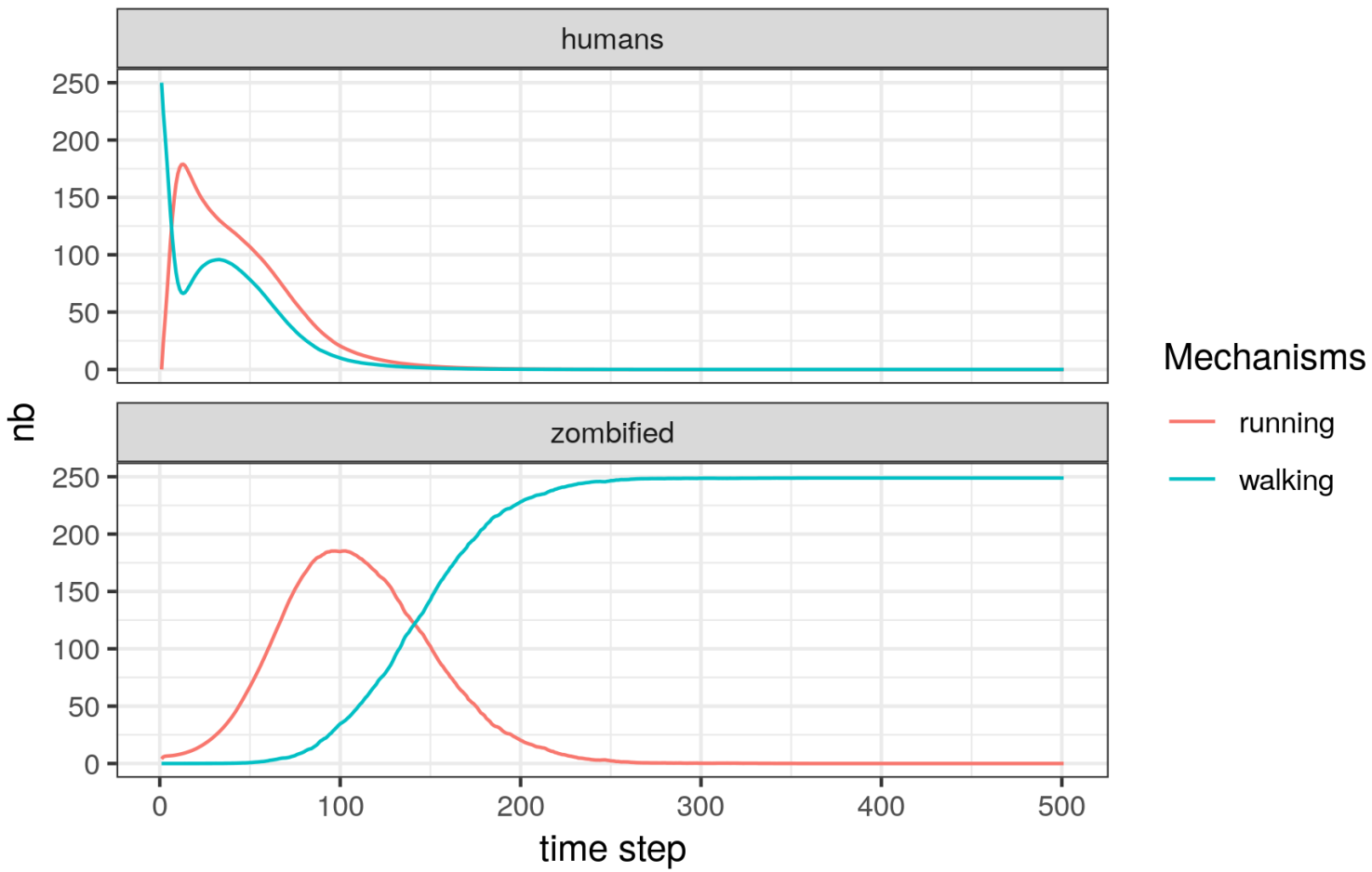
Exploration

First step: Calibrate

We have some real time series of zombie invasion

$\rightarrow$ find the parameter values to best fit them

We have some real time series of zombie invasion
$\rightarrow$ find the parameter values to best fit them



Process

Process

- Embed the model in OpenMOLE

Process

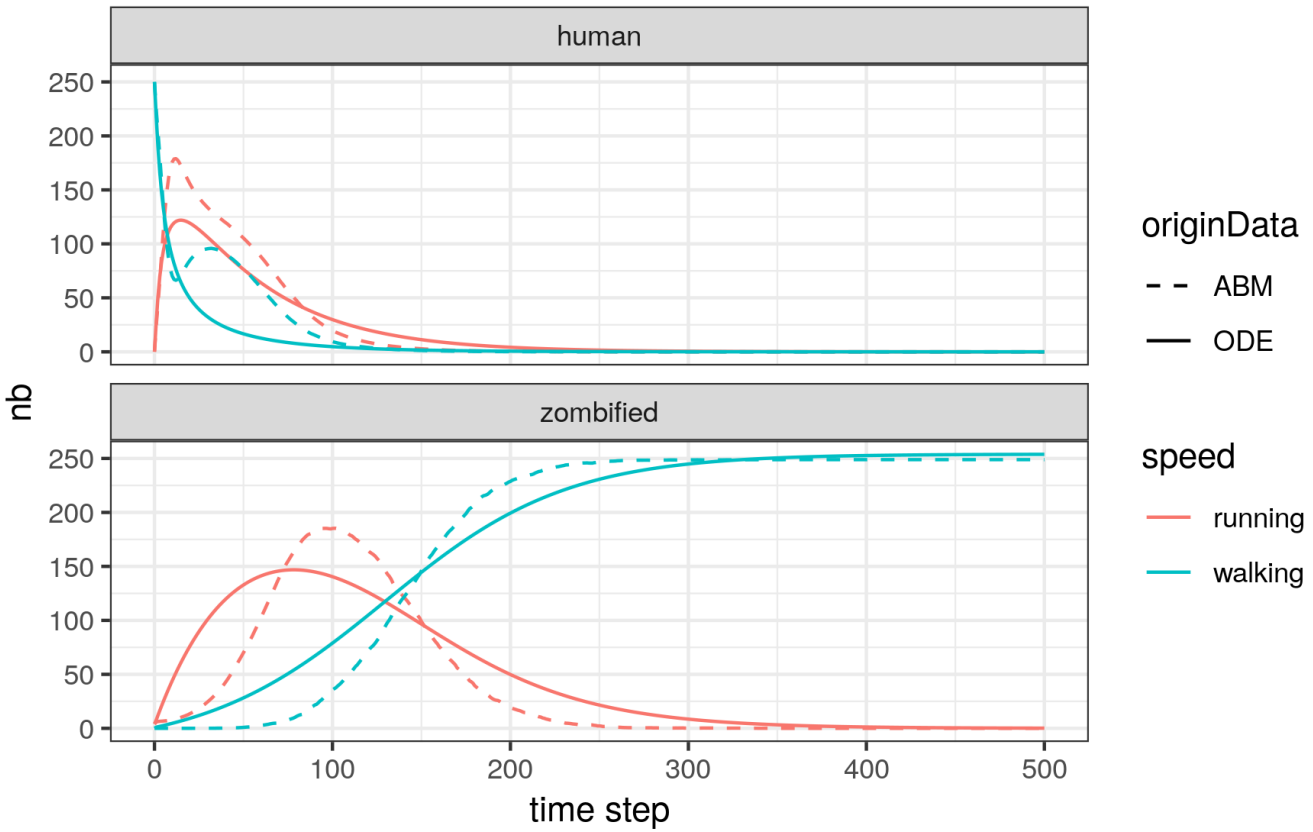
- Embed the model in OpenMOLE
- Define a fitness function

Process

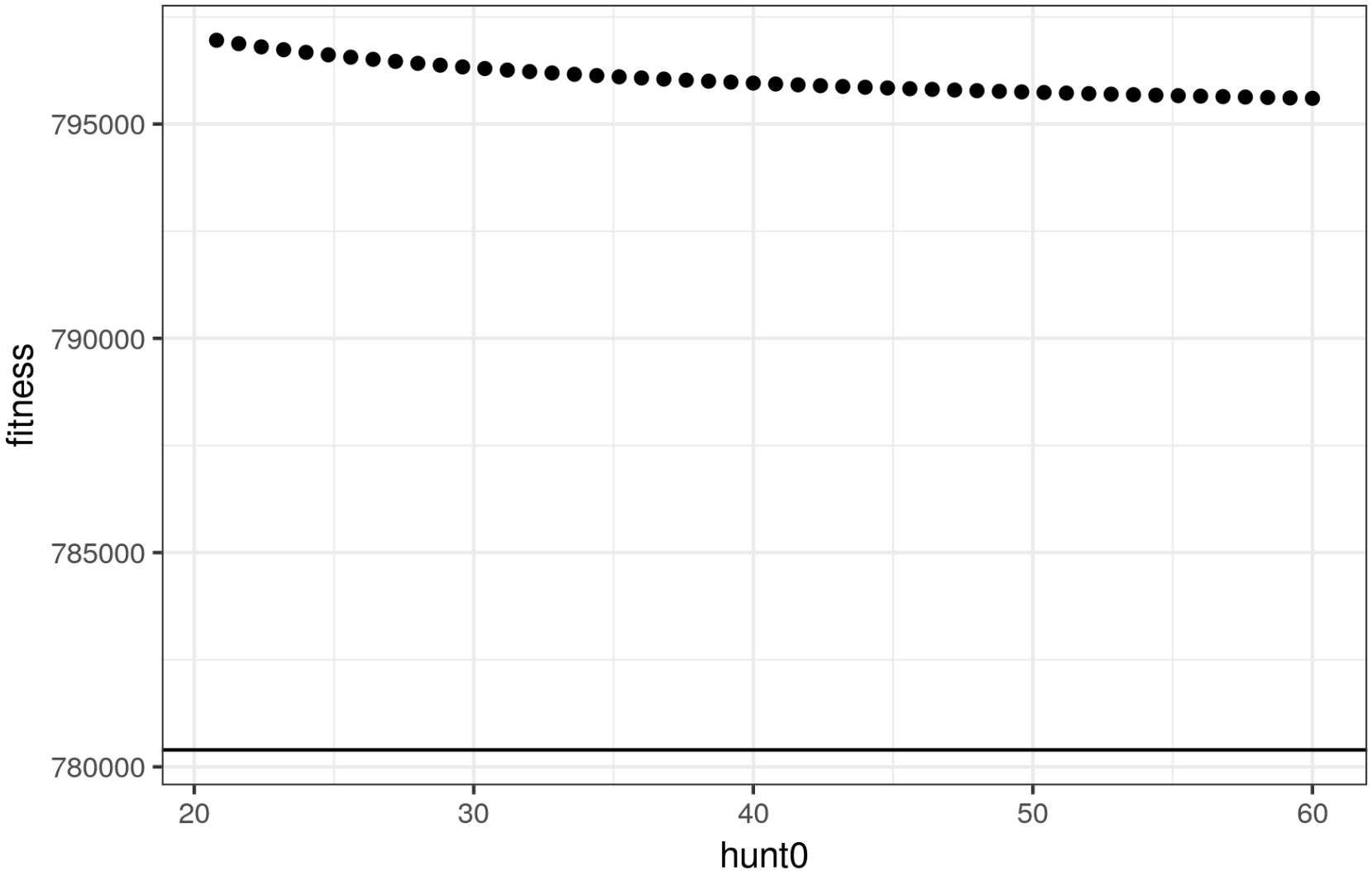
- Embed the model in OpenMOLE
- Define a fitness function
- Write a calibration task

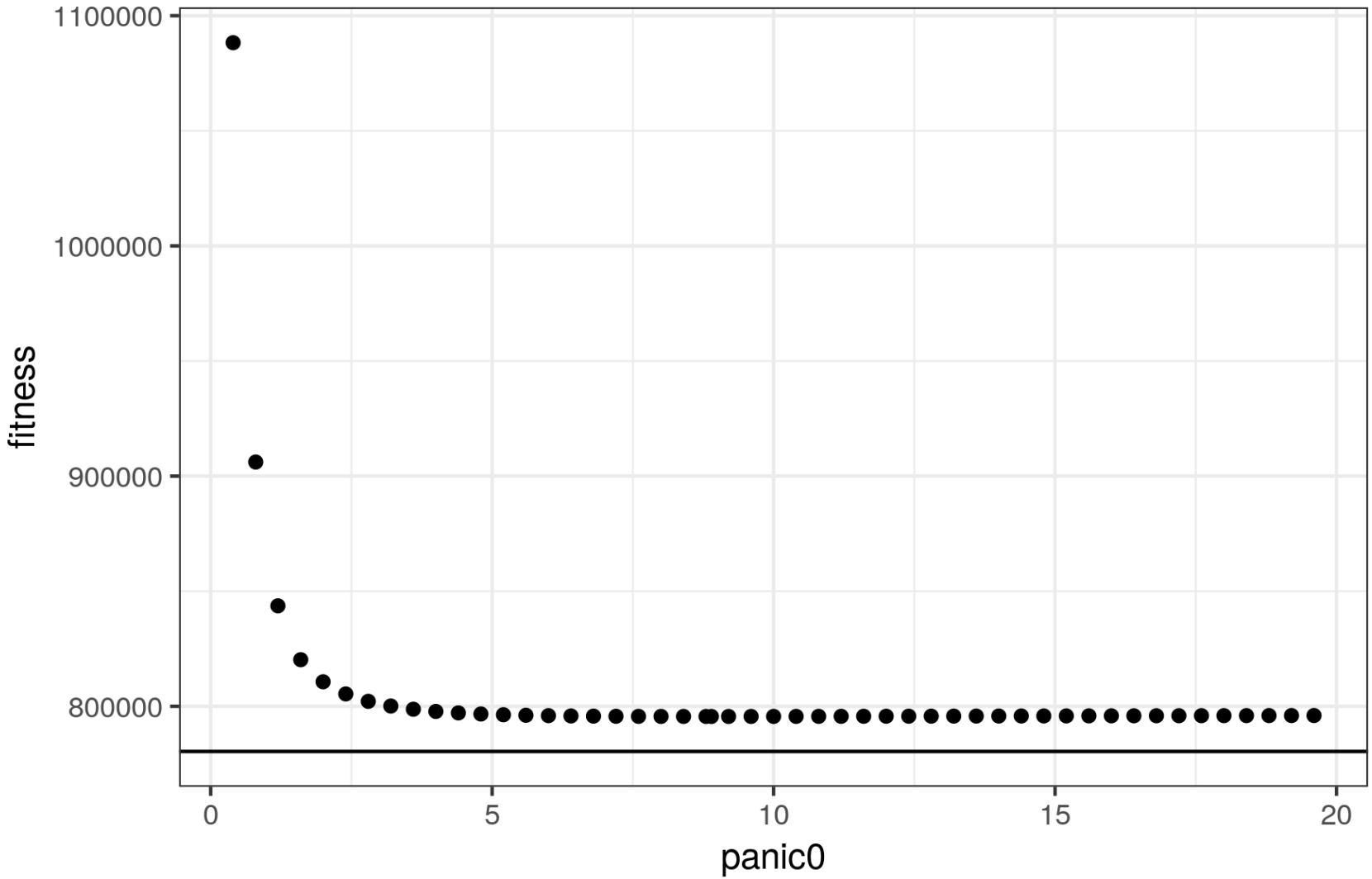
Parameter set

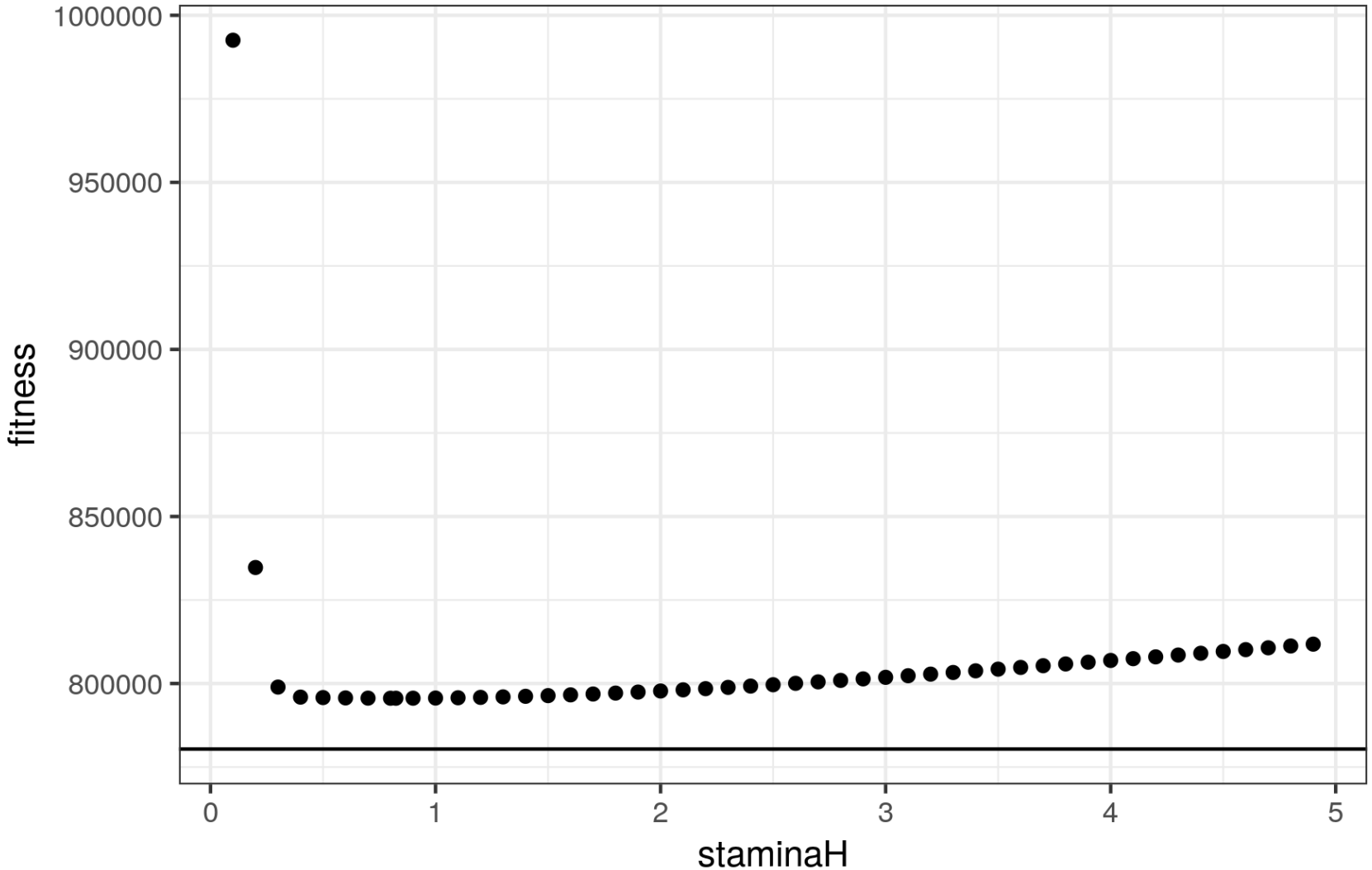
$panic_0$	7.25
$stamina_H$	0.99
$hunt_0$	10.15
$stamina_Z$	1.28
inf	0.02
fitness	780 394

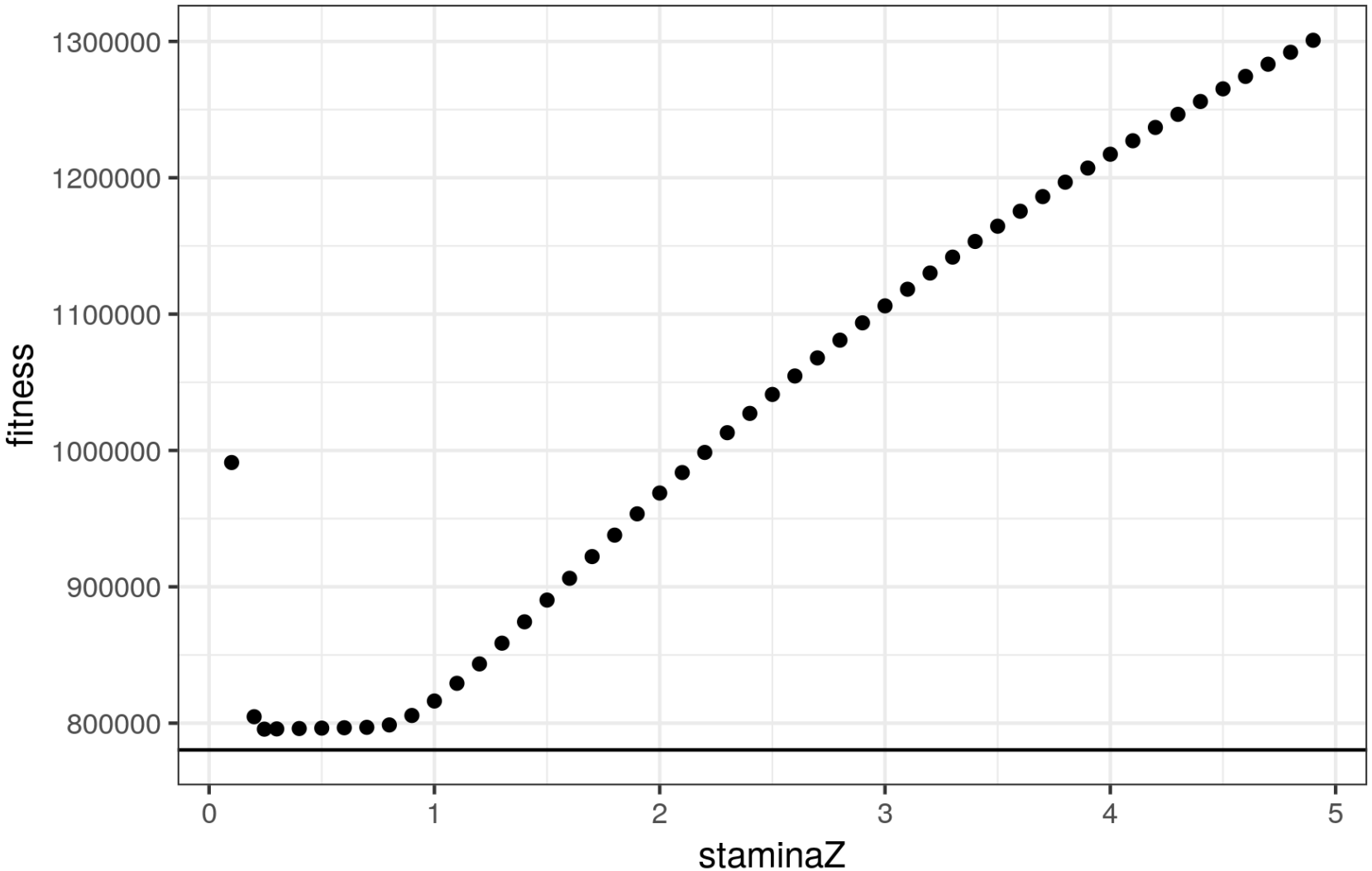


Profiles for each of the 5 parameters:
rightarrow $panic_0$, $stamina_H$, $hunt_0$, $stamina_Z$, inf_0

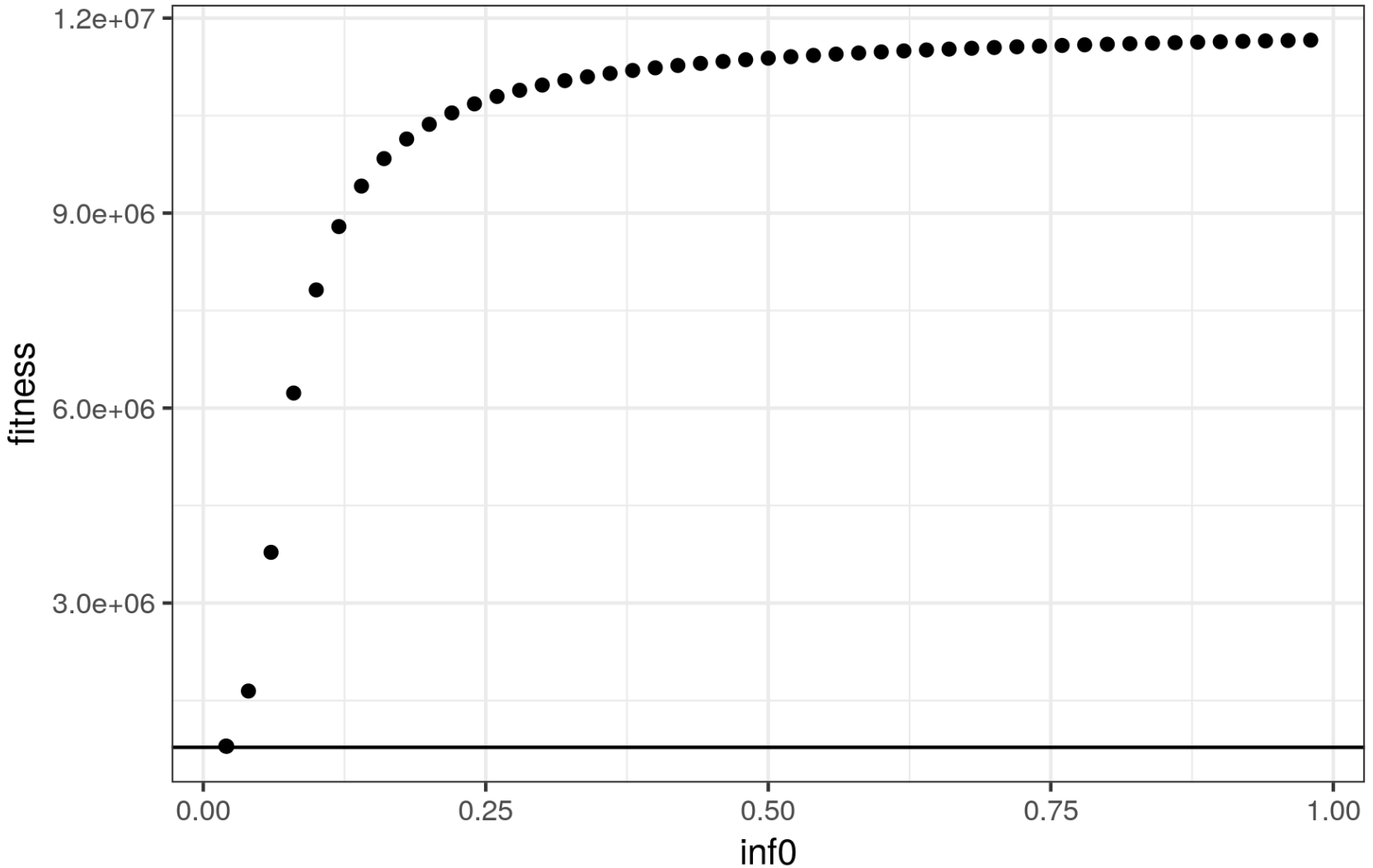

795000
790000
785000
780000
fitness
20
30
40
50
60
hunt0


1100000
1000000
900000
800000
fitness
0
5
10
15
20
panic0


1000000
950000
900000
850000
800000
fitness
0
1
2
3
4
5
staminaH


1300000
1200000
1100000
1000000
900000
800000
fitness
0
1
2
3
4
5
staminaZ

Profiles



Adding complexity

What mechanisms could we add to better represent the complexity of our Zombie situation?

What mechanisms could we add to better represent the complexity of our Zombie situation?

The parcimony issue

What mechanisms could we add to better represent the complexity of our Zombie situation?

The parcimony issue

- Do the new mechanisms really improve the fitness?

What mechanisms could we add to better represent the complexity of our Zombie situation?

The parcimony issue

- Do the new mechanisms really improve the fitness?
- Do we need them all?

What mechanisms could we add to better represent the complexity of our Zombie situation?

The parcimony issue

- Do the new mechanisms really improve the fitness?
- Do we need them all?
- What are the best combinations?

Study our model's parcimony

Process

- Embed the model in OpenMOLE **DONE**

Study our model's parcimony

Process

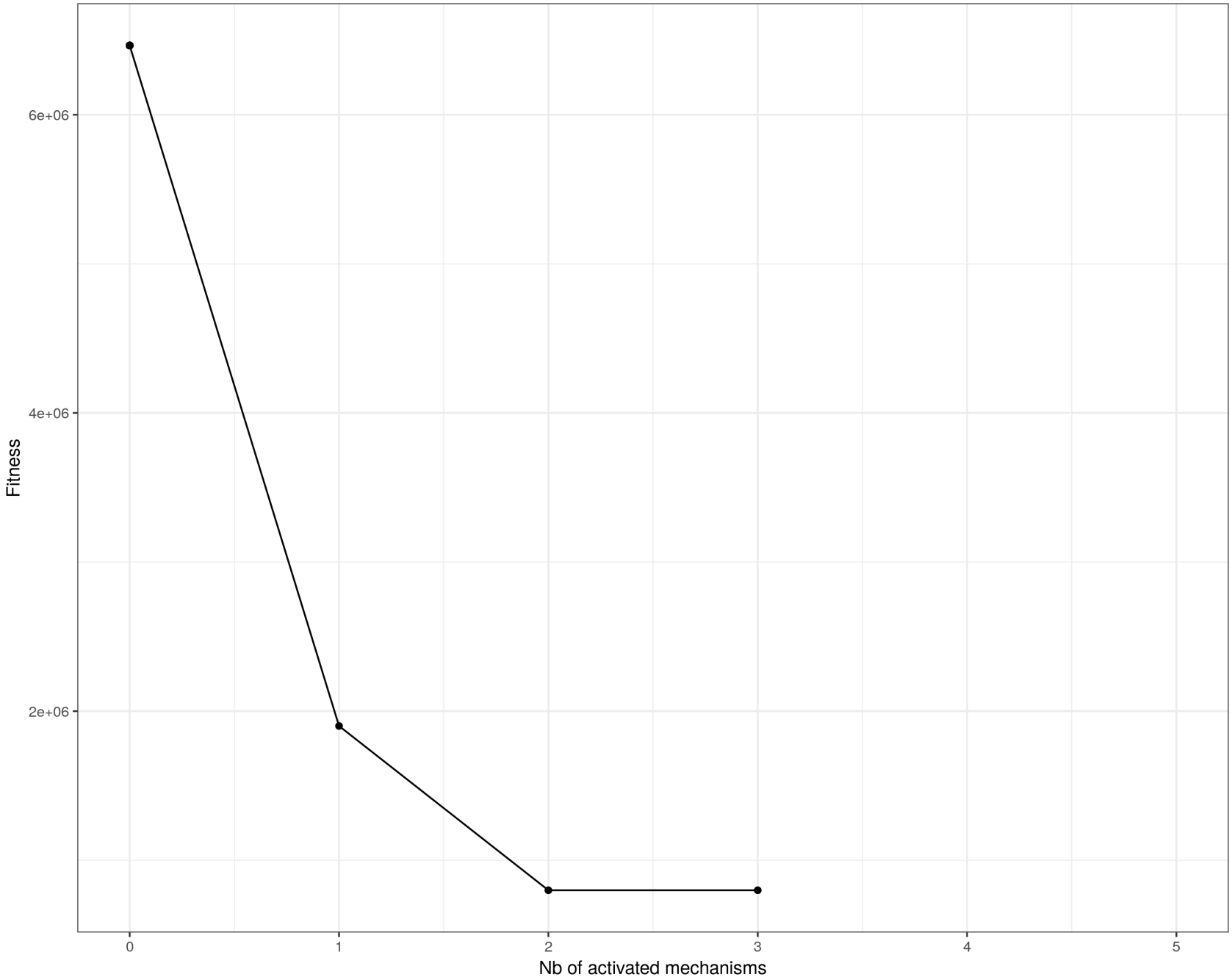
- Embed the model in OpenMOLE **DONE**
- Define a **second** fitness function

Study our model's parcimony

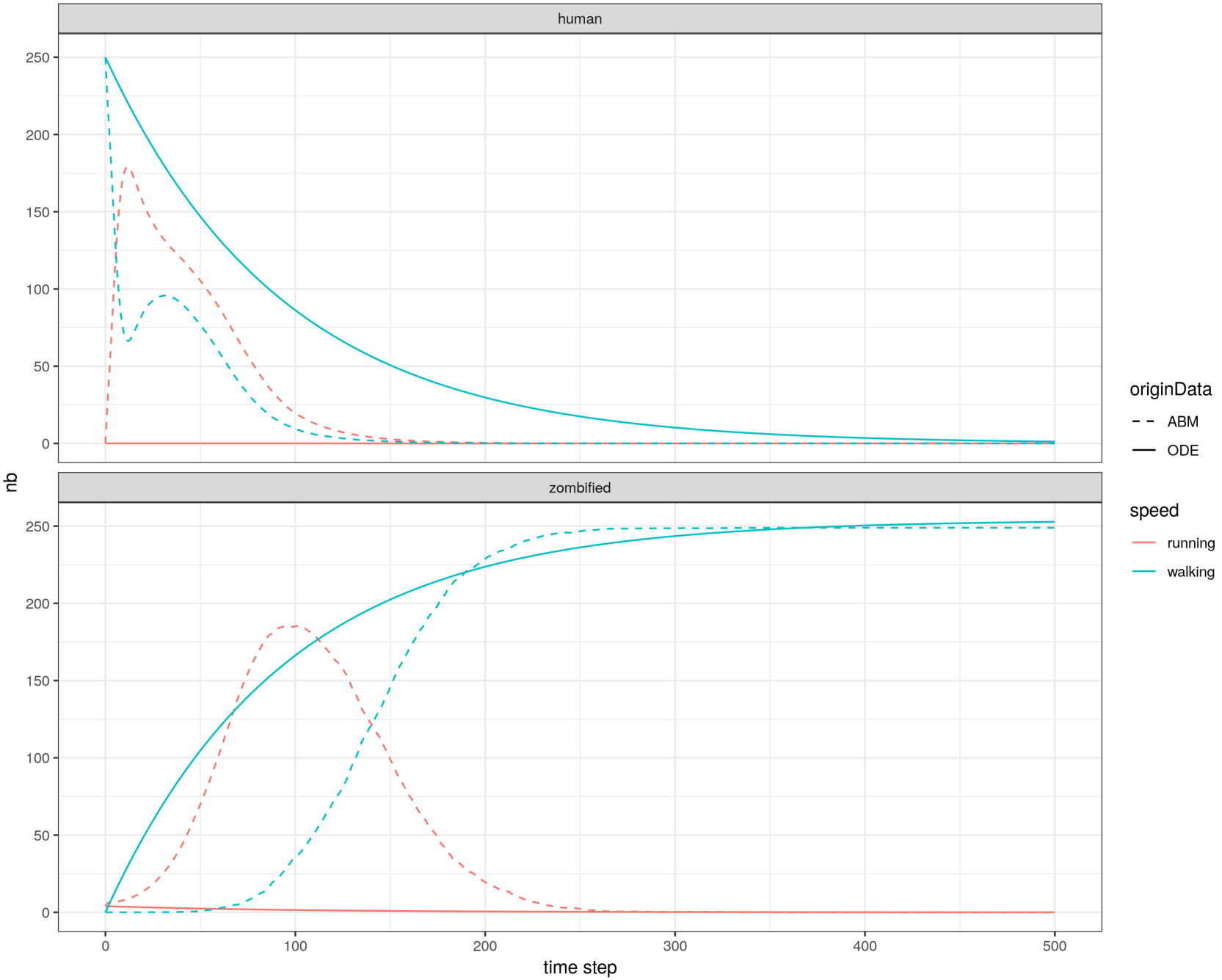
Process

- Embed the model in OpenMOLE **DONE**
- Define a **second** fitness function
- **Modify** the calibration task

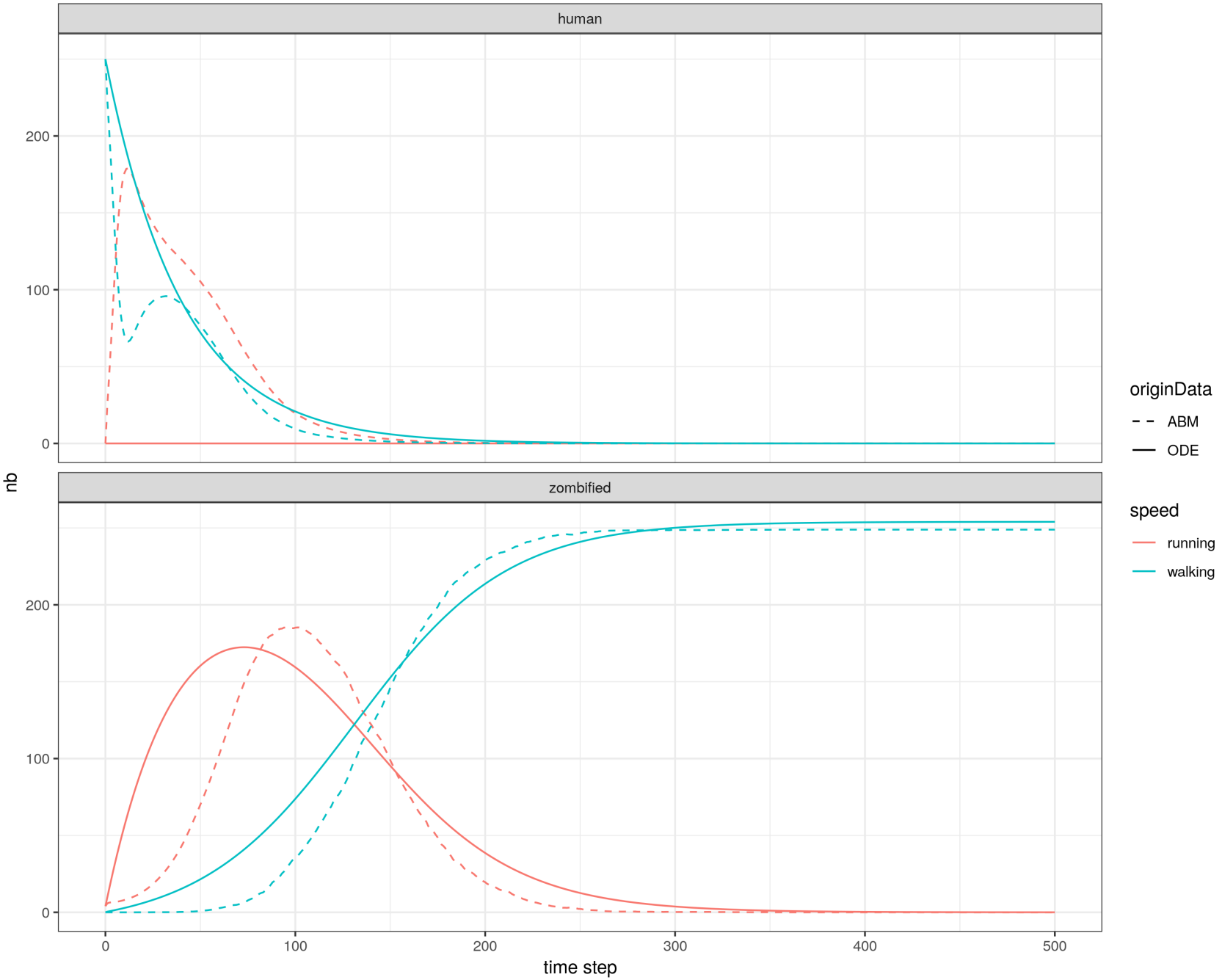
Pareto front



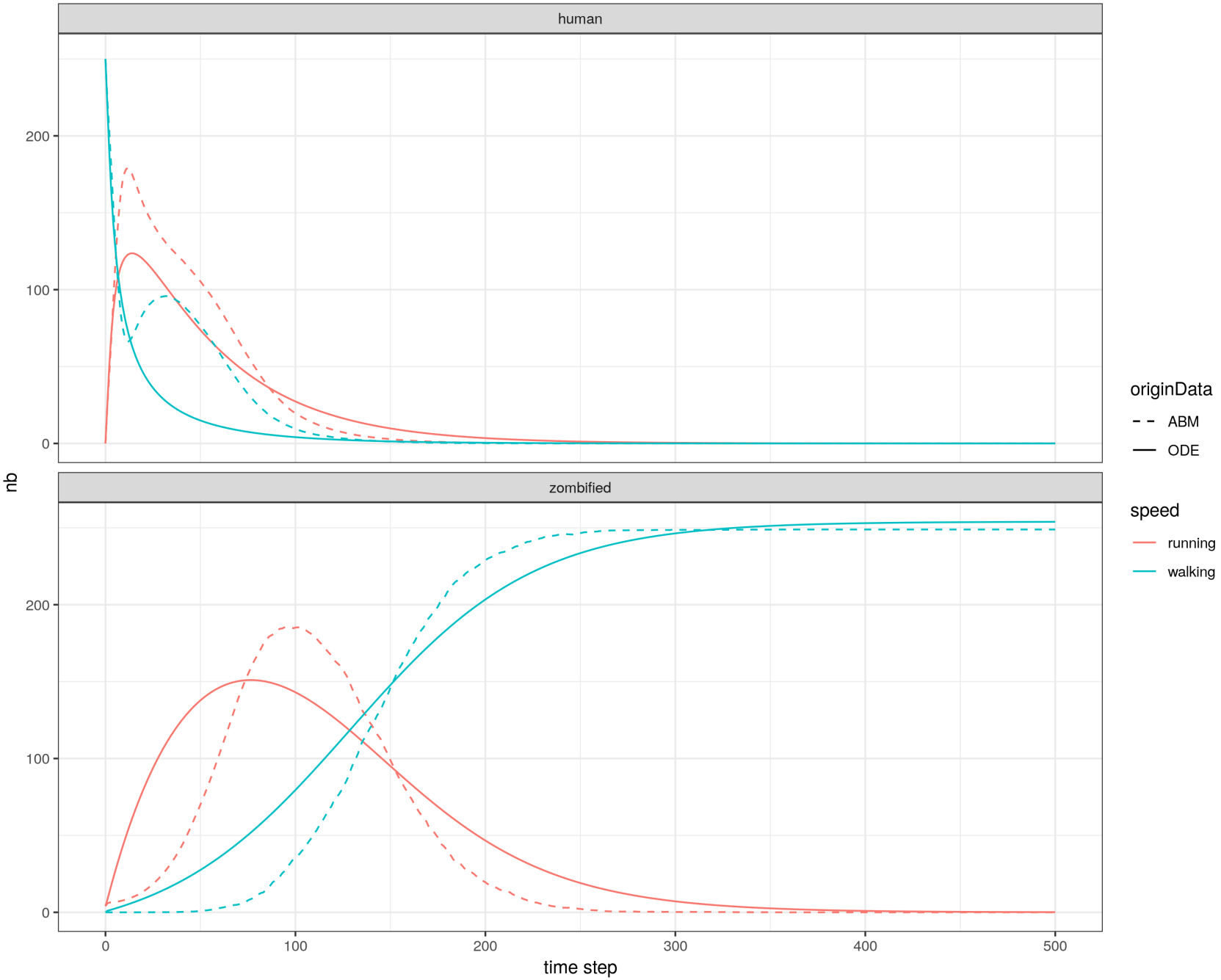
Dynamics for 0 mechanism activated



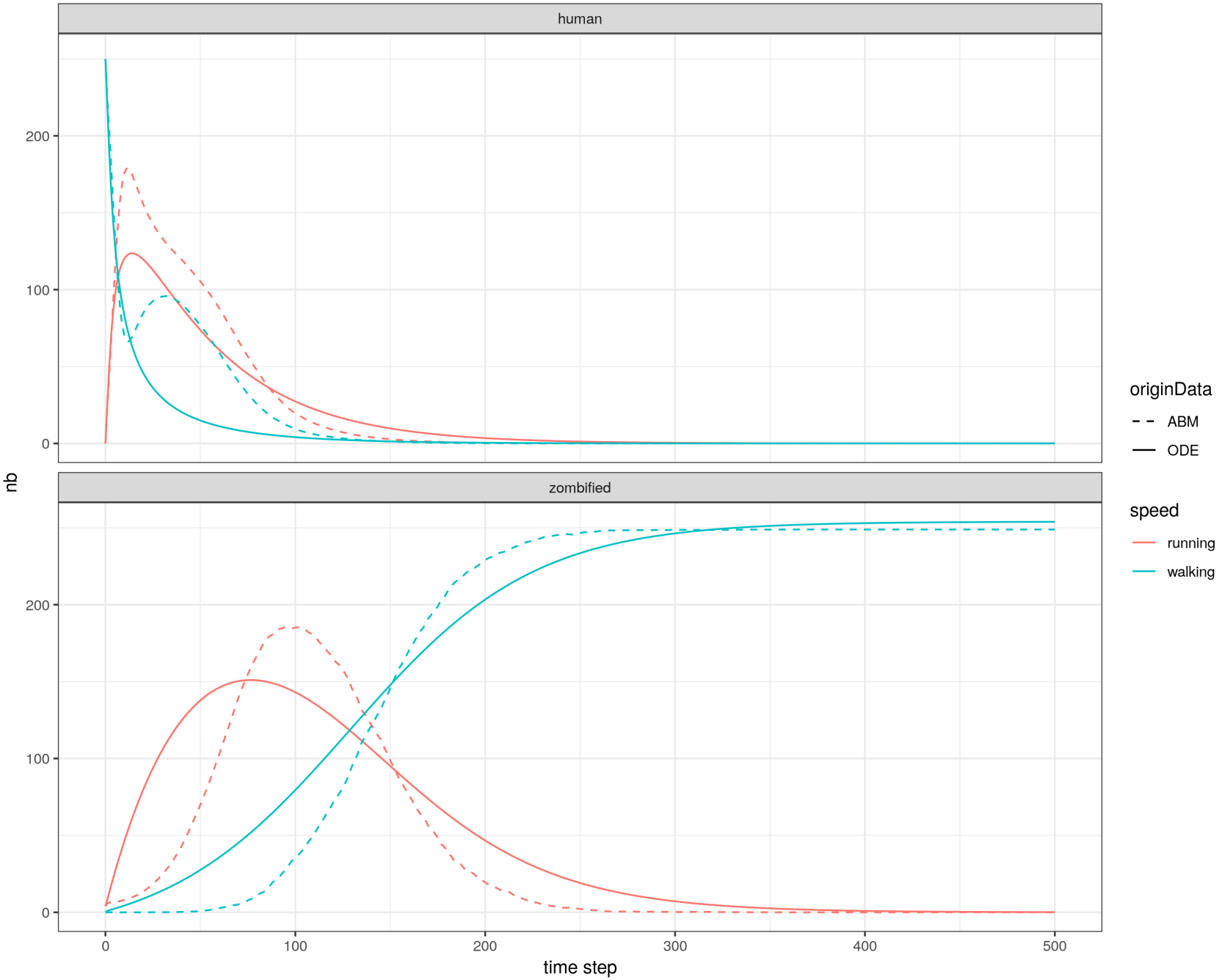
Dynamics for 1 mechanism activated



Dynamics for 2 mechanisms activated



Dynamics for 3 mechanisms activated



Some mathematics

Recall the ODE system:

$$\begin{cases} \frac{\mathrm{d}H_{walk}}{\mathrm{d}t} &= -(panic0 * \frac{Z_{walk} + Z_{run}}{N} + inf) * H_{walk} + exhaustH * H_{run} \\ \frac{\mathrm{d}H_{run}}{\mathrm{d}t} &= panic0 * \frac{Z_{walk} + Z_{run}}{N} * H_{walk} - (exhaustH + inf) * H_{run} \\ \frac{\mathrm{d}Z_{walk}}{\mathrm{d}t} &= inf * (H_{walk} + H_{run}) - hunt0 * \frac{H_{walk} + H_{run}}{N} * Z_{walk} + exhaustZ * Z_{run} \\ \frac{\mathrm{d}Z_{run}}{\mathrm{d}t} &= hunt0 * \frac{H_{walk} + H_{run}}{N} * Z_{walk} - exhaustZ * Z_{run} \end{cases}$$

Where $N := H_{walk} + H_{run} + Z_{walk} + Z_{run}$

First-order **nonlinear** (autonomous) ordinary differential equation... a priori no explicit solution, hence numerical solutions.

Let's note $X(t) := (H_{walk}(t), H_{run}(t), Z_{walk}(t), Z_{run}(t))$
We have the following Cauchy problem:

$$\begin{cases} X'(t) &= F(X(t)) \\ X(0) &= (x_0, y_0, z_0, w_0) \quad \text{initial condition} \end{cases}$$

where

$$(x_0, y_0, z_0, w_0) \in \mathbb{R}_+^*$$

and

$$\begin{array}{rccl} F & : & \mathbb{R}^4 & \to & \mathbb{R}^4 \\ & & (x, y, z, w) & \mapsto & F(x, y, z, w) := ((panic0 * \dfrac{z + w}{x + y + z + w} + inf) * x + exhaustH * y, \\ & & & & \ldots, \ldots, \ldots) \end{array}$$

Cauchy-Lipschitz theorem : Existence and uniqueness to a solution of the Cauchy problem.

Summing the 4 equations, we have:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\mathrm{d}H_{walk}}{\mathrm{d}t} + \frac{\mathrm{d}H_{run}}{\mathrm{d}t} + \frac{\mathrm{d}Z_{walk}}{\mathrm{d}t} + \frac{\mathrm{d}Z_{run}}{\mathrm{d}t} = 0$$

So $N(t)$ is constant:

$$N(t) = N(0), \text{ for all } t.$$

We find back that the population size (human + zombies) is constant : natural !

Let's note $H := H_{walk} + H_{run}$
Summing the first equations, we have:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\mathrm{d}H_{walk}}{\mathrm{d}t} + \frac{\mathrm{d}H_{run}}{\mathrm{d}t} = -inf(H_{walk}(t) + H_{run}(t)) = -infH(t)$$

First-order linear ordinary differential equation with constant coefficient: explicit solution !

$$H(t) = H(0) * e^{-inf*t}$$

Likewise,

$$Z(t) := Z_{walk} + Z_{run} = N - H(t)$$

Definition: The point $x \in \mathbb{R}^4$ is an equilibrium point for the differential equation $X' = F(X)$ if $F(X) = 0$.

Here, there is a unique equilibrium point: $(0, 0, 0, N(0))$: all the population is composed of walking zombies.

Theses 2 facts : $N(t)$ constant and solutions for H and Z are still valid for the system with additional mechanism.
Not able to change the shape of the solutions of the equation of H and Z with such system modifications.