Ordinary Differential Equations

as an alternative to agent-based modelling

eX Modelo school

OpenMOLE

June 26, 2019

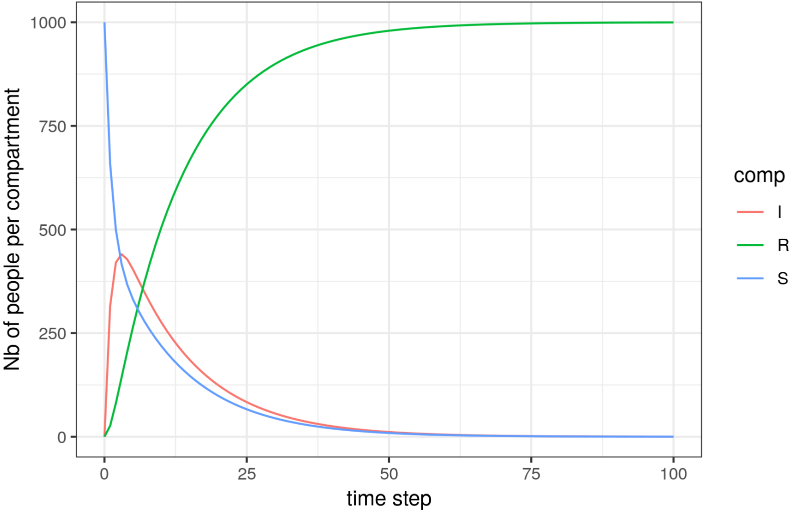
ODE systems

The ODE framework

$\rightarrow$ widely used to model transmission phenomena

- population split into compartments
- system of ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} & = & -\beta S + \lambda I \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} & = & \beta S - (\lambda + \gamma) I \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} & = & \gamma I \end{cases}$$


1000
750
500
250
0
Nb of people per compartment
0
25
50
75
100
time step
comp
I
R
S

ODE vs ABM

ODE	ABM
Equation-based	Individual-based
Generic mechanisms	Precise mechanisms
Population scale	Individual scale
Needs less resources	Computationally expensive

A Zombie situation

How could we model the Zombie invasion?

- Which mechanisms?
- Which parameters?

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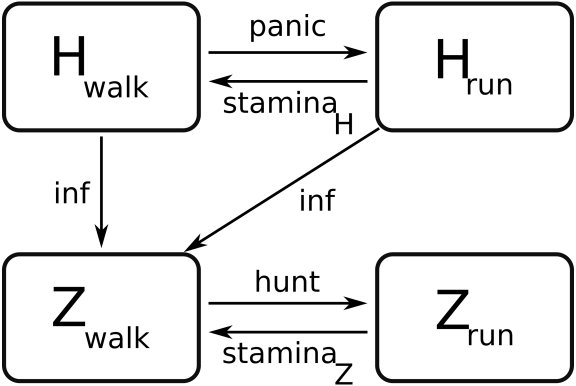
- Which mechanisms?
- Which parameters?

How can we assess our model's ability to reproduce the real data?

- Which metrics?
- Which fitness function?

A very simple ODE model



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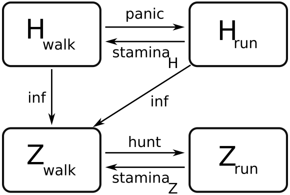

$$\begin{cases} \dfrac{\mathrm{d}H_{walk}}{\mathrm{d}t} &= -(panic + inf) * H_{walk} + exhaustH * H_{run} \\ \dfrac{\mathrm{d}H_{run}}{\mathrm{d}t} &= panic * H_{walk} - (exhaustH + inf) * H_{run} \\ \dfrac{\mathrm{d}Z_{walk}}{\mathrm{d}t} &= inf * (H_{walk} + H_{run}) - hunt * Z_{walk} + exhaustZ * Z_{run} \\ \dfrac{\mathrm{d}Z_{run}}{\mathrm{d}t} &= hunt * Z_{walk} - exhaustZ * Z_{run} \end{cases}$$

A very simple ODE model

$$
\begin{cases}
N & = & H_{walk} + H_{run} + Z_{walk} + Z_{run} \\
panic & = & panic_0 * (Z_{walk} + Z_{run})/N \\
hunt & = & hunt0 * (H_{walk} + H_{run})/N
\end{cases}
$$

Exploration

First step: Calibrate

We have some real time series of zombie invasion
$\rightarrow$ find the parameter values to best fit them

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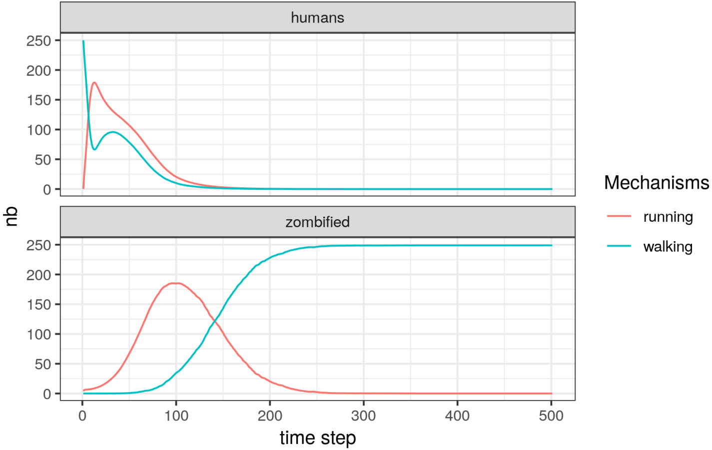

Process

First step: Calibrate

Process

- Embed the model in OpenMOLE

First step: Calibrate

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- Define a fitness function

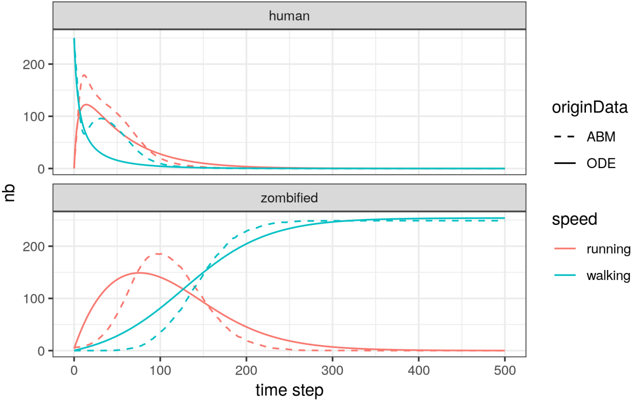
First step: Calibrate

Process

- Embed the model in OpenMOLE
- Define a fitness function
- Write a calibration task

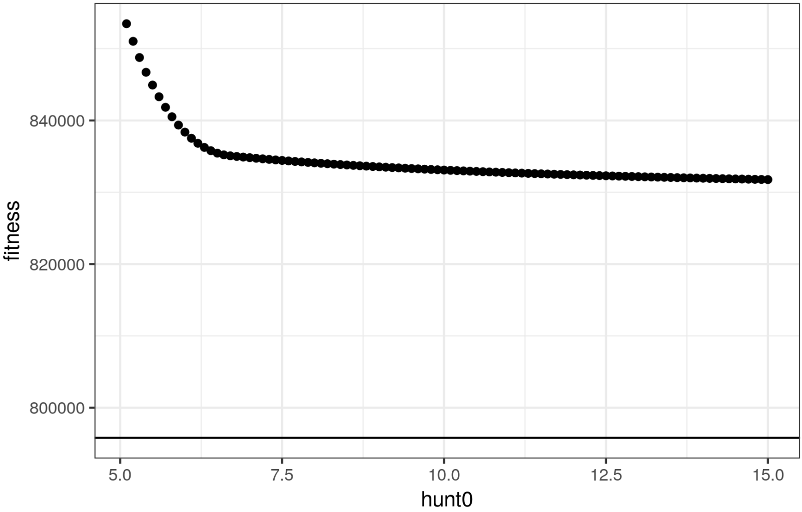
Parameter set

$panic_0$	7.25
$stamina_H$	0.99
$hunt_0$	10.15
$stamina_Z$	1.28
inf	0.02
fitness	780 394

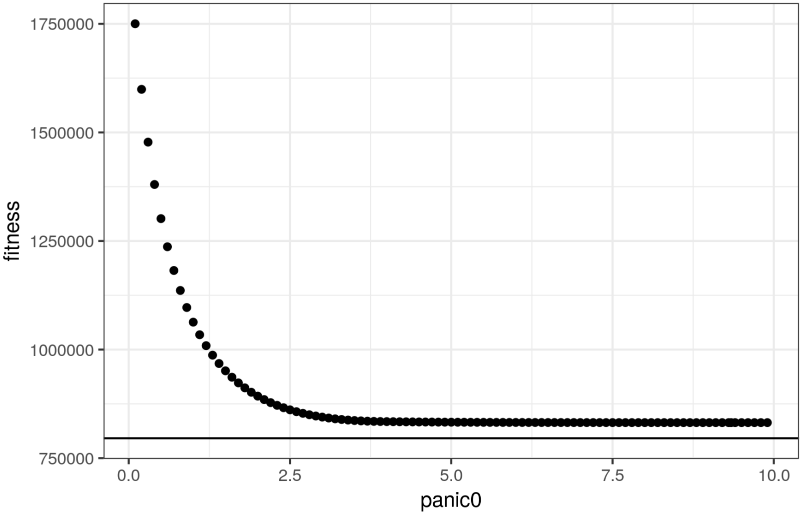


Second step: Profiles

Profiles for each of the 5 parameters:
$rightarrow\ panic_0, stamina_H, hunt_0, stamina_Z, inf_0$

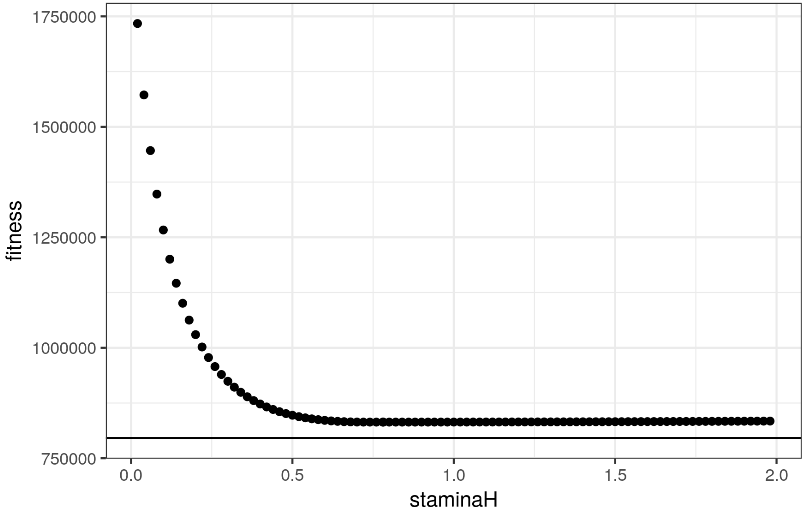

840000
820000
800000
fitness
5.0
7.5
10.0
12.5
15.0
hunt0

Profiles



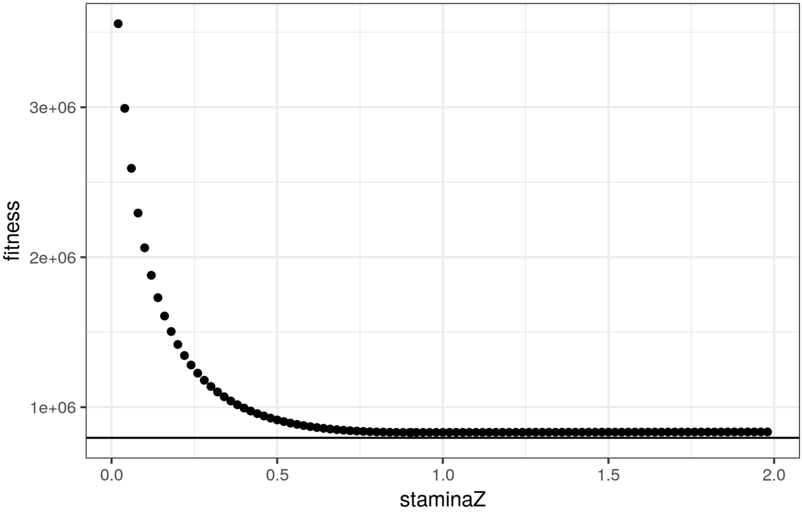
1750000
1500000
1250000
1000000
750000
fitness
0.0
2.5
5.0
7.5
10.0
panic0

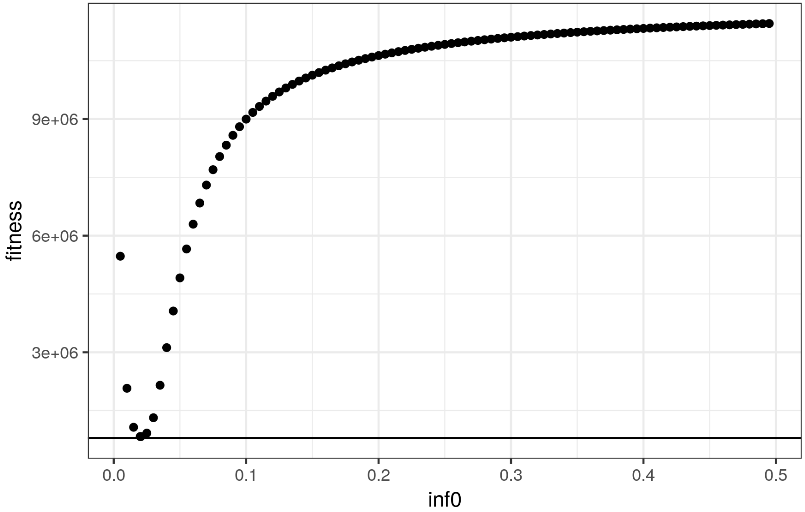
Profiles



1750000
1500000
1250000
1000000
750000
fitness
0.0
0.5
1.0
1.5
2.0
staminaH

Profiles




fitness
9e+06
6e+06
3e+06
0.0
0.1
0.2
0.3
0.4
0.5
inf0

Adding complexity

What mechanisms could we add to better represent the complexity of our Zombie situation?

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The parcimony issue

- Do the new mechanisms really improve the fitness?
- Do we need them all?
- What are the best combinations?

Process

- Embed the model in OpenMOLE **DONE**

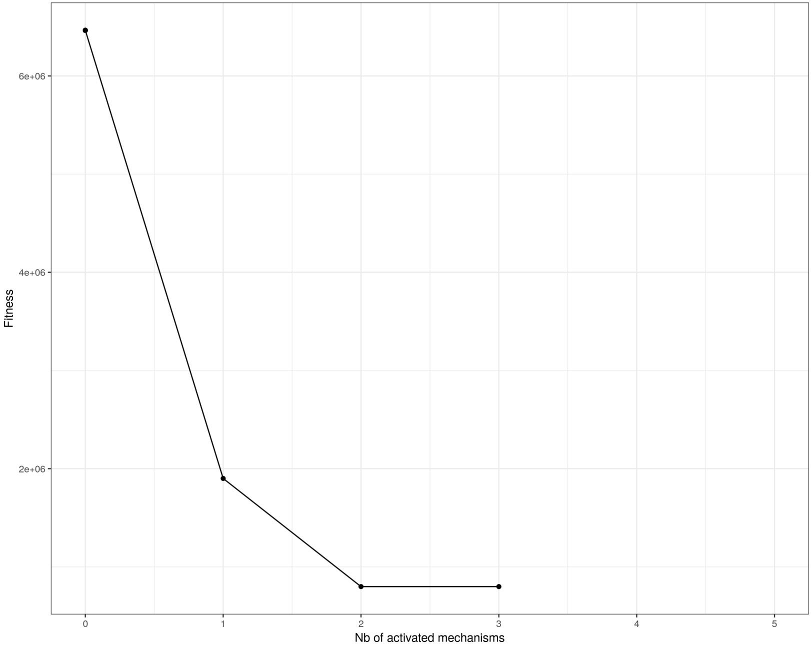
Process

- Embed the model in OpenMOLE **DONE**
- Define a **second** fitness function

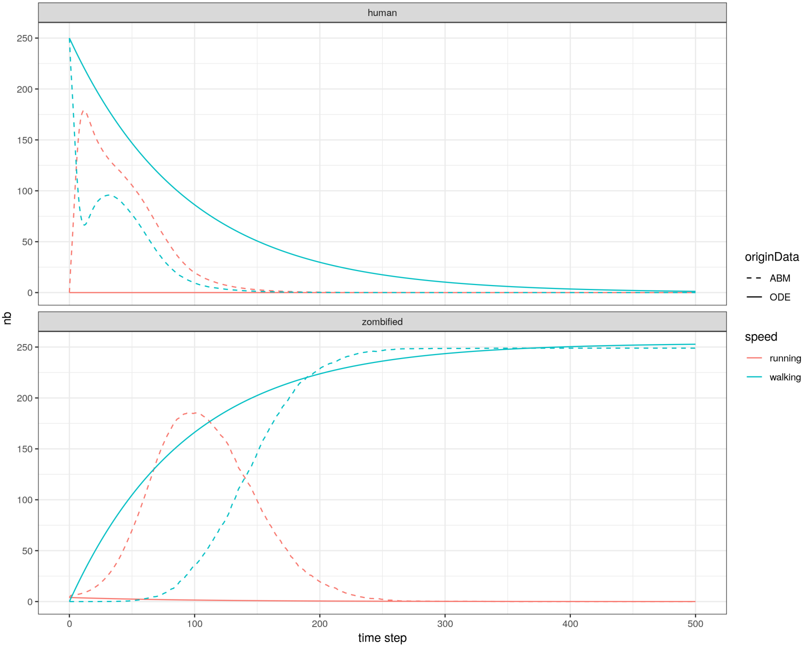
Process

- Embed the model in OpenMOLE **DONE**
- Define a **second** fitness function
- **Modify** the calibration task

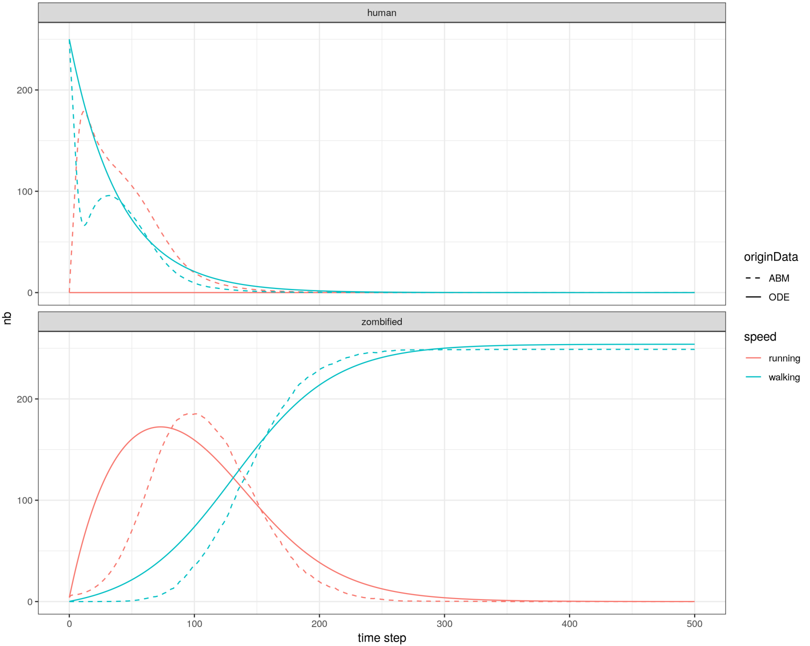
Pareto front



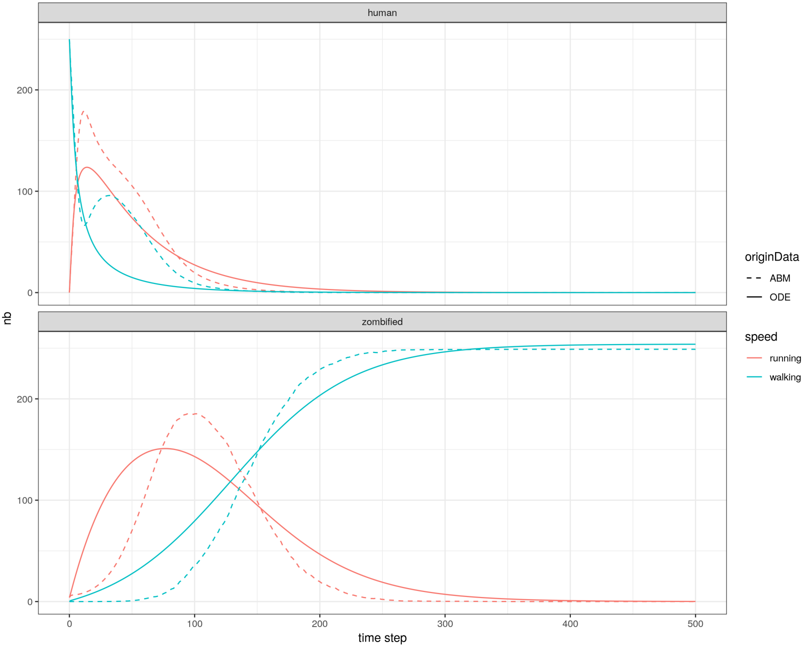
Dynamics for 0 mechanism activated



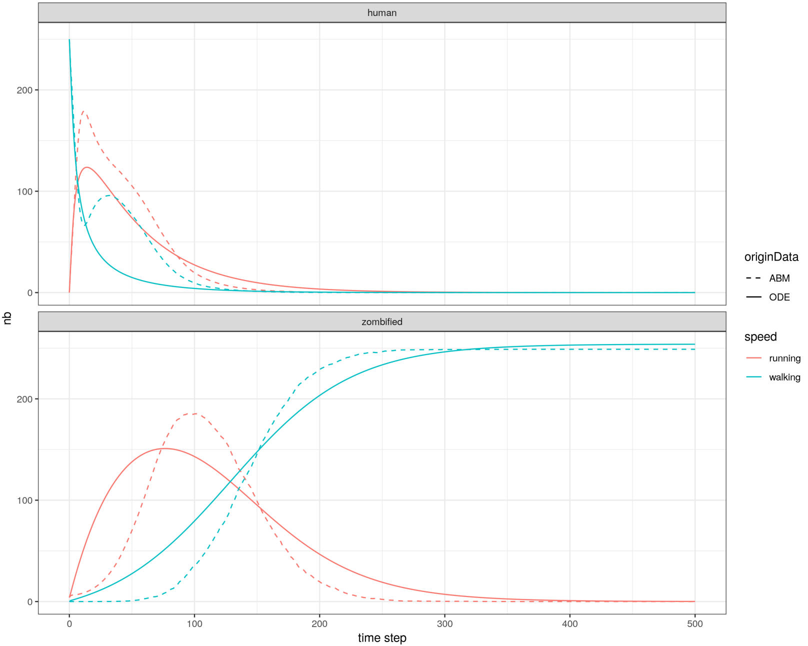
Dynamics for 1 mechanism activated



Dynamics for 2 mechanisms activated



Dynamics for 3 mechanisms activated



Some mathematics

Recall the ODE system:

$$\begin{cases} \frac{\mathrm{d}H_{walk}}{\mathrm{d}t} & = & -(panic0 * \frac{Z_{walk} + Z_{run}}{N} + inf) * H_{walk} + exhaustH * H_{run} \\ \frac{\mathrm{d}H_{run}}{\mathrm{d}t} & = & panic0 * \frac{Z_{walk} + Z_{run}}{N} * H_{walk} - (exhaustH + inf) * H_{run} \\ \frac{\mathrm{d}Z_{walk}}{\mathrm{d}t} & = & inf * (H_{walk} + H_{run}) - hunt0 * \frac{H_{walk} + H_{run}}{N} * Z_{walk} + exhaustZ * Z_{run} \\ \frac{\mathrm{d}Z_{run}}{\mathrm{d}t} & = & hunt0 * \frac{H_{walk} + H_{run}}{N} * Z_{walk} - exhaustZ * Z_{run} \end{cases}$$

Where $N := H_{walk} + H_{run} + Z_{walk} + Z_{run}$

First-order **nonlinear** (autonomous) ordinary differential equation... a priori no explicit solution, hence numerical solutions.

Cauchy Problem

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Let's note $X(t) := (H_{walk}(t), H_{run}(t), Z_{walk}(t), Z_{run}(t))$
We have the following Cauchy problem:

$$\begin{cases} X'(t) & = & F(X(t)) \\ X(0) & = & (x_0, y_0, z_0, w_0) \quad \text{initial condition} \end{cases}$$

where

$$(x_0, y_0, z_0, w_0) \in \mathbb{R}_+^*$$

and

$$\begin{array}{rcll} F & : & \mathbb{R}^4 & \to & \mathbb{R}^4 \\ & & (x, y, z, w) & \mapsto & F(x, y, z, w) := ((panic0 * \frac{z + w}{x + y + z + w} + inf) * x + exhaustH * y, \\ & & & & \ldots, \ldots, \ldots) \end{array}$$

Cauchy-Lipschitz theorem: Existence and uniqueness to a solution of the Cauchy problem.

Total population size

Summing the 4 equations, we have:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\mathrm{d}H_{walk}}{\mathrm{d}t} + \frac{\mathrm{d}H_{run}}{\mathrm{d}t} + \frac{\mathrm{d}Z_{walk}}{\mathrm{d}t} + \frac{\mathrm{d}Z_{run}}{\mathrm{d}t} = 0$$

So $N(t)$ is constant:

$$N(t) = N(0), \text{ for all } t.$$

We find back that the population size (human + zombies) is constant: natural !

Let's note $H := H_{walk} + H_{run}$
Summing the first equations, we have:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\mathrm{d}H_{walk}}{\mathrm{d}t} + \frac{\mathrm{d}H_{run}}{\mathrm{d}t} = -inf * (H_{walk}(t) + H_{run}(t)) = -inf * H(t)$$

First-order linear ordinary differential equation with constant coefficient: explicit solution !

$$H(t) = H(0) * e^{-inf * t}$$

Likewise,

$$Z(t) := Z_{walk} + Z_{run} = N - H(t)$$

Equilibrium points

Definition: The point $x \in \mathbb{R}^4$ is an *equilibrium point* for the differential equation $X' = F(X)$ if $F(X) = 0$.

Here, there is a unique equilibrium point: $(0, 0, 0, N(0))$: all the population is composed of walking zombies.

Theses 2 facts :

- $N(t)$ constant
- solutions for H and Z

  … are still valid for the system with additional mechanism.

  $\rightarrow$ Not able to change the shape of the solutions of the equation of H and Z with such system modifications.