Day 1, Lecture 4

Targeted nonparametric inference

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Key (technical) concepts we cannot avoid talking about:

* asympotically linear estimation
* efficient influence curve
* second-order remainders

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* efficient influence curve
* second-order remainders

While TMLE can be applied without knowing about these concepts ...

... we cannot understand the purpose of TMLE without knowing a bit about the efficient influence curve.

... to understand what conditions are required for TMLE inference, we need to understand the second-order remainder.

Targeted nonparametric inference

Important notation:[1]

- For a function $h : \mathcal{O} \to \mathbb{R}$ and distribution P

$$Ph = \mathbb{E}_P[h(O)] = \int h dP = \int_{\mathcal{O}} h(o) dP(o)$$

  where $\mathcal{O} = \mathbb{R}^d \times \{0,1\} \times \{0,1\}$ is the sample space of $O = (X, A, Y)$.

- For the empirical measure $\mathbb{P}_n$ of the sample $O_1, \ldots, O_n$:

$$\mathbb{P}_n h = \int h d\mathbb{P}_n = \frac{1}{n} \sum_{i=1}^{n} h(O_i);$$

  note: the right-hand-side is really just the empirical average.

- $X_n = o_P(1)$ means that $X_n \xrightarrow{P} 0$; $X_n = o_P(n^{-1/2})$ means that $n^{1/2} X_n \xrightarrow{P} 0$.

[1] van der Vaart, A. W. (2000). Asymptotic statistics (Vol. 3). Cambridge university press.

Asymptotic linearity

A very desirable property —

[2] recall: $o_P(1)$ denotes a sequence which is converges to zero in probability.

Asymptotic linearity

The empirical measure $\mathbb{P}_n$ of the sample $O_1, \ldots, O_n$:

$$\mathbb{P}_n h = \int h d\mathbb{P}_n = \frac{1}{n}\sum_{i=1}^{n} h(O_i).$$

A very desirable property —

An estimator $\hat{\psi}_n$ is $\sqrt{n}$-consistent and asymptotically linear with influence function $\phi(P_0)(O)$ if [2]

$$\sqrt{n}\left(\hat{\psi}_n - \psi_0\right) = \sqrt{n}\,\mathbb{P}_n\phi(P_0) + o_P(1),$$

where $\mathbb{E}_{P_0}[\phi(P_0)(O)] = 0$ and $\mathbb{E}_{P_0}[\{\phi(P_0)(O)\}^2] < \infty$.

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Then CLT + Slutsky implies:

$$\hat{\psi}_n \overset{as}{\sim} N(\Psi(P_0), \mathrm{Var}(\phi(P_0))/n).$$

The estimator behaves asymptotically as an average of the influence function.

[2] recall: $o_P(1)$ denotes a sequence which is converges to zero in probability.

Asymptotic linearity

Simple example: Estimator for the mean $\psi_0 = \mathbb{E}[X]$:

$$\hat{\psi}_{n,0} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Then

$$\sqrt{n}(\hat{\psi}_n - \psi_0) = \sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\underbrace{(X_i - \psi_0)}_{=\phi(P_0)(O_i)} = \sqrt{n}\mathbb{P}_n\phi(P_0)$$

$\hat{\psi}_{n,0}$ is linear and thus asymptotically linear.

Asymptotic linearity

Simple example: Estimator for the mean $\psi_0 = \mathbb{E}[X]$:

$$\hat{\psi}_{n,1} = \frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{n}$$

Then

$$\sqrt{n}(\hat{\psi}_n - \psi_0) = \sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\underbrace{(X_i - \psi_0)}_{=\phi(P_0)(O_i)} + \frac{\sqrt{n}}{n} = \sqrt{n}\mathbb{P}_n\phi(P_0) + \underbrace{\frac{1}{\sqrt{n}}}_{=o(1)}$$

$\hat{\psi}_{n,1}$ is asymptotically linear.

Asymptotic linearity

Simple example: Estimator for the mean $\psi_0 = \mathbb{E}[X]$:

$$\hat{\psi}_{n,2} = \frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{n^{1/2+0.1}}$$

Then

$$\sqrt{n}(\hat{\psi}_n - \psi_0) = \sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\underbrace{(X_i - \psi_0)}_{=\phi(P_0)(O_i)} + \frac{\sqrt{n}}{n^{1/2+0.1}} = \sqrt{n}\mathbb{P}_n\phi(P_0) + \underbrace{\frac{1}{n^{0.1}}}_{=o(1)}$$

$\hat{\psi}_{n,2}$ is asymptotically linear.

Asymptotic linearity

Simple example: Estimator for the mean $\psi_0 = \mathbb{E}[X]$:

$$\hat{\psi}_{n,3} = \frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{n^{1/2-0.1}}$$

Then

$$\sqrt{n}(\hat{\psi}_n - \psi_0) = \sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\underbrace{(X_i - \psi_0)}_{=\phi(P_0)(O_i)} + \frac{\sqrt{n}}{n^{1/2-0.1}} = \sqrt{n}\mathbb{P}_n\phi(P_0) + \underbrace{n^{0.1}}_{\to\infty}$$

$\hat{\psi}_{n,3}$ is **not** asymptotically linear.

Asymptotic linearity

An estimator $\hat{\psi}_n$ has rate of convergence $r_n \to \infty$ if [3]

$$r_n(\hat{\psi}_n - \psi_0) = O_P(1), \quad \text{i.e.,} \quad \hat{\psi}_n - \psi_0 = O_P(1/r_n).$$

The convergence rate r_n tells us how fast $\hat{\psi}_n$ centers around ψ_0, with the difference $\hat{\psi}_n - \psi_0$ behaving like $1/r_n$.

- One wants negligible bias such as to obtain reliable confidence intervals for ψ_0.
- The bias of an asymptotically linear estimator converges to zero at a rate faster than $1/\sqrt{n}$.

Data-adaptive machine learning estimators rarely achieve this rate.

[3] recall: $O_P(1)$ denotes a sequence which is bounded in probability.

Asymptotic linearity

$$\sqrt{n}\hat{\psi}_{n,1} = \underbrace{\sqrt{n}\frac{1}{n}\sum_{i=1}^{n} X_i}_{\xrightarrow{P}\psi_0} + \underbrace{\frac{\sqrt{n}}{n}}_{\to 0}, \qquad \text{i.e.,} \quad \sqrt{n}(\hat{\psi}_{n,1} - \psi_0) = o_P(1).$$

$$\sqrt{n}\hat{\psi}_{n,2} = \underbrace{\sqrt{n}\frac{1}{n}\sum_{i=1}^{n} X_i}_{\xrightarrow{P}\psi_0} + \underbrace{\frac{\sqrt{n}}{n^{1/2+0.1}}}_{\to 0}, \quad \text{i.e.,} \quad \sqrt{n}(\hat{\psi}_{n,3} - \psi_0) = o_P(1).$$

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Asymptotic linearity

$$\sqrt{n}\hat{\psi}_{n,1} = \sqrt{n}\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}_{\xrightarrow{P}\psi_0} + \underbrace{\frac{\sqrt{n}}{n}}_{\to 0}, \qquad \text{i.e.,} \quad \sqrt{n}(\hat{\psi}_{n,1} - \psi_0) = o_P(1).$$

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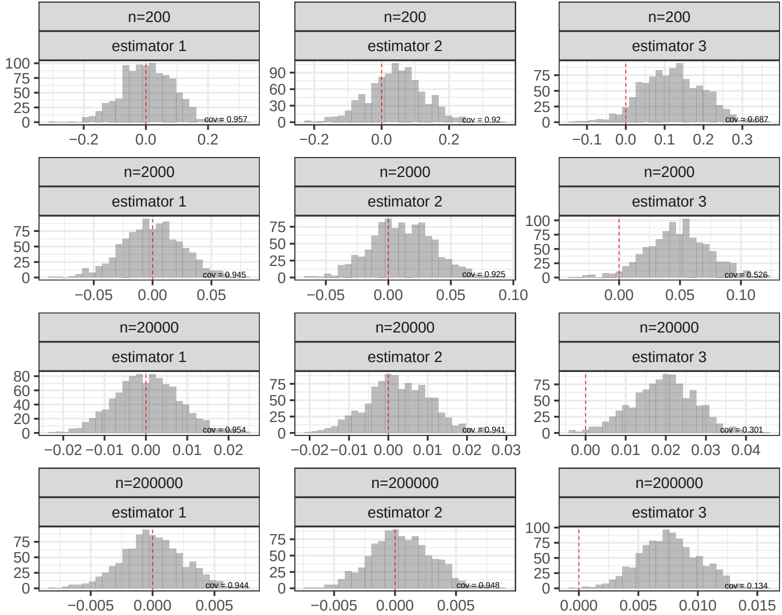
[The remainder term that determines the asymptotic bias the estimator].

Asymptotic linearity



Estimator expansion and the efficient influence curve

A key component in constructing a $\sqrt{n}$-consistent and asymptotically linear estimator, *even when using machine learning estimation*, is the so-called the efficient influence curve.[4]

[4] also known as the efficient influence function, the pathwise derivative, the Neyman orthogonal score, the canonical gradient.

Estimator expansion and the efficient influence curve

Repetition — our goal is $\sqrt{n}$-consistency and asymptotic linearity.

An estimator $\hat{\psi}_n$ is $\sqrt{n}$-consistent and asymptotically linear with influence function $\phi(P_0)(O)$ if

$$\sqrt{n}\big(\hat{\psi}_n - \psi_0\big) = \sqrt{n}\,\mathbb{P}_n\phi(P_0) + o_P(1),$$

where $\mathbb{E}_{P_0}[\phi(P_0)(O)] = 0$ and $\mathbb{E}_{P_0}[\{\phi(P_0)(O)\}^2] < \infty$.

Then CLT + Slutsky implies:

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The estimator behaves asymptotically as an average of the influence function.

Estimator expansion and the efficient influence curve

The von Mises expansion:

A sufficiently smooth functional (as a map from distributions to the real line) $\Psi : \mathcal{M} \to \mathbb{R}$ (the target parameter) admits a certain distributional Taylor expansion:

$$\Psi(P) - \Psi(P') = \int \phi(P)(o)d(P - P')(o) + R_2(P, P'), \tag{1}$$

for distributions $P, P' \in \mathcal{M}$ and a function ϕ satisfying $P\phi(P) = 0$ (mean zero) and $P\phi(P)^2 < \infty$ (finite variance).

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- when the model $\mathcal{M}$ is assumed properly nonparametric, there exists *one* function $\phi(P)$ fulfilling (1). This is called the efficient influence curve; we also denote it $\phi^*(P)$.

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- The efficient influence curve in nonparametric models indicates how to construct asymptotically linear (and efficient) estimators.

Estimator expansion

An estimator $\hat{\psi}_n$ is asymptotically linear if,

$$\sqrt{n}\big(\hat{\psi}_n - \Psi(P_0)\big) = \sqrt{n}\,\mathbb{P}_n\phi^*(P_0) + o_P(1). \quad (*)$$

Evaluating the von Mises expansion in an estimator $\hat{P}_n^*$ and the true data-generating P_0:

$$\Psi(\hat{P}_n^*) - \Psi(P_0) = (\hat{P}_n^* - P_0)\phi^*(\hat{P}_n^*) + R_2(\hat{P}_n^*, P_0)$$

$(*1)$

$(*2)$

$(*3)$

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&= \mathbb{P}_n\phi^*(P_0) + o_P(n^{-1/2}) \\
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i.e., need $(*1)$–$(*3)$ to be $o_P(n^{-1/2})$.

Estimator expansion

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\end{aligned}$$

- $(*1)$ is an empirical process term.
- $(*2)$ second-order bias term.
- $(*3)$ is called the efficient influence curve equation.

Estimator expansion

... about the empirical process term $(*1)$:

1. can be handled by empirical process theory, if $(\phi^*(P) : P \in \mathcal{M})$ is assumed Donsker.[5]
2. otherwise can handled by extra sample splitting.[6, 7]

[5] Lemma 19.24 of van der Vaart, A. W. (2000): Asymptotic statistics yields then that $(\mathbb{P}_n - P_0)(\phi^*(\hat{P}_n) - \phi^*(P_0)) = o_P(n^{-1/2})$.

[6] Zheng, W., & van der Laan, M. J. (2010). Asymptotic theory for cross-validated targeted maximum likelihood estimation.

[7] Chernozhukov, V., Chetverikov, D., Demirer, M., Duflo, E., Hansen, C., Newey, W., & Robins, J. (2018). Double/debiased machine learning for treatment and structural parameters.

Estimator expansion

Side note: Usually, we will assume the Donsker class condition.

- this is a way of nonparametrically characterizing the complexity of nuisance parameters.
- classes of functions that are Donsker: Indicator functions, bounded monotone functions, Lipschitz parametric functions, smooth functions, ...

Donsker classes also include traditional parametric functions.

We will not discuss this further. For a nice intro see Sections 4.2 and 4.3 of Kennedy, E. H. (2016): Semiparametric theory and empirical processes in causal inference.

Estimator expansion

This is basically what is needed to understand the construction of TMLE.

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Conditions (asymptotic linearity and efficiency)

(C1) Solve the efficient influence curve equation: $\mathbb{P}_n \phi^*(\hat{P}_n) = o_P(n^{-1/2})$.

(C2) Remainder $R(\hat{P}_n, P_0) = o_P(n^{-1/2})$.

(C3) Donsker class conditions for $\{\phi^*(P) : P \in \mathcal{M}\}$.

Then: $\Psi(\hat{P}_n) \overset{as}{\sim} N(\Psi(P_0), P_0 \phi^*(P_0)^2/n)$.

Construction of estimators

$$\Psi(\hat{P}_n) - \Psi(P_0) = \mathbb{P}_n \phi^*(P_0) + o_P(n^{-1/2})$$
$$+ R(\hat{P}_n, P_0)$$
$$- \mathbb{P}_n \phi^*(\hat{P}_n)$$

For **a given target parameter** $\Psi : \mathcal{M} \to \mathbb{R}$, we need to

1. Know the efficient influence curve, so that we can solve the efficient influence curve equation.
2. Analyze the remainder $R(P, P_0) := \Psi(P) - \Psi(P_0) + P_0 \phi^*(P)$.

Repetition: These are solely properties of the estimation problem, but also tell us how to construct estimators such as TMLE.

Example: ATE estimation

Analysis of a concrete estimation problem

EXAMPLE: Average treatment effect (ATE)

Observed data $O = (X, A, Y) \in \mathbb{R}^d \times \{0,1\} \times \{0,1\} = \mathcal{O}$

* $X \in \mathbb{R}^d$ are covariates
* $A \in \{0,1\}$ is a binary exposure variable (treatment decision)
* $Y \in \{0,1\}$ is a binary outcome variable

$O \sim P_0$ where P_0 assumed to belong to nonparametric model $\mathcal{M}$.

We are interested in estimating the ATE:

$$\Psi(P) = \mathbb{E}_P\big[\mathbb{E}_P[Y \mid A = 1, X] - \mathbb{E}_P[Y \mid A = 0, X]\big].$$

Analysis of a concrete estimation problem

EXAMPLE: Average treatment effect (ATE)

1. The efficient influence function:

$$\begin{aligned}\phi^*(P)(O) &= \tilde{\phi}^*(f,\pi)(O)\\ &= \left(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)}\right)\big(Y - f(A,X)\big) + f(1,X) - f(0,X) - \Psi(P)\end{aligned}$$

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2. The remainder:

$$\begin{aligned}R(P,P_0) &= \tilde{R}(f,\pi,f_0,\pi_0)\\ &= \int_{\mathbb{R}^d} \sum_{a=0,1} (2a-1)\frac{\pi_0(a \mid x) - \pi(a \mid x)}{\pi(a \mid x)}\big(f_0(a,x) - f(a,x)\big)d\mu_{0,X}(x)\end{aligned}$$

Analysis of a concrete estimation problem

$$f(A, X) = \mathbb{E}_P[Y \mid A, X],\ \pi(A \mid X) = P(A = a \mid X)$$
$$f_0(A, X) = \mathbb{E}_{P_0}[Y \mid A, X],\ \pi_0(A \mid X) = P(A = a \mid X)$$

$$R(P, P_0) := \Psi(P) - \Psi(P_0) + P_0\phi^*(P).$$

2. Deriving the remainder for the ATE:

$$\begin{aligned}
R(P, P_0) &= \mathbb{E}_P[f(1, X) - f(0, X)] - \mathbb{E}_{P_0}[f_0(1, X) - f_0(0, X)] \\
&+ \mathbb{E}_{P_0}\left[\left(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)}\right)(Y - f(A, X))\right] \\
&+ \mathbb{E}_{P_0}\left[f(1, X) - f(0, X)\right] - \Psi(P) \\
&\overset{*}{=} \int_{\mathbb{R}^d} \sum_{a=0,1} (2a-1)\left(\frac{\pi_0(a \mid x)}{\pi(a \mid x)} - 1\right)(f_0(a, x) - f(a, x))\, d\mu_{0,X}(x) \\
&= \int_{\mathbb{R}^d} \sum_{a=0,1} (2a-1)\frac{\pi_0(a \mid x) - \pi(a \mid x)}{\pi(a \mid x)}(f_0(a, x) - f(a, x))\, d\mu_{0,X}(x)
\end{aligned}$$

the equality marked by $*$ is detailed on the next slide.

Analysis of a concrete estimation problem

We used that:

$$
\begin{aligned}
&\mathbb{E}_{P_0}\left[\left(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)}\right)\big(Y - f(A, X)\big)\right] \\
&\quad = \mathbb{E}_{P_0}\left[\frac{2A-1}{\pi(A \mid X)}\big(Y - f(A, X)\big)\right] \\
&\quad = \mathbb{E}_{P_0}\left[\mathbb{E}_{P_0}\left[\frac{2A-1}{\pi(A \mid X)}\big(Y - f(A, X)\big)\,\middle|\, A, X\right]\right] \\
&\quad = \mathbb{E}_{P_0}\left[\frac{2A-1}{\pi(A \mid X)}\big(f_0(A, X) - f(A, X)\big)\right] \\
&\quad = \int_{\mathbb{R}^d} \sum_{a=0,1} \frac{2a-1}{\pi(a \mid x)}\big(f_0(a, x) - f(a, x)\big)\pi_0(a \mid x)\, d\mu_{0,X}(x) \\
&\quad = \int_{\mathbb{R}^d} \sum_{a=0,1} (2a-1)\frac{\pi_0(a \mid x)}{\pi(a \mid x)}\big(f_0(a, x) - f(a, x)\big)\, d\mu_{0,X}(x)
\end{aligned}
$$

Analysis of a concrete estimation problem

The remainder determines the asymptotic bias.

For the ATE, the remainder has a really nice structure.

$$
\begin{aligned}
R(P, P_0) &= \tilde{R}(f, \pi, f_0, \pi_0) \\
&= \int_{\mathbb{R}^d} \sum_{a=0,1} (2a-1) \frac{\pi_0(a \mid x) - \pi(a \mid x)}{\pi(a \mid x)} \big(f_0(a, x) - f(a, x) \big) d\mu_{0,X}(x)
\end{aligned}
$$

A "double robust" structure, which has some important implications.

Analysis of a concrete estimation problem

$$
\begin{aligned}
&|R(P, P_0)| = |\tilde{R}(f, \pi, f_0, \pi_0)| \\
&\le \sum_{a=0,1} \int_{\mathbb{R}^d} \frac{|\pi_0(a \mid x) - \pi(a \mid x)|}{\pi(a \mid x)} |f_0(a, x) - f(a, x)| d\mu_{0,X}(x)
\end{aligned}
$$

Analysis of a concrete estimation problem

$$
\begin{aligned}
|R(P, P_0)| &= |\tilde{R}(f, \pi, f_0, \pi_0)| \\
&\leq \sum_{a=0,1} \int_{\mathbb{R}^d} \frac{|\pi_0(a \mid x) - \pi(a \mid x)|}{\pi(a \mid x)} |f_0(a,x) - f(a,x)| d\mu_{0,X}(x) \\
&\overset{*}{\leq} \sum_{a=0,1} \frac{1}{\pi(a \mid x)} \sqrt{\int_{\mathbb{R}^d} \left\{\pi_0(a \mid x) - \pi(a \mid x)\right\}^2 d\mu_{0,X}(x)} \\
&\qquad\qquad \times \sqrt{\int_{\mathbb{R}^d} \left\{f_0(a,x) - f(a,x)\right\}^2 d\mu_{0,X}(x)}
\end{aligned}
$$

$*$ uses Cauchy-Schwarz.

Analysis of a concrete estimation problem

$$
\begin{aligned}
|R(P, P_0)| &= |\tilde{R}(f, \pi, f_0, \pi_0)| \\
&\leq \sum_{a=0,1} \int_{\mathbb{R}^d} \frac{|\pi_0(a \mid x) - \pi(a \mid x)|}{\pi(a \mid x)} |f_0(a, x) - f(a, x)| d\mu_{0,X}(x) \\
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&\qquad\qquad \times \sqrt{\int_{\mathbb{R}^d} \left\{f_0(a, x) - f(a, x)\right\}^2 d\mu_{0,X}(x)}
\end{aligned}
$$

Thus, if $\pi(a \mid X) > \delta > 0$ a.s., then:

$$
|\tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0)| \leq \sum_{a=0,1} \delta^{-1} \left\|\pi_0(a \mid \cdot) - \hat{\pi}_n(a \mid \cdot)\right\|_{\mu_0} \left\|f_0(a \mid \cdot) - \hat{f}_n(a \mid \cdot)\right\|_{\mu_0}
$$

$*$ uses Cauchy-Schwarz.

Analysis of a concrete estimation problem

What does this imply for estimation?

Double robustness in consistency

$$|\tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0)| \leq \sum_{a=0,1} \delta^{-1} \underbrace{\|\pi_0(a \mid \cdot) - \hat{\pi}_n(a \mid \cdot)\|_{\mu_0}}_{o_P(1), \text{ or}} \underbrace{\|f_0(a \mid \cdot) - \hat{f}_n^*(a \mid \cdot)\|_{\mu_0}}_{o_P(1)}$$

then $\tilde{\Psi}(\hat{f}_n^*) - \tilde{\Psi}(f_0) = o_P(1)$.

Asymptotic linearity (easier to establish due to double robust structure)

$$|\tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0)| \leq \sum_{a=0,1} \delta^{-1} \underbrace{\|\pi_0(a \mid \cdot) - \hat{\pi}_n(a \mid \cdot)\|_{\mu_0}}_{=o_P(n^{-1/4})} \underbrace{\|f_0(a \mid \cdot) - \hat{f}_n^*(a \mid \cdot)\|_{\mu_0}}_{=o_P(n^{-1/4})}$$

i.e., $\tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0) = o_P(n^{-1/2})$.

I.e., bias is converging at fast enough rate for reliable confidence intervals.

Analysis of a concrete estimation problem

Side note: Showing the double robustness in consistency ...

Say we have estimators $(\hat{f}_n^*, \hat{\pi}_n)$;

- converging to (f, π)
- solving the efficient influence curve equation.

Per definition, $\tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0) = \tilde{\Psi}(\hat{f}_n^*) - \tilde{\Psi}(f_0) + P_0\tilde{\phi}^*(\hat{f}_n^*, \hat{\pi}_n)$.

I.e.,
$$\begin{aligned}\tilde{\Psi}(\hat{f}_n^*) - \tilde{\Psi}(f_0) &= -P_0\tilde{\phi}^*(\hat{f}_n^*, \hat{\pi}_n) + \tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0)\\ &= (\mathbb{P}_n - P_0)\phi^*(\hat{f}_n^*, \hat{\pi}_n) + \tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0)\end{aligned}$$

The first term is an empirical process term which equals $(\mathbb{P}_n - P_0)\tilde{\phi}^*(f, \pi)$ plus an $o_P(n^{-1/2})$-term.

This then gives
$$\tilde{\Psi}(\hat{f}_n^*) - \tilde{\Psi}(f_0) = \underbrace{(\mathbb{P}_n - P_0)\tilde{\phi}^*(f, \pi)}_{\text{LLN applies}} + \tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0) + o_P(n^{-1/2})$$

which yields that $\tilde{\Psi}(\hat{f}_n^*) - \tilde{\Psi}(f_0) = o_P(1)$ if $\tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0) = o_P(1)$.

Analysis of a concrete estimation problem

EXAMPLE: Average treatment effect (ATE)

1. The efficient influence function:

$$\begin{aligned}\phi^*(P)(O) &= \tilde{\phi}^*(f,\pi)(O)\\ &= \left(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)}\right)\big(Y - f(A,X)\big) + f(1,X) - f(0,X) - \Psi(P)\end{aligned}$$

2. The remainder:

$$\begin{aligned}R(P,P_0) &= \tilde{R}(f,\pi,f_0,\pi_0)\\ &= \int_{\mathbb{R}^d} \sum_{a=0,1} (2a-1)\frac{\pi_0(a \mid x) - \pi(a \mid x)}{\pi(a \mid x)}\big(f_0(a,x) - f(a,x)\big) d\mu_{0,X}(x)\end{aligned}$$

Deriving these is done once for a given target parameter $\Psi : \mathcal{M} \to \mathbb{R}$.

TMLE as an estimation procedure

TMLE is a two-step procedure:

Step 1 Construct initial estimator $\hat{P}_n$ for P such that $R(\hat{P}_n, P_0) = o_P(n^{-1/2})$.

Step 2 Update the estimator $\hat{P}_n \mapsto \hat{P}_n^*$ such that $\hat{P}_n^*$ solves the efficient influence curve equation.

TMLE and the estimating equation (EE) estimator both solve the efficient influence curve equation ... that is why they share the same asymptotic (large-sample) properties.

Practical 2: Continued explorations based on simulated data

In this exercise we continue the simulation setting of Practical 1, now to explore —

1. Inference for estimators based on the efficient influence curve;
2. Variance estimation and coverage of confidence intervals;
3. Small-sample properties, particularly under positivity violations.

This is described in detail in: **day1-practical2.pdf**.

Practical 2: Continued explorations based on simulated data

NB —

- these exercises emphasize the asymptotic equivalence of TMLE and estimating equation (EE) estimation;

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Practical 2: Continued explorations based on simulated data

NB —

- these exercises emphasize the asymptotic equivalence of TMLE and estimating equation (EE) estimation;
- there may be small-sample differences in performance (often argued in favor of TMLE, but this is not necessarily a general result);
- otherwise, the differences are not so important for the ATE estimation problem. BUT, for other problems (e.g., in survival analysis), the substitution property of TMLE may be crucial.

Aside: deriving the efficient influence curve

We will talk more later on deriving efficient influence curves, but just note for now that it is very much tied to the notion of derivates along smooth parametric submodels.

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The von Mises expansion (1) implies a related notion of smoothness called pathwise differentiability, i.e.,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Psi(P_\varepsilon) = \int \phi(P)(o) b(o) dP(o), \tag{2}$$

for every smooth submodel $\{P_\varepsilon : \varepsilon \in \mathbb{R}\} \subset \mathcal{M}$, where P_ε has density density p_ε, and for which $P_\varepsilon = P_0$.

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for every smooth submodel $\{P_\varepsilon : \varepsilon \in \mathbb{R}\} \subset \mathcal{M}$, where P_ε has density density p_ε, and for which $P_\varepsilon = P_0$.

Equation (2) can be used to derive the efficient influence curve.

Aside: deriving the efficient influence curve

For example one evaluates the pathwise derivative along submodels defined for a mean-zero function $h : \mathcal{O} \to \mathbb{R}$ as:

$$p_\varepsilon(o) = p(o)(1 + \varepsilon h(o)).$$

and solves the integral equation on the right hand side of (2).

Note that this submodel has score function $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \log p_\varepsilon(o) = h(o)$.

(And there are many other useful tricks in this process).

Aside: deriving the efficient influence curve

Another common strategy is to assume the data are discrete and then compute the Gateaux derivative:

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Psi((1-\varepsilon)p(o) + \varepsilon\delta_{o'}),$$

which equals the influence curve $\phi(P)(o')$ directly.

Note that this really corresponds to computing the pathwise derivative along the particular submodel of the form $(1-\varepsilon)p(o) + \varepsilon\delta_{o'}$ for which the right hand side of (2) is $\phi(P)(o')$.