Day 2, Lecture 2

A tiny overview

Summary of TMLE

"Classical" causal inference estimators:

- consistency of g-formula estimators rely on correct specification of the outcome regression $f(a,x) = \mathbb{E}[Y \mid A = a, X = x]$
- consistency of ipw estimators rely on correct specification of the propensity score $\pi(a \mid x) = P(A = a \mid X = x)$
- both types of estimators work poorly with machine learning (cross-validation does not help)

TMLE (and other estimators based on the efficient influence curve):

- built-in bias correction
- double robustness in consistency
- inference based on the efficient influence curve
- even when incorporating machine learning (under conditions!!)

Summary of TMLE

TMLE is a two-step procedure:

Step 1 Construct initial estimator $\hat{P}_n$ for P.

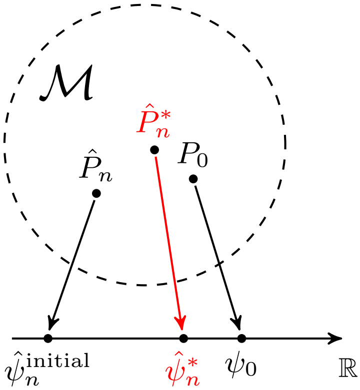
Step 2 Update the estimator $\hat{P}_n \mapsto \hat{P}_n^*$ such that $\hat{P}_n^*$ solves the efficient influence curve equation, i.e.,

$$\mathbb{P}_n \phi^*(\hat{P}_n^*) = \frac{1}{n}\sum_{i=1}^{n} \phi^*(\hat{P}_n^*)(O_i) \approx 0.$$

Step 1 = "initial estimation step"

Step 2 = "targeting step"

Summary of TMLE



- in contrast to the estimating equation (EE) estimator (and the IPW estimator), TMLE is a substitution estimator.
- TMLE *may* show better small-sample performance in settings with unstable weights.

Summary of TMLE

$$\Psi(\hat{P}_n) - \Psi(P_0) = \mathbb{P}_n\phi^*(P_0) + o_P(n^{-1/2})$$
$$+ R(\hat{P}_n, P_0)$$
$$- \mathbb{P}_n\phi^*(\hat{P}_n)$$

- The role of the targeting step (Step 2):
  - Gain double robustness in consistency.
  - Easier to achieve asymptotic lineariy (amounts to getting rid of second-order remainder).
- The role of the initial estimation step (Step 1):
  - This should be done well enough to get rid of the second-order remainder.
  - The second-order remainder tells us if/how machine learning estimators can be incorporated while still achieving asymptotic linearity.

Summary of TMLE

Specifically for the ATE:

$$\tilde{\Psi}(\hat{f}_n^*) - \tilde{\Psi}(f_0) = \mathbb{P}_n \tilde{\phi}^*(f_0, \pi_0) + o_P(n^{-1/2})$$
$$+ \ \tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0) - \underbrace{\mathbb{P}_n \tilde{\phi}^*(\hat{f}_n^*, \hat{\pi}_n)}_{\text{=0, by targeting.}}$$

When $\mathbb{P}_n \tilde{\phi}^*(\hat{f}_n^*, \hat{\pi}_n) = 0$, recall that:

$$|\tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0)| \leq \sum_{a=0,1} \delta^{-1} \|\pi_0(a \mid \cdot) - \hat{\pi}_n(a \mid \cdot)\|_{\mu_0} \|f_0(a \mid \cdot) - \hat{f}_n^*(a \mid \cdot)\|_{\mu_0}$$

What this tells us:

- Asymptotic linearity when π_0 and f_0 are estimated at rate at least $n^{-1/4}$.

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We often see this structure of the second-order remainder, but note that it has to be verified on a case-by-case basis.

Summary of TMLE

How can we perform estimation of π_0 and f_0 such as to achieve rate at least $n^{-1/4}$?

- Correctly specified parametric models
  - although consistency is guaranteed, inference cannot be based on the efficient influence curve when one is misspecified!
- There are no results on this being the case for generic implementations of, for example, random forests.
- Lasso, highly adaptive lasso (HAL), ...
- Loss-based "super learning"
  - oracle property: the super learner achieves the rate of convergence of the *best* estimator in its library.

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  - oracle property: the super learner achieves the rate of convergence of the *best* estimator in its library.
  - this is about minimizing expected loss; tuning is still important.

Summary of TMLE

$f(A, X) = \mathbb{E}_P[Y \mid A, X]$

A **loss function** $\mathscr{L}(f)(O)$ measuring the distance between an estimator f and the observed outcome Y, e.g., the negative log-likelihood:

$$\mathscr{L}(\hat{f}_n)(Y_i, A_i, X_i) = -(Y_i \log(\hat{f}_n(A_i, X_i)) + (1 - Y_i) \log(1 - \hat{f}_n(A_i, X_i))).$$

- The estimator $\hat{f}_n$ closest to the true f_0 minimizes the risk:

$$\mathbb{E}_{P_0}[\mathscr{L}(\hat{f}_n)(Y_i, A_i, X_i)].$$

- Loss-based super learning: Minimizing the cross-validated empirical risk with respect to the loss function $\mathscr{L}$ over the statistical model.

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Again, this does not usually by simple plug-in yield a good estimator for the particular feature of interest (the target parameter). But, combined with targeting, it is used to get rid of the second-order remainder.

Summary of targeted learning and TMLE

1. Scientific question $\Rightarrow$ causal parameter
2. Causal parameter $\Rightarrow$ statistical parameter
3. Statistical estimation problem = statistical parameter + statistical model
   - Efficient influence function
   - Second-order remainder
4. Identify relevant components that need targeting
   - Submodel + loss function
   - Targeting algorithm
5. Construct strong initial learners!!
   - *a priori* specified
6. Inference based on the efficient influence function

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... from a theoretical perspective, for any new type of problem.

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... from a more applied perspective.

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Comment — substitution estimation

The g-formula estimator and the TMLE are substitution estimators:

$$\hat{\psi}_n^{\text{g-formula}} = \tilde{\Psi}(\hat{f}_n, \hat{\mu}_X) = \int_{\mathbb{R}^d} \big(\hat{f}_n(1,x) - \hat{f}_n(0,x)\big) d\hat{\mu}_X(x)$$

$$\hat{\psi}_n^{\mathrm{tmle}} = \tilde{\Psi}(\hat{f}_n^*, \hat{\mu}_X) = \frac{1}{n}\sum_{i=1}^{n}\big\{\hat{f}_n^*(1,X_i) - \hat{f}_n^*(0,X_i)\big\}$$

The IPW estimator and the EE estimator are <u>not</u> substitution estimators:

$$\hat{\psi}_n^{\mathrm{ipw}} = \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{A_i Y_i}{\hat{\pi}_n(A_i \mid X_i)} - \frac{(1-A_i)Y_i}{\hat{\pi}_n(A_i \mid X_i)}\right\}$$

$$\hat{\psi}_n^{\mathrm{ee}} = \hat{\psi}_n^{\mathrm{ee}} = \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{A_i}{\hat{\pi}_n(A_i \mid X_i)} - \frac{(1-A_i)}{\hat{\pi}_n(A_i \mid X_i)}\big(Y_i - \hat{f}_n(A_i, X_i)\big) + \hat{f}_n(1,X_i) - \hat{f}_n(0,X_i)\right\}$$

Comment — substitution estimation

The g-formula estimator and the TMLE are both based on the parametrization of the ATE as follows:

$$\tilde{\Psi}(f, \mu_X) = \mathbb{E}_P[f(1, X) - f(0, X)] = \int_{\mathbb{R}^d} \big(f(1, x) - f(0, x)\big)\mu_X(x) \quad (1)$$

corresponding to an average outcome under the post-interventional distribution.

- This will always have a statistical interpretation as a covariate-adjusted difference of treatment-specific risks (but may have no causal validity).
- Estimators based on plugging estimators for f (respecting the model constraints for f) into (1) will always respect the global constraints of the observed data model.

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