Day 2, Lecture 1

Targeted Minimum Loss-based Estimation (TMLE)

Overview of today

Before lunch (9 – 12):

- Targeted Minimum Loss-based Estimation (TMLE).
- The targeting step: updating/modifying initial nuisance parameter estimators.
- The ATE as a concrete example; the ATT as a different example.
- Valid inference still requires strong initial learners.

* TMLE as a two-step procedure with involving an initial estimation step followed by a targeting step.
* Implementation of the targeting step.
* The link between the theoretical decomposition from yesterday, and TMLE as a practical estimation method.

After lunch (13 – 15): Super learning.

Targeted Minimum Loss-based Estimation (TMLE)

In this lecture, our goal is to:

1. Explain the relationship between the two-step procedure of Targeted Minimum Loss-based Estimation (TMLE) and the decomposition guiding the construction of asymptotically linear estimators.
2. Describe the process of constructing the targeting step in TMLE, highlighting the role of the efficient influence curve in informing the construction of the targeting step, and differentiate TMLE from other estimation methods solving the efficient influence curve.

Targeted learning recap

1. Data is a random variable O with a probability P_0
2. P_0 belongs to a statistical model $\mathcal{M}$
3. Our target is a parameter $\Psi : \mathcal{M} \to \mathbb{R}$
4. Construct estimator $\hat{P}_n$ for (relevant part of) P_0 and estimate the target parameter by $\hat{\psi}_n = \Psi(\hat{P}_n)$
5. Quantify uncertainty for the estimator $\hat{\psi}_n = \Psi(\hat{P}_n)$

Estimation paradigm

1. P_0 is assumed to belong to a nonparametric model $\mathcal{M}$
2. Construction of $\sqrt{n}$-consistent and asymptotically linear estimation of $\psi_0 = \Psi(P_0)$ based the efficient influence function.

Targeted learning recap: defining the target

Observed data $O = (X, A, Y) \in \mathbb{R}^d \times \{0, 1\} \times \{0, 1\} = \mathcal{O}$.

$O \sim P_0$ belonging to a statistical model $\mathcal{M}$.

The observed-data density p of $P \in \mathcal{M}$ can be factorized into:

$$p(o) = \mu_Y(y \mid a, x)\pi(a \mid x)\mu_X(x),$$

- $\mu_Y(y \mid A, X) = P(Y = y \mid A, X)$
- $\pi(a \mid X) = P(A = a \mid X)$
- μ_X is the marginal density of X (with respect to an appropriate dominating measure)

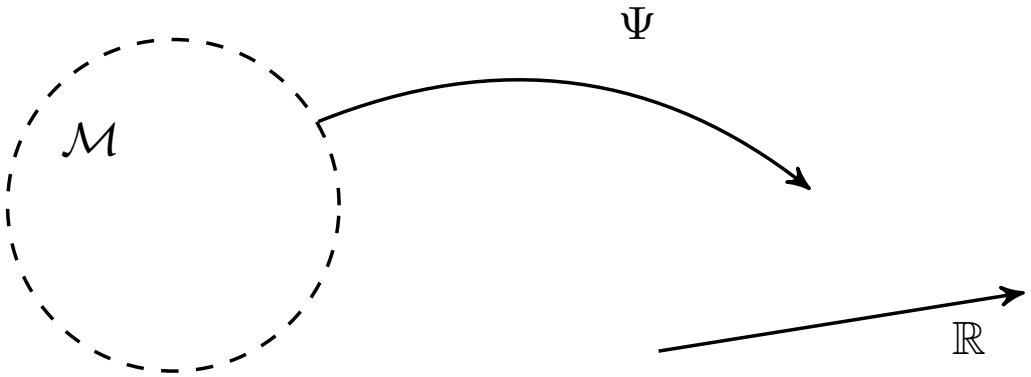
Targeted learning recap: defining the target

The counterfactual distribution $P^{a'}$ with density given by:

$$p^{a'}(o) = \mu_Y(y \mid a, x) 1\{a = a'\} \mu_X(x)$$

Statistical target parameter:

$$\begin{aligned}\Psi(P) &= \int_{\mathcal{O}} y dP^1(o) - \int_{\mathcal{O}} y dP^0(o) \\ &= \mathbb{E}_P[\mathbb{E}_P[Y \mid A = 1, X]] - \mathbb{E}_P[\mathbb{E}_P[Y \mid A = 0, X]].\end{aligned}$$



Targeted Minimum Loss-based Estimation (TMLE)

We have seen that —

Conditions (asymptotic linearity and efficiency)

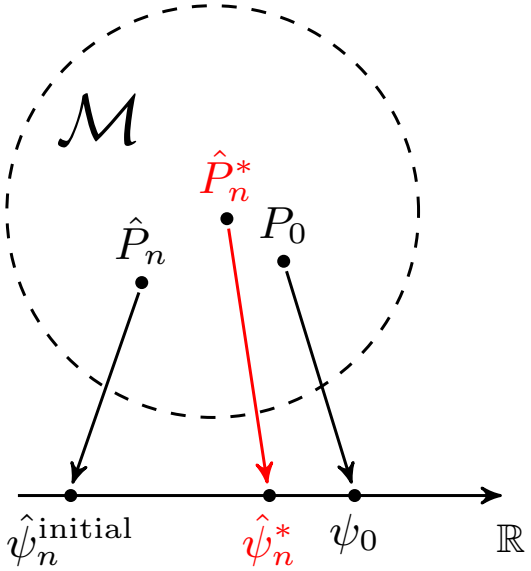
(C1) Solve the efficient influence curve equation: $\mathbb{P}_n \phi^*(\hat{P}_n) = o_P(n^{-1/2})$

(C2) Remainder $R(\hat{P}_n, P_0) = o_P(n^{-1/2})$

(C3) Donsker class conditions for $\{\phi^*(P) : P \in \mathcal{M}\}$

Then: $\Psi(\hat{P}_n) \overset{as}{\sim} N(\Psi(P_0), P_0 \phi^*(P_0)^2/n)$

Targeted Minimum Loss-based Estimation (TMLE)



TMLE is a two-step procedure:

Step 1 Construct initial estimator $\hat{P}_n$ for P.

Step 2 Update the estimator $\hat{P}_n \mapsto \hat{P}_n^*$ such that $\hat{P}_n^*$ solves the efficient influence curve equation, i.e.,

$$\mathbb{P}_n \phi^*(\hat{P}_n^*) = \frac{1}{n} \sum_{i=1}^{n} \phi^*(\hat{P}_n^*)(O_i) \approx 0.$$

Step 1 = "initial estimation step"

Step 2 = "targeting step"

Targeted Minimum Loss-based Estimation (TMLE)

$$\Psi(\hat{P}_n) - \Psi(P_0) = \mathbb{P}_n \phi^*(P_0) + o_P(n^{-1/2})$$
$$+ R(\hat{P}_n, P_0)$$
$$- \mathbb{P}_n \phi^*(\hat{P}_n)$$

- The role of the targeting step (Step 2):
  - Gain double robustness in consistency.
  - Easier to achieve asymptotic lineariy (amounts to getting rid of second-order remainder).
- The role of the initial estimation step (Step 1):
  - This should be done well enough to get rid of the second-order remainder.

Targeted Minimum Loss-based Estimation (TMLE)

$$f(A, X) = \mathbb{E}_P[Y \mid A, X]$$

A **loss function** $\mathscr{L}(f)(O)$ measuring the distance between an estimator f and the observed outcome Y, e.g., the negative log-likelihood:

$$\mathscr{L}(\hat{f}_n)(Y_i, A_i, X_i) = -(Y_i \log(\hat{f}_n(A_i, X_i)) + (1 - Y_i) \log(1 - \hat{f}_n(A_i, X_i))).$$

- The estimator $\hat{f}_n$ closest to the true f_0 minimizes the risk:

$$\mathbb{E}_{P_0}[\mathscr{L}(\hat{f}_n)(Y_i, A_i, X_i)].$$

- Loss-based super learning: Minimizing the cross-validated empirical risk with respect to the loss function $\mathscr{L}$ over the statistical model.

Targeted Minimum Loss-based Estimation (TMLE)

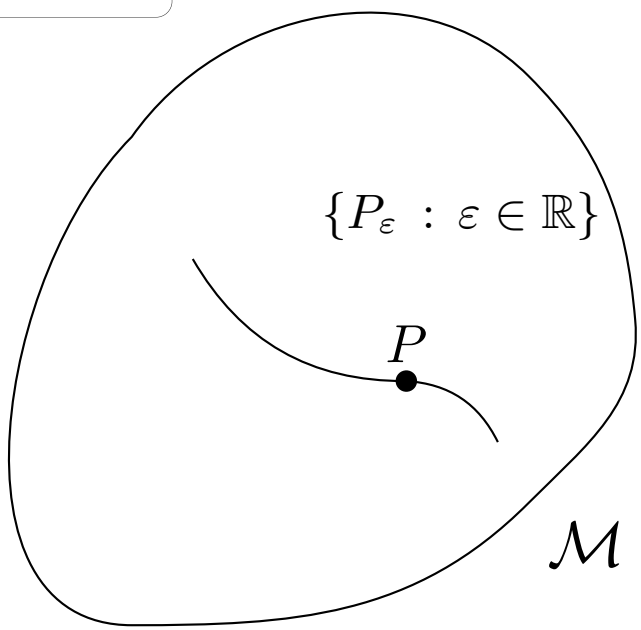
This is all about constructing a good estimator for the conditional expectation f;

- does not necessarily yield a good estimator for the particular feature of interest, the target parameter.

This is Step 1.

Targeted Minimum Loss-based Estimation (TMLE)

Step 2: We can minimize along a loss function in a certain way *that results in a good estimator for the target*.



Loss function $\mathscr{L}(f)(O)$ + clever choice of a **parametric submodel** $\{P_\varepsilon : \varepsilon \in \mathbb{R}\} \subset \mathcal{M}$.

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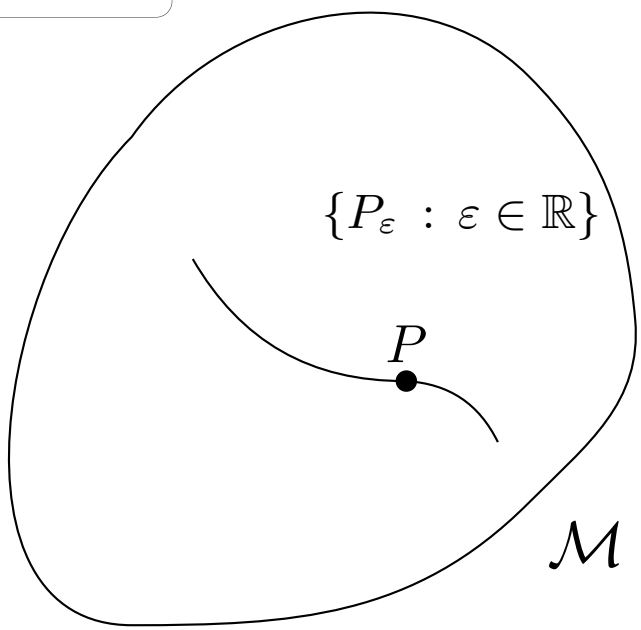

Loss function $\mathscr{L}(f)(O)$ + clever choice of a **parametric submodel** $\{P_\varepsilon : \varepsilon \in \mathbb{R}\} \subset \mathcal{M}$.

$\Rightarrow$ minimize the loss along the submodel, given the estimator $\hat{f}_n$ from Step 1.

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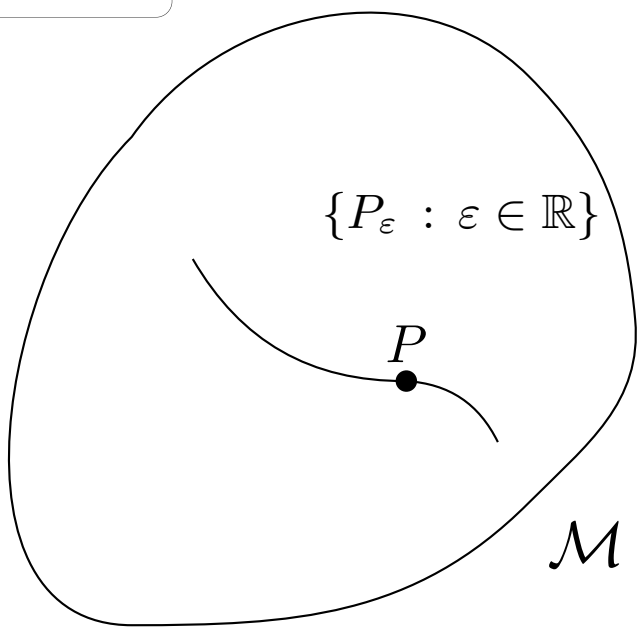

Loss function $\mathscr{L}(f)(O)$ + clever choice of a **parametric submodel** $\{P_\varepsilon : \varepsilon \in \mathbb{R}\} \subset \mathcal{M}$.

- $\Rightarrow$ minimize the loss along the submodel, given the estimator $\hat{f}_n$ from Step 1.
- $\Rightarrow$ update $\hat{f}_n$ along the path defined by P_ε: moving by $\hat{\varepsilon}_n$ that minimizes the loss.

The targeting step (Step 2)

Construction of the targeting step for a given target parameter $\Psi : \mathcal{M} \to \mathbb{R}$ with efficient influence function $\phi^*(P)$ requires:

(i) A parametric submodel $\{P_\varepsilon : \varepsilon \in \mathbb{R}\} \subset \mathcal{M}$

(ii) A loss function $(O, P) \mapsto \mathscr{L}(P)(O)$

such that: (1) $P_{\varepsilon=0} = P$, and, (2) $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathscr{L}(P_\varepsilon)(O) = \phi^*(P)(O)$

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- Initial estimator $\hat{P}_n^0$
- Minimizer $\hat{\varepsilon}_{n,0}$ of $\varepsilon \mapsto \mathbb{P}_n \mathscr{L}(\hat{P}_{n,\varepsilon}^0)$
- Update: $\hat{P}_n^1 := \hat{P}_{\hat{\varepsilon}_{n,0}}^0$

Then: $\mathbb{P}_n \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=\hat{\varepsilon}_{n,0}} \mathscr{L}(\hat{P}_{n,\varepsilon}^0)(O) = 0$

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- Updated estimator $\hat{P}_n^1$
- Minimizer $\hat{\varepsilon}_{n,1}$ of $\varepsilon \mapsto \mathbb{P}_n \mathscr{L}(\hat{P}_{n,\varepsilon}^1)$
- Update: $\hat{P}_n^2 := \hat{P}_{\hat{\varepsilon}_{n,1}}^1$

Then: $\mathbb{P}_n \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=\hat{\varepsilon}_{n,1}} \mathscr{L}(\hat{P}_{n,\varepsilon}^1)(O) = 0$

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- kth updated estimator $\hat{P}_n^k$
- Minimizer $\hat{\varepsilon}_{n,k}$ of $\varepsilon \mapsto \mathbb{P}_n \mathscr{L}(\hat{P}_{n,\varepsilon}^k)$
- Update: $\hat{P}_n^{k+1} := \hat{P}_{\hat{\varepsilon}_{n,k}}^k$

Then: $\mathbb{P}_n \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=\hat{\varepsilon}_{n,k}} \mathscr{L}(\hat{P}_{n,\varepsilon}^k)(O) = 0$

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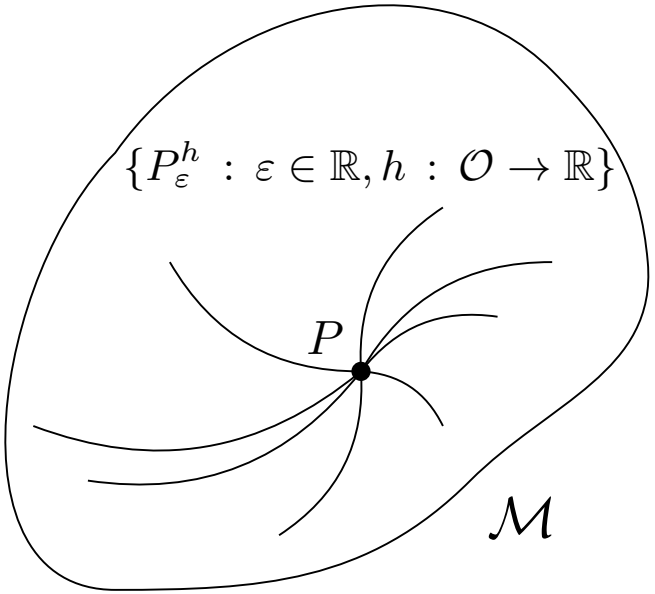
The targeting step (Step 2) *ASIDE*

What happens?



Parametric submodels $\{P_\varepsilon : \varepsilon \in \mathbb{R}\} \subset \mathcal{M}$ are also what we use to:

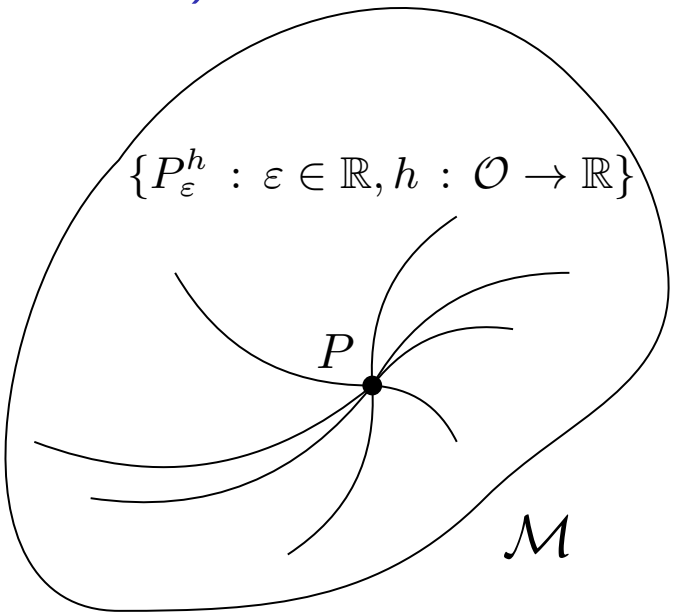
- define pathwise differentiability:[1]

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Psi(P_\varepsilon) = \int \phi(P)(o)b(o)dP(o), \tag{1}$$

- derive a nonparametric lower bound on the variance.

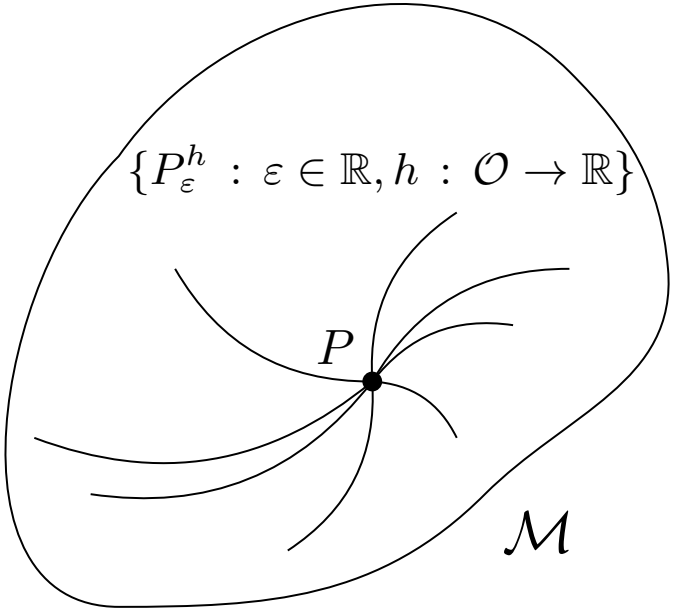
[1] (1) should hold across any smooth submodel $\{P_\varepsilon : \varepsilon \in \mathbb{R}\} \subset \mathcal{M}$.

The targeting step (Step 2) *ASIDE*



- Index submodel by its score function: $\{P_\varepsilon^h : \varepsilon \in \mathbb{R}, h : \mathcal{O} \to \mathbb{R}\}$.
  - i.e., $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \log p_\varepsilon^h(o) = h(o)$.
- Easier to estimate Ψ in the smaller model $\{P_\varepsilon^h : \varepsilon \in \mathbb{R}\}$ than in $\mathcal{M}$.
- The supremum over Cramér-Rao bounds over all submodels $\{P_\varepsilon^h : \varepsilon \in \mathbb{R}\}$ for estimating $\varepsilon \mapsto \Psi(P_\varepsilon^h)$ at $\varepsilon = 0$ provides a lower bound on the variance for estimating Ψ in $\mathcal{M}$:

The targeting step (Step 2) *ASIDE*



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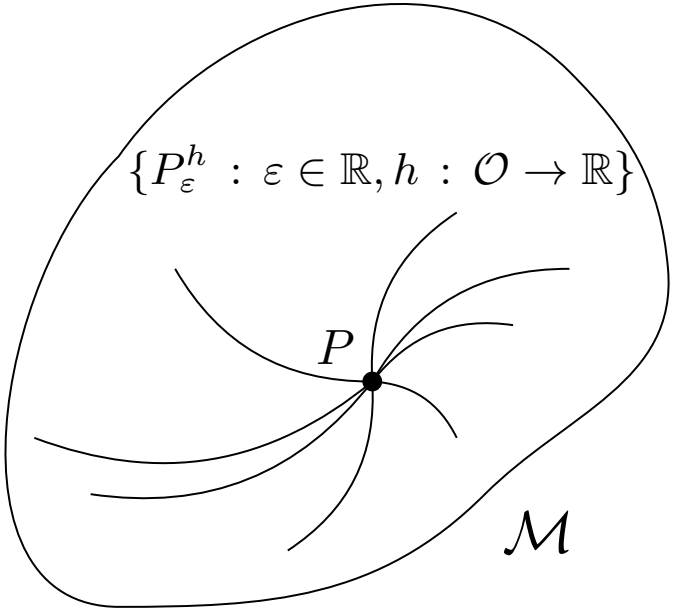
$$\frac{\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Psi(P_\varepsilon^h)\right)^2}{Ph^2} \qquad (*)$$

The targeting step (Step 2) *ASIDE*



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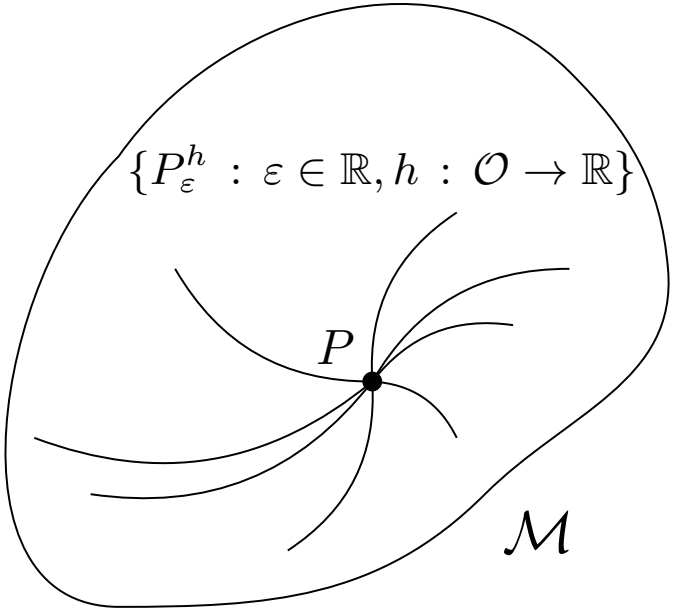
$$\frac{\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Psi(P_\varepsilon^h)\right)^2}{Ph^2} \overset{\text{PD}}{=} \frac{(P\phi^*(P)h)^2}{Ph^2} \qquad (*)$$

The targeting step (Step 2) *ASIDE*



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The targeting step (Step 2) *ASIDE*

The submodel which attains the supremum of the Cramér-Rao bounds over all parametric submodels is called the least favorable submodel;

- ▷ It is the submodel for which the score is equal to the efficient influence function $\phi^*(P)$.

The targeting step (Step 2) *ASIDE*

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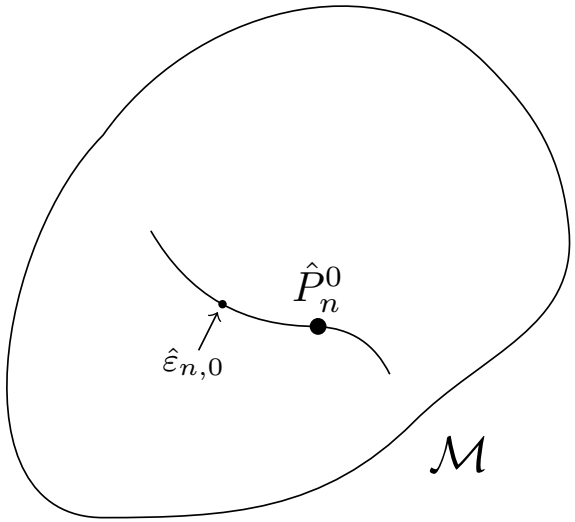

The TMLE step uses the least favorable submodel as a fluctuation model

- given a current estimator $\hat{P}_n^k$ the updated estimator is found by fluctuating along the least favorable submodel;
- the Cramér-Rao bound is reached when no further fluctuation is needed ($\varepsilon \approx 0$);

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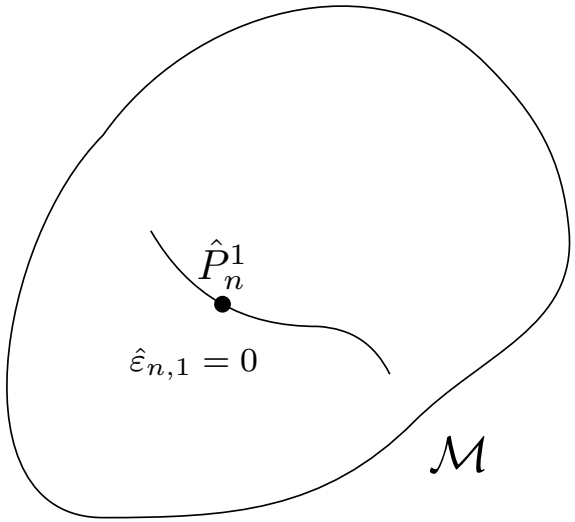

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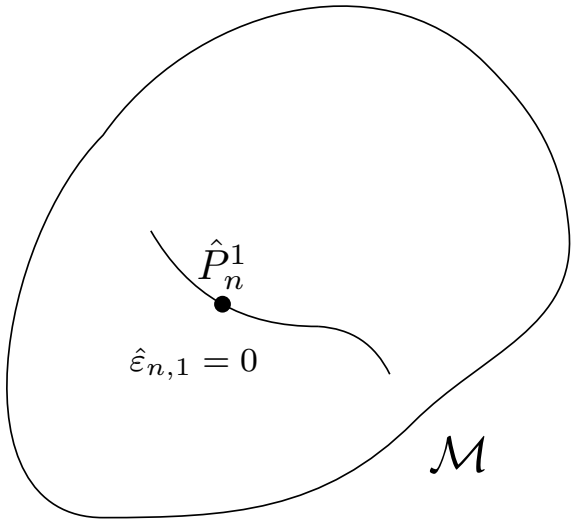

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- given a current estimator $\hat{P}_n^k$ the updated estimator is found by fluctuating along the least favorable submodel;
- the Cramér-Rao bound is reached when no further fluctuation is needed ($\varepsilon \approx 0$); the estimator solves the efficient influence curve equation.

The targeting step (Step 2)

Conditions (asymptotic linearity and efficiency)

(C1) Solve the efficient influence curve equation: $\mathbb{P}_n \phi^*(\hat{P}_n) = o_P(n^{-1/2})$

(C2) Remainder $R(\hat{P}_n, P_0) = o_P(n^{-1/2})$

(C3) Donsker class conditions for $\{\phi^*(P) : P \in \mathcal{M}\}$

Then: $\Psi(\hat{P}_n) \overset{as}{\sim} N(\Psi(P_0), P_0 \phi^*(P_0)^2/n)$

- The targeting step ensures that (C1) holds.
- Assume that (C2) and (C3) hold.

We can use the efficient influence function to compute an estimator for the standard error of the TMLE estimator:

$$\hat{\sigma}_n = \sqrt{\frac{\mathbb{P}_n\{\phi^*(\hat{P}_n)\}^2}{n}}$$

Targeting the average treatment effect (ATE)

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EXAMPLE: Average treatment effect (ATE)

Observed data $O = (X, A, Y) \in \mathbb{R}^d \times \{0,1\} \times \{0,1\} = \mathcal{O}$

* $X \in \mathbb{R}^d$ are covariates
* $A \in \{0,1\}$ is a binary exposure variable (treatment decision)
* $Y \in \{0,1\}$ is a binary outcome variable

$O \sim P_0$ where P_0 assumed to belong to nonparametric model $\mathcal{M}$.

We are interested in estimating the ATE:

$$\Psi(P) = \mathbb{E}_P\Big[\mathbb{E}_P[Y \mid A = 1, X] - \mathbb{E}_P[Y \mid A = 0, X]\Big].$$

Targeting the average treatment effect (ATE)

EXAMPLE: Average treatment effect (ATE)

For the ATE, as we have seen, we can also write the target parameter $\Psi : \mathcal{M} \to \mathbb{R}$ as

$$\Psi(P) = \tilde{\Psi}(f, \mu_X) = \int_{\mathbb{R}^d} \big(f(1,x) - f(0,x)\big) d\mu_X(x) \qquad (*)$$

where

$$f(a,x) = \mathbb{E}[Y \mid A = a, X = x]$$

and μ_X is the marginal distribution of X.

I.e., $\hat{\psi}_n = \tilde{\Psi}(\hat{f}_n, \hat{\mu}_n)$.

Targeting the average treatment effect (ATE)

EXAMPLE: Average treatment effect (ATE)

Step 1 Construct initial estimators $\hat{f}_n, \hat{\pi}_n$ for f, π.

Step 2 Update the estimator $\hat{f}_n \mapsto \hat{f}_n^*$ for f such that $\hat{f}_n^*$ for the fixed $\hat{\pi}_n$ solves the efficient influence curve equation.

For the ATE, Step 2 is simply just an additional logistic regression step.

Targeting the average treatment effect (ATE)

EXAMPLE: Average treatment effect (ATE)

We need:

0. The efficient influence function:

$$\tilde{\phi}^*(f,\pi)(O) = \left(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)}\right)\big(Y - f(A,X)\big)$$

$$+\ f(1,X) - f(0,X) - \tilde{\Psi}(f)$$

Targeting the average treatment effect (ATE)

EXAMPLE: Average treatment effect (ATE)

We need:

0. The efficient influence function:

$$\tilde{\phi}^*(f,\pi)(O) = \underbrace{\left(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)}\right)\big(Y - f(A,X)\big)}_{=\tilde{\phi}^*_f(f,\pi)(O)} + f(1,X) - f(0,X) - \tilde{\Psi}(f)$$

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Further, we need:

(i) A parametric submodel $\{f_\varepsilon : \varepsilon \in \mathbb{R}\} \subset \mathcal{M}$

(ii) A loss function $(O, f) \mapsto \mathscr{L}(f)(O)$

such that

$$(1) \quad f_{\varepsilon=0} = f \qquad (2) \quad \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathscr{L}(f_\varepsilon)(O) = \tilde{\phi}^*_f(f,\pi)(O)$$

Targeting the average treatment effect (ATE)

(i) Log-likelihood loss function:

$$\boxed{\mathrm{logit}(p) = \mathrm{expit}^{-1}(p) = \log\left(\frac{p}{1-p}\right)}$$

$$\mathscr{L}(f)(O) = -\big(Y \log(f(A,X)) + (1-Y)\log(1-f(A,X))\big)$$

(ii) Logistic regression model:

$$f_\varepsilon(A,X) = \mathrm{expit}\big(\mathrm{logit}(f(A,X)) + \varepsilon H(A,X)\big)$$

with the so-called "clever covariate": $H(A,X) := \dfrac{2A-1}{\pi(A \mid X)}$.

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To show this, we verify that (i)–(ii) fulfill

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SMALL EXERCISE: To show this, we verify that (i)–(ii) fulfill

$$(1) \quad f_{\varepsilon=0} = f \qquad (2) \quad \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathscr{L}(f_\varepsilon)(O) = \tilde{\phi}^*_f(f,\pi)(O)$$

Targeting the average treatment effect (ATE)

- Initial estimators $\hat{f}_n$, $\hat{\pi}_n$.
- Estimate clever covariate by:

$$\hat{H}_n(A, X) = \frac{2A - 1}{\hat{\pi}_n(A \mid X)}.$$

- The minimizer $\hat{\varepsilon}_n$ of $\varepsilon \mapsto \mathbb{P}_n \mathscr{L}(\hat{f}_{n,\varepsilon})$ equals the maximum likelihood estimator for ε in the fixed-intercept logistic regression:

$$\operatorname{logit} \mathbb{E}[Y \mid A, X] = \operatorname{logit}(\hat{f}_n(A, X)) + \varepsilon \hat{H}_n(A, X)$$

- Update: $\hat{f}_n^* := \hat{f}_{n,\hat{\varepsilon}_n}$.

Then: $\mathbb{P}_n \frac{d}{d\varepsilon}\Big|_{\varepsilon=\hat{\varepsilon}_n} \mathscr{L}(\hat{f}_{n,\varepsilon}) = 0$, i.e.,

$$\mathbb{P}_n \tilde{\phi}_f^*(\hat{f}_{n,\hat{\varepsilon}_n}, \hat{\pi}_n) = \mathbb{P}_n \tilde{\phi}_f^*(\hat{f}_n^*, \hat{\pi}_n) = 0.$$

Targeting the average treatment effect (ATE)

$$\tilde{\phi}^*(f,\pi)(O) = \underbrace{\left(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)}\right)\big(Y - f(A,X)\big)}_{=\tilde{\phi}^*_f(f,\pi)(O)} + \underbrace{f(1,X) - f(0,X) - \tilde{\Psi}(f)}_{=\tilde{\phi}^*_{\mu_X}(f)(O)}$$

Per construction we already have: $\mathbb{P}_n \phi^*_\mu(\hat{f}^*_n) = 0$,

since: $\tilde{\Psi}(\hat{f}^*_n) = \frac{1}{n}\sum_{i=1}^{n}\big(\hat{f}^*_n(1,X_i) - \hat{f}^*_n(0,X_i)\big) = \mathbb{P}_n\big(\hat{f}^*_n(1,\cdot) - \hat{f}^*_n(0,\cdot)\big)$.

The targeting step thus yields:

$$\mathbb{P}_n \tilde{\phi}^*(\hat{f}^*_n, \hat{\pi}_n) = \mathbb{P}_n \tilde{\phi}^*_f(\hat{f}^*_n, \hat{\pi}_n) + \mathbb{P}_n \phi^*_\mu(\hat{f}^*_n) = 0.$$

Targeting the average treatment effect (ATE)

Doing the targeting in practice using the simulated dataset:

```
set.seed(5)
n <- 500
X <- runif(n, -2, 2)
A <- rbinom(n, 1, prob=plogis(-0.25 + 1.2*X))
Y <- rbinom(n, 1, prob=plogis(-0.9 + 1.9*X^2 + 0.5*A))
(sim.data <- data.table(id=1:n,X=X,A=A,Y=Y))
```

```
     id          X A Y
  1:  1 -1.1991422 0 1
  2:  2  0.7408744 1 1
  3:  3  1.6675031 1 1
  4:  4 -0.8624022 0 1
  5:  5 -1.5813995 0 1
 ---
496: 496 -0.3978523 1 0
497: 497 -1.5069379 0 1
498: 498  1.8340120 1 1
499: 499  0.6349484 1 0
500: 500 -0.5214807 0 1
```

Targeting the average treatment effect (ATE)

Initial estimation:

```
#-- treatment distribution;
glm.A <- glm(A~X, data=sim.data, family=binomial)
pi.1 <- predict(glm.A, type="response")

#-- outcome distribution (misspecified);
glm.Y <- glm(Y~A+X, data=sim.data, family=binomial)
sim.data[, f:=predict(glm.Y, type="response")]
sim.data[, f.A1:=predict(glm.Y, type="response",
             newdata=copy(sim.data)[, A:=1])]
sim.data[, f.A0:=predict(glm.Y, type="response",
             newdata=copy(sim.data)[, A:=0])]

#-- initial estimate of the ATE;
fit.ate.initial <- sim.data[, mean(f.A1 - f.A0)]
```

Targeting the average treatment effect (ATE)

Targeting step:

```
#-- tmle;
sim.data[, clever.covariate:=((A==1)/pi.1 - (A==0)/(1-pi.1))]
eps <- coef(glm(Y ~ offset(qlogis(f))+clever.covariate-1,
         data=sim.data, family=binomial()))
```

```
eps = -0.0157708436790858
```

Targeting the average treatment effect (ATE)

Targeting step:

```
#-- tmle;
sim.data[, clever.covariate:=((A==1)/pi.1 - (A==0)/(1-pi.1))]
eps <- coef(glm(Y ~ offset(qlogis(f))+clever.covariate-1,
         data=sim.data, family=binomial()))
```

```
eps = -0.0157708436790858
```

```
#-- tmle update;
sim.data[, f.A1.tmle:=plogis(qlogis(f.A1) + eps/pi.1)]
sim.data[, f.A0.tmle:=plogis(qlogis(f.A0) - eps/(1-pi.1))]
```

i.e., `f.A1.tmle` is the estimate of $f(1, X) = \mathbb{E}[Y \mid A = 1, X]$, obtained via the submodel:

$$\hat{f}_n^*(1, X) = \hat{f}_{n,\hat{\varepsilon}_n}(1, X) = \mathrm{expit}\big(\mathrm{logit}(\hat{f}_n(1, X)) + \hat{\varepsilon}_n \hat{H}_n(1, X)\big),$$

and likewise with `f.A0.tmle`.

Targeting the average treatment effect (ATE)

```
     id          X A Y      f.A1      f.A0 f.A1.tmle f.A0.tmle
  1:  1 -1.1991422 0 1 0.7655621 0.6713853 0.7488795 0.6755825
  2:  2  0.7408744 1 1 0.7396070 0.6399080 0.7349584 0.6504368
  3:  3  1.6675031 1 1 0.7265721 0.6244167 0.7228545 0.6481588
  4:  4 -0.8624022 0 1 0.7611886 0.6660214 0.7488197 0.6705960
  5:  5 -1.5813995 0 1 0.7704590 0.6774205 0.7463439 0.6813231
 ---
496: 496 -0.3978523 1 0 0.7550638 0.6585507 0.7464799 0.6639337
497: 497 -1.5069379 0 1 0.7695108 0.6762494 0.7471142 0.6802008
498: 498  1.8340120 1 1 0.7241872 0.6216047 0.7205492 0.6495635
499: 499  0.6349484 1 0 0.7410712 0.6416611 0.7362345 0.6513868
500: 500 -0.5214807 0 1 0.7567041 0.6605467 0.7472996 0.6656728
```

Targeting the average treatment effect (ATE)

```
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```
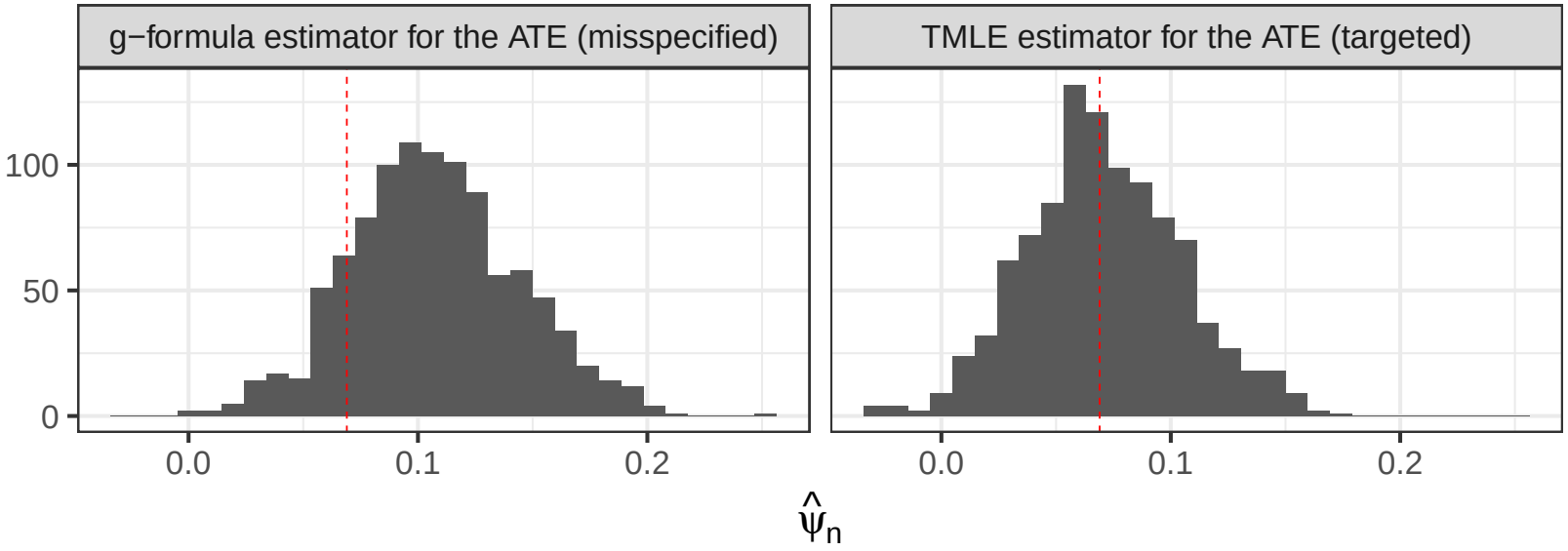

```
fit.ate.tmle <- sim.data[, mean(f.A1.tmle - f.A0.tmle)]
```

```
initial ate est = 0.0975
tmle ate est    = 0.0768
```

Targeting the average treatment effect (ATE)

With 500 repeated simulations:



Practical 1: Implementing the targeting step

Practical Part 1 Implementing the targeting step.

Practical Part 2 Computing the variances of the ATE, the $\log \mathrm{RR}$ and the $\log \mathrm{OR}$.

Practical Part 3 Large-sample properties (simulation study).

The exercise is described in detail in: **day2-practical1.pdf**.

[More comments on the following slides].

Comments for practical

We focused on the ATE as an example of a causal parameter.

But note that other simple causal parameters can be constructed from $\mathbb{E}_P[Y^1]$ and $\mathbb{E}_P[Y^0]$.

Like:

$$\Psi_{\mathrm{RR}}(P) = \frac{\mathbb{E}_P[Y^1]}{\mathbb{E}_P[Y^0]},$$

or,

$$\Psi_{\mathrm{OR}}(P) = \frac{\mathbb{E}_P[Y^1]/(1 - \mathbb{E}_P[Y^1])}{\mathbb{E}_P[Y^0]/(1 - \mathbb{E}_P[Y^0])},$$

Comments for practical

For the targeting step, we can choose to target $\Psi_1(P) = \mathbb{E}_P[Y^1]$ and $\Psi_0(P) = \mathbb{E}_P[Y^0]$ *separately*.

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The efficient influence function for the treatment-specific mean $\Psi_a(P) = \mathbb{E}_P[Y^a]$:

$$\tilde{\phi}_a^*(f,\pi)(O) = \underbrace{\frac{1\{A=a\}}{\pi(a \mid X)}}_{\text{clever covar.}} \big(Y - f(A,X)\big) + f(a,X) - \Psi_a(P)$$

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$$\tilde{\phi}_a^*(f,\pi)(O) = \underbrace{\frac{1\{A=a\}}{\pi(a \mid X)}}_{\text{clever covar.}} \big(Y - f(A,X)\big) + f(a,X) - \Psi_a(P)$$

If we target $\Psi_1(P)$ and $\Psi_0(P)$ separately, we obtain two sets of updated estimators $\hat{f}_n \mapsto \hat{f}_{n,1}^*$ and $\hat{f}_n \mapsto \hat{f}_{n,0}^*$

- one to construct a targeted estimator $\hat{\psi}_{1,n}^*$ for $\Psi_1(P)$;
- and the other to construct a targeted estimator $\hat{\psi}_{0,n}^*$ for $\Psi_0(P)$.

Comments for practical

We can then compute an estimate for the ATE as

$$\hat{\psi}_n^* = \hat{\psi}_{n,1}^* - \hat{\psi}_{n,0}^*,$$

and we can estimate the variance of this estimator by

$$\mathbb{P}_n\{\tilde{\phi}_1^*(\hat{f}_{n,1}^*, \hat{\pi}_n) - \tilde{\phi}_0^*(\hat{f}_{n,0}^*, \hat{\pi}_n)\}^2;$$

since efficient influence function for the ATE is

$$\tilde{\phi}^*(f, \pi) = \tilde{\phi}_1^*(f, \pi) - \tilde{\phi}_0^*(f, \pi).$$

Comments for practical

Similarly we can construct estimators for the RR and the OR by simple plug-in:

$$\hat{\psi}^*_{\mathrm{RR},n} = \frac{\hat{\psi}^*_{1,n}}{\hat{\psi}^*_{0,n}},$$

and,

$$\hat{\psi}^*_{\mathrm{OR},n} = \frac{\hat{\psi}^*_{1,n}/(1-\hat{\psi}^*_{1,n})}{\hat{\psi}^*_{0,n}/(1-\hat{\psi}^*_{0,n})}.$$

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We can use the delta method to derive the efficient influence functions of $\Psi_{\mathrm{RR}}(P)$ and $\Psi_{\mathrm{OR}}(P)$.

Comments for practical

Let $\phi^*(P)$ be the efficient influence function for a parameter $\Psi(P)$. Say that interest is in $h(\Psi(P))$ for a function h.

The delta method yields that:

If the first derivative $h'(\psi) = \frac{d}{d\psi}h(\psi)$ of h exists and is non-zero, then the efficient influence function of $h(\Psi(P))$ is:

$$\phi_h^*(P) = h'(\Psi(P))\phi^*(P).$$

Comments for practical

So, once we have TMLE (targeted) estimators for $\Psi_1(P) = \mathbb{E}[Y^1]$ and $\Psi_0(P) = \mathbb{E}[Y^0]$:

- We can construct estimators for the ATE, the RR and the OR.
- We can compute the variance of the ATE estimator, the $\log \mathrm{RR}$ estimator and the $\log \mathrm{OR}$ estimator.

Practical 1: Implementing the targeting step

Practical Part 1 Implementing the targeting step.

Practical Part 2 Computing the variances of the ATE, the $\log \mathrm{RR}$ and the $\log \mathrm{OR}$.

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