Day 1, Lecture 2

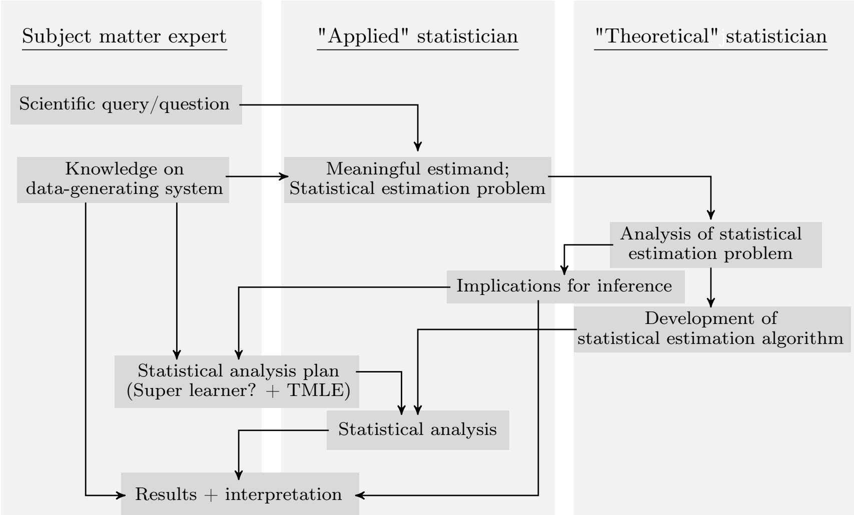
Properly defining the target parameter

Properly defining the target parameter

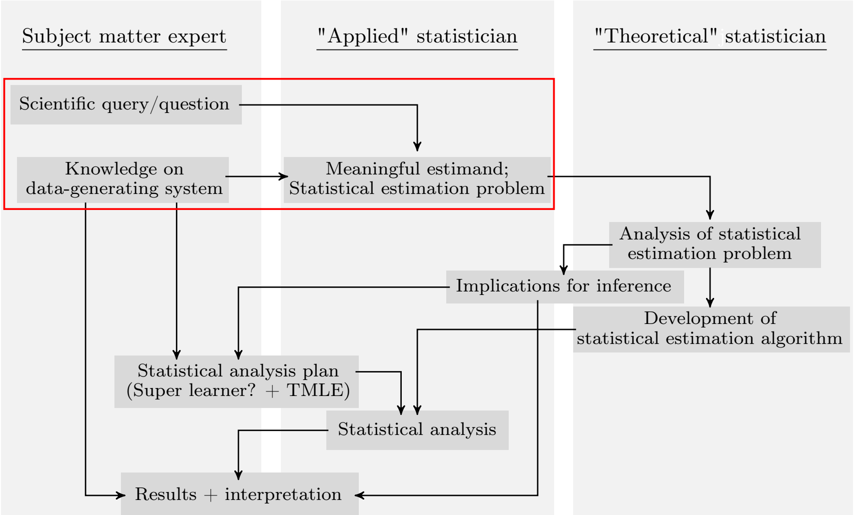
In this lecture, our goal is to:

1. Develop an understanding of the utilization of causal inference tools to define nonparametric targets of estimation, highlighting the role of counterfactual reasoning and hypothetical interventions.
2. Differentiate between causal and statistical parameters, and list the assumptions necessary for the translation between them.

Properly defining the target parameter



Properly defining the target parameter



Properly defining the target parameter

- A clearly defined goal as a starting-point for any analysis
  - necessary to talk about estimator performance
  - semiparametric/nonparametric efficiency theory (and TMLE) requires a clearly defined goal
- Brief introduction to the setting of a typical causal inference problem
  - example: average treatment effect (ATE)
  - *model-free* and *estimator-free* definition of parameters

Moving targets with different logistic regression models

- $X \sim \text{Unif}(-2, 2)$
- $A \sim \text{Bernouilli}(0.5)$ (no confounding)
- $Y \in \{0, 1\}$

Say that the distribution of Y given X and A follows the parametric model:

$$\text{logit}\, \mathbb{E}[\, Y \mid A, X\,] = \beta_0 + \beta_A A + \beta_X^\top X^2$$

Moving targets with different logistic regression models

- $X \sim \text{Unif}(-2, 2)$
- $A \sim \text{Bernouilli}(0.5)$ (no confounding)
- $Y \in \{0, 1\}$

Say that the distribution of Y given X and A follows the parametric model:

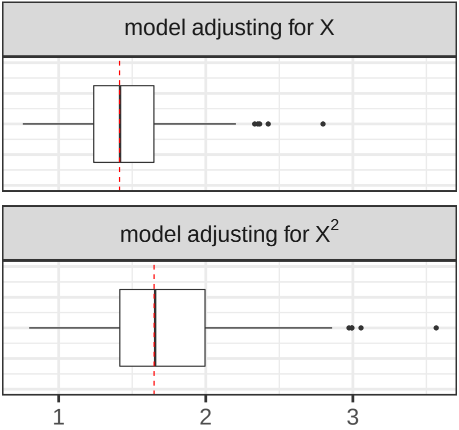
$$\text{logit}\,\mathbb{E}[Y \mid A, X] = \beta_0 + \beta_A A + \beta_X^\top X^2$$

The odds ratio $\exp(\beta_A)$ is a different parameter than $\exp(\alpha_A)$ in a different model:

$$\text{logit}\,\mathbb{E}[Y \mid A, X] = \alpha_0 + \alpha_A A + \alpha_X^\top X$$

Moving targets with different logistic regression models

- The variables X we include in the model to assess the effect of A on Y changes the parameter (conditional OR).
- Only one of the models can be true at a time.



The upper panel does not show a biased estimator, just an estimator targeting a different parameter (dashed red line).

Causal inference

What we obtain moving on to a causal inference setting: 1) An interpretable and relevant target of estimation, and 2) a model-free definition of a target parameter.

[1] And, if you are already familiar, consider this a small repetition and introduction to the notation.

Causal inference

What we obtain moving on to a causal inference setting: 1) An interpretable and relevant target of estimation, and 2) a model-free definition of a target parameter.

- We are only going to go briefly over the "causal inference concepts",[1] but we need this part to very clear about with it is we are estimating.
  - For today and tomorrow we consider just the simple example where the target of estimation is the average treatment effect (ATE).
- For the causal inference notation, we follow the book by Hernán and Robins (which, if you are interested, you can find here: `https://cdn1.sph.harvard.edu/wp-content/uploads/sites/1268/2021/03/ciwhatif_hernanrobins_30mar21.pdf`).
  - I will leave out DAGs/SCMs.

[1] And, if you are already familiar, consider this a small repetition and introduction to the notation.

Steps of the roadmap

Step 1 Go from scientific question to target causal estimand (stated in the language of counterfactuals)

Step 2 Assess whether we can go from target causal estimand to target statistical estimand = assess "identifiability"

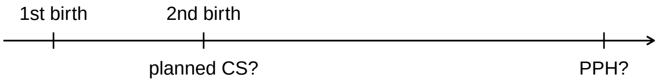
In a given data situation, we want to explicitly clarify:

1. Observed data
2. Causal model
3. Causal question and target causal estimand
4. Identifiability

An example we can have in the back of our minds

Scientific question:

Does having a planned cesarian section (intended cesarian section) among women who gave birth twice change the risk of postpartum haemorrhage (PPH) during the second delivery?



Goal: Translate this is into a precise formulation of a statistical estimation problem.

Observed data

Observed data $O = (X, A, Y) \in \mathbb{R}^d \times \{0,1\} \times \{0,1\} = \mathcal{O}$

* $X \in \mathbb{R}^d$ are covariates
  ex: age at 2nd delivery, information of PPH at first delivery, ...
* $A \in \{0,1\}$ is a binary exposure variable (treatment decision)
  ex: decision to have a planned cesarian section.
* $Y \in \{0,1\}$ is a binary outcome variable
  ex: PPH (postpartum haemorrhage).

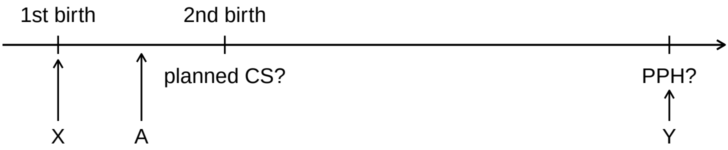
We observe a sample $O_1, \ldots, O_n \overset{iid}{\sim} P_0 \in \mathcal{M},\ n \in \mathbb{N}$.

$\mathcal{M}$ is the set of all possible probability distributions for our data.

Observed data

Implicit assumptions for the data structure:[2]

- X are covariates known before the treatment decision A was made
- Outcome Y is observed after treatment decision was made



[2] This ordering could also be encoded in a structural causal model.

Observed data

Our statistical model $\mathcal{M}$ for P_0 contains possible distributions P for the observed data O.

The density p of $P \in \mathcal{M}$ can be factorized into:

$$p(o) = \mu_Y(y \mid a, x)\pi(a \mid x)\mu_X(x),$$

- $\mu_Y(y \mid A, X) = P(Y = y \mid A, X)$
- $\pi(a \mid X) = P(A = a \mid X)$
- μ_X is the marginal density of X (with respect to an appropriate dominating measure)

Observed data

Our statistical model $\mathcal{M}$ for P_0 contains possible distributions P for the observed data O.

The density p of $P \in \mathcal{M}$ can be factorized into:

$$p(o) = \mu_Y(y \mid a, x)\pi(a \mid x)\mu_X(x),$$

- $\mu_Y(y \mid A, X) = P(Y = y \mid A, X)$
- $\pi(a \mid X) = P(A = a \mid X)$
- μ_X is the marginal density of X (with respect to an appropriate dominating measure)

We assume that $\mathcal{M}$ is a nonparametric model.

* Throughout, we make *no parametric restrictions on* μ_Y, μ_X.
* We could impose some parametric structure on π, but let us assume that we do not.

Operators on functions of the observed data[3]

For a function $h : \mathcal{O} \to \mathbb{R}$ and distribution P

$$Ph = \mathbb{E}_P[h(O)] = \int hdP = \int_{\mathcal{O}} h(o)dP(o)$$

where $\mathcal{O} = \mathbb{R}^d \times \{0,1\} \times \{0,1\}$ is the sample space of $O = (X, A, Y)$.

[3] van der Vaart, A. W. (2000). Asymptotic statistics (Vol. 3). Cambridge university press.

Confounding

How can we define a causal effect?

The contrast $\mathbb{E}_P[Y \mid A = 1] - \mathbb{E}_P[Y \mid A = 0]$ tells us about the risk difference in the two exposure groups.

Any such difference is likely due to other factors than the decision to initiate treatment or not

- the exposure decision is confounded.

Counterfactuals

To answer a causal question, we ideally want to know

Scenario 1 What happened to a subject had they been exposed?

Scenario 2 What would have happened to the same subject had they not been exposed?

We imagine a model with two outcomes for each subject:

- a variable Y^1 corresponding to scenario 1, and
- a variable Y^0 corresponding to scenario 2

= the "counterfactuals" (aka potential outcomes).

Counterfactuals

* Y^1 = outcome if exposed
* Y^0 = outcome if not exposed

We use the counterfactual outcomes to define precisely what a causal effect is:

→ on the individual level, $Y^1 = 1$ and $Y^0 = 0$ for a particular subject would tell us that this subject would experience outcome under exposure and not otherwise

→ on the population level, $\mathbb{E}_P[Y^1] \neq \mathbb{E}_P[Y^0]$ tells us that the risk changes depending on whether exposed or not

Target causal estimand: Average causal effect (ATE)

The average causal effect[4] (ATE/ACE) measures the average effect in the population

$$\mathrm{ATE} = \mathbb{E}_P[Y^1] - \mathbb{E}_P[Y^0]$$

→ It is interpreted as the difference in risk had everyone in the population been exposed and had everyone in the population been unexposed.

[4] or average treatment effect.

Identifiability (estimating the causal effect from observational data)

Can we estimate the causal effect from the observed data?

- ▷ only Y^1 or Y^0 is observed for each individual.

Identifying $\mathbb{E}_P[Y^1] - \mathbb{E}_P[Y^0]$

= write $\mathbb{E}_P[Y^1] - \mathbb{E}_P[Y^0]$ as a parameter of the observed data distribution.

requires three overall assumptions (identifiability assumptions).

Identifiability (estimating the causal effect from observational data)

1. Consistency: $Y^a = Y$ if $A = a$, $a = 0, 1$

- Requires that the "treatment intervention" is well-defined and no interference between subjects.
- Example of a violation: effect of vaccines (one subject's effect of a vaccine depends on whether other subjects are vaccinated or not).

Identifiability (estimating the causal effect from observational data)

1. Consistency: $Y^a = Y$ if $A = a$, $a = 0, 1$

- Requires that the "treatment intervention" is well-defined and no interference between subjects.
- Example of a violation: effect of vaccines (one subject's effect of a vaccine depends on whether other subjects are vaccinated or not).

2. Exchangeability: $Y^a \perp\!\!\!\perp A \mid X$, for $a = 0, 1$

- Conditional on covariates, the exposed group tells us what would happen to the unexposed if they had been exposed and vice versa.
- Requires that there is no unmeasured confounding.

Identifiability (estimating the causal effect from observational data)

1. Consistency: $Y^a = Y$ if $A = a$, $a = 0, 1$

- Requires that the "treatment intervention" is well-defined and no interference between subjects.
- Example of a violation: effect of vaccines (one subject's effect of a vaccine depends on whether other subjects are vaccinated or not).

2. Exchangeability: $Y^a \perp\!\!\!\perp A \mid X$, for $a = 0, 1$

- Conditional on covariates, the exposed group tells us what would happen to the unexposed if they had been exposed and vice versa.
- Requires that there is no unmeasured confounding.

3. Positivity: $P(A = a \mid X) > 0$ for $a = 0, 1$ and almost surely all X

- We cannot investigate the effect of an intervention that was never "tested" in the observed data (conditional on covariates X).

Identifiability (estimating the causal effect from observational data)

Under these assumptions:

$$\begin{aligned}
&\mathbb{E}_P[Y^1] - \mathbb{E}_P[Y^0] \\
&\quad = \mathbb{E}_P\big[\mathbb{E}_P[Y^1 \mid X] - \mathbb{E}_P[Y^0 \mid X]\big] \\
&\quad \overset{2.}{=} \mathbb{E}_P\big[\mathbb{E}_P[Y^1 \mid A = 1, X] - \mathbb{E}_P[Y^0 \mid A = 0, X]\big] \\
&\quad \overset{1.}{=} \mathbb{E}_P\big[\mathbb{E}_P[Y \mid A = 1, X] - \mathbb{E}_P[Y \mid A = 0, X]\big] \\
&\quad = \Psi(P)
\end{aligned}$$

(3. (positivity) ensures that the conditional expectations are well-defined).

Goal achieved: Right hand side is expressed only in terms of observable quantities.

Identifiability (estimating the causal effect from observational data)

Under the assumptions:

$$\mathbb{E}_P[Y^1] - \mathbb{E}_P[Y^0] = \underbrace{\mathbb{E}_P\Big[\mathbb{E}_P[Y \mid A = 1, X] - \mathbb{E}_P[Y \mid A = 0, X]\Big]}_{(*)} = \Psi(P),$$

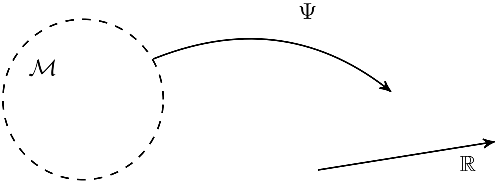
for any $P \in \mathcal{M}$.

In our statistical analysis, **we proceed with** $(*)$.

"Causal inference part" is over.

Target statistical estimand

Now we are exactly in the situation we wanted:



Average treatment effect (ATE)

- $O = (X, A, Y) \in \mathbb{R}^d \times \{0, 1\} \times \{0, 1\}$
- The ATE is defined for $P \in \mathcal{M}$ as

$$\Psi(P) = \mathbb{E}_P[\mathbb{E}_P[Y \mid A = 1, X] - \mathbb{E}_P[Y \mid A = 0, X]]$$

Target statistical estimand: g-formula

We can write the target parameter as:

$$\begin{aligned}\Psi(P) &= \mathbb{E}_P[\mathbb{E}_P[Y \mid A = 1, X] - \mathbb{E}_P[Y \mid A = 0, X]] \\ &= \mathbb{E}_P[f(1, X) - f(0, X)] \\ &= \int_{\mathbb{R}^d} \big(f(1, x) - f(0, x)\big) d\mu_X(x) = \tilde{\Psi}(f, \mu_X) \qquad (*)\end{aligned}$$

where

$$f(a, x) = \mathbb{E}[Y \mid A = a, X = x]$$

and μ_X is the marginal distribution of X.

We refer to this as the g-formula.

Target statistical estimand: IP-weighting

We can also rewrite the target parameter as:

$$
\begin{aligned}
\Psi(P) &= \int_{\mathbb{R}^d} \big(f(1,x) - f(0,x)\big) d\mu_X(x) && (*) \\
&= \int_{\mathbb{R}^d} \sum_{y=0,1} y\big(\mu_Y(y \mid 1, x) - \mu_Y(y \mid 0, x)\big) d\mu_X(x) \\
&= \int_{\mathbb{R}^d} \sum_{y=0,1} \sum_{a=0,1} y\big(a\mu_Y(y \mid a, x) - (1-a)\mu_Y(y \mid a, x)\big) d\mu_X(x) \\
&= \int_{\mathbb{R}^d} \sum_{y=0,1} \sum_{a=0,1} \left(\frac{ay}{\pi(a \mid x)} - \frac{(1-a)y}{\pi(a \mid x)}\right) \mu_Y(y \mid a, x)\pi(a \mid x) d\mu_X(x) \\
&= \tilde{\Psi}_{\mathrm{ipw}}(\pi, p) && (**)
\end{aligned}
$$

where $\pi(a \mid x) = P(A = 1 \mid X = x)$.

Target statistical estimand

The g-formula:

$$\tilde{\Psi}(f, \mu_X) = \int_{\mathbb{R}^d} \big(f(1,x) - f(0,x)\big) d\mu_X(x)$$
$$= \mathbb{E}_P\big[f(1,X) - f(0,X)\big]. \qquad (*)$$

The IP-weighted formula:

$$\tilde{\Psi}_{\text{ipw}}(\pi, p) = \int_{\mathbb{R}^d} \sum_{a=0,1} \sum_{y=0,1} \left(\frac{ay}{\pi(a \mid x)} - \frac{(1-a)y}{\pi(a \mid x)} \right) dP(o)$$
$$= \mathbb{E}_P\left[\frac{AY}{\pi(1 \mid X)} - \frac{(1-A)Y}{\pi(0 \mid X)} \right] \qquad (**)$$

- f and (the average over) μ_X are nuisance parameters for the g-formula.
- π and (the average over) p are nuisance parameters for the IP-weighted formula.

Target statistical estimand

Yet another representation of the target parameter is

$$
\begin{aligned}
\tilde{\Psi}_{\mathrm{ee}}(f,\pi,p) &= \int_{\mathbb{R}^d} \sum_{a=0,1} \sum_{y=0,1} \Bigg\{ \left(\frac{a}{\pi(a \mid x)} - \frac{1-a}{\pi(a \mid x)} \right) \big(y - f(a,x) \big) \\
&\qquad + f(1,x) - f(0,x) \Bigg\} p_Y(y \mid a,x) \pi(a \mid x) d\mu_X(x) \\
&= \mathbb{E}_P \left[\left(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)} \right) \big(Y - f(A,X) \big) + f(1,X) - f(0,X) \right]
\end{aligned}
$$

- f, π and (the average over) p are nuisance parameters for this parametrization.

Target statistical estimand

Yet another representation of the target parameter is

$$\tilde{\Psi}_{\mathrm{ee}}(f,\pi,p) = \int_{\mathbb{R}^d} \sum_{a=0,1} \sum_{y=0,1} \left\{ \left(\frac{a}{\pi(a \mid x)} - \frac{1-a}{\pi(a \mid x)} \right) \big(y - f(a,x)\big) \right.$$
$$\left. + f(1,x) - f(0,x) \right\} p_Y(y \mid a,x) \pi(a \mid x) d\mu_X(x)$$
$$= \mathbb{E}_P\left[\left(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)} \right) \big(Y - f(A,X)\big) + f(1,X) - f(0,X) \right]$$

- f, π and (the average over) p are nuisance parameters for this parametrization.

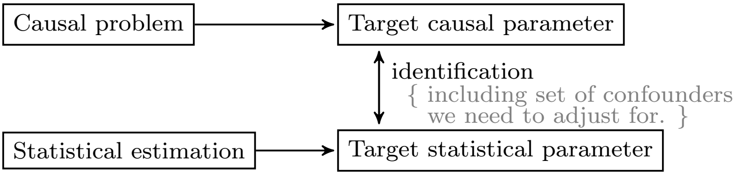
SMALL EXERCISE:

1. Note that $\tilde{\Psi}_{\mathrm{ee}}(f,\pi,p) = \tilde{\Psi}(f,\mu_X) + [\text{an extra term}]_1$. Show that it can also be written as $\Psi_{\mathrm{ee}}(f,\pi,p) = \tilde{\Psi}_{\mathsf{ipw}}(\pi,p) + [\text{an extra term}]_2$.
2. Show that $\tilde{\Psi}_{\mathrm{ee}}(f,\pi,p) = \mathbb{E}_P[Y^1] - \mathbb{E}_P[Y^0]$ under the identifiability assumptions (consistency, exchangeability and positivity).

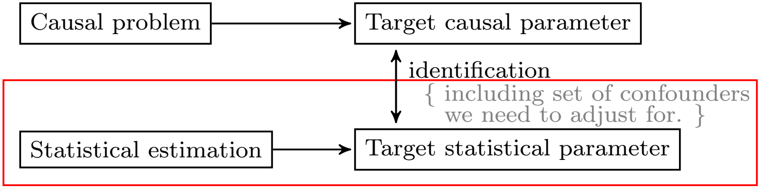
Summary: Properly defining the target

- Causal parameter (now fixed).
- Causal model: How do the observed variables affect one another?
  - are the covariates we observe sufficient to remove confounding? which variables **do we need to adjust for** to make treatment groups comparable?
- Identifiability
  - identifiability assumptions allow us to write causal parameter as statistical parameter.
  - the assumptions may not hold, but we can state and discuss them.
- Statistical parameter
  - Statistical interpretation: The average effect in the population, standardized to the distribution of covariates.

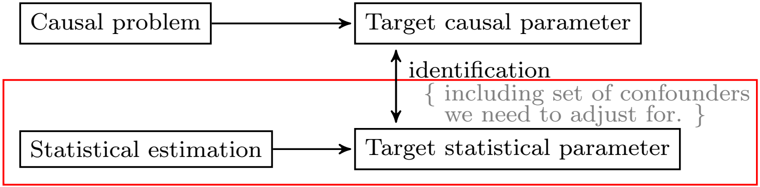
Summary: Properly defining the target



Summary: Properly defining the target



Summary: Properly defining the target



- one estimator is not more causal than another.
- different estimators are based on different nuisance parameters and have different statistical properties (bias/variance).

On a sidenote: Other simple causal parameters

We focus on the ATE as an example of a causal parameter.

But note that other simple causal parameters can be constructed from $\mathbb{E}[Y^1]$ and $\mathbb{E}[Y^0]$.

Like:

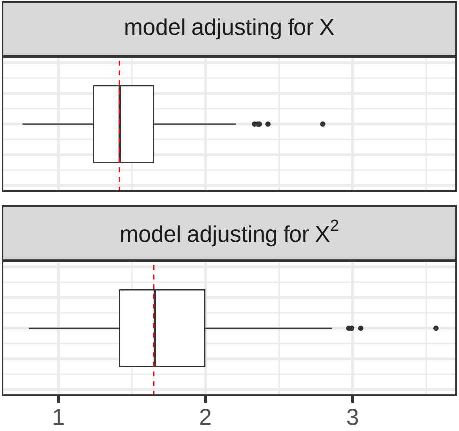
$$\Psi^{\mathrm{RR}}(P) = \frac{\mathbb{E}[Y^1]}{\mathbb{E}[Y^0]},$$

or,

$$\Psi^{\mathrm{OR}}(P) = \frac{\mathbb{E}[Y^1]/(1-\mathbb{E}[Y^1])}{\mathbb{E}[Y^0]/(1-\mathbb{E}[Y^0])}.$$

On a sidenote: Other simple causal parameters

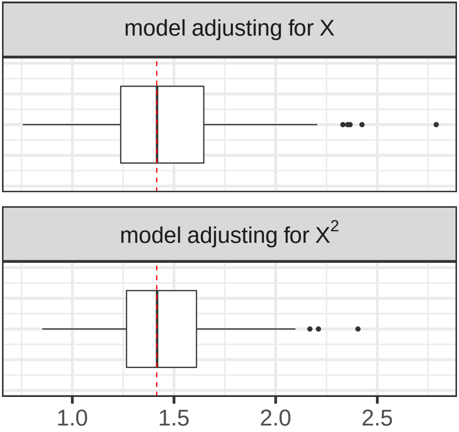
log(OR) as a regression coefficient is a moving target in different logistic regression models:



The upper panel does not show a biased estimator, just an estimator targeting a different parameter (dashed red line).

On a sidenote: Other simple causal parameters

The corresponding causal odds ratio is a fixed target — and the target does not change depending on adjustment for X or X^2:



... but these are different statistical estimators, and they have different statistical properties.

Last slide of this lecture

Summarizing this lecture:

- take 5 minutes to write down 3–10 keywords/concepts/formulas from this lecture;
- discuss the keywords with the person sitting next to you, and explain their significance in the overall targeted learning framework (5 mins).