Targeted nonparametric inference

Asymptotic linearity

A very desirable property —

[1] $o_P(1)$ denotes a sequence which is converges to zero in probability.

Asymptotic linearity

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The empirical measure $\mathbb{P}_n$ of the sample $O_1, \ldots, O_n$:

$$\mathbb{P}_n h = \int h d\mathbb{P}_n = \frac{1}{n}\sum_{i=1}^{n} h(O_i).$$

An estimator $\hat{\psi}_n$ is $\sqrt{n}$-consistent and asymptotically linear with influence function $\phi(P_0)(O)$ if [1]

$$\sqrt{n}\big(\hat{\psi}_n - \psi_0\big) = \sqrt{n}\,\mathbb{P}_n\phi(P_0) + o_P(1),$$

where $\mathbb{E}_{P_0}[\phi(P_0)(O)] = 0$ and $\mathbb{E}_{P_0}[\{\phi(P_0)(O)\}^2] < \infty$.

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Asymptotic linearity

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Then CLT + Slutsky implies:

$$\hat{\psi}_n \overset{as}{\sim} N(\Psi(P_0), \mathrm{Var}(\phi(P_0))/n).$$

The estimator behaves asymptotically as an average of the influence function.

[1] $o_P(1)$ denotes a sequence which is converges to zero in probability.

Asymptotic linearity

Simple example: Estimator for the mean $\psi_0 = \mathbb{E}[X]$:

$$\hat{\psi}_{n,0} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Then

$$\sqrt{n}(\hat{\psi}_n - \psi_0) = \sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\underbrace{(X_i - \psi_0)}_{=\phi(P_0)(O_i)} = \sqrt{n}\mathbb{P}_n\phi(P_0)$$

$\hat{\psi}_{n,0}$ is linear and thus asymptotically linear.

Asymptotic linearity

Simple example: Estimator for the mean $\psi_0 = \mathbb{E}[X]$:

$$\hat{\psi}_{n,1} = \frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{n}$$

Then

$$\sqrt{n}(\hat{\psi}_n - \psi_0) = \sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\underbrace{(X_i - \psi_0)}_{=\phi(P_0)(O_i)} + \frac{\sqrt{n}}{n} = \sqrt{n}\mathbb{P}_n\phi(P_0) + \underbrace{\frac{1}{\sqrt{n}}}_{=o(1)}$$

$\hat{\psi}_{n,1}$ is asymptotically linear.

Asymptotic linearity

Simple example: Estimator for the mean $\psi_0 = \mathbb{E}[X]$:

$$\hat{\psi}_{n,2} = \frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{n^{1/2+0.1}}$$

Then

$$\sqrt{n}(\hat{\psi}_n - \psi_0) = \sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\underbrace{(X_i - \psi_0)}_{=\phi(P_0)(O_i)} + \frac{\sqrt{n}}{n^{1/2+0.1}} = \sqrt{n}\mathbb{P}_n\phi(P_0) + \underbrace{\frac{1}{n^{0.1}}}_{=o(1)}$$

$\hat{\psi}_{n,2}$ is asymptotically linear.

Asymptotic linearity

Simple example: Estimator for the mean $\psi_0 = \mathbb{E}[X]$:

$$\hat{\psi}_{n,3} = \frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{n^{1/2-0.1}}$$

Then

$$\sqrt{n}(\hat{\psi}_n - \psi_0) = \sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\underbrace{(X_i - \psi_0)}_{=\phi(P_0)(O_i)} + \frac{\sqrt{n}}{n^{1/2-0.1}} = \sqrt{n}\mathbb{P}_n\phi(P_0) + \underbrace{n^{0.1}}_{\to\infty}$$

$\hat{\psi}_{n,3}$ is **not** asymptotically linear.

Asymptotic linearity

An estimator $\hat{\psi}_n$ has rate of convergence $r_n \to \infty$ if [2]

$$r_n(\hat{\psi}_n - \psi_0) = O_P(1), \quad \text{i.e.,} \quad \hat{\psi}_n - \psi_0 = O_P(1/r_n).$$

The convergence rate r_n tells us how fast $\hat{\psi}_n$ centers around ψ_0, with the difference $\hat{\psi}_n - \psi_0$ behaving like $1/r_n$.

- One wants negligible bias such as to obtain reliable confidence intervals for ψ_0.
- The bias of an asymptotically linear estimator converges to zero at a rate faster the $1/\sqrt{n}$.

Data-adaptive machine learning estimators rarely achieve this rate.

[2] $O_P(1)$ denotes a sequence which is bounded in probability.

Asymptotic linearity

$$\sqrt{n}\hat{\psi}_{n,1} = \sqrt{n}\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}_{\xrightarrow{P}\psi_0} + \underbrace{\frac{\sqrt{n}}{n}}_{\to 0}, \qquad \text{i.e.,} \quad \sqrt{n}(\hat{\psi}_{n,1} - \psi_0) = o_P(1).$$

$$\sqrt{n}\hat{\psi}_{n,2} = \sqrt{n}\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}_{\xrightarrow{P}\psi_0} + \underbrace{\frac{\sqrt{n}}{n^{1/2+0.1}}}_{\to 0}, \quad \text{i.e.,} \quad \sqrt{n}(\hat{\psi}_{n,3} - \psi_0) = o_P(1).$$

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Asymptotic linearity

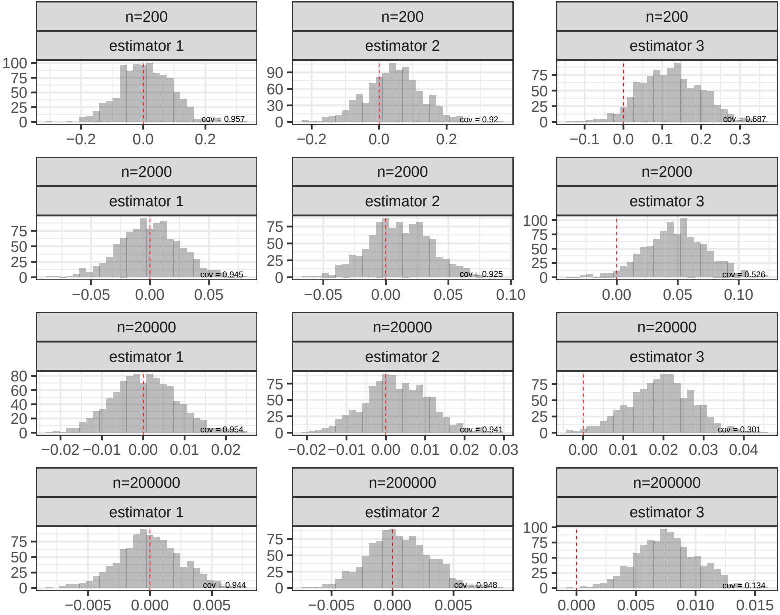
$$\sqrt{n}\hat{\psi}_{n,1} = \sqrt{n}\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}_{\xrightarrow{P}\psi_0} + \underbrace{\frac{\sqrt{n}}{n}}_{\to 0}, \qquad \text{i.e.,} \quad \sqrt{n}(\hat{\psi}_{n,1} - \psi_0) = o_P(1).$$

$$\sqrt{n}\hat{\psi}_{n,2} = \sqrt{n}\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}_{\xrightarrow{P}\psi_0} + \underbrace{\frac{\sqrt{n}}{n^{1/2+0.1}}}_{\to 0}, \quad \text{i.e.,} \quad \sqrt{n}(\hat{\psi}_{n,3} - \psi_0) = o_P(1).$$

$$\sqrt{n}\hat{\psi}_{n,3} = \sqrt{n}\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}_{\xrightarrow{P}\psi_0} + \underbrace{\frac{\sqrt{n}}{n^{1/2-0.1}}}_{\to \infty}, \qquad \text{i.e.,} \quad \sqrt{n}(\hat{\psi}_{n,3} - \psi_0) \xrightarrow{P} \infty.$$

[The remainder term that determines the asymptotic bias the estimator].

Asymptotic linearity



A quick run-through of the theory

The decomposition that guides the construction of TMLE.

The von Mises expansion and the efficient influence curve

A key component in constructing a $\sqrt{n}$-consistent and asymptotically linear estimator, *even when using machine learning estimation*, is the so-called the efficient influence function (also known as the canonical gradient).

The von Mises expansion and the efficient influence curve

The von Mises expansion:

Suppose the functional (the target parameter) $\Psi : \mathcal{M} \to \mathbb{R}$ is sufficiently smooth (as a map from distributions to the real line), in the sense that it admits a certain distributional Taylor expansion

$$\Psi(P) - \Psi(P') = \int \phi(P)(o) d(P - P')(o) + R_2(P, P'), \tag{1}$$

for distributions $P, P' \in \mathcal{M}$ for a function ϕ satisfying $P\phi(P) = 0$ (mean zero) and $P\phi(P)^2 < \infty$ (finite variance).

Intuitively, the von Mises expansion is just a distributional analogue of a Taylor expansion, with the function $\phi(P)$ acting as a usual derivative term; it describes how the functional Ψ changes locally when the distribution changes from P to P'.

The von Mises expansion and the efficient influence curve

When the model $\mathcal{M}$ is assumed properly nonparametric, there exists *one* function $\phi(P)$. This is called the efficient influence curve; we also denote it $\phi^*(P)$.[3]

The efficient influence curve in nonparametric models indicates how to construct asymptotically linear (and efficient) estimators.

[3] This may be confusing here, but it is useful in restricted (semi)parametric models, where multiple ϕ's can satisfy (1). For these situations we by the way have that $P_0\phi(P_0)^2 \geq P_0\phi^*(P_0)^2$.

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The efficient influence curve in nonparametric models indicates how to construct asymptotically linear (and efficient) estimators.

Also note that the efficient curve has many names (also called: influence function, pathwise derivative, Neyman orthogonal score, canonical gradient).

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$\sqrt{n}$-consistency and asymptotically linearity

An estimator $\hat{\psi}_n$ is $\sqrt{n}$-consistent and asymptotically linear with influence function $\phi(P_0)(O)$ if

$$\sqrt{n}\big(\hat{\psi}_n - \psi_0\big) = \sqrt{n}\,\mathbb{P}_n\phi(P_0) + o_P(1),$$

where $\mathbb{E}_{P_0}[\phi(P_0)(O)] = 0$ and $\mathbb{E}_{P_0}[\{\phi(P_0)(O)\}^2] < \infty$.

Then CLT + Slutsky implies:

$$\hat{\psi}_n \overset{as}{\sim} N(\Psi(P_0), \mathrm{Var}(\phi(P_0))/n).$$

The estimator behaves asymptotically as an average of the influence function.[4]

[4] One may also note that the efficient influence curve characterizes the estimator with the smallest variance.

Estimator expansion

An estimator $\hat{\psi}_n$ is asymptotically linear if,

$$\sqrt{n}\big(\hat{\psi}_n - \Psi(P_0)\big) = \sqrt{n}\,\mathbb{P}_n\phi^*(P_0) + o_P(1). \quad (*)$$

Evaluating the von Mises expansion in an estimator $\hat{P}_n^*$ and the true data-generating P_0:

$$\Psi(\hat{P}_n^*) - \Psi(P_0) = (\hat{P}_n^* - P_0)\phi^*(\hat{P}_n^*) + R_2(\hat{P}_n^*, P_0)$$

$(*1)$

$(*2)$

$(*3)$

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An estimator $\hat{\psi}_n$ is asymptotically linear if,

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$$\begin{aligned}\Psi(\hat{P}_n^*) - \Psi(P_0) &= (\hat{P}_n^* - P_0)\phi^*(\hat{P}_n^*) + R_2(\hat{P}_n^*, P_0)\\ &= -P_0\phi^*(\hat{P}_n^*) + R_2(P_0, \hat{P}_n^*)\end{aligned}$$

$(*1)$

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$$\begin{aligned}\Psi(\hat{P}_n^*) - \Psi(P_0) &= (\hat{P}_n^* - P_0)\phi^*(\hat{P}_n^*) + R_2(\hat{P}_n^*, P_0) \\ &= -P_0\phi^*(\hat{P}_n^*) + R_2(P_0, \hat{P}_n^*) \\ &\qquad + \mathbb{P}_n\phi^*(\hat{P}_n^*) - \mathbb{P}_n\phi^*(\hat{P}_n^*) \\ &\qquad + (\mathbb{P}_n - P_0)\phi^*(P_0)\end{aligned}$$

$(*1)$

$(*2)$

$(*3)$

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Evaluating the von Mises expansion in an estimator $\hat{P}_n^*$ and the true data-generating P_0:

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&\qquad + (\mathbb{P}_n - P_0)\phi^*(P_0) \\
&= (\mathbb{P}_n - P_0)\phi^*(P_0) \\
&\qquad + (\mathbb{P}_n - P_0)\big(\phi^*(\hat{P}_n^*) - \phi^*(P_0)\big) && (*1) \\
&\qquad + R_2(\hat{P}_n^*, P_0) && (*2) \\
&\qquad - \mathbb{P}_n\phi^*(\hat{P}_n^*) && (*3)
\end{aligned}$$

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&\qquad + (\mathbb{P}_n - P_0)\phi^*(P_0) \\
&= \mathbb{P}_n\phi^*(P_0) + o_P(n^{-1/2}) \\
&\qquad + (\mathbb{P}_n - P_0)\big(\phi^*(\hat{P}_n^*) - \phi^*(P_0)\big) && (*1) \\
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\end{aligned}$$

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&= \mathbb{P}_n\phi^*(P_0) + o_P(n^{-1/2}) \\
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&\qquad + R_2(\hat{P}_n^*, P_0) && (*2) \\
&\qquad - \mathbb{P}_n\phi^*(\hat{P}_n^*) && (*3)
\end{aligned}$$

i.e., need $(*1)$–$(*3)$ to be $o_P(n^{-1/2})$.

Estimator expansion

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$$\sqrt{n}\big(\hat{\psi}_n - \Psi(P_0)\big) = \sqrt{n}\,\mathbb{P}_n\phi^*(P_0) + o_P(1). \quad (*)$$

$$\begin{aligned} \Psi(\hat{P}_n^*) - \Psi(P_0) = \mathbb{P}_n\phi^*(P_0) + o_P(n^{-1/2}) & \\ + (\mathbb{P}_n - P_0)\big(\phi^*(\hat{P}_n^*) - \phi^*(P_0)\big) & \qquad (*1) \\ + R_2(\hat{P}_n^*, P_0) & \qquad (*2) \\ - \mathbb{P}_n\phi^*(\hat{P}_n^*) & \qquad (*3) \end{aligned}$$

- $(*1)$ is an empirical process term.
- $(*2)$ second-order bias term.
- $(*3)$ is called the efficient influence curve equation.

Estimator expansion

... about the empirical process term $(*1)$:

1. can be handled by empirical process theory, if $(\phi^*(P) : P \in \mathcal{M})$ is assumed Donsker.[5]
2. otherwise can handled by extra sample splitting.

[5] Lemma 19.24 of van der Vaart, A. W. (2000): Asymptotic statistics yields then that $(\mathbb{P}_n - P_0)(\phi^*(\hat{P}_n) - \phi^*(P_0)) = o_P(n^{-1/2})$.

Estimator expansion

Side note: Usually, we will assume the Donsker class condition.

- this is a way of nonparametrically characterizing the complexity of nuisance parameters.
- classes of functions that are Donsker: Indicator functions, bounded monotone functions, Lipschitz parametric functions, smooth functions, . . .

Donsker classes also include traditional parametric functions.

We will not discuss this further. For a nice intro see Sections 4.2 and 4.3 of Kennedy, E. H. (2016): Semiparametric theory and empirical processes in causal inference.

Estimator expansion

That is it.

Estimator expansion

That is it.

Conditions (asymptotic linearity and efficiency)

(C1) Solve the efficient influence curve equation: $\mathbb{P}_n \phi^*(\hat{P}_n) = o_P(n^{-1/2})$.

(C2) Remainder $R(\hat{P}_n, P_0) = o_P(n^{-1/2})$.

(C3) Donsker class conditions for $\{\phi^*(P) : P \in \mathcal{M}\}$.

Then: $\Psi(\hat{P}_n) \overset{as}{\sim} N(\Psi(P_0), P_0\phi^*(P_0)^2/n)$.

Construction of estimators

$$\begin{aligned}\Psi(\hat{P}_n) - \Psi(P_0) = \mathbb{P}_n \phi^*(P_0) + o_P(n^{-1/2}) \\ + R(\hat{P}_n, P_0) \\ - \mathbb{P}_n \phi^*(\hat{P}_n)\end{aligned}$$

For **a given target parameter** $\Psi : \mathcal{M} \to \mathbb{R}$, we need to

1. Know the efficient influence curve, so that we can solve the efficient influence curve equation.
2. Analyze the remainder $R(P, P_0) := \Psi(P) - \Psi(P_0) + P_0 \phi^*(P)$.

NB: These are solely properties of the estimation problem, but also tell us how to construct estimators such as TMLE.

Some comments on deriving the efficient influence curve

Purely as a technical device, we define parametric submodels $\mathcal{M}_\varepsilon = (p_\varepsilon : \varepsilon \in \mathbb{R})$ that satifies (a) $\mathcal{M}_\varepsilon \subseteq \mathcal{M}$ and (b) $p_\varepsilon = p_0$.

The von Mises expansion (1) implies a related notion of smoothness called *pathwise differentiability*, i.e.,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Psi(P_\varepsilon) = \int \phi(P)(o)b(o)dP(o), \tag{2}$$

for every smooth submodel P_ε with density p_ε.

Some comments on deriving the efficient influence curve

Equation (2) can be used to derive the efficient influence curve, for example(!!) as follows.

One evaluates the pathwise derivative along submodels defined for a mean-zero function $b : \mathcal{O} \to \mathbb{R}$ as

$$p_\varepsilon(o) = p(o)(1 + \varepsilon b(o)).$$

and solves the integral equation on the right hand side of (2).

Note that this submodel has score function $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \log p_\varepsilon(o) = b(o)$.

Aaaaand, there are many other useful tricks in this process, which we will not cover.

Some comments on deriving the efficient influence curve

Another common strategy is to assume the data are discrete and then compute the Gateaux derivative

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Psi((1-\varepsilon)p(o)+\varepsilon\delta_{o'}),$$

which equals the influence curve $\phi(P)(o')$ directly.

Note that this really corresponds to computing the pathwise derivative along the particular submodel of the form $(1-\varepsilon)p(o)+\varepsilon\delta_{o'}$ for which the right hand side of (2) is $\phi(P)(o')$.

Analysis of the ATE estimation problem

Analysis of the ATE estimation problem

EXAMPLE: Average treatment effect (ATE)

Observed data $O = (X, A, Y) \in \mathbb{R}^d \times \{0,1\} \times \{0,1\} = \mathcal{O}$

- $X \in \mathbb{R}^d$ are covariates
- $A \in \{0,1\}$ is a binary exposure variable (treatment decision)
- $Y \in \{0,1\}$ is a binary outcome variable

$O \sim P_0$ where P_0 assumed to belong to nonparametric model $\mathcal{M}$.

We are interested in estimating the ATE:

$$\Psi(P) = \mathbb{E}_P\big[\mathbb{E}_P[Y \mid A = 1, X] - \mathbb{E}_P[Y \mid A = 0, X]\big].$$

Analysis of the ATE estimation problem

EXAMPLE: Average treatment effect (ATE)

1. The efficient influence function:

$$\begin{aligned}\phi^*(P)(O) &= \tilde{\phi}^*(f,\pi)(O) \\ &= \left(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)}\right)\big(Y - f(A,X)\big) + f(1,X) - f(0,X) - \Psi(P)\end{aligned}$$

Analysis of the ATE estimation problem

EXAMPLE: Average treatment effect (ATE)

1. The efficient influence function:

$$\phi^*(P)(O) = \tilde{\phi}^*(f,\pi)(O)$$
$$= \left(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)}\right)\big(Y - f(A,X)\big) + f(1,X) - f(0,X) - \Psi(P)$$

2. The remainder:

$$R(P,P_0) = \tilde{R}(f,\pi,f_0,\pi_0)$$
$$= \int_{\mathbb{R}^d} \sum_{a=0,1} (2a-1) \frac{\pi_0(a \mid x) - \pi(a \mid x)}{\pi(a \mid x)} \big(f_0(a,x) - f(a,x)\big) d\mu_{0,X}(x)$$

Analysis of the ATE

$$f(A,X) = \mathbb{E}_P[Y \mid A,X],\ \pi(A \mid X) = P(A = a \mid X)$$
$$f_0(A,X) = \mathbb{E}_{P_0}[Y \mid A,X],\ \pi_0(A \mid X) = P(A = a \mid X)$$

$$R(P,P_0) := \Psi(P) - \Psi(P_0) + P_0\phi^*(P).$$

2. Deriving the remainder for the ATE:

$$\begin{aligned}
R(P,P_0) &= \mathbb{E}_P[f(1,X) - f(0,X)] - \mathbb{E}_{P_0}[f_0(1,X) - f_0(0,X)] \\
&+ \mathbb{E}_{P_0}\left[\left(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)}\right)(Y - f(A,X))\right] \\
&+ \mathbb{E}_{P_0}\left[f(1,X) - f(0,X)\right] - \Psi(P) \\
&\overset{*}{=} \int_{\mathbb{R}^d} \sum_{a=0,1} (2a-1)\left(\frac{\pi_0(a \mid x)}{\pi(a \mid x)} - 1\right)(f_0(a,x) - f(a,x))d\mu_{0,X}(x) \\
&= \int_{\mathbb{R}^d} \sum_{a=0,1} (2a-1)\frac{\pi_0(a \mid x) - \pi(a \mid x)}{\pi(a \mid x)}(f_0(a,x) - f(a,x))d\mu_{0,X}(x)
\end{aligned}$$

the equality marked by $*$ is detailed on the next slide.

Analysis of the ATE estimation problem

We used that:

$$
\begin{aligned}
&\mathbb{E}_{P_0}\left[\left(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)}\right)\big(Y - f(A,X)\big)\right] \\
&\quad = \mathbb{E}_{P_0}\left[\frac{2A-1}{\pi(A \mid X)}\big(Y - f(A,X)\big)\right] \\
&\quad = \mathbb{E}_{P_0}\left[\mathbb{E}_{P_0}\left[\frac{2A-1}{\pi(A \mid X)}\big(Y - f(A,X)\big)\,\middle|\, A, X\right]\right] \\
&\quad = \mathbb{E}_{P_0}\left[\frac{2A-1}{\pi(A \mid X)}\big(f_0(A,X) - f(A,X)\big)\right] \\
&\quad = \int_{\mathbb{R}^d} \sum_{a=0,1} \frac{2a-1}{\pi(a \mid x)}\big(f_0(a,x) - f(a,x)\big)\pi_0(a \mid x)\, d\mu_{0,X}(x) \\
&\quad = \int_{\mathbb{R}^d} \sum_{a=0,1} (2a-1)\frac{\pi_0(a \mid x)}{\pi(a \mid x)}\big(f_0(a,x) - f(a,x)\big)\, d\mu_{0,X}(x)
\end{aligned}
$$

Analysis of the ATE estimation problem

The remainder determines the asymptotic bias.

For the ATE, the remainder has a really nice structure!

$$
\begin{aligned}
R(P, P_0) &= \tilde{R}(f, \pi, f_0, \pi_0) \\
&= \int_{\mathbb{R}^d} \sum_{a=0,1} (2a-1) \frac{\pi_0(a \mid x) - \pi(a \mid x)}{\pi(a \mid x)} \big(f_0(a, x) - f(a, x) \big) d\mu_{0,X}(x)
\end{aligned}
$$

A "double robust" structure, which has some important implications.

Analysis of the ATE estimation problem

$$
\begin{aligned}
|R(P, P_0)| &= |\tilde{R}(f, \pi, f_0, \pi_0)| \\
&\leq \sum_{a=0,1} \int_{\mathbb{R}^d} \frac{|\pi_0(a \mid x) - \pi(a \mid x)|}{\pi(a \mid x)} |f_0(a, x) - f(a, x)| d\mu_{0,X}(x)
\end{aligned}
$$

Analysis of the ATE estimation problem

$$
\begin{aligned}
|R(P, P_0)| &= |\tilde{R}(f, \pi, f_0, \pi_0)| \\
&\leq \sum_{a=0,1} \int_{\mathbb{R}^d} \frac{|\pi_0(a \mid x) - \pi(a \mid x)|}{\pi(a \mid x)} |f_0(a, x) - f(a, x)| d\mu_{0,X}(x) \\
&\overset{*}{\leq} \sum_{a=0,1} \frac{1}{\pi(a \mid x)} \sqrt{\int_{\mathbb{R}^d} \left\{\pi_0(a \mid x) - \pi(a \mid x)\right\}^2 d\mu_{0,X}(x)} \\
&\qquad \times \sqrt{\int_{\mathbb{R}^d} \left\{f_0(a, x) - f(a, x)\right\}^2 d\mu_{0,X}(x)}
\end{aligned}
$$

$*$ uses Cauchy-Schwarz.

Analysis of the ATE estimation problem

$$
\begin{aligned}
|R(P,P_0)| &= |\tilde{R}(f,\pi,f_0,\pi_0)| \\
&\leq \sum_{a=0,1}\int_{\mathbb{R}^d}\frac{|\pi_0(a\mid x)-\pi(a\mid x)|}{\pi(a\mid x)}|f_0(a,x)-f(a,x)|d\mu_{0,X}(x) \\
&\overset{*}{\leq} \sum_{a=0,1}\frac{1}{\pi(a\mid x)}\sqrt{\int_{\mathbb{R}^d}\left\{\pi_0(a\mid x)-\pi(a\mid x)\right\}^2 d\mu_{0,X}(x)} \\
&\qquad\qquad \times\sqrt{\int_{\mathbb{R}^d}\left\{f_0(a,x)-f(a,x)\right\}^2 d\mu_{0,X}(x)}
\end{aligned}
$$

Thus, if $\pi(a\mid X)>\delta>0$ a.s., then:

$$
\boxed{|\tilde{R}(\hat{f}_n^*,\hat{\pi}_n,f_0,\pi_0)|\leq\sum_{a=0,1}\delta^{-1}\left\|\pi_0(a\mid\cdot)-\hat{\pi}_n(a\mid\cdot)\right\|_{\mu_0}\left\|f_0(a\mid\cdot)-\hat{f}_n(a\mid\cdot)\right\|_{\mu_0}}
$$

$*$ uses Cauchy-Schwarz.

Analysis of the ATE estimation problem

What does this imply for estimation?

Double robustness in consistency

$$|\tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0)| \leq \sum_{a=0,1} \delta^{-1} \underbrace{\|\pi_0(a \mid \cdot) - \hat{\pi}_n(a \mid \cdot)\|_{\mu_0}}_{o_P(1), \text{ or}} \underbrace{\|f_0(a \mid \cdot) - \hat{f}_n^*(a \mid \cdot)\|_{\mu_0}}_{o_P(1)}$$

then $\tilde{\Psi}(\hat{f}_n^*) - \tilde{\Psi}(f_0) = o_P(1)$.

Asymptotic linearity (easier to establish due to double robust structure)

$$|\tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0)| \leq \sum_{a=0,1} \delta^{-1} \underbrace{\|\pi_0(a \mid \cdot) - \hat{\pi}_n(a \mid \cdot)\|_{\mu_0}}_{=o_P(n^{-1/4})} \underbrace{\|f_0(a \mid \cdot) - \hat{f}_n^*(a \mid \cdot)\|_{\mu_0}}_{=o_P(n^{-1/4})}$$

i.e., $\tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0) = o_P(n^{-1/2})$.

I.e., bias is converging at fast enough rate for reliable confidence intervals.

Analysis of the ATE estimation problem

Side note: Showing the double robustness in consistency ...

Say we have estimators $(\hat{f}_n^*, \hat{\pi}_n)$;

- converging to (f, π)
- solving the efficient influence curve equation.

Per definition, $\tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0) = \tilde{\Psi}(\hat{f}_n^*) - \tilde{\Psi}(f_0) + P_0\tilde{\phi}^*(\hat{f}_n^*, \hat{\pi}_n)$.

I.e.,

$$\begin{aligned}\tilde{\Psi}(\hat{f}_n^*) - \tilde{\Psi}(f_0) &= -P_0\tilde{\phi}^*(\hat{f}_n^*, \hat{\pi}_n) + \tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0) \\ &= (\mathbb{P}_n - P_0)\phi^*(\hat{f}_n^*, \hat{\pi}_n) + \tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0)\end{aligned}$$

The first term is an empirical process term which equals $(\mathbb{P}_n - P_0)\tilde{\phi}^*(f, \pi)$ plus an $o_P(n^{-1/2})$-term.

This then gives

$$\tilde{\Psi}(\hat{f}_n^*) - \tilde{\Psi}(f_0) = \underbrace{(\mathbb{P}_n - P_0)\tilde{\phi}^*(f, \pi)}_{\text{LLN applies}} + \tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0) + o_P(n^{-1/2})$$

which yields that $\tilde{\Psi}(\hat{f}_n^*) - \tilde{\Psi}(f_0) = o_P(1)$ if $\tilde{R}(\hat{f}_n^*, \hat{\pi}_n, f_0, \pi_0) = o_P(1)$.

Analysis of a concrete estimation problem

EXAMPLE: Average treatment effect (ATE)

1. The efficient influence function:

$$\begin{aligned}\phi^*(P)(O) &= \tilde{\phi}^*(f,\pi)(O) \\ &= \left(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)}\right)\big(Y - f(A,X)\big) + f(1,X) - f(0,X) - \Psi(P)\end{aligned}$$

2. The remainder:

$$\begin{aligned}R(P,P_0) &= \tilde{R}(f,\pi,f_0,\pi_0) \\ &= \int_{\mathbb{R}^d} \sum_{a=0,1} (2a-1)\frac{\pi_0(a \mid x) - \pi(a \mid x)}{\pi(a \mid x)}\big(f_0(a,x) - f(a,x)\big) d\mu_{0,X}(x)\end{aligned}$$

Deriving these is done once for a given target parameter $\Psi : \mathcal{M} \to \mathbb{R}$.

TMLE

Conditions (asymptotic linearity and efficiency)

(C1) Solve the efficient influence curve equation: $\mathbb{P}_n \phi^*(\hat{P}_n) = o_P(n^{-1/2})$

(C2) Remainder $R(\hat{P}_n, P_0) = o_P(n^{-1/2})$

(C3) Donsker class conditions for $\{\phi^*(P) : P \in \mathcal{M}\}$

Then: $\Psi(\hat{P}_n) \overset{as}{\sim} N(\Psi(P_0), P_0\phi^*(P_0)^2/n)$

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TMLE is a two-step procedure:

Step 1 Construct initial estimator $\hat{P}_n$ for P such that $R(\hat{P}_n, P_0) = o_P(n^{-1/2})$.

Step 2 Update the estimator $\hat{P}_n \mapsto \hat{P}_n^*$ such that $\hat{P}_n^*$ solves the efficient influence curve equation.

TMLE far

- The role of the targeting step (Step 2):
  - Gaining double robustness in consistency.
  - Easier to get rid of second-order remainder.
- The role of the initial estimation step (Step 1):
  - This should be done well enough to get rid of the second-order remainder.