One solution to small exercises day 1 (double robustness)

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Part 1 (in Lecture 2). We can write

$$\tilde{\Psi}_{\text{ee}}(f,\pi,p) = \mathbb{E}_P\left[\left(\frac{A}{\pi(1 \mid X)} - \frac{1-A}{\pi(0 \mid X)}\right)\big(Y - f(A,X)\big) + f(1,X) - f(0,X)\right]$$

equivalently as

$$\begin{aligned} &= \mathbb{E}_P\left[\left(\frac{AY}{\pi(1 \mid X)} - \frac{(1-A)Y}{\pi(0 \mid X)}\right) + \left(1 - \frac{A}{\pi(1 \mid X)}\right) f(1,X) \right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad \left. - \left(1 - \frac{1-A}{\pi(0 \mid X)}\right) f(0,X)\right] \end{aligned}$$

Showing that $\tilde{\Psi}_{\text{ee}}(f,\pi,p) = \mathbb{E}_P[Y^1] - \mathbb{E}_P[Y^0]$ follows trivially since

$$\begin{aligned} \tilde{\Psi}_{\text{ee}}(f,\pi,p) &= \underbrace{\tilde{\Psi}(f,\mu_X)}_{=\mathbb{E}_P[Y^1]-\mathbb{E}_P[Y^0]} + \mathbb{E}_P\left[\left(\frac{A}{\pi(1 \mid X)} - \frac{1-A}{\pi(0 \mid X)}\right)\big(Y - f(A,X)\big)\right] \\ &= \mathbb{E}_P[Y^1] - \mathbb{E}_P[Y^0] + \underbrace{\mathbb{E}_P\left[\left(\frac{A}{\pi(1 \mid X)} - \frac{1-A}{\pi(0 \mid X)}\right)\big(f(A,X) - f(A,X)\big)\right]}_{=0}, \end{aligned}$$

using iterated expectations at the second equality.

Part 2 (in Lecture 3). Consider first $f = f_0$ and use iterated expectations to write:

$$\begin{aligned} &\mathbb{E}_{P_0}\left[\left(\frac{A}{\pi(1 \mid X)} - \frac{1-A}{\pi(0 \mid X)}\right)\big(Y - f(A,X)\big) + f(1,X) - f(0,X)\right] \\ &\quad = \mathbb{E}_{P_0}\left[\frac{A}{\pi(A \mid X)} \underbrace{\big(f_0(A,X) - f(A,X)\big)}_{=0} + \underbrace{f(1,X) - f(0,X)}_{=f_0(1,X)-f_0(0,X)}\right] = \Psi(P_0). \end{aligned}$$

Then consider $\pi = \pi_0$ and proceed with the rewritten expression from **Task 1**:

$$\begin{aligned}
&\mathbb{E}_{P_0}\left[\left(\frac{AY}{\pi(1 \mid X)} - \frac{(1-A)Y}{\pi(0 \mid X)}\right) + \left(1 - \frac{A}{\pi(0 \mid X)}\right) f(1,X) - \left(1 - \frac{1-A}{\pi(0 \mid X)}\right) f(0,X)\right] \\
&\quad = \int_{\mathbb{R}^d} \sum_{y=0,1} \sum_{a=0,1} \left(\frac{ay}{\pi(1 \mid x)} - \frac{(1-a)y}{\pi(0 \mid x)}\right) \mu_{Y,0}(y \mid a, x) \pi_0(a \mid x) d\mu_{X,0}(x) \\
&\qquad\qquad\qquad + \left(1 - \underbrace{\frac{\pi_0(1 \mid X)}{\pi(1 \mid X)}}_{=1}\right) f(1,X) - \left(1 - \underbrace{\frac{\pi_0(0 \mid X)}{\pi(0 \mid X)}}_{=1}\right) f(0,X)\Bigg] \\
&\quad = \int_{\mathbb{R}^d} \sum_{y=0,1} \left(y \underbrace{\frac{\pi_0(1 \mid x)}{\pi(1 \mid x)}}_{=1} \mu_{Y,0}(y \mid 1, x) - y \underbrace{\frac{\pi_0(0 \mid x)}{\pi(0 \mid x)}}_{=1} \mu_{Y,0}(y \mid 0, x)\right) d\mu_{X,0}(x) \\
&\quad = \int_{\mathbb{R}^d} \sum_{y=0,1} \big(f_0(1,x) - f_0(0,x)\big) d\mu_{X,0}(x) = \Psi(P_0).
\end{aligned}$$