

Day 2, Lecture 3

Targeting: Changing the target

# Changing the target

## ATE: Statistical estimation problem

$O_1, \dots, O_n \stackrel{iid}{\sim} P_0$ ,  $O_i$  is the observation for individual  $i$  of the dataset, consists of

- ▶ Covariates:  $X_i \in \mathcal{X} \subseteq \mathbb{R}^d$
- ▶ Exposure/treatment:  $A_i \in \{0, 1\}$
- ▶ Outcome:  $Y_i \in \{0, 1\}$  or  $Y \in \mathbb{R}$

We are interested in:

$$\Psi(P) = \tilde{\Psi}(f, \mu_X) = \int_{\mathbb{R}} (f(1, x) - f(0, x)) d\mu_X(x),$$

where  $f(a, x) = \mathbb{E}_P[Y \mid A = a, X = x]$ .

A plug-in estimator requires an estimator  $\hat{f}_n$  for  $f$ :

$$\hat{\psi}_n = \tilde{\Psi}(\hat{f}_n, \mathbb{P}_n) = \frac{1}{n} \sum_{i=1}^n (\hat{f}_n(1, X_i) - \hat{f}_n(0, X_i)).$$

# Changing the target

What is the interpretation?

**Causal interpretation:** The risk difference, had everyone in the population been treated versus had everyone in the population been untreated.

## Changing the target

In an observational study, the de facto treated and the de facto untreated groups may differ quite a lot.

Sometimes we may be interested in the effect averaged with respect to the distribution of covariates *in the treated population*.

⇒ the average treatment effect among the treated.

Changing the target: Average treatment effect among the treated

Causal interpretation: The risk difference, had everyone in the treated population been treated versus had everyone in the treated population been untreated.

## Changing the target: Average treatment effect among the treated

### Average treatment effect (ATE)

- ▶  $O = (X, A, Y) \in \mathbb{R}^d \times \{0, 1\} \times \{0, 1\}$
- ▶ The ATE is defined for  $P \in \mathcal{M}$  as

$$\Psi(P) = \mathbb{E}_P[\mathbb{E}_P[Y \mid A = 1, X] - \mathbb{E}_P[Y \mid A = 0, X]]$$

- ▶ Under causal assumptions:

$$\Psi(P) = \mathbb{E}_P[Y^1] - \mathbb{E}_P[Y^0]$$

## Changing the target: Average treatment effect among the treated

### Average treatment effect among the treated (ATT)

- ▶  $O = (X, A, Y) \in \mathbb{R}^d \times \{0, 1\} \times \{0, 1\}$
- ▶ The ATT is defined for  $P \in \mathcal{M}$  as

$$\Psi(P) = \mathbb{E}_P[\mathbb{E}_P[Y \mid A = 1, X] - \mathbb{E}_P[Y \mid A = 0, X] \mid A = 1]$$

- ▶ Under causal assumptions:

$$\Psi(P) = \mathbb{E}_P[Y^1 \mid A = 1] - \mathbb{E}_P[Y^0 \mid A = 1]$$

This changes the statistical estimation problem and thus the TMLE.

## Changing the target: Average treatment effect among the treated

We can identify the causal parameter under the causal assumptions (consistency, exchangeability and positivity):

$$\begin{aligned}\Psi(P) &= \mathbb{E}[Y^1 \mid A = 1] - \mathbb{E}[Y^0 \mid A = 1] \\ &= \mathbb{E}_P[\mathbb{E}_P[Y \mid A = 1, X] - \mathbb{E}_P[Y \mid A = 0, X] \mid A = 1] \\ &= \int_{\mathbb{R}^d} (f(1, x) - f(0, x)) d\mu_{X|A}(x \mid 1) \\ &= \int_{\mathbb{R}^d} (f(1, x) - f(0, x)) \frac{\pi(1 \mid x)}{\bar{\pi}(1)} d\mu_X(x) \\ &= \tilde{\Psi}(\mu_X, \bar{\pi}, \pi, f)\end{aligned}$$



## Changing the target: Average treatment effect among the treated

Thus, the ATT can be identified as the statistical parameter:

$$\begin{aligned}\Psi(P) &= \mathbb{E}_P[\mathbb{E}_P[Y \mid A = 1, X] - \mathbb{E}_P[Y \mid A = 0, X] \mid A = 1] \\ &= \int_{\mathbb{R}^d} (f(1, x) - f(0, x)) \frac{\pi(1 \mid x)}{\bar{\pi}(1)} d\mu_X(x) \\ &= \tilde{\Psi}(\mu_X, \bar{\pi}, \pi, f)\end{aligned}$$

where:

- ▶  $f(a, x) = \mathbb{E}_P[Y \mid A = a, X = x]$
- ▶  $\pi(a \mid x) = P(A = a \mid X = x)$
- ▶  $\bar{\pi}(a) = P(A = a)$  is the marginal distribution of  $A$
- ▶  $\mu_X$  is the marginal distribution of  $X$

## Changing the target: Average treatment effect among the treated

A substitution estimator:

$$\hat{\psi}_n = \tilde{\Psi}(\hat{\mu}_X, \hat{\pi}_n, \hat{\pi}_n, \hat{f}_n) = \frac{1}{n} \sum_{i=1}^n \frac{1\{A_i = 1\}}{\hat{\pi}_n(1)} (\hat{f}_n(1, X_i) - \hat{f}_n(0, X_i)),$$

$$\text{where, } \hat{\pi}_n(1) = \frac{1}{n} \sum_{i=1}^n A_i.$$

## Targeting step: Average treatment effect on the treated

### EXAMPLE: Average treatment effect (ATE)

Step 1 Construct initial estimators  $\hat{f}_n, \hat{\pi}_n$  for  $f, \pi$

Step 2 Update the estimator  $\hat{f}_n \mapsto \hat{f}_n^*$  for  $f$  such that  $\hat{f}_n^*$  for the fixed  $\hat{\pi}_n$  solves the efficient influence curve equation

For the ATE, Step 2 is simply just an additional logistic regression step.

## Targeting step: Average treatment effect on the treated

EXAMPLE: Average treatment effect among the treated (ATT)

Step 1 Construct initial estimators  $\hat{f}_n, \hat{\pi}_n$  for  $f, \pi$

Step 2 Update the estimator  $\hat{f}_n \mapsto \hat{f}_n^*$  for  $f$  and the estimator  $\hat{\pi}_n \mapsto \hat{\pi}_n^*$  for  $\pi$  such that  $\hat{f}_n^*, \hat{\pi}_n^*$  solves the efficient influence curve equation

For the ATE, Step 2 is simply just an additional logistic regression step.

For the ATT, Step 2 is an iterative algorithm with recursive steps of additional logistic regressions.

## Targeting step: Average treatment effect on the treated

EXAMPLE: Average treatment effect among the treated (ATT)

The efficient influence function:

$$\begin{aligned}\tilde{\phi}^*(f, \pi, \bar{\pi})(O) = & \left( \frac{A}{\bar{\pi}(1)} - \frac{(1-A)\pi(1|X)}{\bar{\pi}(1)\pi(0|X)} \right) (Y - f(A, X)) \\ & + \frac{A}{\bar{\pi}(1)} (f(1, X) - f(0, X) - \Psi(P))\end{aligned}$$

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EXAMPLE: Average treatment effect among the treated (ATT)

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## Targeting step: Average treatment effect on the treated

We need:

- (i) Parametric submodel  $\{f_\varepsilon, \pi_\varepsilon : \varepsilon \in \mathbb{R}\} \subset \mathcal{M}$
- (ii) Loss function  $(O, (f, \pi)) \mapsto \mathcal{L}(f, \pi)(O)$

such that

$$(1) \quad f_{\varepsilon=0} = f, \pi_\varepsilon = \pi \qquad (2) \quad \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathcal{L}(f_\varepsilon, \pi_\varepsilon)(O) = \tilde{\phi}^*(f, \pi, \bar{\pi})(O)$$

## Targeting step: Average treatment effect on the treated

$$\text{logit}(p) = \text{expit}^{-1}(p) = \log\left(\frac{p}{1-p}\right)$$

(i) Sum loss function  $\mathcal{L}(f, \pi) = \mathcal{L}_1(f) + \mathcal{L}_2(\pi)$ , where

$$\mathcal{L}_1(f)(O) = -(Y \log(f(A, X)) + (1 - Y) \log(1 - f(A, X)))$$

$$\mathcal{L}_2(\pi)(O) = -(A \log(\pi(1 | X)) + (1 - A) \log(1 - \pi(1 | X)))$$

(ii) Logistic regression models:

$$f_{\varepsilon}(A, X) = \text{expit}(\text{logit}(f(A, X)) + \varepsilon H_1(\pi, \bar{\pi})(A, X))$$

$$\pi_{\varepsilon}(X) = \text{expit}(\text{logit}(\pi(1 | X)) + \varepsilon H_2(f, \pi, \bar{\pi})(A, X))$$

with the "clever covariates":

$$H_1(\pi, \bar{\pi})(A, X) = \left( \frac{A}{\bar{\pi}(1)} - \frac{(1 - A)\pi(1 | X)}{\bar{\pi}(1)\pi(0 | X)} \right), \quad \text{and,}$$

$$H_2(f, \pi, \bar{\pi})(A, X) = \frac{f(1, X) - f(0, X) - \Psi(P)}{\bar{\pi}(1)}$$

## Targeting step: Average treatment effect on the treated

Iterative algorithm:

1. Given initial estimators  $\hat{f}_n^0, \hat{\pi}_n^0$ :

- ▶ Obtain estimate  $\hat{\varepsilon}_Y^0$  for  $\varepsilon$ :

$$f_\varepsilon(A, X) = \text{expit}(\text{logit}(\hat{f}_n^0(A, X)) + \varepsilon H_1(\hat{\pi}_n^0, \bar{\pi})(A, X))$$

(i.e., regress  $Y$  on covariate  $H_1(\hat{\pi}_n^0, \bar{\pi})(A, X)$  with offset  $\text{logit}(\hat{f}_n^0(A, X))$ )

- ▶ Update:  $\hat{f}_n^1 := \hat{f}_{n, \hat{\varepsilon}_Y^0}^0$ .

- ▶ Obtain estimate  $\hat{\varepsilon}_A^0$  for  $\varepsilon$ :

$$\pi_\varepsilon(X) = \text{expit}(\text{logit}(\hat{\pi}_n^0(1 | X)) + \varepsilon H_2(\hat{f}_n^1, \hat{\pi}_n^0, \bar{\pi})(A, X))$$

(i.e., regress  $A$  on covariate  $H_2(\hat{f}_n^1, \hat{\pi}_n^0, \bar{\pi})(A, X)$  with offset  $\text{logit}(\hat{\pi}_n^0(1 | X))$ )

- ▶ Update:  $\hat{\pi}_n^1 := \hat{\pi}_{n, \hat{\varepsilon}_A^0}^0$ .

## Targeting step: Average treatment effect on the treated

Iterative algorithm:

2. Iteratively from  $k$  to  $k + 1$ , given current estimators  $\hat{f}_n^k, \hat{\pi}_n^k$ :

- ▶ Obtain estimate  $\hat{\varepsilon}_Y^k$  for  $\varepsilon$ :

$$f_\varepsilon(A, X) = \text{expit}(\text{logit}(\hat{f}_n^k(A, X)) + \varepsilon H_1(\hat{\pi}_n^k, \bar{\pi})(A, X))$$

(i.e., regress  $Y$  on covariate  $H_1(\hat{\pi}_n^k, \bar{\pi})(A, X)$  with offset  $\text{logit}(\hat{f}_n^k(A, X))$ )

- ▶ Update:  $\hat{f}_n^{k+1} := \hat{f}_{n, \hat{\varepsilon}_Y^k}^k$ .

- ▶ Obtain estimate  $\hat{\varepsilon}_A^k$  for  $\varepsilon$ :

$$\pi_\varepsilon(X) = \text{expit}(\text{logit}(\hat{\pi}_n^k(1 | X)) + \varepsilon H_2(\hat{f}_n^{k+1}, \hat{\pi}_n^k, \bar{\pi})(A, X))$$

(i.e., regress  $A$  on covariate  $H_2(\hat{f}_n^{k+1}, \hat{\pi}_n^k, \bar{\pi})(A, X)$  with offset  $\text{logit}(\hat{\pi}_n^k(1 | X))$ )

- ▶ Update:  $\hat{\pi}_n^{k+1} := \hat{\pi}_{n, \hat{\varepsilon}_A^k}^k$ .

## Targeting step: Average treatment effect on the treated

This is continued until we solve:

$$\frac{1}{n} \sum_{i=1}^n \left( \frac{A_i}{\hat{\pi}_n(1)} - \frac{(1 - A_i) \hat{\pi}_n^{k*}(1 | X_i)}{\hat{\pi}_n(1) \hat{\pi}_n^{k*}(0 | X_i)} \right) (Y - \hat{f}_n^{k*}(A_i, X_i)) \approx 0$$

and,

$$\frac{1}{n} \sum_{i=1}^n \frac{\hat{f}_n^{k*}(1, X_i) - \hat{f}_n^{k*}(0, X_i) - \tilde{\Psi}(\hat{\mu}_X, \hat{\pi}_n, \hat{\pi}_n^{k*}, \hat{f}_n^{k*})}{\hat{\pi}_n(1)} (A_i - \hat{\pi}_n^{k*}(1 | X_i)) \approx 0;$$

These are the different parts of the efficient influence curve equation.

## Targeting step: Average treatment effect on the treated

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and,

$$\frac{1}{n} \sum_{i=1}^n \frac{\hat{f}_n^{k*}(1, X_i) - \hat{f}_n^{k*}(0, X_i) - \tilde{\Psi}(\hat{\mu}_X, \hat{\pi}_n, \hat{\pi}_n^{k*}, \hat{f}_n^{k*})}{\hat{\pi}_n(1)} (A_i - \hat{\pi}_n^{k*}(1 | X_i)) \approx 0;$$

note that we already solve:

$$\frac{1}{n} \sum_{i=1}^n \frac{\hat{\pi}_n^{k*}(1 | X_i)}{\hat{\pi}_n(1)} (\hat{f}_n^{k*}(1, X_i) - \hat{f}_n^{k*}(0, X_i) - \tilde{\Psi}(\hat{\mu}_X, \hat{\pi}_n, \hat{\pi}_n^{k*}, \hat{f}_n^{k*})) = 0.$$

These are the different parts of the efficient influence curve equation.

## Average treatment effect on the treated

This was the targeting step: What we need procedurally to carry out the TMLE estimation.

## Average treatment effect on the treated

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To finish the analysis, it remains to analyze the remainder term:

$$R(P, P_0) = \Psi(P) - \Psi(P_0) + P_0 \phi^*(P).$$



## Average treatment effect on the treated

This was the targeting step: What we need procedurally to carry out the TMLE estimation.

To finish the analysis, it remains to analyze the remainder term:

$$R(P, P_0) = \Psi(P) - \Psi(P_0) + P_0\phi^*(P).$$

To derive this: Start from  $P_0\phi^*(P) = \mathbb{E}_{P_0}[\phi^*(P)(O)]$  and show that this can be written as [ something ] plus  $\Psi(P_0) - \Psi(P)$ . This [ something ] is the remainder term.

## Average treatment effect on the treated

For the ATT we can derive that:

$$\begin{aligned}\tilde{R}(f, \pi, \bar{\pi}, f_0, \pi_0, \bar{\pi}_n) = & \frac{1}{\bar{\pi}(1)} \left( \frac{\pi_0(1 | X) - \pi(1 | X)}{1 - \pi(1 | X)} \right) (f_0(0, X) - f(0, X)) \\ & + \left( \frac{\bar{\pi}_0(1) - \bar{\pi}(1)}{\bar{\pi}(1)} \right) (\psi(P_0) - \psi(P))\end{aligned}$$

Again we see the **double robust structure**.

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Again we see the **double robust structure**.

This is a particularly nice result, since the parameter depends on both the outcome regression  $f$  and the propensity score  $\pi$ .

## Final comments — changing the target

It depends very much on the target parameter and the structure of its efficient influence function how easy/hard estimation, and particularly targeting, becomes.

For many target parameters, all this work has already been done!

## Final comments — changing the target

The average treatment effect among the treated is implemented in the `tmle` package:

```
set.seed(15)
sim.data <- sim.fun(n=1000)
```

```
library(tmle)
fit.tmle <- tmle(Y=sim.data$Y, A=sim.data$A,
  cbind(X1=sim.data$X1,X2=sim.data$X2,
    X3=sim.data$X3),
  gform=A~X1+X2+X3, ## treatment model
  Qform=Y~A+X1+X2+X3, ## outcome model
  family="binomial",
  cvQinit=FALSE)
```

## Final comments — changing the target

### Additive Effect

Parameter Estimate: 0.066263  
Estimated Variance: 0.00085811  
p-value: 0.023694  
95% Conf Interval: (0.0088482, 0.12368)

### Additive Effect among the Treated

Parameter Estimate: 0.072104  
Estimated Variance: 0.0009739  
p-value: 0.020862  
95% Conf Interval: (0.010938, 0.13327)

### Additive Effect among the Controls

Parameter Estimate: 0.059976  
Estimated Variance: 0.0009839  
p-value: 0.055869  
95% Conf Interval: (-0.0015039, 0.12146)

### Relative Risk

Parameter Estimate: 1.0954

## Final comments — changing the target

Many other (!! ) interesting parameters<sup>1</sup>

- ▶ Controlled and natural direct and indirect effects (mediation analysis parameters)
- ▶ Effects among groups defined by specific covariate characteristics (effect modification)
- ▶ Dynamic interventions, stochastic interventions

⋮

We get back to examples of target parameters in longitudinal settings.

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<sup>1</sup>Newer software ecosystem: <https://tlverse.org/>.