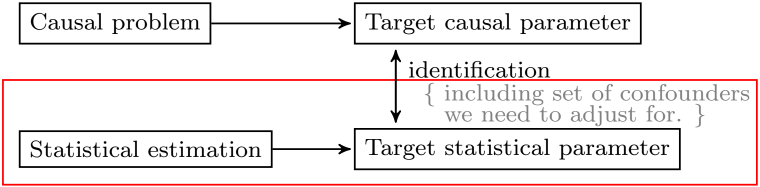
Estimating the target

Estimating the target


Causal problem
Target causal parameter
identification
{ including set of confounders we need to adjust for. }
Statistical estimation
Target statistical parameter

- one estimator is not more causal than another.
- different estimators are based on different nuisance parameters and have different statistical properties (bias/variance).

G-formula versus IP-weighting

G-formula

1. Estimate nuisance parameters $f(a,x) = \mathbb{E}[Y \mid A = a, X = x]$ and the average over the marginal distribution μ_X of X
2. Plug in to estimate the ATE:

$$\hat{\psi}_n^{\text{g-formula}} = \tilde{\Psi}(\hat{f}_n, \hat{\mu}_X) = \int_{\mathbb{R}^d} \big(\hat{f}_n(1,x) - \hat{f}_n(0,x)\big) d\hat{\mu}_X(x)$$

IP-weighting

1. Estimate nuisance parameters $\pi(a \mid x) = P(A = a \mid X = x)$ and the average over the distribution P of O
2. Plug in to estimate the ATE:

$$\hat{\psi}_n^{\text{ipw}} = \tilde{\Psi}_{\text{ipw}}(\hat{\pi}_n, \hat{P}_n) = \int_{\mathbb{R}^d} \left(\frac{ay}{\hat{\pi}_n(a \mid x)} - \frac{(1-a)y}{\hat{\pi}_n(a \mid x)} \right) d\hat{P}_n(x)$$

One-step estimation

One-step

1. Estimate nuisance parameters
   $f(a,x) = \mathbb{E}[Y \mid A = a, X = x]$, $\pi(a \mid x) = \mathbb{E}[A \mid X = x]$
   and the average over the distribution P of O
2. Plug in to estimate the ATE:

$$\hat{\psi}_n^{\text{one}} = \tilde{\Psi}_{\text{one}}(\hat{f}_n, \hat{\pi}_n, \hat{P}_n) = \int_{\mathbb{R}^d} \sum_{a=0,1} \sum_{y=0,1} \Bigg\{ \left(\frac{a}{\hat{\pi}_n(a \mid x)} - \frac{1-a}{\hat{\pi}_n(a \mid x)} \right) \big(y - \hat{f}_n(a,x) \big) \\ + \hat{f}_n(1,x) - \hat{f}_n(0,x) \Bigg\} d\hat{P}_n(o)$$

G-formula versus IP-weighting

Estimation of the averages over μ_X and P is straightforward using the empirical average over the observed data.

This yields:

G-formula estimator: $$\hat{\psi}_n^{\text{g-formula}} = \frac{1}{n}\sum_{i=1}^{n}\left\{\hat{f}_n(1, X_i) - \hat{f}_n(0, X_i)\right\}$$

IP-weighted estimator: $$\hat{\psi}_n^{\text{ipw}} = \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{A_i Y_i}{\hat{\pi}_n(A_i \mid X_i)} - \frac{(1 - A_i) Y_i}{\hat{\pi}_n(A_i \mid X_i)}\right\}$$

One-step estimator: $$\hat{\psi}_n^{\text{one}} = \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{A_i}{\hat{\pi}_n(A_i \mid X_i)} - \frac{(1 - A_i)}{\hat{\pi}_n(A_i \mid X_i)}\big(Y_i - \hat{f}_n(A_i, X_i)\big) + \hat{f}_n(1, X_i) - \hat{f}_n(0, X_i)\right\}.$$

G-formula versus IP-weighting versus one-step/TMLE

Properties of the different estimators:

G-formula versus IP-weighting versus one-step/TMLE

Properties of the different estimators:

G-formula estimator requires estimator $\hat{f}_n$ for conditional expectation f.

- consistent if $\hat{f}_n$ is consistent.

IP-weighted estimator requires estimator $\hat{\pi}_n$ for the propensity score π.

- consistent if $\hat{\pi}_n$ is consistent.

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One-step estimator requires estimators $\hat{f}_n$ and $\hat{\pi}_n$ for conditional expectation f and propensity score π.

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- the one-step estimator and the TMLE estimator share the same large-sample properties.

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- the one-step estimator and the TMLE estimator share the same large-sample properties.
- one thing they have in common is an additional bias correction leads to consistency if either $\hat{f}_n$ or $\hat{\pi}_n$ is consistent (commonly known as double robustness).

G-formula versus IP-weighting versus TMLE

SMALL EXERCISE:
By the law of large numbers, the one-step estimator converges in probability to:

$$\mathbb{E}_{P_0}\left[\left(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)}\right)\big(Y - f(A,X)\big) + f(1,X) - f(0,X)\right] \qquad (1)$$

where (f,π) denotes the limit of $(\hat{f}_n, \hat{\pi}_n)$. Compute the right hand side of (1) when

1. $f = f_0$ (i.e., the outcome regression is consistently estimated), and
2. $\pi = \pi_0$ (i.e., the propensity score is consistently estimated).

G-formula versus IP-weighting versus TMLE

TMLE estimator requires estimators $\hat{f}_n$ and $\hat{\pi}_n$ for conditional expectation f and propensity score π.

- consistent if either $\hat{f}_n$ or $\hat{\pi}_n$ is consistent.

G-formula versus IP-weighting versus TMLE

TMLE estimator requires estimators $\hat{f}_n$ and $\hat{\pi}_n$ for conditional expectation f and propensity score π.

- consistent if either $\hat{f}_n$ or $\hat{\pi}_n$ is consistent.

Some other things we will see/explore:

- if $\hat{\pi}_n$ is modeled correctly, the TMLE estimator will have lower variance than the IP-weighted estimator.

G-formula versus IP-weighting versus TMLE

TMLE estimator requires estimators $\hat{f}_n$ and $\hat{\pi}_n$ for conditional expectation f and propensity score π.

- consistent if either $\hat{f}_n$ or $\hat{\pi}_n$ is consistent.

Some other things we will see/explore:

- if $\hat{\pi}_n$ is modeled correctly, the TMLE estimator will have lower variance than the IP-weighted estimator.
- the TMLE estimator may have larger variance than the g-formula estimator based on a correctly modeled $\hat{f}_n$ (but it gives protection against the case that it is not).

G-formula versus IP-weighting versus TMLE

Can't we just construct a good g-formula estimator???

G-formula versus IP-weighting versus TMLE

Can't we just construct a good g-formula estimator???

- a logistic regression — great if correctly specified, but horrible if not.
- a random forest — properly tuned?

A random forest — properly tuned?

Predictive performance of an estimator can be measured in terms of some distance[1] between:

1) the observed outcome: Y_i

2) and the predicted conditional expectation: $\hat{f}_n(A_i, X_i)$

[1] Measured in terms of a *loss function*.

A random forest — properly tuned?

Predictive performance of an estimator can be measured in terms of some distance[1] between:

1) the observed outcome:	Y_i
2) and the predicted conditional expectation:	$\hat{f}_n(A_i, X_i)$

One example of a loss function $\mathscr{L}(f)(O)$ is the negative log-likelihood loss:

$$\mathscr{L}(\hat{f}_n)(Y_i, A_i, X_i) = -(Y_i \log(\hat{f}_n(A_i, X_i)) + (1 - Y_i)\log(1 - \hat{f}_n(A_i, X_i))).$$

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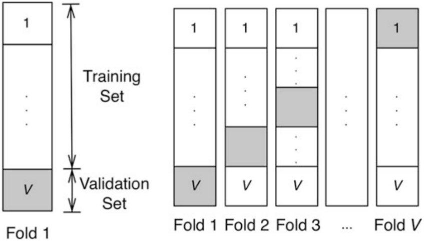
The estimator $\hat{f}_n$ closest to the true f_0 minimizes the risk:

$$\mathbb{E}_{P_0}[\mathscr{L}(\hat{f}_n)(Y_i, A_i, X_i)].$$

[1] Measured in terms of a *loss function*.

A random forest — properly tuned?



The risk can be estimated in a cross-validation scheme.[a]

I.e., for each sample split:

1. Each model is created and fitted on the training data: $\hat{f}_n^{\text{train}}$.
2. The quality of the model is checked on the validation data
   - Average of $\mathscr{L}(\hat{f}_n^{\text{train}})(O_i)$ in the validation sample.

[a] To measure performance on independent data.

A random forest — properly tuned?

Simulated example

- $X \sim \text{Unif}(-2, 2)$
- $X_1^{\text{noise}}, \ldots, X_5^{\text{noise}} \sim N(0, 1)$
- $A \in \{0, 1\}$ with distribution given X given by:

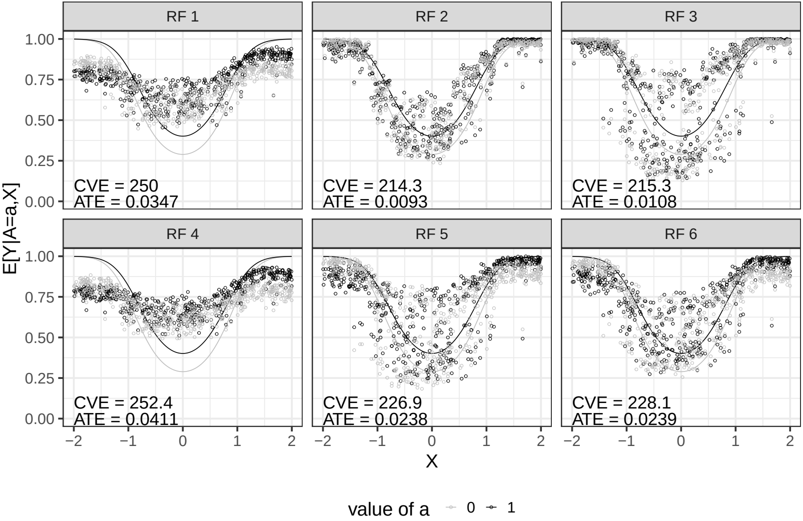
$$\operatorname{logit} \mathbb{E}[A \mid X] = \gamma_0 + \gamma_X^\top X$$

- $Y \in \{0, 1\}$ with distribution given X and A given by:

$$\operatorname{logit} \mathbb{E}[Y \mid A, X] = \beta_0 + \beta_A A + \beta_X^\top X^2$$

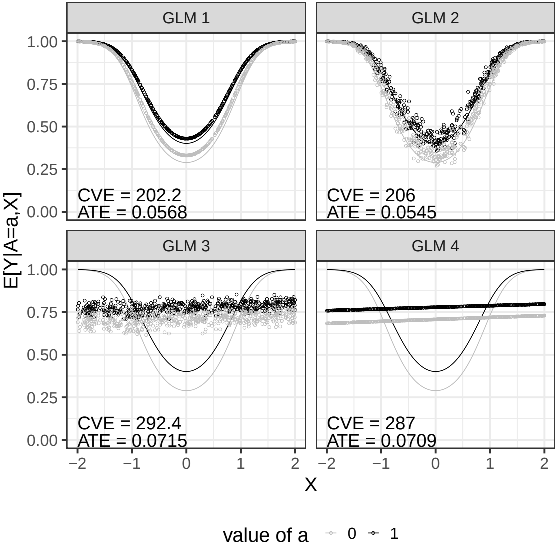
A random forest — properly tuned?

RF fitted with different values of tuning parameters (`nodesize`, `mtry`):



Different GLM models

GLM models fitted with different covariates and functional form of covariates:



A random forest — properly tuned?

This is all about constructing a good estimator for the conditional expectation f.

This does not necessarily translate into a good estimator for the target $\tilde{\Psi}(f, \mu_X)$.

A random forest — properly tuned?

This is all about constructing a good estimator for the conditional expectation f.

This does not necessarily translate into a good estimator for the target $\tilde{\Psi}(f, \mu_X)$.

TMLE is all about constructing a g-formula estimator which is a good estimator for *the target*.

Simulating simple data structure in R

Fix randomness:

```r
set.seed(5)
```

Fix a sample size:

```r
n <- 500
```

Generate covariate $X \in [-2, 2]$:

```r
X <- runif(n, -2, 2)
```

Generate binary treatment decision A:

```r
A <- rbinom(n, 1, prob=plogis(-0.25 + 1.2*X))
```

(corresponding to $\operatorname{logit} \mathbb{E}[A \mid X] = \gamma_0 + \gamma_X X$)

Simulating simple data structure in R

Generate binary outcome Y according to

$$\operatorname{logit} \mathbb{E}[\, Y \mid A, X \,] = \beta_0 + \beta_A A + \beta_X X^2$$

Simulating simple data structure in R

Generate binary outcome Y according to

$$\operatorname{logit} \mathbb{E}[\,Y \mid A, X\,] = \beta_0 + \beta_A A + \beta_X X^2$$

First generate counterfactuals:

```
Y1 <- rbinom(n, 1, prob=plogis(-0.9 + 1.9*X^2 + 0.5*1))
Y0 <- rbinom(n, 1, prob=plogis(-0.9 + 1.9*X^2 + 0.5*0))
```

Simulating simple data structure in R

Generate binary outcome Y according to

$$\operatorname{logit} \mathbb{E}[\, Y \mid A, X\,] = \beta_0 + \beta_A A + \beta_X X^2$$

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Y0 <- rbinom(n, 1, prob=plogis(-0.9 + 1.9*X^2 + 0.5*0))
```

We only observe the counterfactual outcome corresponding to the observed treatment level:

```r
Y <- A*Y1 + (1-A)*Y0
```

Simulating simple data structure in R

Observed data:

```
             X A Y
  1: -1.1991422 0 0
  2:  0.7408744 1 0
  3:  1.6675031 1 1
  4: -0.8624022 0 1
  5: -1.5813995 0 1
 ---
496: -0.3978523 1 0
497: -1.5069379 0 1
498:  1.8340120 1 1
499:  0.6349484 1 1
500: -0.5214807 0 1
```

Simulating simple data structure in R

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              X A Y
  1: -1.1991422 0 0
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498:  1.8340120 1 1
499:  0.6349484 1 1
500: -0.5214807 0 1
```

Counterfactual data:

```
              X Y1 Y0
  1: -1.1991422  0  1
  2:  0.7408744  1  0
  3:  1.6675031  1  1
  4: -0.8624022  0  1
  5: -1.5813995  1  1
 ---
496: -0.3978523  0  1
497: -1.5069379  0  1
498:  1.8340120  1  1
499:  0.6349484  0  0
500: -0.5214807  0  0
```

Simulating simple data structure in R

Simulating many observations of counterfactuals allows us to approximate the true ATE:

```
X <- runif(1e6, -2, 2)
Y1 <- rbinom(1e6, 1, prob=plogis(-0.9 + 1.9*X^2 + 0.5*1))
Y0 <- rbinom(1e6, 1, prob=plogis(-0.9 + 1.9*X^2 + 0.5*0))
```

The true ATE is then approximately:

```
(true.ate <- mean(Y1 - Y0))
```

```
[1] 0.070292
```

since ATE $= \mathbb{E}_{P_0}[Y^1] - \mathbb{E}_{P_0}[Y^0]$.

Simulating simple data structure in R

Fit correctly specified parametric model:

```r
fit.glm <- glm(Y~A+X.squared, data=dt[, X.squared:=X^2],
    family=binomial)
```

Use model to estimate $f(1, X)$ for all subjects:

```r
dt[, pred.glm.A1:=predict(fit.glm, type="response", newdata=
    copy(dt)[, A:=1])]
```

And similarly $f(0, X)$ for all subjects:

```r
dt[, pred.glm.A0:=predict(fit.glm, type="response", newdata=
    copy(dt)[, A:=0])]
```

Then we can estimate the ATE by:

```r
(fit.glm <- dt[, mean(pred.glm.A1-pred.glm.A0)])
```

```
[1] 0.04322891
```

Simulating simple data structure in R

Using a random forest (no tuning):

```r
library(randomForestSRC)
fit.rf <- rfsrc(Y~A+X, data=dt)
dt[, pred.rf.A1:=predict(fit.rf, type="response", newdata=
    copy(dt)[, A:=1])$predicted]
dt[, pred.rf.A0:=predict(fit.rf, type="response", newdata=
    copy(dt)[, A:=0])$predicted]
(fit.rf <- dt[, mean(pred.rf.A1-pred.rf.A0)])
```

```
[1] 0.07005063
```

Simulating simple data structure in R

Using a misspecified parametric model:

```r
fit.glm.mis <- glm(Y~A+X, data=dt, family=binomial)
dt[, pred.glm.mis.A1:=predict(fit.glm.mis, type="response",
    newdata=copy(dt)[, A:=1])]
dt[, pred.glm.mis.A0:=predict(fit.glm.mis, type="response",
    newdata=copy(dt)[, A:=0])]
(fit.glm.mis <- dt[, mean(pred.glm.mis.A1-pred.glm.mis.A0)])
```

```
[1] 0.09127889
```

Simulating simple data structure in R

We can investigate the properties of different estimators —

- We know the true value of ATE: $\psi_0 \approx 0.0702$
- We have generated the outcome Y according to

$$\operatorname{logit} \mathbb{E}[\, Y \mid A, X \,] = \beta_0 + \beta_A A + \beta_X X^2$$

- We have generated the treatment A according to

$$\operatorname{logit} \mathbb{E}[\, A \mid X \,] = \gamma_0 + \gamma_X X$$

If we repeat the experiment of drawing n observations we would every time end up with a different realization of the particular estimator.

Different estimators

G-formula estimator Using an estimator $\hat{f}_n$ for $f(a, X) = \mathbb{E}[Y \mid A = a, X]$, estimate the ATE by:

$$\hat{\psi}_n^{\text{g–formula}} = \frac{1}{n}\sum_{i=1}^{n}\left\{\hat{f}_n(1, X_i) - \hat{f}_n(0, X_i)\right\}$$

Inverse probability weighted estimator Using an estimator $\hat{\pi}_n$ for $\pi(a \mid X) = P(A = a \mid X)$, estimate the ATE by:

$$\hat{\psi}_n^{\text{ipw}} = \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{A_i Y_i}{\hat{\pi}_n(A_i \mid X_i)} - \frac{(1 - A_i) Y_i}{\hat{\pi}_n(A_i \mid X_i)}\right\}$$

TMLE estimator Update the estimator $\hat{f}_n \mapsto \hat{f}_n^*$ in a "targeted way" using the information from the estimator $\hat{\pi}_n$, then estimate the ATE by:

$$\hat{\psi}_n^{\text{tmle}} = \frac{1}{n}\sum_{i=1}^{n}\left\{\hat{f}_n^*(1, X_i) - \hat{f}_n^*(0, X_i)\right\}$$

Different estimators — `tmle` implementation

Today we will just (more or less blindly) use software to estimate TMLE.

Different estimators — `tmle` implementation

Today we will just (more or less blindly) use software to estimate TMLE.

```r
library(tmle)
```

```r
tmle(Y, A, X,
     gform,
     Qform,
     SL.library,
     family="binomial",
     cvQinit=FALSE,
     ...
     )
```

- $Y \in \mathbb{R}$ or $Y \in \{0, 1\}$
- $A \in \{0, 1\}$
- X a vector, matrix or a data frame

Different estimators — `tmle` implementation

- `gform`
  - optional regression formula for the propensity score π
  - on the form `A~X1+X2`
  - (overrides call to `SuperLearner`)

- `Qform`
  - optional regression formula for the conditional expectation f
  - on the form `Y~X1+X2`
  - (overrides call to `SuperLearner`)

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  - (overrides call to `SuperLearner`)

- `cvQinit=FALSE`
  - default is `TRUE` which means cross-validated predicted values are estimated

- `gbound`
  - truncation of predicted probabilities of treatment

Different estimators — `tmle` implementation

On a sidenote — tomorrow

- `Q.SL.library`
  - optional vector of prediction algorithms to use for `SuperLearner` in initial estimation of f
- `g.SL.library`
  - optional vector of prediction algorithms to use for `SuperLearner` in initial estimation of π
- `Q.discreteSL`
  - if `TRUE`, a discrete super learner is used (rather than ensemble)
  - default is `FALSE`
- `g.discreteSL`
  - if `TRUE`, a discrete super learner is used (rather than ensemble)
  - default is `FALSE`

Note: The discrete super learner simply picks an algorithm from its library by minimizing the cross-validated empirical risk with respect a loss function.

Different estimators — `tmle` implementation

What were the estimated IP weights?

```
summary(fit.tmle$g$g1W)
```

```
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
0.04751 0.19441 0.49405 0.49400 0.79710 0.94109
```

Note that weights close to 0 or to 1 would indicate positivity issues.

Different estimators — tmle implementation

What were the estimated IP weights?

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summary(fit.tmle$g$g1W)
```

```
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
0.04751 0.19441 0.49405 0.49400 0.79710 0.94109
```

Note that weights close to 0 or to 1 would indicate positivity issues.

What truncation level was used?

```
fit.tmle$gbound
```

```
[1] 0.03598084 1.00000000
```

I.e., no weights were truncated.

Explorations based on simulated data

As part of the exercise we want to explore —

1. Comparing g-formula estimators for different estimators for f; either different logistic regressions or different machine learning algorithms.
2. Properties of the g-formula estimator and the IP-weighted estimator, compared to the TMLE estimator.
3. Double robustness: Misspecification of the outcome regression (f).

The exercise is described in detail in: **day1-practical1.pdf**.

Some relevant concepts linking what we have seen by now to the rest of today + tomorrow.

Some relevant concepts of asymptotic theory

A very desirable property —

[2] $o_P(1)$ denotes a sequence which is converges to zero in probability.

Some relevant concepts of asymptotic theory

The empirical measure $\mathbb{P}_n$ of the sample $O_1, \ldots, O_n$:

$$\mathbb{P}_n h = \int h d\mathbb{P}_n = \frac{1}{n}\sum_{i=1}^{n} h(O_i).$$

A very desirable property —

An estimator $\hat{\psi}_n$ is $\sqrt{n}$-consistent and asymptotically linear with influence function $\phi(P_0)(O)$ if [2]

$$\sqrt{n}\Big(\hat{\psi}_n - \psi_0\Big) = \sqrt{n}\,\mathbb{P}_n\phi(P_0) + o_P(1),$$

where $\mathbb{E}_{P_0}[\phi(P_0)(O)] = 0$ and $\mathbb{E}_{P_0}[\{\phi(P_0)(O)\}^2] < \infty$.

[2] $o_P(1)$ denotes a sequence which is converges to zero in probability.

Some relevant concepts of asymptotic theory

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$$\sqrt{n}\left(\hat{\psi}_n - \psi_0\right) = \sqrt{n}\,\mathbb{P}_n \phi(P_0) + o_P(1),$$

where $\mathbb{E}_{P_0}[\phi(P_0)(O)] = 0$ and $\mathbb{E}_{P_0}[\{\phi(P_0)(O)\}^2] < \infty$.

Then CLT + Slutsky implies:

$$\hat{\psi}_n \overset{as}{\sim} N(\Psi(P_0), \mathrm{Var}(\phi(P_0))/n).$$

The estimator behaves asymptotically as an average of the influence function.

[2] $o_P(1)$ denotes a sequence which is converges to zero in probability.

Some relevant concepts of asymptotic theory

Simple example: Estimator for the mean $\psi_0 = \mathbb{E}[X]$:

$$\hat{\psi}_{n,0} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Then

$$\sqrt{n}(\hat{\psi}_n - \psi_0) = \sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\underbrace{(X_i - \psi_0)}_{=\phi(P_0)(O_i)} = \sqrt{n}\mathbb{P}_n\phi(P_0)$$

$\hat{\psi}_{n,0}$ is linear and thus asymptotically linear.

Some relevant concepts of asymptotic theory

Simple example: Estimator for the mean $\psi_0 = \mathbb{E}[X]$:

$$\hat{\psi}_{n,1} = \frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{n}$$

Then

$$\sqrt{n}(\hat{\psi}_n - \psi_0) = \sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\underbrace{(X_i - \psi_0)}_{=\phi(P_0)(O_i)} + \frac{\sqrt{n}}{n} = \sqrt{n}\mathbb{P}_n\phi(P_0) + \underbrace{\frac{1}{\sqrt{n}}}_{=o_P(1)}$$

$\hat{\psi}_{n,1}$ is asymptotically linear.

Some relevant concepts of asymptotic theory

Simple example: Estimator for the mean $\psi_0 = \mathbb{E}[X]$:

$$\hat{\psi}_{n,2} = \frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{n^{1/2+0.1}}$$

Then

$$\sqrt{n}(\hat{\psi}_n - \psi_0) = \sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\underbrace{(X_i - \psi_0)}_{=\phi(P_0)(O_i)} + \frac{\sqrt{n}}{n^{1/2+0.1}} = \sqrt{n}\mathbb{P}_n\phi(P_0) + \underbrace{\frac{1}{n^{0.1}}}_{=o_P(1)}$$

$\hat{\psi}_{n,2}$ is asymptotically linear.

Some relevant concepts of asymptotic theory

Simple example: Estimator for the mean $\psi_0 = \mathbb{E}[X]$:

$$\hat{\psi}_{n,3} = \frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{n^{1/2-0.1}}$$

Then

$$\sqrt{n}(\hat{\psi}_n - \psi_0) = \sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\underbrace{(X_i - \psi_0)}_{=\phi(P_0)(O_i)} + \frac{\sqrt{n}}{n^{1/2-0.1}} = \sqrt{n}\mathbb{P}_n\phi(P_0) + \underbrace{n^{0.1}}_{\to\infty}$$

$\hat{\psi}_{n,3}$ is **not** asymptotically linear.

Some relevant concepts of asymptotic theory

An estimator $\hat{\psi}_n$ has rate of convergence $r_n \to \infty$ if [3]

$$r_n(\hat{\psi}_n - \psi_0) = O_P(1), \quad \text{i.e.,} \quad \hat{\psi}_n - \psi_0 = O_P(1/r_n).$$

The convergence rate r_n tells us how fast $\hat{\psi}_n$ centers around ψ_0, with the difference $\hat{\psi}_n - \psi_0$ behaving like $1/r_n$.

- One wants negligible bias such as to obtain reliable confidence intervals for ψ_0.
- The bias of an asymptotically linear estimator converges to zero at a rate faster the $1/\sqrt{n}$.

Data-adaptive machine learning estimators rarely achieve this rate.

[3] $O_P(1)$ denotes a sequence which is bounded in probability.

Some relevant concepts of asymptotic theory

$$\sqrt{n}\hat{\psi}_{n,1} = \sqrt{n}\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}_{\overset{P}{\to}\psi_0} + \underbrace{\frac{\sqrt{n}}{n}}_{\to 0}, \qquad \text{i.e.,} \quad \sqrt{n}(\hat{\psi}_{n,1} - \psi_0) = o_P(1).$$

$$\sqrt{n}\hat{\psi}_{n,2} = \sqrt{n}\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}_{\overset{P}{\to}\psi_0} + \underbrace{\frac{\sqrt{n}}{n^{1/2+0.1}}}_{\to 0}, \quad \text{i.e.,} \quad \sqrt{n}(\hat{\psi}_{n,3} - \psi_0) = o_P(1).$$

$$\sqrt{n}\hat{\psi}_{n,3} = \sqrt{n}\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}_{\overset{P}{\to}\psi_0} + \underbrace{\frac{\sqrt{n}}{n^{1/2-0.1}}}_{\to\infty}, \qquad \text{i.e.,} \quad \sqrt{n}(\hat{\psi}_{n,3} - \psi_0) \overset{P}{\to} \infty.$$

Some relevant concepts of asymptotic theory

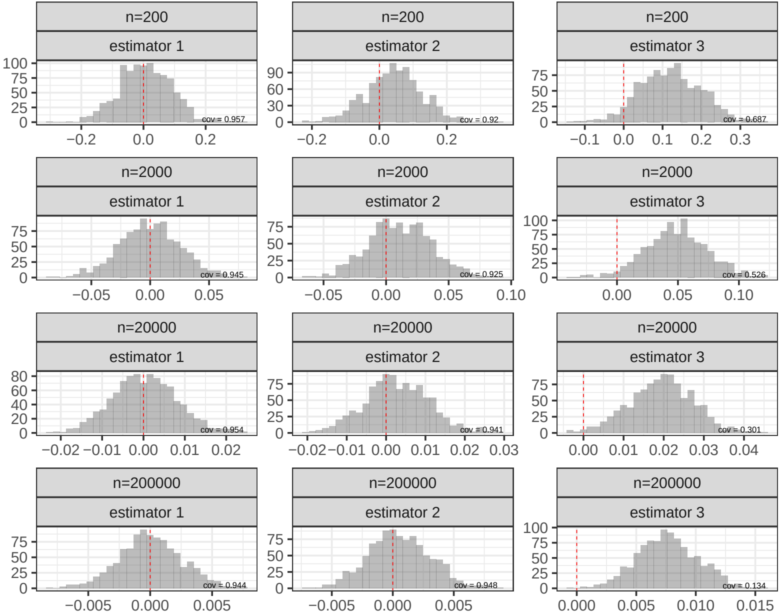
$$\sqrt{n}\hat{\psi}_{n,1} = \sqrt{n}\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}_{\overset{P}{\to}\psi_0} + \underbrace{\frac{\sqrt{n}}{n}}_{\to 0}, \qquad \text{i.e.,} \quad \sqrt{n}(\hat{\psi}_{n,1} - \psi_0) = o_P(1).$$

$$\sqrt{n}\hat{\psi}_{n,2} = \sqrt{n}\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}_{\overset{P}{\to}\psi_0} + \underbrace{\frac{\sqrt{n}}{n^{1/2+0.1}}}_{\to 0}, \quad \text{i.e.,} \quad \sqrt{n}(\hat{\psi}_{n,3} - \psi_0) = o_P(1).$$

$$\sqrt{n}\hat{\psi}_{n,3} = \sqrt{n}\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}_{\overset{P}{\to}\psi_0} + \underbrace{\frac{\sqrt{n}}{n^{1/2-0.1}}}_{\to \infty}, \qquad \text{i.e.,} \quad \sqrt{n}(\hat{\psi}_{n,3} - \psi_0) \overset{P}{\to} \infty.$$

[The remainder term that determines the asymptotic bias the estimator].

Some relevant concepts of asymptotic theory



Some relevant concepts of asymptotic theory

A key component in constructing a $\sqrt{n}$-consistent and asymptotically linear estimator, *even when using machine learning estimation*, is the so-called the efficient influence function (also known as the canonical gradient).

Some relevant concepts of asymptotic theory

A key component in constructing a $\sqrt{n}$-consistent and asymptotically linear estimator, *even when using machine learning estimation*, is the so-called the efficient influence function (also known as the canonical gradient).

- The efficient influence function provides a nonparametric lower bound for the estimation problem.
- Tells us how to do bias-correction.
- With the bias-correction, the remainder term that we need to control to have $\sqrt{n}$-consistency and asymptotic linearity admits a nice structure that we *can* control.

Rest of today + tomorrow (TMLE).