Day 1, Lecture 2

Properly defining the target parameter

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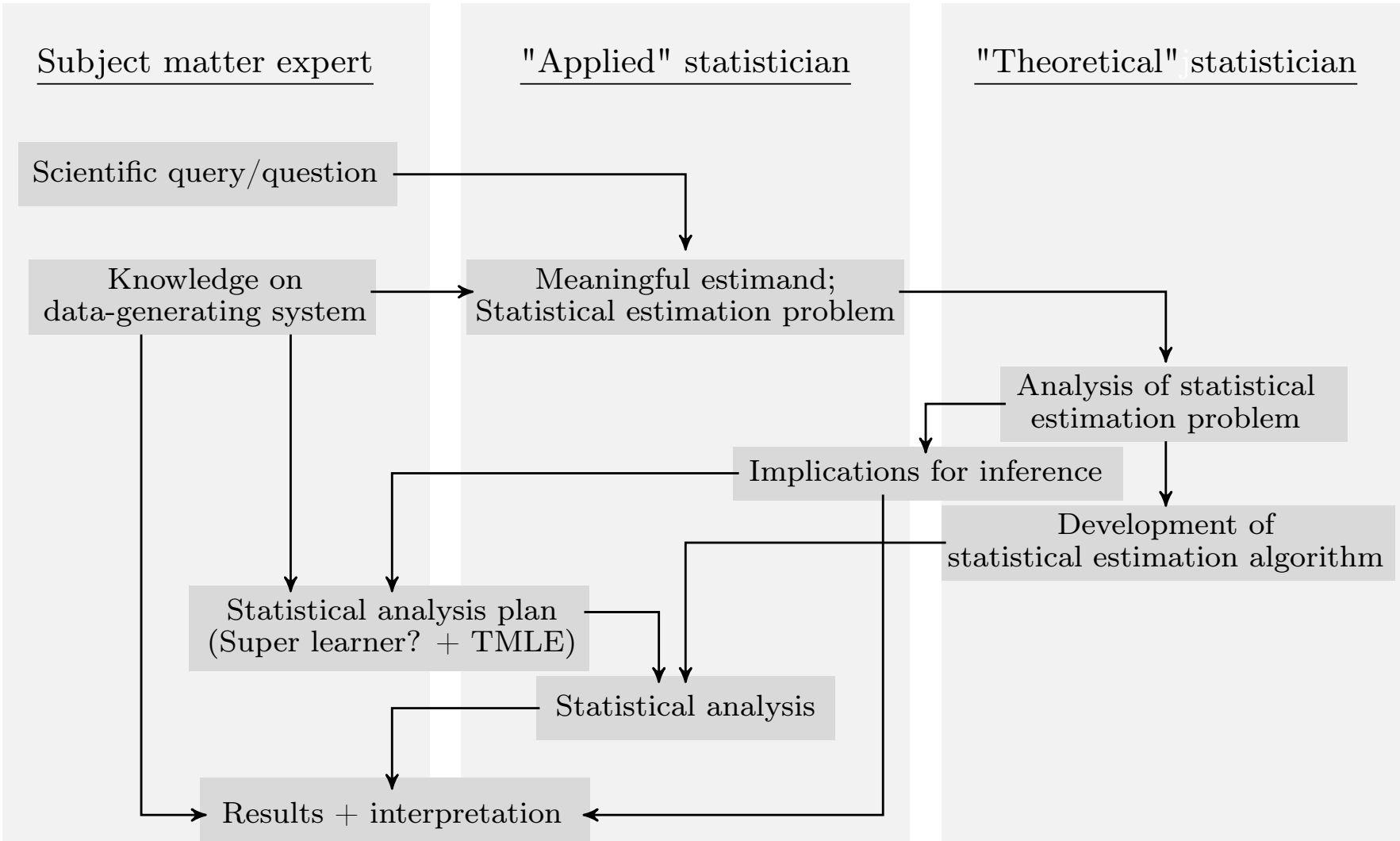

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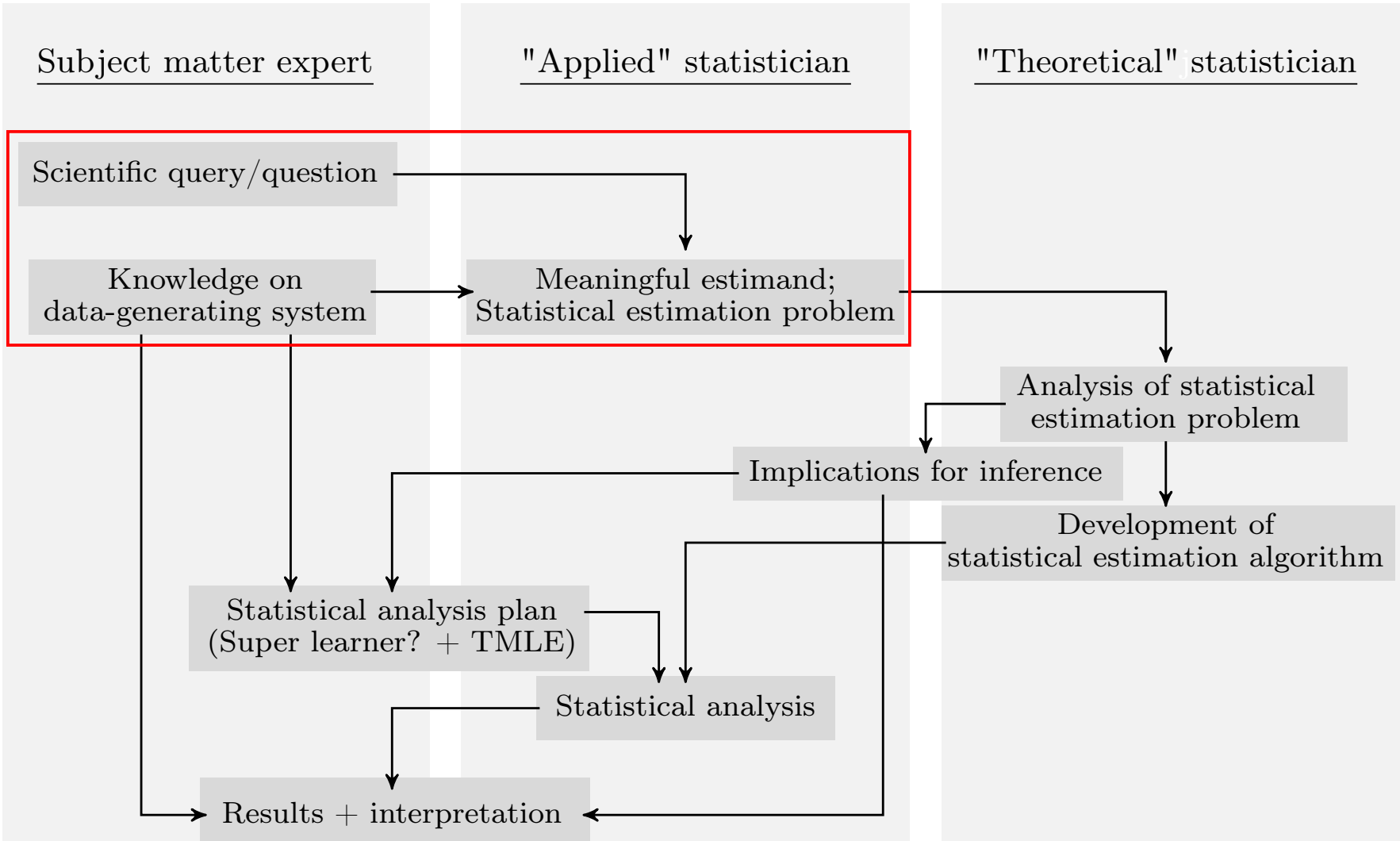

Properly defining the target parameter

- A clearly defined goal as a starting-point for any analysis
  - necessary to talk about estimator performance
  - semiparametric/nonparametric efficiency theory (and TMLE) requires a clearly defined goal
- Brief introduction to the setting of a typical causal inference problem
  - example: average treatment effect
  - *model-free* and *estimator-free* definition of parameters

Moving targets with different logistic regression models

- $X \sim \text{Unif}(-2, 2)$
- $A \sim \text{Bernouilli}(0.5)$ (no confounding)
- $Y \in \{0, 1\}$

Say that the distribution of Y given X and A follows the parametric model:

$$\text{logit}\, \mathbb{E}[Y \mid A, X] = \beta_0 + \beta_A A + \beta_X^\top X^2$$

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Say that the distribution of Y given X and A follows the parametric model:

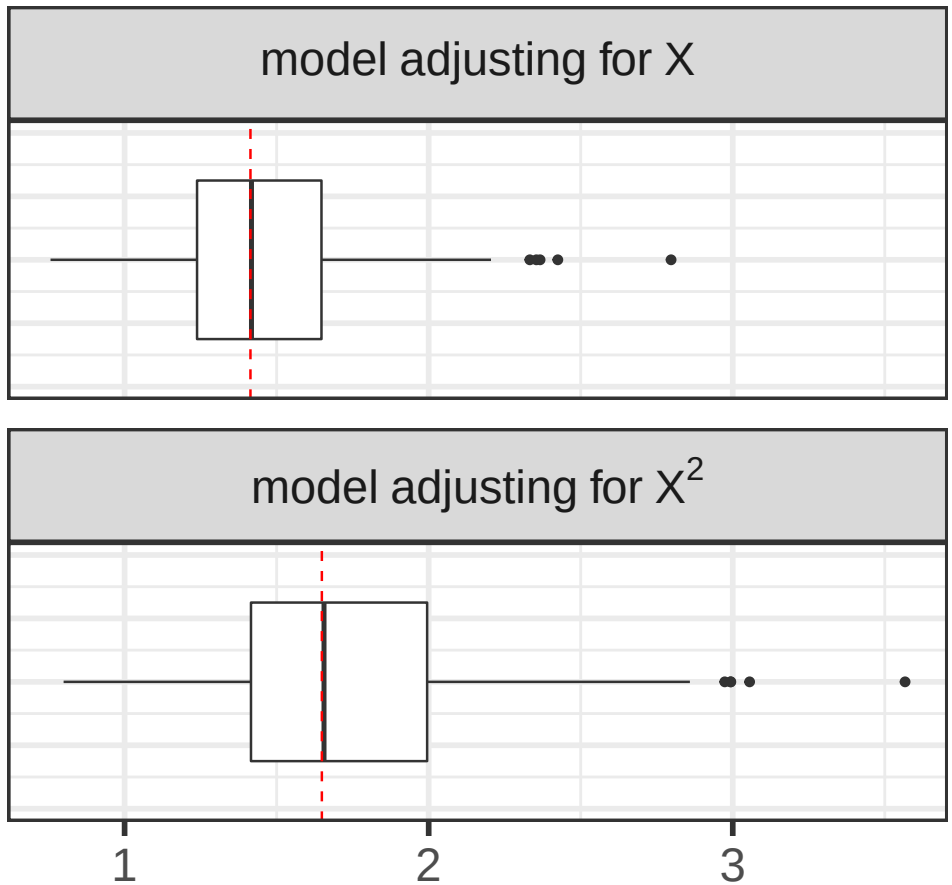
$$\text{logit}\, \mathbb{E}[Y \mid A, X] = \beta_0 + \beta_A A + \beta_X^\top X^2$$

The odds ratio $\exp(\beta_A)$ is a different parameter than $\exp(\alpha_A)$ in a different model:

$$\text{logit}\, \mathbb{E}[Y \mid A, X] = \alpha_0 + \alpha_A A + \alpha_X^\top X$$

Moving targets with different logistic regression models

- The variables X we include in the model to assess the effect of A on Y changes the parameter (conditional OR).
- Only one of the models can be true at a time.



The upper panel does not show a biased estimator, just an estimator targeting a different parameter (dashed red line).

Causal inference

What we obtain moving on to a causal inference setting: 1) An interpretable and relevant target of estimation, and 2) a model-free definition of a target parameter.

[1] And, if you are already familiar, consider this a small repetition and introduction to the notation.

Causal inference

What we obtain moving on to a causal inference setting: 1) An interpretable and relevant target of estimation, and 2) a model-free definition of a target parameter.

- We are only going to go briefly over the "causal inference concepts",[1] but we need this part to very clear about with it is we are estimating.
  - For today and tomorrow we consider just the simple example where the target of estimation is the average treatment effect (ATE).
- For the causal inference notation, we follow the book by Hernán and Robins (which, if you are interested, you can find here: `https://cdn1.sph.harvard.edu/wp-content/uploads/sites/1268/2021/03/ciwhatif_hernanrobins_30mar21.pdf`).
  - I will leave out DAGs/SCMs.

[1] And, if you are already familiar, consider this a small repetition and introduction to the notation.

Steps of the roadmap

Step 1 Go from scientific question to target causal estimand (stated in the language of counterfactuals)

Step 2 Assess whether we can go from target causal estimand to target statistical estimand = assess "identifiability"

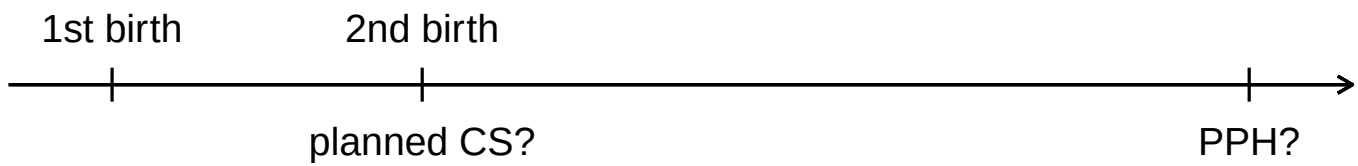
In a given data situation, we want to explicitly clarify:

1. Observed data
2. Causal model
3. Causal question and target causal estimand
4. Identifiability

An example we can have in the back of our minds

Scientific question:

Does having a planned cesarian section (intended cesarian section) among women who gave birth twice change the risk of postpartum haemorrhage (PPH) during the second delivery?



Goal: Translate this is into a precise formulation of a statistical estimation problem.

Observed data

Observed data $O = (X, A, Y) \in \mathbb{R}^d \times \{0,1\} \times \{0,1\} = \mathcal{O}$

* $X \in \mathbb{R}^d$ are covariates
  ex: age at 2nd delivery, information of PPH at first delivery, ...
* $A \in \{0,1\}$ is a binary exposure variable (treatment decision)
  ex: decision to have a planned cesarian section.
* $Y \in \{0,1\}$ is a binary outcome variable
  ex: PPH (postpartum haemorrhage).

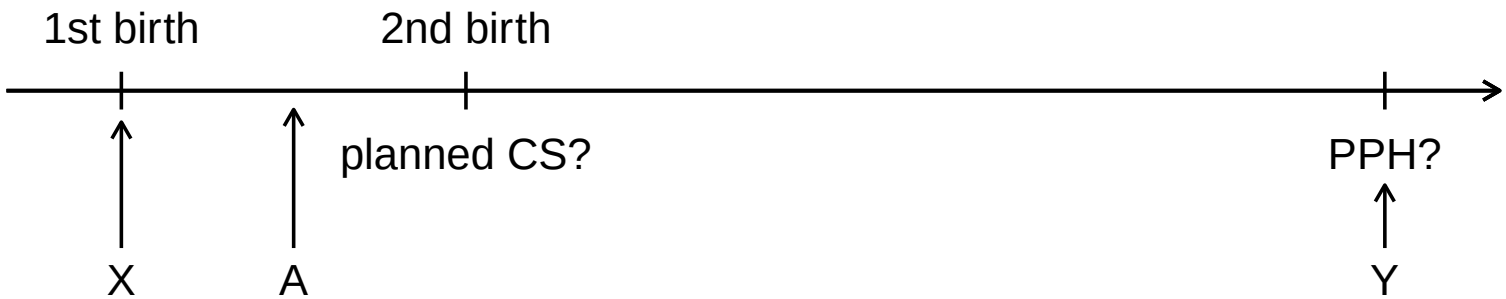
We observe a sample $O_1, \ldots, O_n \overset{iid}{\sim} P_0 \in \mathcal{M}$, $n \in \mathbb{N}$.

$\mathcal{M}$ is the set of all possible probability distributions for our data.

Observed data

Implicit assumptions for the data structure:[2]

- X are covariates known before the treatment decision A was made
- Outcome Y is observed after treatment decision was made



[2]This ordering could also be encoded in a structural causal model.

Observed data

Our statistical model $\mathcal{M}$ for P_0 contains possible distributions P for the observed data O.

The density p of $P \in \mathcal{M}$ can be factorized into:

$$p(o) = \mu_Y(y \mid a, x)\pi(a \mid x)\mu_X(x),$$

- $\mu_Y(y \mid A, X) = P(Y = y \mid A, X)$
- $\pi(a \mid X) = P(A = a \mid X)$
- μ_X is the marginal density of X (with respect to an appropriate dominating measure)

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We assume that $\mathcal{M}$ is a nonparametric model.

* Throughout, we make *no parametric restrictions on* μ_Y, μ_X.
* We could impose some parametric structure on π, but let us assume that we do not.

Operators on functions of the observed data[3]

For a function $h : \mathcal{O} \to \mathbb{R}$ and distribution P

$$Ph = \mathbb{E}_P[h(O)] = \int h dP = \int_{\mathcal{O}} h(o) dP(o)$$

where $\mathcal{O} = \mathbb{R}^d \times \{0, 1\} \times \{0, 1\}$ is the sample space of $O = (X, A, Y)$.

[3] van der Vaart, A. W. (2000). Asymptotic statistics (Vol. 3). Cambridge university press.

Confounding

How can we define a causal effect?

The contrast $\mathbb{E}_P[Y \mid A = 1] - \mathbb{E}_P[Y \mid A = 0]$ tells us about the risk difference in the two exposure groups.

Any such difference is likely due to other factors than the decision to initiate treatment or not

* the exposure decision is confounded.

Counterfactuals

To answer a causal question, we ideally want to know

Scenario 1 What happened to a subject had they been exposed?

Scenario 2 What would have happened to the same subject had they not been exposed?

We imagine a model with two outcomes for each subject:

- a variable Y^1 corresponding to scenario 1, and
- a variable Y^0 corresponding to scenario 2

= the "counterfactuals" (aka potential outcomes).

Counterfactuals

* Y^1 = outcome if exposed
* Y^0 = outcome if not exposed

We use the counterfactual outcomes to define precisely what a causal effect is:

→ on the individual level, $Y^1 = 1$ and $Y^0 = 0$ for a particular subject would tell us that this subject would experience outcome under exposure and not otherwise

→ on the population level, $\mathbb{E}_P[Y^1] \neq \mathbb{E}_P[Y^0]$ tells us that the risk changes depending on whether exposed or not

Target causal estimand: Average causal effect (ATE)

The average causal effect[4] (ATE/ACE) measures the average effect in the population

$$\text{ATE} = \mathbb{E}_P[Y^1] - \mathbb{E}_P[Y^0]$$

→ It is interpreted as the difference in risk had everyone in the population been exposed and had everyone in the population been unexposed.

[4] or average treatment effect.

Identifiability (estimating the causal effect from observational data)

Can we estimate the causal effect from the observed data?

- ▷ only Y^1 or Y^0 is observed for each individual.

Identifying $\mathbb{E}_P[Y^1] - \mathbb{E}_P[Y^0]$

- = write $\mathbb{E}_P[Y^1] - \mathbb{E}_P[Y^0]$ as a parameter of the observed data distribution.

requires three overall assumptions (identifiability assumptions).

Identifiability (estimating the causal effect from observational data)

1. Consistency: $Y^a = Y$ if $A = a$, $a = 0, 1$

- Requires that the "treatment intervention" is well-defined and no interference between subjects.
- Example of a violation: effect of vaccines (one subject's effect of a vaccine depends on whether other subjects are vaccinated or not).

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2. Exchangeability: $Y^a \perp\!\!\!\perp A \mid X$, for $a = 0, 1$

- Conditional on covariates, the exposed group tells us what would happen to the unexposed if they had been exposed and vice versa.
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- Requires that there is no unmeasured confounding.

3. Positivity: $P(A = a \mid X) > 0$ for $a = 0, 1$ and almost surely all X

- We cannot investigate the effect of an intervention that was never "tested" in the observed data (conditional on covariates X).

Identifiability (estimating the causal effect from observational data)

Under these assumptions:

$$
\begin{aligned}
&\mathbb{E}_P[Y^1] - \mathbb{E}_P[Y^0] \\
&\quad = \mathbb{E}_P\Big[\mathbb{E}_P[Y^1 \mid X] - \mathbb{E}_P[Y^0 \mid X]\Big] \\
&\quad \overset{2.}{=} \mathbb{E}_P\Big[\mathbb{E}_P[Y^1 \mid A=1, X] - \mathbb{E}_P[Y^0 \mid A=0, X]\Big] \\
&\quad \overset{1.}{=} \mathbb{E}_P\Big[\mathbb{E}_P[Y \mid A=1, X] - \mathbb{E}_P[Y \mid A=0, X]\Big] \\
&\quad = \Psi(P)
\end{aligned}
$$

(3. (positivity) ensures that the conditional expectations are well-defined).

Goal achieved: Right hand side is expressed only in terms of observable quantities.

Identifiability (estimating the causal effect from observational data)

Under the assumptions:

$$\mathbb{E}_P[Y^1] - \mathbb{E}_P[Y^0] = \underbrace{\mathbb{E}_P\Big[\mathbb{E}_P[Y \mid A=1, X] - \mathbb{E}_P[Y \mid A=0, X]\Big]}_{(*)} = \Psi(P),$$

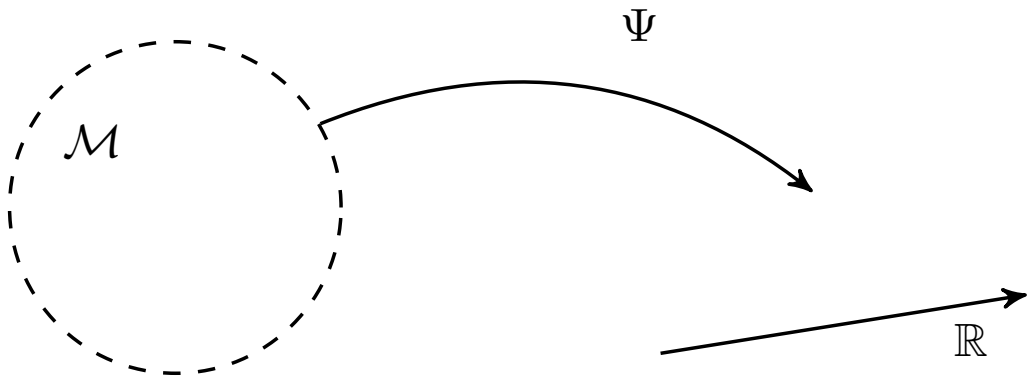
for any $P \in \mathcal{M}$.

In our statistical analysis, **we proceed with** $(*)$.

"Causal inference part" is over.

Target statistical estimand

Now we are exactly in the situation we wanted:



Average treatment effect (ATE)

- $O = (X, A, Y) \in \mathbb{R}^d \times \{0, 1\} \times \{0, 1\}$
- The ATE is defined for $P \in \mathcal{M}$ as

$$\Psi(P) = \mathbb{E}_P[\mathbb{E}_P[Y \mid A = 1, X] - \mathbb{E}_P[Y \mid A = 0, X]]$$

Target statistical estimand: g-formula

We can write the target parameter as:

$$\begin{aligned}\Psi(P) &= \mathbb{E}_P[\mathbb{E}_P[Y \mid A=1, X] - \mathbb{E}_P[Y \mid A=0, X]] \\ &= \mathbb{E}_P[f(1,X) - f(0,X)] \\ &= \int_{\mathbb{R}^d} \big(f(1,x) - f(0,x)\big) d\mu_X(x) = \tilde{\Psi}(f, \mu_X) \qquad (*)\end{aligned}$$

where

$$f(a,x) = \mathbb{E}[Y \mid A=a, X=x]$$

and μ_X is the marginal distribution of X.

We refer to this as the g-formula.

Target statistical estimand: IP-weighting

We can also rewrite the target parameter as:

$$
\begin{aligned}
\Psi(P) &= \int_{\mathbb{R}^d} \Big(f(1,x) - f(0,x)\Big) d\mu_X(x) && (*) \\
&= \int_{\mathbb{R}^d} \sum_{y=0,1} y\Big(\mu_Y(y \mid 1,x) - \mu_Y(y \mid 0,x)\Big) d\mu_X(x) \\
&= \int_{\mathbb{R}^d} \sum_{y=0,1} \sum_{a=0,1} y\Big(a\mu_Y(y \mid a,x) - (1-a)\mu_Y(y \mid a,x)\Big) d\mu_X(x) \\
&= \int_{\mathbb{R}^d} \sum_{y=0,1} \sum_{a=0,1} \left(\frac{ay}{\pi(a \mid x)} - \frac{(1-a)y}{\pi(a \mid x)}\right) \mu_Y(y \mid a,x)\pi(a \mid x) d\mu_X(x) \\
&= \tilde{\Psi}_{\text{ipw}}(\pi, p) \quad (**)
\end{aligned}
$$

where $\pi(a \mid x) = P(A = 1 \mid X = x)$.

Target statistical estimand

The g-formula:

$$\begin{aligned}\tilde{\Psi}(f, \mu_X) &= \int_{\mathbb{R}^d} \big(f(1,x) - f(0,x)\big) d\mu_X(x) \\ &= \mathbb{E}_P\big[f(1,X) - f(0,X)\big]. \qquad (*)\end{aligned}$$

The IP-weighted formula:

$$\begin{aligned}\tilde{\Psi}_{\text{ipw}}(\pi, p) &= \int_{\mathbb{R}^d} \sum_{a=0,1} \sum_{y=0,1} \left(\frac{ay}{\pi(a \mid x)} - \frac{(1-a)y}{\pi(a \mid x)} \right) dP(o) \\ &= \mathbb{E}_P\left[\frac{AY}{\pi(1 \mid X)} - \frac{(1-A)Y}{\pi(0 \mid X)} \right] \qquad (**)\end{aligned}$$

- f and (the average over) μ_X are nuisance parameters for the g-formula.
- π and (the average over) p are nuisance parameters for the IP-weighted formula.

Target statistical estimand

Yet another representation of the target parameter is

$$\tilde{\Psi}_{\mathrm{ee}}(f,\pi,p) = \int_{\mathbb{R}^d} \sum_{a=0,1} \sum_{y=0,1} \left\{ \left(\frac{a}{\pi(a \mid x)} - \frac{1-a}{\pi(a \mid x)} \right) \big(y - f(a,x)\big) + f(1,x) - f(0,x) \right\} p_Y(y \mid a,x)\pi(a \mid x) d\mu_X(x)$$

$$= \mathbb{E}_P\left[\left(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)} \right) \big(Y - f(A,X)\big) + f(1,X) - f(0,X) \right]$$

- f, π and (the average over) p are nuisance parameters for this parametrization.

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$$= \mathbb{E}_P \Big[\Big(\frac{A}{\pi(A \mid X)} - \frac{1-A}{\pi(A \mid X)} \Big) \big(Y - f(A,X) \big) + f(1,X) - f(0,X) \Big]$$

- f, π and (the average over) p are nuisance parameters for this parametrization.

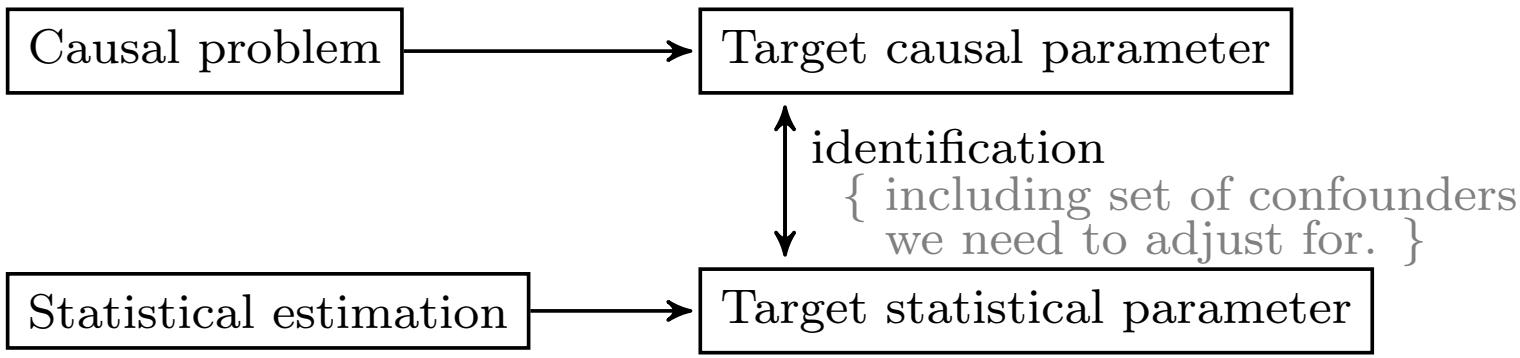
SMALL EXERCISE:

1. Note that $\tilde{\Psi}_{\text{ee}}(f,\pi,p) = \tilde{\Psi}(f,\mu_X) + [\text{an extra term}]_1$. Show that it can also be written as $\Psi_{\text{ee}}(f,\pi,p) = \tilde{\Psi}_{\text{ipw}}(\pi,p) + [\text{an extra term}]_2$.
2. Show that $\tilde{\Psi}_{\text{ee}}(f,\pi,p) = \mathbb{E}_P[Y^1] - \mathbb{E}_P[Y^0]$ under the identifiability assumptions (consistency, exchangeability and positivity).

Summary: Properly defining the target

- Causal parameter (now fixed).
- Causal model: How do the observed variables affect one another?
  - are the covariates we observe sufficient to remove confounding? which variables **do we need to adjust for** to make treatment groups comparable?
- Identifiability
  - identifiability assumptions allow us to write causal parameter as statistical parameter.
  - the assumptions may not hold, but we can state and discuss them.
- Statistical parameter
  - Statistical interpretation: The average effect in the population, standardized to the distribution of covariates.

Summary: Properly defining the target



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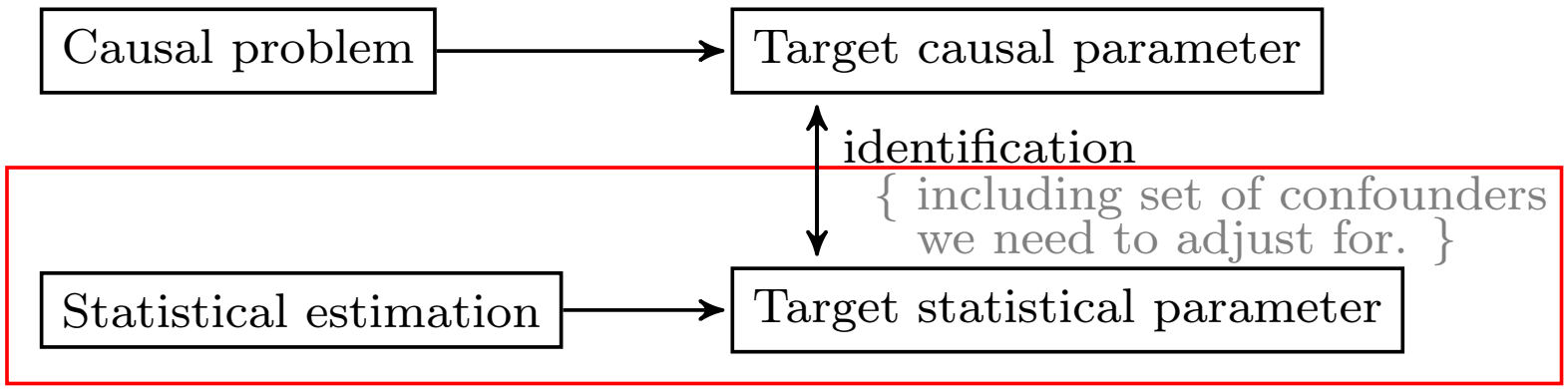

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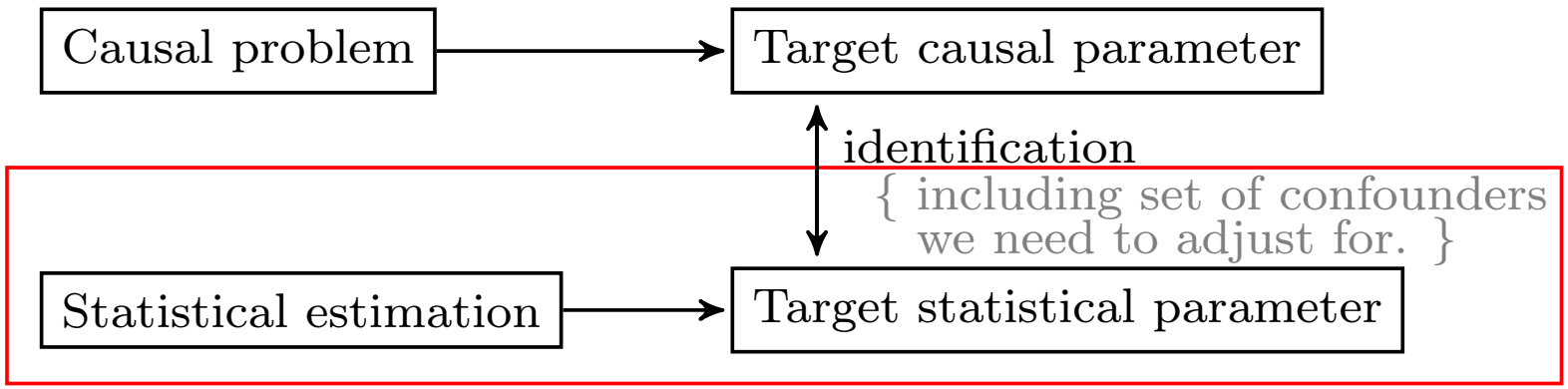


- one estimator is not more causal than another.
- different estimators are based on different nuisance parameters and have different statistical properties (bias/variance).

On a sidenote: Other simple causal parameters

We focus on the ATE as an example of a causal parameter.

But note that other simple causal parameters can be constructed from $\mathbb{E}[Y^1]$ and $\mathbb{E}[Y^0]$.

Like:

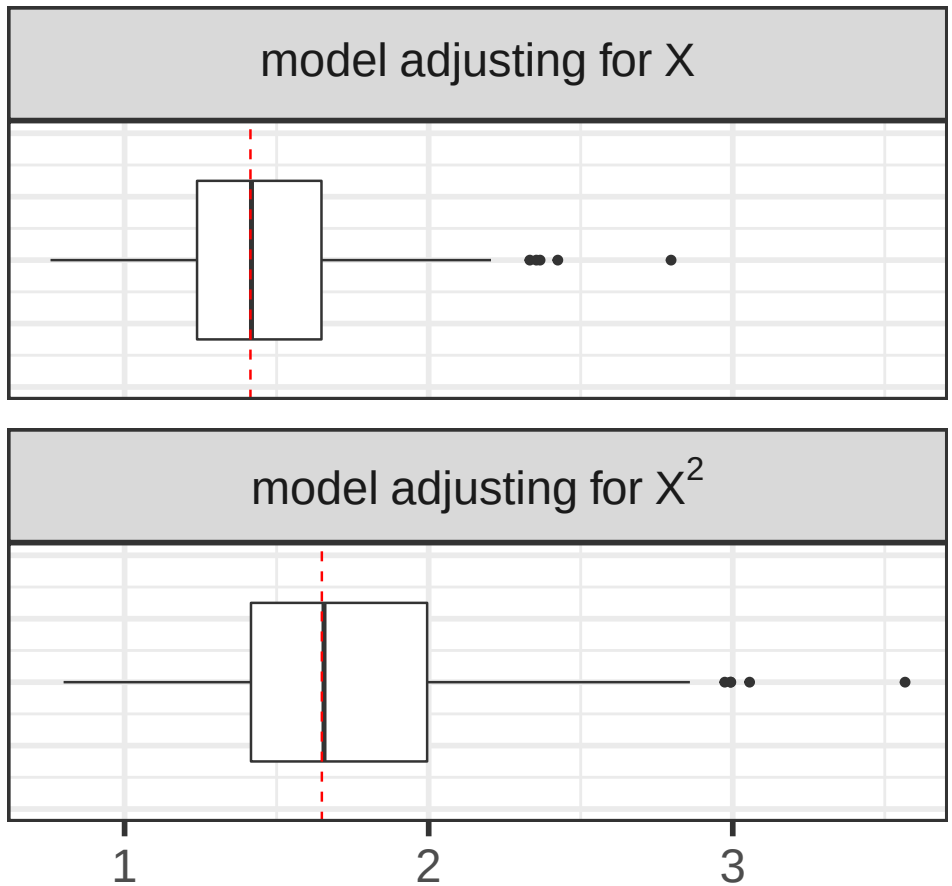
$$\psi^{\mathrm{RR}}(P) = \frac{\mathbb{E}[Y^1]}{\mathbb{E}[Y^0]},$$

or,

$$\psi^{\mathrm{OR}}(P) = \frac{\mathbb{E}[Y^1]/(1-\mathbb{E}[Y^1])}{\mathbb{E}[Y^0]/(1-\mathbb{E}[Y^0])}.$$

On a sidenote: Other simple causal parameters

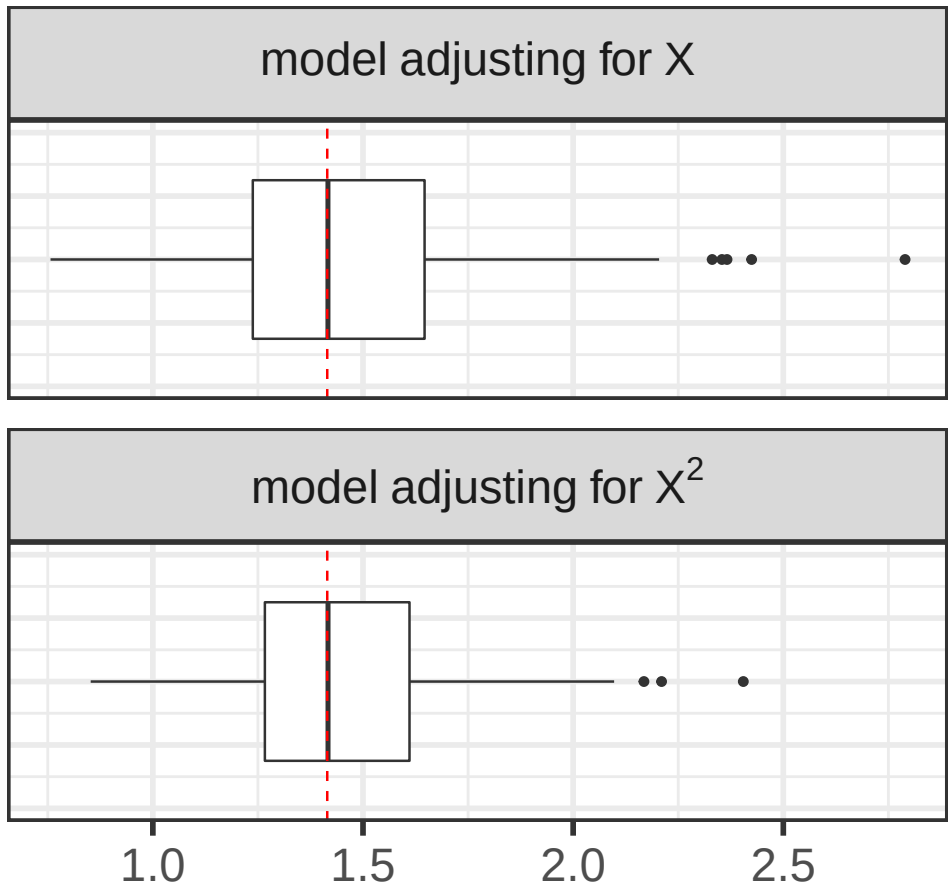
log(OR) as a regression coefficient is a moving target in different logistic regression models:



The upper panel does not show a biased estimator, just an estimator targeting a different parameter (dashed red line).

On a sidenote: Other simple causal parameters

The corresponding causal odds ratio is a fixed target — and the target does not change depending on adjustment for X or X^2:



... but these are different statistical estimators, and they have different statistical properties.

Last slide of this lecture

Summarizing this lecture:

- take 5 minutes to write down 3–10 keywords/concepts/formulas from this lecture;
- discuss the keywords with the person sitting next to you, and explain their significance in the overall targeted learning framework (5 min).