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Par


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**Comportement en temps long et étude de modèles probabilistes pour des phénomènes physiques, biologiques et actuariels.**

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# Introduction

This document is an overview of my different works since my PhD thesis. I have considered probabilistic models for different applications, in physics, biology or actuarial science, always with the aim of studying carefully the behavior and properties of the underlying process. Most of my results investigates the question of the long-time behavior of stochastic processes.
The first model I studied is about some rarefied homogeneous gas in kinetic theory. The behavior of the particles is given either by the well-known Boltzmann equation or by the Landau equation, depending on the nature of the collision between particles. During my thesis, I worked on the existence of solutions to the Landau equation by a probabilistic way, their regularity and on the link between the Boltzmann solutions and the Landau solutions. I continued to explore this subject during the first years after my PhD thesis. With Nicolas Fournier, we studied the dependence on the initial condition of the solutions of the Boltzmann and the Landau equation, [FG08, FG09]. With Joaquin Fontbona and Sylvie Méléard, we have been interested in constructing an easily simulable diffusive interacting particle system converging towards the nonlinear Landau process and in obtaining an explicit speed of convergence, [FGM09, FGM10]. The results on the Boltzmann and Landau equations are presented in Chapter 1 of this document.

With Florent Malrieu and Joaquin Fontbona, we have been interested in the study of a model in biology, more precisely a model describing the behavior of a flagellated bacterium in its environment. The general model introduced by Erban and Othmer being quite complex with many parameters, we considered a simplified model in order to obtain interesting and quantitative results on the long time behavior of the bacterium using a probabilistic approach, [FGM12, FGM16]. The behavior of a such bacteria is modelized by a piecewise deterministic Markov process (PDMP). This kind of processes has been extensively studied during the last decade, and can also have many applications, especially for effective Monte-Carlo algorithms. Although we have considered a fairly simple model, the study of

its long time-behavior has highlighted the important role of jump times, especially when the deterministic flows are not contractive. Using a geometrical coupling, we obtained an explicit speed of convergence to equilibrium. These results are presented in Chapter 2 of this document.

During a long stay in Montréal in 2012-2014, I have met some researchers in actuarial science which led to fruitful collaborations in ruin theory for the spectrally negative Lévy framework, [BSGMF15, GR16, GR17]. We worked on new ruin measures, which extend the classical probability to have a negative surplus in the case of spectrally negative Lévy model. We can relate those results to the work on Ornstein-Ulhenbeck diffusions with Markov switching, in collaboration with Florent Malrieu and Jean-Baptiste Bardet, [BGM10], which has applications in finance and in insurance. The results related to these topics are presented in Chapter 3 of this document.

Each chapter of this work can be read independently from the others. In the next sections of this introduction, we give some common notations (the devoted notation of each chapter being defined therein), and we recall some well-known definitions and results in probability which will be used in different chapters of this document.

# 1 Notations

Let $d \geq 1$. Let us denote by $C^2_\infty\big(\mathbb{R}^d\big)$ (resp. $C^2_b\big(\mathbb{R}^d\big)$, resp. $C^2_c\big(\mathbb{R}^d\big)$) the set of $C^2$-functions $\phi : \mathbb{R}^d \mapsto \mathbb{R}$ of which the second derivative is bounded (resp. of which the derivatives of order 0 to 2 are bounded, resp. which are compactly supported).
Let also $L^p\big(\mathbb{R}^d\big)$ be the space of measurable functions $f$ with

$$\|f\|_{L^p\left(\mathbb{R}^d\right)} := \left(\int_{\mathbb{R}^d} f^p(v)\, dv\right)^{1/p} < +\infty.$$

Let us introduce $\Big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\Big)$ a filtered probability space.
The distribution of a random variable $X$ will be denoted by $\mathcal{L}(X)$.
Let $\mathcal{P}\big(\mathbb{R}^d\big)$ be the set of probability measures on $\mathbb{R}^d$, and for $k \geq 1$,

$$\mathcal{P}_k\Big(\mathbb{R}^d\Big) = \Big\{f \in \mathcal{P}\Big(\mathbb{R}^d\Big),\ m_k(f) < \infty\Big\} \quad \text{with} \quad m_k(f) := \int_{\mathbb{R}^d} |v|^k\, f(dv).$$

We denote by $L^\infty\big([0,T], \mathcal{P}_k\big(\mathbb{R}^d\big)\big)$ and $L^1\big(0,T], L^p\big(\mathbb{R}^d\big)\big)$ the sets of measurable families $(f_t)_{t\in[0,T]}$ of probability measures on $\mathbb{R}^d$ such that $\sup_{t\in[0,T]} m_k(f_t) < \infty$, $\int_0^T ||f_t||_{L^p} dt < \infty$ respectively.
$I\!D\big(\mathbb{R}_+, \mathbb{R}^d\big)$ will denote the Skorohod space of càdlàg functions from $\mathbb{R}_+$ into $\mathbb{R}^d$. This space endowed with the Skorohod topology is a Polish space.

Finally we denote $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$ for $x, y \in \mathbb{R}_+$, and $z.\tilde{z}$ the scalar product of $z, \tilde{z} \in \mathbb{R}^d$ and $M^t$ and $M^*$ are respectively the transpose matrix and the adjoint matrix of the matrix $M$.
For some set $A$ we write $\mathbb{1}_A$ the usual indicator function of $A$.

# 2 Some probabilistic tools

We introduce in this section some classical probabilistic tools which are used throughout this document. Specific tools for each model will be introduced in the devoted chapter.

## 2.1 The notion of infinitesimal generator of a Feller process

Let $(E, \mathcal{E})$ be a measurable space. On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, we consider $(X_t)_{t\geq 0}$ a Feller process with transition $(P_t)_{t\geq 0}$ with values in $(E, \mathcal{E})$, which means that $X$ is a Markov process with transition $(P_t)_{t\geq 0}$, i.e. for any nonnegative measurable function $f$ and $s < t$,

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = P_{t-s}f(X_s) \quad \mathbb{P} - ps,$$

such that for any $t \geq 0$, any continuous function $f$, $P_t f$ is a continuous function and $\lim_{t\downarrow 0} P_t f(x) = f(x)$.
The infinitesimal generator of a Feller process $X$ is the operator defined, for $f$ a continuous function, by

$$Lf = \lim_{t\downarrow 0} \frac{1}{t}(P_t f - f).$$

The domain $D(L)$ of the operator $L$ is the subset of continuous functions such that the previous limit exists.
Consequently, for $f \in D(L)$,

$$\mathbb{E}[f(X_{t+h}) - f(X_t)|\mathcal{F}_t] = hLf(X_t) + o(h).$$

The operator $L$ describes the infinitesimal behavior of the process on a small time interval. We can observe that if $f \in D(L)$, the process $M_t^f = f(X_t) - f(X_0) - \int_0^t Lf(X_s)\mathrm{d}s$ is a $(\mathcal{F}_t, \mathbb{P}_\nu)$ martingale for any initial measure $\nu$, see [RY94].
A measure $\mu$ is said to be an invariant measure for the Markov process $X$ if for any $t \geq 0$, any smooth function $f$,

$$\int P_t f \mathrm{d}\mu = \int f \mathrm{d}\mu,$$

or equivalently,

$$\int Lf \mathrm{d}\mu = 0.$$

## 2.2 Some distances in probability

### The Wasserstein distance

Let us consider processes in dimension $d \geq 1$.
For two probability measures $\mu, \nu \in \mathcal{P}_p\big(\mathbb{R}^d\big)$ and $\pi \in \mathcal{P}_p(\mathbb{R}^d \times \mathbb{R}^d)$, we write $\pi <^{\mu}_{\nu}$ if the marginals of $\pi$ are $\pi_1 = \mu$ and $\pi_2 = \nu$. The Wasserstein distance $\mathcal{W}_p$ on $\mathcal{P}_p\big(\mathbb{R}^d\big)$ is defined as follows: for any $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$,

$$\mathcal{W}_p(\mu,\nu) = \left(\inf_{\pi <^{\mu}_{\nu}} \left\{ \int |x-y|^p \, \pi(\mathrm{d}x, \mathrm{d}y) \right\}\right)^{1/p} \tag{0.1}$$

$$= \inf\Big\{ \mathbb{E}[|X-Y|^p] : (X,Y) \in \mathbb{R}^d \times \mathbb{R}^d \text{ random vectors with } \mathcal{L}(X) = \mu \text{ and } \mathcal{L}(Y) = \nu \Big\}^{1/p}.$$

It is well-known that $\big(\mathcal{P}_p\big(\mathbb{R}^d\big), \mathcal{W}_p\big)$ is a complete metric space (see [Vil02]).
For $p = 2$, $\mathcal{W}_2$ is called the Wasserstein distance with quadratic cost $c(x,y) = |x-y|^2$. The topology endowed by the Wasserstein distance is stronger than the usual weak topology (for more details, see Villani, [Vil03] Theorem 7.12). It is well-known that the infimum is actually a minimum, and that if $\mu$ (or $\nu$) has a density with respect to the Lebesgue measure on $\mathbb{R}^d$, then there is a unique $\pi <^{\mu}_{\nu}$ such that $\mathcal{W}_2^2(\mu,\nu) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 \pi(dx,dy)$. In section 4.3 of Chapter 1, the question of optimal transportation for the Wasserstein distance with general costs will be considered.

### The total variation distance

The total variation distance between two probability measures $\mu$ and $\nu$ in some measurable space $(\mathcal{X}, \mathcal{B})$ is given by

$$\|\mu - \nu\|_{\mathrm{TV}} = \sup_{A \in \mathcal{B}} \{|\mu(A) - \nu(A)|\} = \frac{1}{2} \sup\{|\mu(\phi) - \nu(\phi)| : \|\phi\|_{\infty} \leq 1\}. \tag{0.2}$$

We notice that

$$\|\mu - \nu\|_{\mathrm{TV}} = \inf\{\mathbb{P}(X \neq Y) \, : \, X, Y \text{ random variables with } \mathcal{L}(X) = \mu, \ \mathcal{L}(Y) = \nu\}, \tag{0.3}$$

where each pair of random elements $(X,Y)$ of $\mathcal{X}$ is simultaneously constructed in some probability space and is called a *coupling*. See [Lin92] for alternative definitions of this distance and its main properties.

Let us notice that the Wasserstein distance is quantitative while the total variation distance is qualitative. Indeed, the Wasserstein distance is small if there exists a coupling $(X,Y)$ with close random variables while this is not enough for the total variation distance. For example,

$$\mathcal{W}_p(\delta_x, \delta_y) = |x-y|, \quad \|\delta_x - \delta_y\|_{\mathrm{TV}} = \mathbb{1}_{x \neq y},$$

where $\delta_x$ denote the Dirac measure at $x$.

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# Chapter 1

# Transport Theory, Particle Systems
# Models in kinetic theory : the Boltzmann and Landau equations

My first steps in research concerned the study of the Bolzmann and Landau equations in kinetic theory by a probabilistic way. I continued to explore this topic during the first years after my PhD thesis. The results obtained during this period are presented in this chapter.

With Nicolas Fournier, Sorbonne université, we have studied the dependence on the initial condition of the solutions of the Boltzmann and of the Landau equations. These results has been published in Journal of Statistical Physics, On the uniqueness for the spatially homogeneous Boltzmann equation with a strong angular singularity, [FG08], and in Journal of Functional Analysis, Well-posedness of the spatially homogeneous Landau equation for soft potentials, [FG09].

With Joaquin Fontbona, Universidad de Chile, and Sylvie Méléard, CMAP, École Polytechnique, we have been interested in constructing an easily simulable diffusive interacting particle system converging towards the nonlinear Landau process and in obtaining an explicit speed of convergence. These results rely on two articles, *Measurability of optimal transportation and convergence rate for Landau type interacting particle systems* published in Probability Theory and Related Fields, [FGM09], and Measurability of optimal transportation and strong coupling of martingale measures published in Electronic Communications in Probability, [FGM10]. The approach was quite new and used in a sophisticated way the optimal transport theory.

Let us first introduce the spatially homogeneous Boltzmann and Landau equations.

# 1 The spatially homogeneous Boltzmann and Landau equations and their probabilistic representation

## 1.1 The Boltzmann equation

The spatially homogeneous Boltzmann equation is an important equation from kinetic theory which describes the evolution of the density $f_t(v)$ of particles with velocity $v \in \mathbb{R}^3$ at time $t$ in a rarefied homogeneous gas. This is a nonlinear differential equation given by

$$\frac{\partial f}{\partial t} = Q_B(f, f), \tag{1.1}$$

with $Q_B$ the quadratic collision kernel defined by

$$Q_B(f,f)(t,v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma B(|v - v_*|, \theta) \big[ f_t(v') f_t(v'_*) - f_t(v) f_t(v_*) \big]$$

where the pre-collisional velocities are given by

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma, \tag{1.2}$$

and $\theta$ is the so-called *deviation angle* defined by $\cos\theta = \frac{(v-v_*)}{|v-v_*|}\cdot\sigma$. The *collision kernel* $B(|v-v_*|,\theta) = B(|v'-v'_*|,\theta)$ depends on the nature of the interactions between particles.

This equation is quite natural: it says that for each $v \in \mathbb{R}^3$, new particles with velocity $v$ appear due to a collision between two particles with velocities $v'$ and $v'_*$, at rate $B(|v'-v'_*|,\theta)$, while particles with velocity $v$ disappear because they collide with another particle with velocity $v_*$, at rate $B(|v-v_*|,\theta)$. See Desvillettes [Des01] and Villani [Vil02] for more details.

Since the collisions are assumed to be elastic, conservation of mass, momentum and kinetic energy hold at least formally for solutions to (1.1), that is for all $t \geq 0$,

$$\int_{\mathbb{R}^3} f_t(v)\,\phi(v)\,dv = \int_{\mathbb{R}^3} f_0(v)\,\phi(v)\,dv, \quad \text{for } \phi(v) = 1, v, |v|^2.$$

We classically assume without loss of generality that $\int_{\mathbb{R}^3} f_0(v)\,dv = 1$.
We also assume that for some functions $\Phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and $\beta : (0,\pi] \mapsto (0,\infty)$, the collision kernel is of the form

$$B(|v-v_*|,\theta)\,\sin\theta = \Phi(|v-v_*|)\,\beta(\theta). \tag{\textbf{A1}}$$

When particles interact through repulsive forces in $1/r^s$, with $s \in (2,\infty)$, one has (see Cercignani [Cer88])

$$\Phi(z) = z^\gamma, \quad \beta(\theta) \underset{0}{\sim} \mathrm{cst}\,\theta^{-1-\nu}, \quad \text{with } \gamma = \frac{s-5}{s-1} \in (-3,1), \quad \nu = \frac{2}{s-1} \in (0,2).$$

One classically names *hard potentials* the case when $\gamma \in (0,1)$ (*i.e.*, $s > 5$), *Maxwellian molecules* the case when $\gamma = 0$ (*i.e.*, $s = 5$), *moderately soft potentials* the case when $\gamma \in (-1,0)$ (*i.e.*, $s \in (3,5)$), and *very soft potentials* the case when $\gamma \in (-3,-1)$ (*i.e.*, $s \in (2,3)$).

In any cases, $\int_{0+} \beta(\theta)d\theta = +\infty$, which expresses the affluence of *grazing collisions*, that is collisions with a very small deviation. We will assume here the general physically reasonnable conditions

$$\int_0^\pi \beta(\theta)d\theta = +\infty, \quad \kappa_1 := \int_0^\pi \theta^2\,\beta(\theta)d\theta < +\infty. \tag{\textbf{A2}}$$

Let us define the notion of weak solutions we shall use.

**Definition 1.1.** *Let $B$ be a collision kernel which satisfies* **(A1-A2)**. *A family $f = (f_t)_{t\in[0,T]} \in L^\infty([0,T],\mathcal{P}_2(\mathbb{R}^3))$ is a weak solution to (1.1) if*

$$\int_0^T dt \int_{\mathbb{R}^3} f_t(dv) \int_{\mathbb{R}^3} f_t(dv_*)\,\Phi(|v-v_*|)\,|v-v_*|^2 < +\infty, \tag{1.5}$$

*and if for any* $\phi \in C^2_\infty\big(\mathbb{R}^3\big)$*, and any* $t \in [0,T]$*,*

$$\int_{\mathbb{R}^3} \phi(v)\, f_t(dv) = \int_{\mathbb{R}^3} \phi(v)\, f_0(dv) + \int_0^t ds \int_{\mathbb{R}^3} f_s(dv) \int_{\mathbb{R}^3} f_s(dv_*)\, \mathcal{A}\phi(v, v_*), \tag{1.6}$$

*where*

$$\mathcal{A}\phi(v, v_*) = \frac{\Phi(|v - v_*|)}{2} \int_0^\pi \beta(\theta) d\theta \int_0^{2\pi} d\varphi \left[\phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*)\right].$$

As noted by Villani [Vil98b, p 291], one has, for all $v, v_* \in \mathbb{R}^3$, all $\theta \in [0,\pi]$, all $\phi \in C^2_\infty\big(\mathbb{R}^3\big)$,

$$\left| \int_0^{2\pi} d\varphi \left[\phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*)\right] \right| \leq C||\phi''||_\infty \theta^2 |v - v_*|^2,$$

so that thanks to assumption **(A2)**, (1.5) ensures that all the terms in (1.6) are well-defined.

Let us now introduce the jump amplitude

$$a := a(v, v_*, \theta, \varphi) := (v' - v) = -(v'_* - v_*). \tag{1.7}$$

One of the main difficulties of this equation is that $a$ is not a Lipschitz continuous function on the variables $v$ and $v_*$. It just satisfies an "almost"-Lipschitz property of $a$ (Lipschitz up to a rotation), as proved in [Tan79] or in its "fine" version in [FM02].

## 1.2 Probabilistic approach of the Boltzmann equation

Since $\int_0^\pi \theta\beta(\theta)d\theta$ may be infinite and taking into account the conservation of the mass, we introduce the following definition of probability measure solutions of (1.1).

**Definition 1.2.** *We say that a probability measure family* $(Q_t)_{t\geq 0}$ *is a measure-solution of the Boltzmann equation (1.1) if for each* $\phi \in C^2_b(\mathbb{R}^3)$

$$\int \phi(v) Q_t(\mathrm{d}v) = \int \phi(v) Q_0(\mathrm{d}v) + \int_0^t \iint K^\phi_{\beta,\gamma}(v, v_*) Q_s(dv)\, Q_s(dv_*)\, ds, \tag{1.8}$$

*where* $K^\phi_{\beta,\gamma}$ *is defined in the compensated form by*

$$\begin{aligned} K^\phi_{\beta,\gamma}(v, v_*) &= -b_\beta \Phi(v - v_*)(v - v_*).\nabla\phi(v) \\ &+ \int_0^{2\pi} \int_0^\pi \Big(\phi(v + a(v, v_*, \theta, \varphi)) - \phi(v) - a(v, v_*, \theta, \varphi).\nabla\phi(v)\Big) \Phi(|v - v_*|)\beta(\theta) d\theta d\varphi \end{aligned}$$

*and where*

$$b_\beta = \pi \int_0^\pi (1 - \cos\theta)\beta(\theta) d\theta.$$

We can consider (1.8) as the evolution equation for the marginals of a Markov process whose law is defined by a martingale problem. Indeed, the distribution $Q$ can be seen as the law of a stochastic process belonging to $I\!D(\mathbb{R}_+, \mathbb{R}^3)$ solution of the nonlinear stochastic differential equation

$$V_t = V_0 - b_\beta \int_0^t \int_{\mathbb{R}^3} \Phi(|V_s - z|)(V_s - z) Q_s(dz) ds$$
$$+ \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_0^\pi \int_0^{2\pi} a(V_{s-}, z, \theta, \varphi) \mathbf{1}_{\{x \leq \Phi(|V_{s-} - z|)\}} \tilde{N}^*(dx, d\theta, d\varphi, dz, ds)$$

where $\tilde{N}^*$ is the compensated martingale of an inhomogeneous Poisson-point measure on $\mathbb{R}_+ \times [0, \pi] \times [0, 2\pi] \times \mathbb{R}^3 \times \mathbb{R}_+$ with intensity $dx\beta(\theta)d\theta d\varphi Q_t(dz)dt$. The nonlinearity appears through $Q_s$, which is the law of $V_s$ for each $s$.
We consider a compensated form of the Poisson-point measure following Definition 1.2. That gives a pathwise mean-field interacting representation of the Boltzmann process: the process jumps following a Poisson-point measure which picks independent colliding particules having the same law as the process itself. The jump takes place if $x \leq \Phi(|V_{s-} - z|)$ and the amplitude of the jump is equal to $a$.

## 1.3 The Landau equation and its associated process

The Landau equation is a *continuous* approximation of the Boltzmann equation: when there are infinitely many infinitesimally small collisions, the particle velocities become continuous in time. The most interesting case is that of Coulomb potential (i.e. when $\gamma = -3$), since then the Boltzmann equation seems to be meaningless. It is also the most difficult case to study mathematically. However, the Landau equation can be derived from the Boltzmann equation with *very soft potentials*, that is $\gamma \in (-3, -1)$ (see see Funaki [Fun85], Goudon [Gou97], Villani [Vil98b, Vil98a] and Guérin-Méléard [GM03b, Gué04]). The main idea is that the more $\gamma$ is negative, the more the Landau equation is physically interesting. We refer to [Vil02] for a detailed survey about such considerations.
Consequently, the Landau equation, also called the Fokker-Planck-Landau equation, describes the collisions of particles in a plasma and is obtained as limit of Boltzmann equations when the grazing collisions prevail. In the spatially homogeneous case, it writes in $I\!R^3$:

$$\frac{\partial f}{\partial t} = Q_L(f, f) \tag{1.9}$$

with

$$Q_L(f, f)(t, v) =$$
$$\frac{1}{2} \sum_{i,j=1}^3 \frac{\partial}{\partial v_i} \left\{ \int_{I\!R^3} dv_* A_{ij}(v - v_*) \left[ f_t(v_*) \frac{\partial f_t}{\partial v_j}(v) - f_t(v) \frac{\partial f_t}{\partial v_{*j}}(v_*) \right] \right\}$$

where $f_t(v) \geq 0$ is the density of particles having velocity $v \in \mathbb{R}^3$ at time $t \in \mathbb{R}^+$, and $(A_{ij}(z))_{1\leq i,j\leq 3}$ is a nonnegative symmetric matrix depending on the interaction between the particles, of the form

$$A(z) = |z|^{\gamma+2}\,\Pi(z) = |z|^{\gamma}\begin{bmatrix} z_2^2+z_3^2 & -z_1z_2 & -z_1z_3 \\ -z_1z_2 & z_1^2+z_3^2 & -z_2z_3 \\ -z_1z_3 & -z_2z_3 & z_1^2+z_2^2 \end{bmatrix} \tag{1.10}$$

where $\Pi(z)$ is the orthogonal projection on $(z)^{\perp}$ and $\gamma \in (-3, 0]$ the potential. As $A$ is a symmetric nonnegative matrix, there exists a matrix $\sigma$ such that $A = \sigma.\sigma^*$, where $\sigma$ is the adjoint matrix of $\sigma$. There is not uniqueness of $\sigma$.

Like for the Boltzmann equation, we observe that the solutions to (1.9) conserve, at least formally, the mass, the momentum and the kinetic energy: for any $t \geq 0$,

$$\int_{\mathbb{R}^3} f_t(v)\phi(v)dv = \int_{\mathbb{R}^3} f_0(v)\phi(v)dv, \quad \text{for} \quad \phi(v) = 1,\, v,\, |v|^2.$$

We classically may assume without loss of generality that $\int_{\mathbb{R}^3} f_0(v)dv = 1$. Another fundammental *a priori* estimate is the decay of entropy, that is solutions satisfy, at least formally, for all $t \geq 0$,

$$\int_{\mathbb{R}^3} f_t(v)\log f_t(v)dv \leq \int_{\mathbb{R}^3} f_0(v)\log f_0(v)dv.$$

By integration by parts, see [Vil98b], a weak formulation of Equation (1.9) writes, at least formally, for any test function $\phi \in C_b^2(\mathbb{R}^3)$,

$$\frac{d}{dt}\int_{\mathbb{R}^3}\phi(v)f_t(v)dv = \iint_{\mathbb{R}^3\times\mathbb{R}^3} f_t(dv)f_t(dv^*)L\phi(v,v^*) \tag{1.11}$$

where the operator $L$ is defined by

$$L\phi(v,v^*) = \frac{1}{2}\sum_{i,j=1}^{3} A_{ij}(v-v^*)\partial_{ij}^2\phi(v) + \sum_{i=1}^{3} b_i(v-v^*)\partial_i\phi(v) \tag{1.12}$$

$$\text{with} \quad b_i(z) = \sum_{j=1}^{3}\partial_j A_{ij}(z) = -2|z|^{\gamma}z_i, \ \text{for } i = 1,2,3. \tag{1.13}$$

Let us mention that existence of weak solutions, under physically reasonnable assumptions on initial conditions, has been obtained by Villani [Vil98b].
Like for the Boltzmann equation, using the conservation of mass, we introduce a definition of probability-measure solutions of the Landau equation :

**Definition 1.3.** *Let $P_0$ belong to $\mathcal{P}_2\left(I\!R^3\right)$. A weak solution of the Landau equation (1.14) with initial data $f_0$ is a family $(f_t)_{t\geq 0}$ of probability measures in $\mathcal{P}_2\left(\mathbb{R}^3\right)$ satisfying*

$$\int \phi(v)\, f_t(\mathrm{d}v) = \int \phi(v)\, f_0(\mathrm{d}v) + \int_0^t \iint L\phi\,(v, v_*)\; f_s\,(dv)\, f_s\,(dv_*)\, ds \tag{1.14}$$

*for any function $\phi \in C_b^2\left(I\!R^3\right)$ where $L\phi$ is the Landau kernel defined by* (1.12).

The probability measure solution of the Landau equation can be seen as the distribution of a Markov processes $X$ of a nonlinear stochastic differential equation of type:

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}^3} \sigma(X_s - y) W_P(dy, ds) + \int_0^t \int_{\mathbb{R}^3} b(X_s - y) P_s(dy) ds \tag{1.15}$$

where $\sigma$ is a "square-root" of the matrix $A$ defined by (1.10), $b$ is defined by (1.13), $P_t$ is the law of $X_t$, and $W_P$ is a $\mathbb{R}^3$ valued space-time white noise on $[0,T]\times\mathbb{R}^3$ with independent coordinates, each of which having covariance measure $P_t(dy)\otimes dt$, in the sense of Walsh [Wal86].

The nonlinear process (1.15) was introduced by Funaki [Fun84], who obtained existence and uniqueness results for Lipschitz coefficients $\sigma : \mathbb{R}^3 \to \mathbb{R}^{3\otimes 3}$ and $b : \mathbb{R}^3 \to \mathbb{R}^3$, see also Guérin [Gué02, Gué03] for a different approach.

## 2 Well-posedness of the spatially homogeneous Boltzmann equation

We investigate in this section the question of the well-posedness of the spatially homogeneous Boltzmann equation for singular collision kernel as introduced above. In particular we are interested in uniqueness and stability with respect to the initial condition.

In the case of a collision kernel with angular cutoff, that is when $\int_0^\pi \beta(\theta)d\theta < +\infty$, there are some optimal existence and uniqueness results: see Mischler-Wennberg [MW99] and Lu-Mouhot [LM12].

The case of collision kernels without cutoff is much more difficult, but is very important, since it corresponds to the previously described physical collision kernels. This difficulty is not surprising: on each compact time interval, each particle collides with infinitely (resp. finitely) many others in the case without (resp. with) cutoff.

In all the previously cited physical situations, global existence of weak solutions has been proved by Villani [Vil98b] by using some compactness methods.

Tanaka [Tan79], Horowitz-Karandikar [HK90], Toscani-Villani [TV99] studied uniqueness for non cutoff collision kernel in the case of Maxwellian molecules: it was proved in [TV99] that uniqueness holds for the Boltzmann equation as soon as $\Phi$ is constant and **(A2)** is met,

for any initial (measure) datum with finite mass and energy, that is $\int_{\mathbb{R}^3}(1+|v|^2)\,f_0(dv) < +\infty$.

Then there has been three papers in the case where $\beta$ is non cutoff and $\Phi$ is not constant. The case where $\Phi$ is bounded (together with additionnal regularity assumptions) was treated in [Fou06], for essentially any initial (measure) datum such that $\int_{\mathbb{R}^d}(1+|v|)f_0(dv) < \infty$. More realistic collision kernels have been treated by Desvillettes-Mouhot [DM09] and Fournier-Mouhot [FM09] (including hard and moderately soft potentials). However, all these results apply only when assuming the following condition, stronger than **(A2)**,

$$\int_0^\pi \theta\beta(\theta)d\theta < \infty. \tag{1.16}$$

In particular, this does not apply to very soft potentials ($\gamma \in (-3,-1]$). Weighted Sobolev spaces were used in [DM09], while the results of [FM09] rely on the Kantorovich-Rubinstein distance.

With Nicolas Fournier in [FG08], we obtained the first uniqueness result which can deal with the case where only **(A2)** is supposed. Our result is based on the use of the Wasserstein distance with quadratic cost. The main interest of our paper concerns very soft potentials, for which we obtain the uniqueness of the solution provided it remains in $L^p(\mathbb{R}^3)$, for some $p$ large enough. Since we are only able to propagate locally such a property, we obtain some local (in time) well-posedness result.

Our method could be applied to the case of hard potentials, but we have not treat this case, since there are already some available uniqueness results, as said previously.

Our proof is probabilistic, and we did not manage to rewrite it in an analytic way. The main idea is quite simple: for two solutions $(f_t)_{t\geq 0}$, $(\tilde{f}_t)_{t\geq 0}$ to the Boltzmann equation, we construct two stochastic processes $(V_t)_{t\geq 0}$ and $(\tilde{V}_t)_{t\geq 0}$ whose time marginal laws are given by $(f_t)_{t\geq 0}$ and $(\tilde{f}_t)_{t\geq 0}$, and which are coupled in such a way that $E[|V_t - \tilde{V}_t|^2]$ is "small" for all times. This bounds from above the Wasserstein distance with quadratic cost between $f_t$ and $\tilde{f}_t$.

## 2.1 Some notations

For $\alpha \in (-3,0)$, we introduce the space $\mathcal{J}_\alpha$ of probability measures $f$ on $\mathbb{R}^3$ such that

$$J_\alpha(f) := \sup_{v\in\mathbb{R}^3} \int_{\mathbb{R}^3} |v-v_*|^\alpha f(dv_*) < \infty, \tag{1.17}$$

We denote by $L^\infty\big([0,T],\mathcal{P}_k\big)$, $L^1\big([0,T],L^p\big)$, $L^1([0,T],\mathcal{J}_\alpha)$ the set of measurable families $(f_t)_{t\in[0,T]}$ of probability measures on $\mathbb{R}^3$ such that $\sup_{t\in[0,T]} m_k(f_t) < \infty$, $\int_0^T ||f_t||_{L^p}dt < \infty$, $\int_0^T J_\alpha(f_t)dt < \infty$ respectively. Observe that (see [FG08, (5.2)]):

**Remark 1.4.** *For $\alpha \in (-3, 0)$ and $p > 3/(3+\alpha)$, there exists a constant $C_{\alpha,p} > 0$ such that for all nonnegative measurable $f : \mathbb{R}^3 \mapsto \mathbb{R}$, any $v^* \in \mathbb{R}^d$*

$$J_\alpha(f) \leq ||f||_{L^1} + C_{\alpha,p}||f||_{L^p}.$$

For a nonnegative function $f \in L^1(\mathbb{R}^3)$, we denote its entropy by

$$H(f) = \int_{\mathbb{R}^3} f(v) \log(f(v)) dv.$$

## 2.2 Assumptions on the collision kernel

We now introduce, for $\theta \in (0, \pi]$,

$$H(\theta) := \int_\theta^\pi \beta(x)dx \quad \text{and} \quad G(z) := H^{-1}(z).$$

Here $H$ is a continuous decreasing bijection from $(0, \pi]$ into $[0, +\infty)$, and its inverse function $G : [0, +\infty) \mapsto (0, \pi]$ is defined by $G(H(\theta)) = \theta$, and $H(G(z)) = z$. We will suppose that there exists $\kappa_2 > 0$ such that for all $x, y \in \mathbb{R}_+$,

$$\int_0^\infty \left(G(z/x) - G(z/y)\right)^2 dz \leq \kappa_2 \frac{(x-y)^2}{x+y}. \tag{\textbf{A3}}$$

Concerning the velocity part of the cross section, we will assume that for all $x, y \in \mathbb{R}_+$,

$$\begin{aligned} \min(x^2, y^2)\frac{[\Phi(x) - \Phi(y)]^2}{\Phi(x) + \Phi(y)} + (x-y)^2[\Phi(x) + \Phi(y)] & \\ + \min(x, y)|x - y||\Phi(x) - \Phi(y)| \quad &\leq \quad (x-y)^2[\Psi(x) + \Psi(y)] \end{aligned}$$

for some function $\Psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, with for some $\gamma \in (-3, 0]$, some $\kappa_3 > 0$, for all $x \in \mathbb{R}_+$,

$$\Psi(x) \leq \kappa_3 x^\gamma. \tag{\textbf{A4}($\gamma$)}$$

Under Assumption (**(A4)**($\gamma$)), we can easily see that necessarily for all $x \in \mathbb{R}_+$, $\Phi(x) \leq \Psi(x)$, and then $\Phi(x) \leq \kappa_3 x^\gamma$.
These assumptions are not very transparent. However, the following lemma, proved in [FG08], shows how they apply. Roughly, **(A3)** is very satisfying, **(A4**(0)**)** corresponds to *regularized* velocity cross sections, while **(A4**($\gamma$)**)** allows us to deal with general soft potentials.

**Lemma 1.5.** *(i) Assume that there are $0 < c < C$ and $\nu \in (0, 2)$ such that for all $\theta \in (0, \pi]$, $c\theta^{-\nu-1} \leq \beta(\theta) \leq C\theta^{-\nu-1}$. Then* **(A2-A3)** *hold.*
*(ii) Assume that $\Phi(x) = \min(x^\alpha, A)$ for some $A > 0$, some $\alpha \in \mathbb{R}$, or that $\Phi(x) = (\varepsilon + x)^\alpha$ for some $\varepsilon > 0$, $\alpha < 0$. Then* **(A4**(0)**)** *holds.*
*(iii) Assume that for some $\gamma \in (-3, 0]$, $\Phi(x) = x^\gamma$. Then* **(A4**($\gamma$)**)** *holds.*

## 2.3 The main results

Our result is based on the use of the Wasserstein distance with quadratic cost. A remarkable result, due to Tanaka [Tan79], is that in the Maxwellian case, that is when $\Phi \equiv 1$, $t \mapsto \mathcal{W}_2(f_t, \tilde{f}_t)$ is nonincreasing for each pair of reasonnable solutions $f, \tilde{f}$ to the Boltzmann equation.
Our main result is the following inequality.

**Theorem 1.6.** *Assume* **(A1-A2-A3-A4**$(\gamma)$**)** *for some* $\gamma \in (-3, 0]$*. Let us consider two weak solutions* $(f_t)_{t\in[0,T]}, (\tilde{f}_t)_{t\in[0,T]}$ *to (1.1) lying in* $L^\infty([0,T], \mathcal{P}_2(\mathbb{R}^3)) \cap L^1([0,T], \mathcal{J}_\gamma)$*. Assume furthermore than for all* $t \in [0,T]$*,* $f_t$ *(or* $\tilde{f}_t$*) has a density with respect to the Lebesgue measure on* $\mathbb{R}^3$*. There exists a constant* $K = K(\kappa_1, \kappa_2, \kappa_3)$ *such that for all* $t \in [0,T]$*,*

$$\mathcal{W}_2(f_t, \tilde{f}_t) \leq \mathcal{W}_2(f_0, \tilde{f}_0) \exp\left( K \int_0^t J_\gamma(f_s + \tilde{f}_s) ds \right),$$

*where* $\mathcal{W}_2(.,.)$ *is the Wasserstein distance with quadratic cost.*

Observe here that the technical assumption that $f_t$ has a density can easily be removed, provided one has some uniform estimates on $J_\gamma(f_t)$, as will be the case in the applications below.

We first give some application to the case of mollified velocity cross sections.

**Corollary 1.7.** *Assume* **(A1-A2-A3-A4**$(0)$**)**. *For any* $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$*, any* $T > 0$*, there exists a unique weak solution* $(f_t)_{t\in[0,T]} \in L^\infty([0,T], \mathcal{P}_2(\mathbb{R}^3))$ *to (1.1). Furthermore, there exists a constant* $K = K(\kappa_1, \kappa_2, \kappa_3)$ *such that for any pair of weak solutions* $(f_t)_{t\in[0,T]}$ *and* $(\tilde{f}_t)_{t\in[0,T]}$ *to (1.1) in* $L^\infty([0,T], \mathcal{P}_2(\mathbb{R}^3))$*, any* $t > 0$*,*

$$\mathcal{W}_2(f_t, \tilde{f}_t) \leq \mathcal{W}_2(f_0, \tilde{f}_0) e^{Kt}.$$

We now apply our inequality to the case of soft potentials.

**Corollary 1.8.** *Assume* **(A1-A2-A3-A4**$(\gamma)$**)** *for some* $\gamma \in (-3, 0]$*, and let* $p \in (3/(3+\gamma), \infty)$*.*
*(i) For any pair of weak solutions* $(f_t)_{t\in[0,T]}$*,* $(\tilde{f}_t)_{t\in[0,T]}$ *to (1.1) lying in* $L^\infty([0,T], \mathcal{P}_2(\mathbb{R}^3)) \cap L^1([0,T], L^p(\mathbb{R}^3))$*, there holds*

$$\forall\, t \in [0,T], \qquad \mathcal{W}_2(f_t, \tilde{f}_t) \leq \mathcal{W}_2(f_0, \tilde{f}_0)\, \exp\left( K_p \int_0^t [1 + \|f_s\|_{L^p(\mathbb{R}^3)} + \|\tilde{f}_s\|_{L^p(\mathbb{R}^3)}] ds \right)$$

*where* $K_p$ *depends only on* $\gamma, p, \kappa_1, \kappa_2, \kappa_3$*. Uniqueness and stability with respect to the initial condition thus hold in* $L^\infty([0,T], \mathcal{P}_2(\mathbb{R}^3)) \cap L^1([0,T], L^p(\mathbb{R}^3))$*.*
*(ii) For any* $f_0 \in \mathcal{P}_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$*, there exists* $T_* = T_*\big(\|f_0\|_{L^p(\mathbb{R}^3)}, p, \gamma, \kappa_1, \kappa_2, \kappa_3\big) > 0$ *such that there exists a unique weak solution* $(f_t)_{t\in[0,T_*)}$ *to (1.1) lying in* $L^\infty_{\mathrm{loc}}\big([0,T_*), \mathcal{P}_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)\big)$*.*

# 3 Well-posedness of the spatially homogeneous Landau equation

We consider now the spatially homogeneous Landau equation of kinetic theory, and provide some well-posedness results from a differential inequality for the Wasserstein distance with quadratic cost between two solutions. The main difficulty is that this equation presents a singularity for small relative velocities. Our uniqueness result is the first one in the important case of soft potentials. Furthermore, it is almost optimal for a class of moderately soft potentials, that is for a moderate singularity. Indeed, in such a case, our result applies for initial conditions with finite mass, energy, and entropy. For the other moderately soft potentials, we assume additionnally some moment conditions on the initial data. For very soft potentials, we obtain only a local (in time) well-posedness result, under some integrability conditions. Our proof is probabilistic, and uses a stochastic version of the Landau equation, in the spirit of Tanaka [Tan79].

## 3.1 Existing results and goals

We study here the question of uniqueness (and stability with respect to the initial condition). This question is of particular importance, since uniqueness is needed to justify the derivation of the equation, the convergence from the Boltzmann equation to the Landau equation, the convergence of some numerical schemes, ...
Villani [Vil98a] has obtained uniqueness for Maxwell molecules, and this was extended by Desvillettes-Villani [DV00] to the case of hard potentials. We adapt for this section the ideas of [FG08] and also [FM09, DM09]. We essentially prove here that uniqueness and stability hold in the following situations:

(a) for $\gamma \in (\gamma_0, 0)$, with $\gamma_0 = 1 - \sqrt{5} \simeq -1.236$, as soon as $f_0$ satifies the sole physical assumptions of finite mass, energy and entropy;
(b) for $\gamma \in (-2, \gamma_0]$, as soon $f_0$ has finite mass, energy, entropy, plus some finite moment of order $q(\gamma)$ large enough;
(c) for $\gamma \in (-3, -2]$, as soon $f_0$ has a finite mass, energy, and belongs to $L^p$, with $p > 3/(3+\gamma)$, and the result is local in time.

Observe that (a) is extremely satisfying, and (c) is quite disappointing.
Observe also that we obtain some much better results than for the Boltzmann equation [FM09, FG08], where well-posedness is proved in the following cases: for $\gamma \in (-0.61, 0)$ and $f_0$ with finite mass, energy, entropy; for $\gamma \in (-1, -0.61)$ and $f_0$ with finite mass, energy, entropy, and a moment of order $q(\gamma)$ sufficiently large, and for $\gamma \in (-3, -1]$ and $f_0 \in L^p$ with finite mass and energy, with $p > 3/(3+\gamma)$, and the result being local in time in the latest case.

In [Vil98b], Villani proved several results on the convergence of the Boltzmann equation to the Landau equation. His results work *up to extraction of a subsequence*. Of course, our

uniqueness result allows us to get a true convergence.

### 3.2 A general inequality

Our main result reads as follows.

**Theorem 1.9.** *Let $T > 0$ and $\gamma \in (-3, 0)$. Consider two weak solutions $f$ and $\tilde{f}$ to the Landau equation (1.11) such that $f, \tilde{f} \in L^\infty\big([0,T], \mathcal{P}_2\big) \cap L^1([0,T], \mathcal{J}_\gamma)$. We also assume that $f_t$ (or $\tilde{f}_t$) has a density with respect the Lebesgue measure on $\mathbb{R}^3$ for each $t \in [0,T]$. Then there exists an explicit constant $C_\gamma > 0$, depending only on $\gamma$, such that for all $t \in [0,T]$,*

$$\mathcal{W}_2^2(f_t, \tilde{f}_t) \leq \mathcal{W}_2^2(f_0, \tilde{f}_0) \exp\left( C_\gamma \int_0^t (J_\gamma(f_s) + J_\gamma(\tilde{f}_s)) dt \right),$$

*where $\mathcal{W}_2(.,.)$ is the Wasserstein distance with quadratic cost.*

Observe that if the initial entropy $H(f_0)$ is finite, then $H(f_t) < \infty$ for all $t \geq 0$, so that $f_t$ has a density for all times. Thus this result always applies for solutions with finite entropy. This result asserts that uniqueness and stability hold in $L^\infty\big([0,T], \mathcal{P}_2\big) \cap L^1([0,T], \mathcal{J}_\gamma)$. Using Remark 1.4, we immediately deduce that uniqueness and stability also hold in $L^\infty\big([0,T], \mathcal{P}_2\big) \cap L^1([0,T], L^p)$, as soon as $p > 3/(3+\gamma)$.

### 3.3 Applications

We now show the existence of solutions in $L^\infty\big([0,T], \mathcal{P}_2\big) \cap L^1([0,T], L^p)$.

**Corollary 1.10.** *(i) Assume that $\gamma \in (-2, 0)$. Let $q(\gamma) := \gamma^2/(2+\gamma)$. Let $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ satisfy also $H(f_0) < \infty$ and $m_q(f_0) < \infty$, for some $q > q(\gamma)$. Consider $p \in (3/(3+\gamma), (3q-3\gamma)/(q-3\gamma)) \subset (3/(3+\gamma), 3)$. Then the Landau equation (1.11) has an unique weak solution in $L^\infty\big([0,T], \mathcal{P}_2\big) \cap L^1([0,T], L^p)$.*
*(ii) Assume that $\gamma \in (-3, 0)$, and let $p > 3/(3+\gamma)$. Let $f_0 \in \mathcal{P}_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$. Then there exists $T^* > 0$ depending on $\gamma, p, ||f_0||_p$ such that there is an unique weak solution in $L^\infty_{loc}([0,T_*), L^p)$.*

Observe that if $\gamma \in (1-\sqrt{5}, 0) \simeq (-1.236, 0)$, then $q(\gamma) < 2$, and thus we obtain the well-posedness for the Landau equation under the physical assumptions of finite mass, energy, and entropy. This is of course extremely satisfying.

## 4 Mean field interacting particle system for the Landau type processes

With Sylvie Méléard and Joaquin Fontbona, in [FGM09], we have worked on the construction of an easily simulable mean field interacting particle system converging in law towards

processes solution of nonlinear differential equation of type Landau processes (1.15), with an explicit pathwise rate. In a second time, we have generalized the result to a class of nonlinear processes in [FGM10]. This work is somehow an adaptation of techniques introduced by Méléard-Rœlly in [MRC88].
Almost simultaneously, Nicolas Fournier, [Fou09] worked also on a mean field interacting particle system towards Landau processes (4.1). The inventiveness of its results was to work with a simpler particle system with a better estimation of its rate of convergence.

This question was of great interest in order to construct a tractable simulation algorithm for the law $P_t$ and thus, in particular, for solutions $f$ of equation (1.9). At that time, there was no result on convergence rates of the deterministic numerical methods used for the Landau equation, which are reviewed in [CM03]. The interest of our approach, [FGM09], and the one of Fournier, [Fou09], is that they are based on the diffusive nature of the equation, and that they address a large class of nonlinear processes.
We consider the following nonlinear differential equation in dimension $d \geq 2$:

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}^d} \sigma(X_s - y) W_P(dy, ds) + \int_0^t \int_{\mathbb{R}^d} b(X_s - y) P_s(dy) ds \tag{4.1}$$

where $P_t$ is the law of $X_t$, and $W_P$ is a $\mathbb{R}^d$ valued space-time white noise on $[0,T] \times \mathbb{R}^d$ with independent coordinates, each of which having covariance measure $P_t(dy) \otimes dt$, in the sense of Walsh [Wal86]. The stochastic process associated to the Landau equation (1.9), defined in (1.15), is solution of a such nonlinear equation for a particular choice of $b$ and $\sigma$. That explains that solutions of equations of the form (4.1) will be called Landau type processes. The nonlinear process (4.1) was introduced by Funaki [Fun84], who obtained existence and uniqueness results for Lipschitz coefficients $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \otimes d}$ and $b : \mathbb{R}^d \to \mathbb{R}^d$, see also Guérin [Gué03, Gué02] for a different approach in the case of the Landau process.

The fact that we want to deal with simulable systems will necessitate a coupling between finite dimensional and infinite dimensional stochastic processes. With Sylvie Méléard and Joaquin Fontbona, we introduce a coupling argument based on results on measurability of the optimal mass transportation problem, [FGM09, FGM10].

## 4.1 An interacting particles system

We consider a particle system which is naturally related to the nonlinear process (4.1). Indeed, we notice that the diffusion matrix associated with the process (4.1) can be written on $\mathbb{R}^d$ as

$$A(x, P_t) := \int_{\mathbb{R}^d} \sigma(x - y) \sigma^*(x - y) P_t(dy) \tag{4.2}$$

where $\sigma^*$ is the adjoint matrix of $\sigma$. Thus, if in order to approximate the white noise driven stochastic differential equation (4.1), we heuristically replace $P_t$ in (4.2) by an empirical

measure of $n \in \mathbb{N}^*$ particles in $\mathbb{R}^d$, we are led to consider the following system driven by $n^2$ independent Brownian motions $(B^{ik})$:

$$X_t^{i,n} = X_0^i + \frac{1}{\sqrt{n}} \int_0^t \sum_{k=1}^n \sigma(X_s^{i,n} - X_s^{k,n}) dB_s^{ik} + \frac{1}{n} \int_0^t \sum_{k=1}^n b(X_s^{i,n} - X_s^{k,n}) ds, \ i \in \{1, \dots, n\}. \tag{4.3}$$

To be more precise, if $\mu_t^n = \frac{1}{n} \sum_{i=1}^n \delta_{X_t^{i,n}}$ is the empirical measure of the system, the mappings

$$g(t, \omega, x) \mapsto \frac{1}{\sqrt{n}} \int_0^t \sum_{k=1}^n g(s, \omega, X_s^{k,n}) dB_s^{ik}, i = 1, \dots, n, \tag{4.4}$$

define (for suitably measurable functions $g$) orthogonal martingale measures in the sense of Walsh [Wal86], with covariance measure $\mu_t^n \otimes dt$.

By adapting techniques of Méléard-Rœlly [MRC88] based on martingale problems, we show propagation of chaos for system (4.3) with as limit the process (4.1). This says in particular that the covariance measure of (4.4) converges in law to $P_t \otimes dt$ when $n$ goes to infinity. But in turn, the arguments of [MRC88] do not give any information on the speed of convergence. To estimate the distance between the law of the particles and the law of the nonlinear process, we need to construct a significant coupling between finitely many Brownian motions and the white noises processes. This problem is much more subtle than in the McKean-Vlasov model (cf. Sznitman [Szn91] or Méléard [Mél96]), where each particle is coupled with a limiting process through a single Brownian motion that drives them both. The well known $\frac{1}{\sqrt{n}}-$ convergence rate in that model is consequence of the standard $L^2$-law of large numbers in $\mathbb{R}^d$ and of the fact that the diffusion and drift coefficients of the nonlinear process depend linearly on the limiting law through expectations with respect to it. In the present model, we have to deal with the space-time random fields (4.4), which have fluctuations of constant order in $n$. This is also reflected in the fact that it is the *squared* diffusion matrix (4.2) of the Landau process (4.1), that depends linearly on $P_t$. It is hence not clear where a convergence rate can be deduced from.

## 4.2 Convergence to the nonlinear Landau type process

We considered $n$ independent copies $(X^i)_{i \in \{1,\dots,n\}}$ of the nonlinear process (4.1) in $\mathbb{R}^d$ in some probability space:

$$X_t^i = X_0^i + \int_0^t \int_{\mathbb{R}^d} \sigma(X_s^i - y) W^i(dy, ds) + \int_0^t \int_{\mathbb{R}^d} b(X_s^i - y) P_s(y) dy ds$$

and $\nu_t^n$ their empirical measure at time $t$ (observe that it samples $P_t$). We construct particles (4.3) on the same probability space, in such way that they will converge pathwise in $L^2$ on finite time intervals, at the same rate at which the Wasserstein distance $\mathcal{W}_2$ between $P_t$ and $\nu_t^n$ goes to 0. Let us present our main result.

**Theorem 1.11.** *Let $n \in \mathbb{N}$ and assume usual Lipschitz hypothesis on $\sigma$ and $b$, and that the law $P_0$ of $X_0^i$ has finite second order moment. Assume moreover that $P_t$ has a density with respect to Lebesgue measure for each $t > 0$.*
*Then, in the same probability space as $(X^1, \dots, X^n)$ there exist independent standard Brownian motions $(B^{ik})_{1 \le i,k \le n}$ such that the particle system $(X^{i,n})_{i=1}^n$ defined in (4.3) satisfies*

$$E\left(\sup_{t\in[0,T]} |X_t^{i,n} - X_t^i|^2\right) \le C \exp(C'T) \int_0^T E\big(\mathcal{W}_2^2(\nu_s^n, P_s)\big) ds$$

*for constants $C, C'$ that do not depend on $n$.*

Thanks to available convergence results for empirical measures of i.i.d samples (see e.g. [RR98a, RR98b]), Theorem 1.11 will allow us to obtain, under some additional moment assumptions on $P_0$, the speed of convergence $n^{\frac{-2}{d+4}}$ for the pathwise law of the system (see [FGM09, Corollary 6.2]). We remark that the absolute continuity condition of Theorem 1.11 can be obtained under non-degeneracy of the matrix $\sigma\sigma^*$ by using for instance Malliavin calculus [Nua06]; it is also true for the specific coefficients of the Landau equation (1.9) despite their degeneracy, and for some generalizations (see Guérin [Gué02]).

The proof of Theorem 1.11 relies on new results on the optimal mass transportation problem. For general background on the theory of mass transportation, we refer to Villani [Vil03]. Recall that if $\mu$ and $\nu$ are probability measures in $\mathbb{R}^d$ with finite second moment, the first of them having a density, then the optimal mass transportation problem with quadratic cost between $\mu$ and $\nu$ has a unique solution, which is a probability measure on $\mathbb{R}^{2d}$ of the form $\pi(dx, dy) = \mu(dx)\delta_{T(x)}(dy)$ . The so-called Brenier or optimal transport map $T(x)$ is ($\mu$ a.s. equal to) the gradient of some convex function in $\mathbb{R}^d$, and pushes forward $\mu$ to $\nu$.

Let now $W_P^i$ be the white noise process driving the $i$-th nonlinear process $X^i$. The key idea in Theorem 1.11 is to construct Brownian motions $(B^{ik})_{k\in\{1,\dots,n\}}$ in an "optimal" pathwise way from $W_P^i$. Heuristically, this will consist in pushing forward the martingale measure $W_P^i$ through the Brenier maps $T^{t,\omega,n}(x)$ realizing the optimal transport between $P_t$ and $\nu_t^n(\omega)$ (this is the reason for the absolute continuity assumption on $P_t$). But to give such a construction a rigorous sense, we must make sure that we can compute stochastic integrals of $T^{t,\omega,n}(x)$ with respect to $W_P^i(dx, dt)$. From the basic definition of stochastic integration with respect to space-time white noise ([Wal86]), this requires the existence of a measurable version of $(t, \omega, x) \mapsto T^{t,\omega,n}(x)$ being moreover predictable in $(t, \omega)$. A striking problem then is that no available result in the mass transportation theory can provide any information about joint measurability properties of the optimal transport map, with respect to the space variable and some parameter making the marginals vary. Nevertheless, we show that a suitable "predictable transportation process" exists:

**Theorem 1.12.** *There exists a measurable process $(t,\omega,x)\mapsto T^n(t,\omega,x)$ that is predictable in $(t,\omega)$ with respect to the filtration associated to $(W_P^1,\dots,W_P^n)$ and $(X_0^1,\dots,X_0^n)$, and such that for $dt\otimes\mathbb{P}(d\omega)$ almost every $(t,\omega)$,*

$$T^n(t,\omega,x)=T^{t,\omega,n}(x)\quad P_t(dx)\text{-almost surely.}$$

This statement is a consequence of a general abstract result about "measurability" of the mass transportation problem. To be more explicit, recall that the optimality of a transfert plan $\pi$ is determined by its support (it is equivalent to the support being cyclically monotone, see McCann [McC95] or Villani [Vil03]). On the other hand, without assumptions (besides moments) on the marginals $\mu$ and $\nu$, the solution $\pi$ of the mass transportation problem may not be unique. A basic question then is how to formulate, in a general setting, the adequate property of "measurability" of the solution(s) $\pi$ with respect to the data $(\mu,\nu)$. The natural formulation requires indeed to introduce notions and techniques from set-valued analysis. Then, we prove the following result.

**Theorem 1.13.** *Let $\mathcal{P}_2(\mathbb{R}^d)$ be the space of Borel probability measures in $\mathbb{R}^2$ with finite second order moment, endowed with the Wasserstein distance and its Borel $\sigma-$field. Denote by $\Pi^*(\mu,\nu)$ the set of solutions of the mass transportation problem with quadratic cost associated with $(\mu,\nu)\in(\mathcal{P}_2(\mathbb{R}^d))^2$. The function assigning to $(\mu,\nu)$ the* **set** *of $\mathbb{R}^{2d}$:*

$$\bigcup_{\pi\in\Pi^*(\mu,\nu)} supp(\pi),$$

*is measurable in the sense of set-valued mappings.*

In particular, this ensures that if $\mu_\lambda$ and $\nu_\lambda$ vary in a measurable way with respect to some parameter $\lambda$, so that in each of the associated optimal transportation problems uniqueness holds, then the support of the solution $\pi_\lambda$ also "varies" in a measurable way.

Theorem 1.12 has been generalized by Fontbona, Guérin, Méléard in [FGM10] for a more general class of cost than the quadratic cost. We just give in the next section the main result.

### 4.3 Mesurability of optimal transportation for a general cost

Let $\mathcal{P}(\mathbb{R}^d)$ be the space of Borel probability measures in $\mathbb{R}^d$ endowed with the usual weak topology, and $\mathcal{P}_p(\mathbb{R}^d)$ the subspace of probability measures having finite $p-$order moment. Given $\pi\in\mathcal{P}(\mathbb{R}^{2d})$, we write

$$\pi<^{\mu}_{\nu}$$

if $\mu,\nu\in\mathcal{P}(\mathbb{R}^d)$ are respectively its first and second marginals. Such $\pi$ is referred to as a "transference plan" between $\mu$ and $\nu$.

Let $c : \mathbb{R}^{2d} \to \mathbb{R}_+$ be a continuous function. The mapping

$$\pi \to I(\pi) := \int_{\mathbb{R}^{2d}} c(x,y)\pi(dx,dy)$$

is then lower semi continuous. The Monge-Kantorovich or optimal mass transportation problem with cost $c$ and marginals $\mu$, $\nu$ consists in finding

$$\inf_{\pi <^{\mu}_{\nu}} I(\pi).$$

It is well known that the infimum is attained as soon as it is finite, see [Vil03, Ch.1]. In this case, we denote by $\Pi_c^*(\mu,\nu)$ the subset of $\mathcal{P}(\mathbb{R}^{2d})$ of minimizers. If otherwise, $I(\pi) = +\infty$ for all $\pi <^{\mu}_{\nu}$, then by convention we set $\Pi_c^*(\mu,\nu) = \emptyset$.

We shall say that **Assumption** $H(\mu,\nu,c)$ holds if

> there exists a unique optimal transference plan $\pi \in \Pi_c^*(\mu,\nu)$, and it has the form
>
> $$\pi(dx,dy) = \mu(dx) \otimes \delta_{T(x)}(dy)$$
>
> for a $\mu(dx) - a.s.$ unique mapping $T : \mathbb{R}^d \to \mathbb{R}^d$.

Such $T$ is called an *optimal transport map between $\mu$ and $\nu$ for the cost function $c$.*
We recall that the condition that $\mu$ does not give mass to sets with Hausdorff dimension smaller than or equal to $d-1$ is optimal both for existence and uniqueness of $T$, see Remark 9.5 in [Vil09]. Moreover, if $\Pi_c^*(\mu,\nu) \neq \emptyset$, then the latter condition implies $H(\mu,\nu,c)$ in the following situations (see Gangbo and McCann [GM96]):

i) $c(x,y) = \tilde{c}(|x-y|)$ with $\tilde{c} : \mathbb{R}_+ \to \mathbb{R}_+$ strictly convex, superlinear and differentiable with locally Lipschitz gradient.

ii) $c(x,y) = \tilde{c}(|x-y|)$ with $\tilde{c}$ strictly concave, and $\mu$ and $\nu$ are mutually singular.

Condition $H(\mu,\nu,c)$ also holds if

iii) $c(x,y) = \tilde{c}(|x-y|)$ with $\tilde{c}$ strictly convex and superlinear, and moreover $\mu$ is absolutely continuous with respect to Lebesgue measure.

When $\mu,\nu \in \mathcal{P}_p(\mathbb{R}^d)$, fundamental examples are the cost function $c(x,y) = |x-y|^p$ with $p \geq 2$ for case i), $p > 1$ for case iii), and $p \in (0,1)$ for case ii).

The main result of [FGM10] is the following generalization of Theorem 1.12.

**Theorem 1.14.** *Let $(E,\Sigma,m)$ be a $\sigma-$finite measure space and consider a measurable function $\lambda \in E \mapsto (\mu_\lambda,\nu_\lambda) \in (\mathcal{P}(\mathbb{R}^d))^2$ such that for $m-$almost every $\lambda$, $H(\mu_\lambda,\nu_\lambda,c)$ holds,*

*with optimal transport map* $T_\lambda : \mathbb{R}^d \to \mathbb{R}^d$*. Then, there exists a function* $(\lambda, x) \mapsto T(\lambda, x)$ *which is measurable with respect to* $\Sigma \otimes \mathcal{B}(\mathbb{R}^d)$ *and such that* $m(d\lambda)-$*almost everywhere,*

$$T(\lambda, x) = T_\lambda(x) \quad \mu_\lambda(dx)\text{-almost surely.}$$

*In particular,* $T_\lambda(x)$ *is measurable with respect to the completion of* $\Sigma \otimes \mathcal{B}(\mathbb{R}^d)$ *with respect to* $m(d\lambda)\mu_\lambda(dx)$*.*

# Chapter 2

# Long time behavior of Piecewise Deterministic Markov Processes A model in biology : chemotaxis of a flagellated bacterium

We consider in this chapter a particular Piecewise Deterministic Markov Process (PDMP) which can be seen as a toy model of chemotaxis. We study the long time behavior of this process in an ergodic situation and obtain a control of the total variation distance to equilibrium at each instant of time. These results rely on an exact description of the excursions of the process away from the origin and on the explicit construction of an original coalescent coupling for both velocity and position.
The results presented in this chapter were obtained in collaboration with Joaquin Fontbona, Universidad de Chile, and Florent Malrieu, UniversitÃl' de Tours. The case of constant jump rates has been published in Advances in Applied Probability, *Quantitative estimates for the long-time behavior of an ergodic variant of the telegraph process*, [FGM12], and the general case has been published in Stochastic Processes and their Applications, *Long time behavior of telegraph processes under convex potentials*, [FGM16].

## 1 The model

Chemotaxis is the movement of an organism in response to a chemical stimulus. Bacteria or cellular organisms direct their movements according to certain chemicals in their environment. This is important for bacteria to find nutriments by swimming toward the highest concentration of food molecules, or to flee from poisons. Generally, the motion of flagellated bacteria consist of a sequence of *run* phases and *tumble* phases.
PDMPs have been introduced by Davis to distinguished those processes from diffusions and have been extensively studied in the last two decades (see [Dav84, Dav93, Jac06] for general background) and have recently received renewed attention, motivated by their natural application in areas such as biology [RS13, DHKR15], toxicology [BCT09, BCT13, Bou15], communication networks [DGR02, BCG$^+$13] or reliability of complex systems, to name a few. Understanding the ergodic properties of these models, in particular the rate at which they stabilize towards equilibrium, has in turn increased the interest in the long-time behavior of PDMPs.
We present here the study of these questions on PDMP models of bacterial chemotaxis, initiated by Othmer and Erban in [EO05a, EO05b] by means of analytic tools, and deepened in [FGM12, FGM16] and [MM13] on simplified versions that can be seen as variants of Kac's classic "telegraph process" [Kac74]. We consider the simple PDMP of kinetic type $(Z_t)_{t\geq 0} = ((Y_t, W_t))_{t\geq 0}$ with values in $\mathbb{R} \times \{-1, +1\}$ and infinitesimal generator

$$Lf(y,w) = w\partial_y f(y,w) + \big(a(y)\mathbb{1}_{\{yw\leq 0\}} + b(y)\mathbb{1}_{\{yw>0\}}\big)(f(y,-w) - f(y,w)), \qquad (1.1)$$

where $a$ and $b$ are nonnegative functions in $\mathbb{R}$. That is, the continuous component $Y$ evolves according to $\frac{dY_t}{dt} = W_t$ and represents the position of a particle on the real line, whereas the discrete component $W$ represents the velocity of the particle and jumps between $+1$ and $-1$, with instantaneous state-dependent rate given by $a(y)$ (resp. $b(y)$) if the particle at position $y$ approaches (resp. goes away from) the origin. This process describes, in a

naive way, the motion of flagellated bacteria as a sequence of linear "runs", the directions of which randomly change at rates that depend on the position of the bacterium. The emergence of macroscopical drift is expected when the response mechanism favors longer runs in specific directions (reflecting the propensity to move for instance towards a source of nutriments). We refer the reader to [RS13] for a scaling limit of the processes introduced in [EO05a, EO05b] that leads to simplified models like (1.1), to [MM13] for the study of related models in the circle, relying on a spectral decomposition, and to [Mon14] for a general approach to some kinetic models including the above one, based on functional inequalities.
More recently, this process has been studied by Bierkens and Roberts, in [BR17], the case of velocities taken their values uniformly in $[-1, 1]$, has been considered by Calvez, Raoul and Schmeiser in [CRS15], and the case of higher dimension has been studied by Bierkens, Fearnhead and Roberts in [BFR16] and in a more general case by FÃl'tique in [Fét17].
In this presentation we will consider position dependent jump-rates which throughout will be assumed to satisfy:

**Hypothesis 2.1.** *Function $b$ (resp. $a$) is measurable, even, non decreasing (resp. non increasing) on $[0, +\infty)$, bounded from below by $\underline{b} > 0$ (resp. $\underline{a} > 0$). Moreover we assume that $b(y) > a(y)$ for all $y \neq 0$.*

In the sequel, $\bar{b}$ stands for $\sup_{y>0} b(y) \in [\underline{b}, \infty]$ and $\mathrm{sgn} : \mathbb{R} \to \{-1, +1\}$ denotes de function

$$\mathrm{sgn}(y) = \mathbb{1}_{\{y \geq 0\}} - \mathbb{1}_{\{y<0\}}.$$

The case where the rates are constant functions with $a = b$ corresponds to the classical *telegraph process* in $\mathbb{R} \times \{-1, +1\}$ introduced by Kac [Kac74], in which case the density of $(Y_t)$ solves the damped wave equation

$$\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} + a\frac{\partial p}{\partial t} = 0$$

called the *telegraph equation*. The telegraph process, as well as its variants and its connections with the so-called persistent random walks have received considerable attention both in the physical and mathematical literature (see e.g. [HV10] for historical references and for some recent probabilistic developments). It is well known that $(Y_t)_{t\geq 0}$ converges when $a = b$ to the standard one dimensional Brownian motion in the suitable scaling limit. This result can be generalised to a process with infinitesimal generator (1.1). Indeed, under the suitable scaling, a such process (1.1) behaves like a diffusion processes, see Theorem 1.5 in [FGM16] in the general case and Proposition 1.1 in [FGM12] for a simpler version in the case constant jump rate.
Under Hypotheses 2.1, a particle driven by (1.1) spends in principle more time moving towards the origin than away from it. Thus, a macroscopic attraction to the origin should take place in the long run, though in a consistent way with the fact that the particle has

constant speed. Our aim is to clarify this picture by determining the invariant measure $\mu$ of $(Y, W)$ under Hypotheses 2.1, and obtaining bounds, as quantitative as possible (*i.e.* estimates that are explicit functions of the parameters $a$ and $b$), for the convergence to its invariant measure $\mu$ of the law of $(Y_t, W_t)$ as $t$ goes to infinity.
We first point out the explicit form of the equilibrium of the process $(Y, W)$ by noting that the following measure satisfies

$$\int Lf(y, w)\, \mu(dy, dw) = 0.$$

**Proposition 2.2** (Invariant distribution, [FGM16, Proposition 1.3])**.** *The invariant distribution of $(Y, W)$ on $\mathbb{R} \times \{-1, +1\}$ is given by*

$$\mu(dy, dw) = \frac{1}{C_F} e^{-F(y)} dy \otimes \frac{1}{2}(\delta_{-1} + \delta_{+1})(dw)$$

*where $C_F := \int_{\mathbb{R}} e^{-F(y)}\, dy < \infty$ and $F$ is the convex function*

$$y \in \mathbb{R} \mapsto F(y) = \int_0^y \mathrm{sgn}(z)(b(z) - a(z))\, dz. \tag{1.2}$$

*The domain of the Laplace transform of $\mu$ is $(-\bar{b} + \underline{a}, \bar{b} - \underline{a}) \times \{-1, +1\}$.*

Since the invariant measure is explicitly known in terms of the jump rates, one can construct Monte-Carlo algorithms based on the Zig-Zag process. Let us mention the works of Bierkens and his co-authors, for example [BR17, BFR16].

**Example 2.3** (Laplace and Gaussian equilibria)**.** *If $a$ and $b$ are constant functions, then*

$$\mu(dy, dw) = \frac{b - a}{2} e^{-(b-a)|y|} dy \otimes \frac{1}{2}(\delta_{-1} + \delta_{+1})(dw).$$

*If $a$ is a constant function and $b$ is the map $y \mapsto a + |y|$, then*

$$\mu(dy, dw) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \otimes \frac{1}{2}(\delta_{-1} + \delta_{+1})(dw).$$

*Figures 2.2 and 2.1 compare in the latter case the empirical law of $Y_t$ to its invariant measure at increasing time instants.*

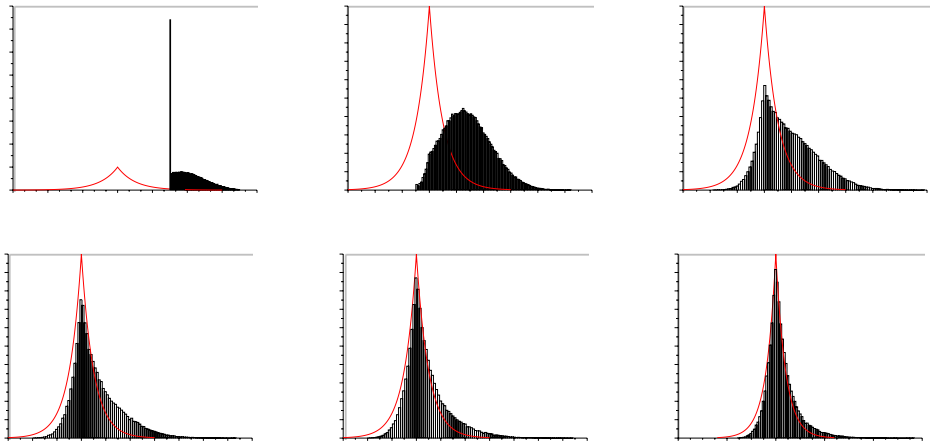


Figure 2.1: Empirical law of $Y_t$ starting at $(5,-1)$ for $t \in \{2,6,10,14,18,22\}$ with $a=1$ and $b=2$.

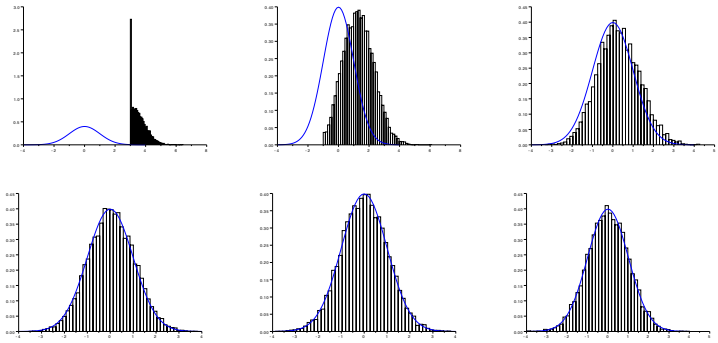


Figure 2.2: Empirical distribution of $Y_t$ starting at $(5,-1)$ for $t \in \{2,6,10,14,18,22\}$ with $a(y)=1$ and $b(y)=1+|y|$.

This chapter is organized as follows: in the following section, we introduce the reflected process, then in Section 3 we introduce a coalescent coupling for two reflected processes starting at different initial data and finally in Section 4 we estimate the coalescent time for two initial processes starting at different initial data and we give an estimate of the speed of convergence to equilibrium.

## 2 The reflected process

From the symmetry of the model with respect to zero, it is easy to see that the reflected process is quite simplest to study. The idea is then to obtain explicit speed of convergence for the reflected process and then deduce estimates for initial process. For constant jump rates, it is quite simple to deduce the speed of convergence for the initial process from the one of the reflected process since the relationship between the reflected process and the initial process is quite explicit. Unfortunately, as we will see in Section 4 of this Chapter, it was more complex than expected to transfer the results to the unreflected process in the general case.
We now introduce the reflected process $(X, V)$. The Markov process $((X_t, V_t))_{t\geq 0}$ is defined by its infinitesimal generator:

$$Af(x,v) = v\partial_x f(x,v) + \left(a(x)\mathbb{1}_{\{v=-1\}} + b(x)\mathbb{1}_{\{v=1\}} + \frac{\mathbb{1}_{\{x=0\}}}{\mathbb{1}_{\{x>0\}}}\right)(f(x,-v) - f(x,v)), \quad (2.3)$$

where the maps $a$ and $b$ satisfy Hypothesis 2.1. The term $\mathbb{1}_{\{x=0\}}(\mathbb{1}_{\{x>0\}})^{-1}$ means that $V$ flips from $-1$ to $+1$ as soon as $X$ hits zero. In other words, $X$ is reflected at zero.
Given a path $((Y_t, W_t))_{t\geq 0}$ driven by (1.1), a path of $((X_t, V_t))_{t\geq 0}$ can be constructed taking

$$X_t = |Y_t|, \quad V_0 = \mathrm{sgn}(Y_0)W_0$$

and defining the set of jump times of $V$ to be

$$\{t > 0 \,:\, \Delta V_t \neq 0\} = \{t > 0 \,:\, \Delta W_t \neq 0\} \cup \{t > 0 \,:\, Y_t = 0\}.$$

Since $W$ does not jump with positive probability when $Y$ hits the origin, one can also construct a path of $((Y_t, W_t))_{t\geq 0}$ from an initial value $y \in \mathbb{R}$ and a path $((X_t, V_t))_{t\geq 0}$ driven by (2.3): set $\sigma_0 = 0$ and $(\sigma_i)_{i\geq 1}$ for the successive hitting times of the origin and

$$(Y_t, W_t) = (-1)^i \mathrm{sgn}(y)(X_t, V_t) \quad \text{if } t \in [\sigma_i, \sigma_{i+1}).$$

## 3 Coalescent coupling for the reflected process

The estimates on speed of convergence to equilibrium relies on a probabilistic coupling argument, reminiscent of Meyn-Tweedie-Foster-Lyapunov techniques, see [MT09, Lin92]. Variants of this type of methods have been developed in several previous works on specific instances of PDMP [BLBMZ12, BCG$^+$13, ABG$^+$14, Mal15]. The model under study in the present paper is harder to deal with, since the vector fields that drive the continuous part are not contractive.

A pair of stochastic processes $(U_t, \tilde{U}_t)_{t\geq 0}$ constructed in the same probability space, for which an almost surely finite random time $T$ satisfying $U_{t+T} = \tilde{U}_{t+T}$ for any $t \geq 0$ exists,

is called a coalescent coupling. The random variable

$$T_* = \inf\Big\{t \geq 0 : U_{t+s} = \tilde{U}_{t+s}\, \forall s \geq 0\Big\}$$

is then called the *coupling time*. It follows in this case that the total variation distance, defined in Section 2.2, is controled from above by

$$\Big\|\mathcal{L}(U_t) - \mathcal{L}(\tilde{U}_t)\Big\|_{\mathrm{TV}} \leq \mathbb{P}(T_* > t).$$

A helpful notion in obtaining an effective control of the distance is *stochastic domination* (see [Lin92] for a complete introduction).

**Definition 2.4.** *Let $S$ and $T$ be two non-negative random variables with respective cumulative distribution functions $F$ and $G$. We say that $S$ is stochastically smaller than $T$ and we write $S \leq_{\mathrm{sto.}} T$, if $F(t) \geq G(t)$ for any $t \in \mathbb{R}$.*

In particular, for a couple $(U_t, \tilde{U}_t)$ as above, Chernoff's inequality yields

$$\Big\|\mathcal{L}(U_t) - \mathcal{L}(\tilde{U}_t)\Big\|_{\mathrm{TV}} \leq \mathbb{P}(T > t) \leq \mathbb{E}\Big(e^{\lambda T}\Big)e^{-\lambda t} \tag{3.4}$$

for any non-negative random variable $T$ such that $T_* \leq_{\mathrm{sto.}} T$, and any $\lambda \geq 0$ in the domain of the Laplace transform $\lambda \mapsto \mathbb{E}\big(e^{\lambda T}\big)$ of $T$.
In the sequel, we explain how to construct a coalescent coupling from two trajectories starting at different initial data in order to obtain the exponential convergence to equilibrium for $(Y, W)$ and its reflected version $(X, V)$. We first work with the reflected process and give some properties of its jump times.

## 3.1 Jump times of the reflected process

Let us denote by $T_{(x,v)}$ the first jump time of the stochastic process $(X, V)$ starting from $(X_0, V_0) = (x, v)$ with infinitesimal generator defined by (2.3). This random time satisfies

$$T_{(x,v)} = \inf\left\{t \geq 0 : \int_0^t c(X_s)ds \geq E\right\},$$

where $E$ is an exponential variable with unit mean and $c$ stands for the function $b$ when $v = +1$ and, when $v = -1$,

$$c(x) = \begin{cases} a(x) & \text{for } x > 0, \\ +\infty & \text{for } x \leq 0. \end{cases}$$

The process $(X, V)$ being deterministic between jump times, we have

$$T_{(x,v)} = \inf\left\{t \geq 0 : \int_0^t c(x + vs)ds \geq E\right\}.$$

Consequently, if $B$ is the primitive of $b$ with $B(0) = 0$, we have

$$T_{(x,+1)} = B^{-1}(E + B(x)) - x, \tag{3.5}$$

and if $A$ is the primitive of $a$ with $A(0) = 0$,

$$T_{(x,-1)} = \begin{cases} x - A^{-1}(A(x) - E) & \text{if } E < A(x), \\ x & \text{otherwise.} \end{cases} \tag{3.6}$$

The functions $A^{-1}$ and $B^{-1}$ are well defined since $a$ and $b$ are positive functions.
The distribution of those jump times is explicit in terms of $a$ and $b$, and the domain of their Laplace transform is known, see [FGM16, Section 2.1]. However the following result on jump times is the keypoint of our construction of a coalescent coupling of two reflected processes starting at two different initial data.

**Lemma 2.5** (Decomposition of jump times, part I, [FGM16, Lemma 2.4])**.** *For $x \geq \tilde{x} \geq 0$, the identity in law*

$$T_{(x,+1)} \overset{\mathcal{L}}{=} T_{(\tilde{x},+1)} \wedge Z_+$$

*holds, with $Z_+$ a random variable with values in $(0, \infty]$ independent of $T_{(\tilde{x},+1)}$, such that*

$$\mathbb{P}(Z_+ > t) = \exp\left(-B(x+t) + B(x) + B(\tilde{x}+t) - B(\tilde{x})\right) \text{ for all } t \in [0, \tilde{x}). \tag{3.7}$$

*Moreover, there exists a coupling $(T_{(x,+1)}, T_{(\tilde{x},+1)})$ such that, almost surely,*

$$T_{(\tilde{x},+1)} \geq T_{(x,+1)}$$

*and, conditionally on $\{T_{(x,+1)} = t\}$,*

$$T_{(\tilde{x},+1)} \overset{\mathcal{L}}{=} t + \xi_t \hat{T}_{(\tilde{x}+t,+1)}$$

*with $\hat{T}_{(\tilde{x}+t,+1)} \overset{\mathcal{L}}{=} T_{(\tilde{x}+t,+1)}$ and $\xi_t$ a Bernoulli r.v. independent of $\hat{T}_{(x+t,+1)}$ of parameter*

$$\beta_t := \frac{b(x+t) - b(\tilde{x}+t)}{b(x+t)} \in [0, 1). \tag{3.8}$$

**Remark 2.6.** *When $b$ is a constant function, $\xi_t = 0$ a.s., i.e. we can use the same jump time for both trajectories whatever the initial position is.*

We observe that if $b(x)$ goes to $+\infty$ as $x \to \infty$, we have $Z_+ < +\infty$ a.s., whereas $Z_+ = +\infty$ a.s. if $b(x) = b$ is constant. The function $a$ being non-increasing, we obtain an analogous result for $T_{(x,-1)}$:

**Lemma 2.7** (Decomposition of jump times, part II, [FGM16, Lemma 2.5]). *For $x > \tilde{x} > 0$, the identity in law*

$$T_{(\tilde{x},-1)} \overset{\mathcal{L}}{=} T_{(x,-1)} \wedge Z_-$$

*holds, with $Z_-$ a random variable with values in $(0, \tilde{x}]$ independent of $T_{(x,+1)}$, such that*

$$\mathbb{P}(Z_- > t) = \exp\left(-A(\tilde{x}) + A(\tilde{x} - t) + A(x) - A(x - t)\right) \text{ for all } t \in [0, \tilde{x}). \tag{3.9}$$

*Moreover, there exists a coupling $(T_{(x,-1)}, T_{(\tilde{x},-1)})$ such that, almost surely,*

$$T_{(x,-1)} \geq T_{(\tilde{x},-1)}$$

*and, conditionally on $\{T_{(\tilde{x},-1)} = t\}$,*

$$T_{(x,-1)} \overset{\mathcal{L}}{=} t + \chi_t \hat{T}_{(x-t,-1)}$$

*with $\hat{T}_{(x-t,-1)} \overset{\mathcal{L}}{=} T_{(x-t,+1)}$ and $\chi_t$ a Bernoulli r.v. independent of $\hat{T}_{(x-t,+1)}$ of parameter*

$$\alpha_t := \frac{a(\tilde{x} - t) - a(x - t)}{a(\tilde{x} - t)} \in [0, 1) \quad \text{if } t < \tilde{x} \quad \text{and} \quad \alpha_{\tilde{x}} := 1.$$

**Remark 2.8.** *When $a$ is a constant function, conditionally on $\{T_{(\tilde{x},-1)} = t\}$ with $t < x$, $\chi_t = 0$ a.s. and conditionally on $\{T_{(\tilde{x},-1)} = x\}$, $\chi_t = 1$ a.s., i.e. both trajectories jump at the same time except when the lower trajectory hits 0.*

The proof of those lemmas, given in [FGM16, Section 2.1], is based on the memoryless property of the exponential distribution and the monotonicity property of the jump rates. Indeed, the result can be extended to any jump times of a PDMP such that the jump rate is monotone along the flow.
The idea of those coupling is highlighted in Figure 2.3 : either the Bernoulli r.v. is a fail and both trajectories jump at the same time, either the Bernoulli r.v. is a success and the difference between both jump times has the same distribution as a jump time. The control of this difference is a key point in the construction of a coupling of two trajectories starting from different initial data, as we will see in Section 3.2.
We now have all the tools to construct a coalescent coupling of two reflected processes starting at different initial data.

## 3.2 A Coupling for the reflected processes

Let us consider two initial data $(x, v)$ and $(\tilde{x}, \tilde{v})$, with $x \geq \tilde{x}$. We let evolve both path until their first crossing time, denoted by $T_c := T_c(x, \tilde{x})$. At that time, both path are at the same position $X_{T_c}$ with different velocities.
We now construct two paths which are equal after a coalescent time $T_*$. The successful coupling consists in producing a jump of (exactly) one of the two velocities $V$ or $\tilde{V}$ at a

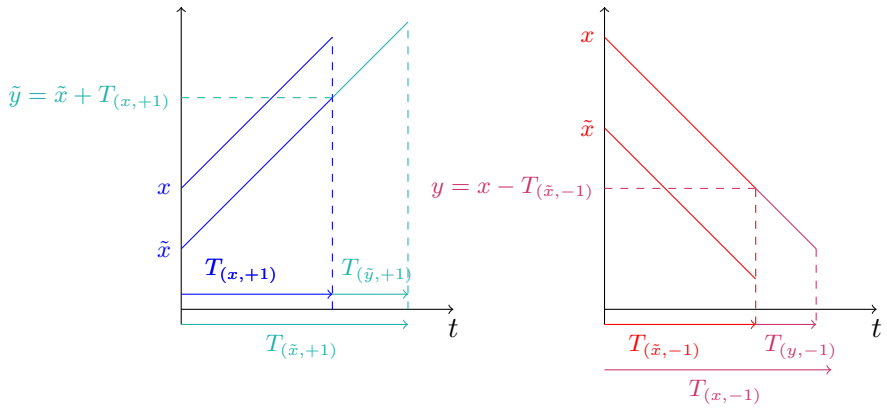



Figure 2.3: Coupling of the jump times.

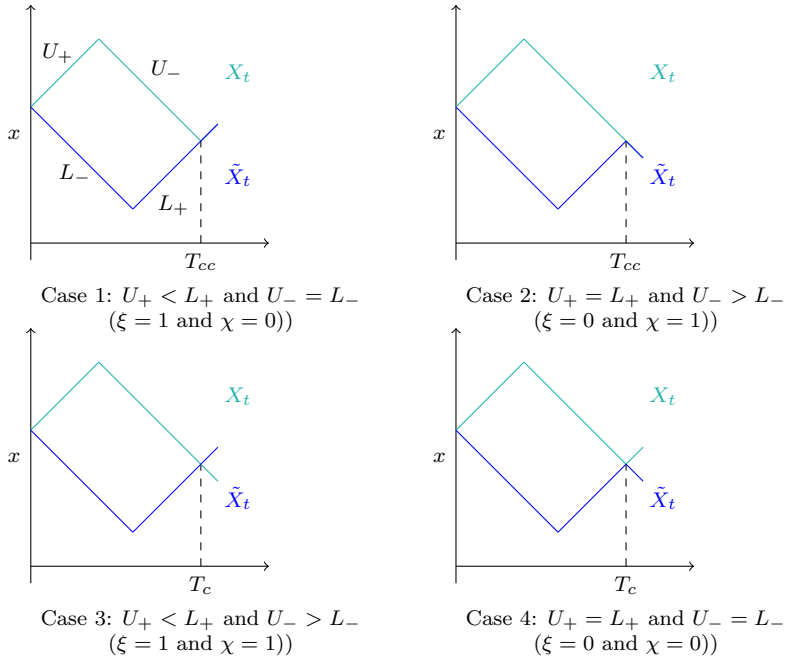
crossing time of their position components $X$ and $\tilde{X}$. We will use the coupled jump-times studied in Lemmas 2.5 and 2.7 in order to minimize the time required to do so.
To be more precise, given $x > 0$ fixed, let us denote by $U_+$ and $U_-$ (respectively $L_-$ and $L_+$) the first and the second inter-jump time lapses of the path starting from $(x, +1)$ (resp. starting from $(x, -1)$). These random variables are constructed as follows. We first choose $U_+$ with distribution $T_{(x,+1)}$ and $L_-$ with distribution $T_{(x,-1)}$ independently. We then define $U_-$ and $L_+$ in such a way that $(U_+, L_+)$ and $(U_-, L_-)$ have the laws of the couplings defined in Lemmas 2.5 and 2.7 respectively and that $L_+ - U_+$ and $U_- - L_-$ are independent conditionally on $(U_+, L_-)$. More precisely, conditionally on $U_+$ and $L_-$, we introduce two independent Bernoulli variables $\xi$ and $\chi$ with

$$\mathbb{P}(\xi = 1 | U_+, L_-) = \frac{b(x + U_+) - b(x - L_- + U_+)}{b(x + U_+)} \quad \text{and}$$
$$\mathbb{P}(\chi = 1 | U_+, L_-) = \frac{a(x - L_-) - a(x + U_+ - L_-)}{a(x - L_-)} \mathbb{1}_{\{L_- < x\}} + \mathbb{1}_{\{L_- = x\}}$$

and two independent random variables $L_+ - U_+$ and $U_- - L_-$ with the same law as $T_{x+U_+-L_-,+1}$ and $T_{x+U_+-L_-,-1}$ respectively. Then we set

$$L_+ := U_+ + \xi(L_+ - U_+) \quad \text{and} \quad U_- := L_- + \chi(U_- - L_-).$$

Figure 2.4 shows the four possible outcomes. Those where exactly one of the Bernoulli variables is equal to 1 allow us to stick the paths at time $U_+ + L_-$ (i.e. on the rightmost corner of the rectangle): the velocities of the two paths are the same right after that instant, and the overshot length (beyond the rectangle's corner) determined by the previous coupling is compatible with the law of the two marginal processes from that moment on (because of their Markov property). We then say that the coupling attempt succeeded,




Case 1: $U_+ < L_+$ and $U_- = L_-$ ($\xi = 1$ and $\chi = 0$))

Case 2: $U_+ = L_+$ and $U_- > L_-$ ($\xi = 0$ and $\chi = 1$))

Case 3: $U_+ < L_+$ and $U_- > L_-$ ($\xi = 1$ and $\chi = 1$))

Case 4: $U_+ = L_+$ and $U_- = L_-$ ($\xi = 0$ and $\chi = 0$))

Figure 2.4: Position of both paths after one step.

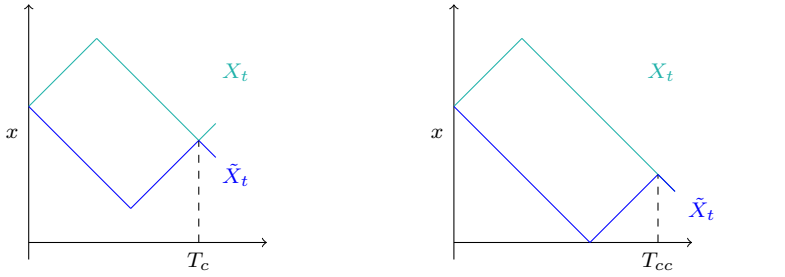
and this happens conditionally on $(U_+, L_-)$ with probability

$$\mathbb{P}(\xi = 0, \chi = 1 | U_+, L_-) + \mathbb{P}(\xi = 1, \chi = 0 | U_+, L_-) = \left[ \frac{b(x - L_- + U_+)}{b(x + U_+)} \left( 1 - \frac{a(x - L_- + U_+)}{a(x - L_-)} \right) + \left( 1 - \frac{b(x - L_- + U_+)}{b(x + U_+)} \right) \frac{a(x - L_- + U_+)}{a(x - L_-)} \right] \mathbb{1}_{\{L_- < x\}} + \frac{b(U_+)}{b(x + U_+)} \mathbb{1}_{\{L_- = x\}}. \tag{3.10}$$

We observe that the success or failure of the coupling attempt is determined by the Bernoulli random variables $\xi$ and $\chi$. If the coupling attempt fails, the two trajectories cross or bounce off of each other at time $U_+ + L_-$ and by similar reason as before the (already determined) lengths $(L_+ - U_+)$ and $(U_- - L_-)$ can be used to restart two (upward and downward) trajectories from $x - L_- + U_+$, independently of each other conditionally on the past and consistently with the pathwise laws of each of the two processes.

A coupling of two reflected processes, starting at the same position with opposite velocities, is now obvious: we repeat this scheme starting from the new crossing or bouncing point, until we succeed in sticking the two paths. Notice that this iterative algorithm is more

efficient than the general procedure of the Meyn-Tweedie method [MT09] since, after a fail, the two processes have already the same position (and still opposite velocities).
Let us note that in the particular case of constant jump rates, the previous coupling is quite more simple than in the general case. Indeed, thanks to Remarks 2.6 and 2.8, there are only two outcomes whereas the four in the general case presented in Figure 2.4.




Case 1: $U_+ = L_+$ and $U_- = L_-$ Case 2: $U_+ = L_+$ and $U_- > L_-$

Figure 2.5: Position of both paths after one step for constant jump rates.

**Remark 2.9.** *When the jump rates are constant functions, we can observe on Figure 2.5 that the upper path stay always above the lower path, at each step. The coalescent coupling succeeds right after the first hitting time of zero of the lower path, and then the coalescent time of both paths is smaller than the hitting time of zero of the upper path.*

Finally, the coalescent time $T_*(x, v, \tilde{x}, \tilde{v})$ of a path $(X, V)$ starting from $(x, v)$ and a path $\left(\tilde{X}, \tilde{V}\right)$ starting from $(\tilde{x}, \tilde{v})$ is equal to the first crossing time $T_c$ of both paths plus the time spent to stick them using the coupling described above. Since trajectories are continuous, the first crossing time of both trajectories can be controlled by the first hitting time $Z_{(x,v)}$ of the origin of the path starting at the upper point $(x, v)$.

## 3.3 Estimate of the speed of convergence to the equilibrium

The jump times having a finite Laplace transform, it is not surprising that this is also true for the coupling time. The constant rates case is quite specific and allows us to have the explicit distribution of the first hitting time of zero of a path and then a quite simple control of the spent time to stick both reflected paths. We just recall here the result on the hitting times of zero in the case of constant jump rates.

**Proposition 2.10** (Hitting time of zero, [FGM12, Lemma 2.2])**.** *We assume that the jump rates are constant functions.*
*For $x > 0$, let $Z_{(x,v)}$ denote the hitting time of 0 starting from $(x, v)$. Then*

$$\mathbb{E}\left(e^{\lambda Z_{(x,-1)}}\right) = e^{xc(\lambda)} \quad \textit{and} \quad \mathbb{E}\left(e^{\lambda Z_{(x,+1)}}\right) = \frac{a+b-2\lambda-\sqrt{(a+b-2\lambda)^2-4ab}}{2a} e^{xc(\lambda)}$$

*with* $c(\lambda) = \frac{b-a-\sqrt{(a+b-2\lambda)^2-4ab}}{2}$, *if* $\lambda \in (-\infty, (\sqrt{b}-\sqrt{a})^2/2]$, *and* $+\infty$ *otherwise.*

Let us consider two path starting from different initial data. As we have seen in the previous section the coupling time is stochastically smaller than the first hitting time of zero starting from the upper position plus the time spent to stick both trajectories. In the constant jump rates case, thanks to Remark 2.9, this second random time is also smaller than the hitting time of zero starting from the first crossing point. Using an explicit coupling in order to control the position of the first crossing point, we obtain an explicit speed of convergence, which is quite optimal.

**Theorem 2.11** (Convergence to equilibrium for the reflected process in the case of constant jumps rates, [FGM12, Theorem 1.2])**.** *If* $\nu_t^{x,v}$ *stands for the law of* $(X_t, V_t)$ *when* $X_0 = x$ *and* $V_0 = v$*, we have, for any* $x, \tilde{x} \geq 0$ *and* $v, \tilde{v} \in \{-1, +1\}$*,*

$$\left\| \nu_t^{x,v} - \nu_t^{\tilde{x},\tilde{v}} \right\|_{\mathrm{TV}} \leq \frac{(a+b)b}{2a^2} e^{r(a,b)(x \vee \tilde{x})} e^{-\lambda_c t}, \tag{3.11}$$

*where*

$$r(a,b) = \frac{3(b-a)}{4} \vee (b - \sqrt{ab}) \quad \textit{and} \quad \lambda_c = \frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{b}-\sqrt{a})^2}{2}.$$

For general jump rates satisfying Hypothesis 2.1, the evolution away from the origin is no longer invariant by translation, and then we do not have a simple estimate of the coupling time and in particular the distribution of the first hitting time of zero is no more known. We won't give in details here the tools used to estimates the coupling time of the reflected process, since this is quite complex and technical. However, we will just mention some key points. One of the main idea is the control of the Laplace transform of the first hitting time of zero of a path, which has been obtained by introducing a Lyapunov function for the infinitesimal generator (see Proposition 2.8 in [FGM16]). But, the most complex part is to estimate the spent time to stick two reflected trajectories. In the general case, we cannot control the coupling time by the hitting time of 0 for the process starting at $(X_{T_c}, +1)$ as it was done in the constant jump rates case since we cannot control the relative positions of both trajectories (see Case 3. in Figure 2.4). On the other hand, contrary to the constant rates case where the coupling could only succeed right after the lower process hits 0 (see Remark 2.9), the coupling can now succeed at an arbitrary step of the scheme, though not with a probability bounded from below uniformly in $x$ (this can be easily seen from formula (3.10) e.g. in the case case when $a$ is constant and $b(x) = a + x$). Therefore, a new approach to estimate the coupling time is needed and is based on the attraction to zero of the process, due to Hypothesis 2.1, (see Section 3.3 in [FGM16]). We just recall here the main result and we refer to Section 3.3 in [FGM16] for the complete (and quite technical) proof.

**Theorem 2.12** (Convergence to equilibrium for the reflected process in the general, [FGM16, Theorem 1.9])**.** *The invariant measure of* $(X, V)$ *is the product measure on* $\mathbb{R}_+ \times \{-1, +1\}$ *given by*

$$\nu(dx, dv) = \frac{2}{C_F} e^{-F(x)}\, dx \otimes \frac{1}{2}(\delta_{-1} + \delta_{+1})(dv).$$

*where* $F$ *and* $C_F$ *are given by* (1.2)*. Moreover, denoting by* $\nu_t^{x,v}$ *the law of* $(X_t, V_t)$ *when* $X_0 = x$ *and* $V_0 = v$*, there exists* $\lambda > 0$*,* $K > 0$ *and* $c > 0$ *such that, for any* $x, \tilde{x} \geq 0$ *and* $v, \tilde{v} \in \{-1, +1\}$*,*

$$\left\| \nu_t^{x,v} - \nu_t^{\tilde{x},\tilde{v}} \right\|_{\mathrm{TV}} \leq K e^{-\lambda t} e^{c(x \vee \tilde{x})}. \tag{3.12}$$

We now give, in the next section, the construction of a coupling for the initial trajectories from a coupling of the reflected trajectories in order to deduce an estimate of the speed of convergence to equilibrium.

# 4 Convergence to equilibrium for the initial process

To estimate the speed of convergence to equilibrium, we have to construct a coalescent coupling of two unreflected processes starting from $(y, w)$ and $(\tilde{y}, \tilde{w})$ respectively. The case of non constant jump rates is quite harder to deal with than the case of constant jump rates, and both cases will be treated separately from this point.

### For constant jump rates

In this case $T_{x,+1}$ is an exponential random variable (r.v. for simplicity) with parameter $b$ and $T_{x,-1}$ is the minimum between an exponential r.v. with parameter $a$ and the starting position $x$. The coupling algorithm of the initial process is the following:

1. Define $(x, v) = (|y|, w\,\mathrm{sgn}(y))$ and $(\tilde{x}, \tilde{v}) = (|\tilde{y}|, \tilde{w}\,\mathrm{sgn}(\tilde{y}))$.

2. Couple two reflected processes starting at $(x, v)$ and $(\tilde{x}, \tilde{v})$ as in Section 3.2.

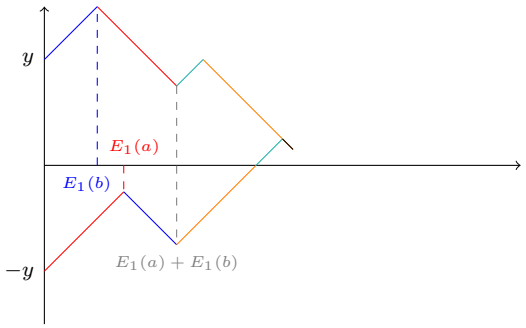
3. Let them evolve until their common hitting time of 0.

4. Construct from that time the two associated unreflected processes $(Y, W)$ and $(\tilde{Y}, \tilde{W})$ respectively starting at $(0, w)$ and $(0, \tilde{w})$ as explained in Section 2. The algorithm stops if, when at the origin, the two copies have the same velocities. Otherwise, go to Step 5.

5. Wait for the first jump time of $(Y, W, \tilde{Y}, \tilde{W})$, which is an exponential r.v. $E$ with parameter $2b$ (as the minimum of two independent exponential r.v.).

6. Try to couple in one step the unreflected processes starting at $(y, w)$ and $(-y, w)$, with $y = E$ and $w \in \{-1, +1\}$. In case of failure, return to Step 6 of the algorithm for different values of $y$ and $w$.

Let us explain the coupling in Step 6, in the case $w = +1$, with a picture (a complete description of this step being given in [FGM12, Section 4.]). We couple the jump times of both trajectories. Let $(E_i(b))_{i\geq 1}$ and $(E_i(a))_{i\geq 1}$ two independent family of independent exponential r.v. with parameter $b$ and $a$ respectively. The algorithm stop as soon as one exponential r.v. with parameter $a$ is larger than the starting point, which means as soon as the lower trajectory cross the horizontal axis.




Using the value of the Laplace transform of the first hitting time of zero of the path in the constant jump rates case given in Proposition 2.10, we finally obtain

**Theorem 2.13** ([FGM12, Theorem 1.1])**.** *The invariant probability measure $\mu$ of $(Y, W)$ is the product measure on $\mathbb{R} \times \{-1, +1\}$ given by*

$$\mu(dy, dw) = \frac{b-a}{2} e^{-(b-a)|y|} dy \otimes \frac{1}{2}(\delta_{-1} + \delta_{+1})(dw).$$

*Moreover, denoting by $\mu_t^{y,w}$ the law of $Z_t = (Y_t, W_t)$ when issued from $Z_0 = (y, w)$, we have, for any $y, \tilde{y} \in \mathbb{R}$ and $w, \tilde{w} \in \{-1, +1\}$,*

$$\left\| \mu_t^{y,w} - \mu_t^{\tilde{y},\tilde{w}} \right\|_{\mathrm{TV}} \leq C(a,b) e^{r(a,b)|y| \vee |\tilde{y}|} e^{-\lambda_c t}, \tag{4.13}$$

*where*

$$C(a,b) = \left(\frac{b}{a}\right)^{5/2} \frac{a+b}{\sqrt{ab}+b}, \quad r(a,b) = \frac{3(b-a)}{4} \vee (b - \sqrt{ab}) \quad \text{and} \quad \lambda_c = \frac{(\sqrt{b} - \sqrt{a})^2}{2}.$$

**For general jump rates satisfying Hypothesis 2.1**

The general case is quite more complex and we have to introduce a fixed time $t_0 > 0$. The coupling algorithm is the following:

1. Define $(x, v) = (|y|, w\,\mathrm{sgn}(y))$ and $(\tilde{x}, \tilde{v}) = (|\tilde{y}|, \tilde{w}\,\mathrm{sgn}(\tilde{y}))$.
2. Couple two reflected processes starting at $(x, v)$ and $(\tilde{x}, \tilde{v})$ as in Section 3.2.
3. Let them evolve until their common hitting time of 0.
4. Construct from that time the two associated unreflected processes $(Y, W)$ and $(\tilde{Y}, \tilde{W})$ respectively starting at $(0, w)$ and $(0, \tilde{w})$ as explained in Section 2. The algorithm stops if, when at the origin, the two copies have the same velocities. Otherwise, go to Step 5.
5. Try to couple the unreflected processes starting from $(0, +1)$ and $(0, -1)$ before a fixed time $t_0$.
6. In case of failure, return to step 1 for two initial conditions in $[-t_0, t_0] \times \{-1, +1\}$.

From the distribution of the jump times, given a fixed time $t_0 > 0$, the probability to succeed to stick both path before the time $t_0$ for initial data in $[-t_0, t_0] \times \{-1, +1\}$ is bounded from below by a positive constant $\varepsilon_{t_0}$. Then we can estimate the number of try of the algorithm is dominated by a geometric r.v. of parameter $\varepsilon_{t_0}$.
Let us denote by $\mu_t^{y,w}$ the law of $Z_t = (Y_t, W_t)$ when issued from $Z_0 = (y, w)$. We finally obtain the main result:

**Theorem 2.14** (Convergence to equilibrium [FGM16, Theorem 1.2])**.** *There exists $\kappa > 0$, $K > 0$, and $\lambda > 0$ such that for any $y, \tilde{y} \in \mathbb{R}$ and $w, \tilde{w} \in \{-1, +1\}$,*

$$\left\| \mu_t^{y,w} - \mu_t^{\tilde{y},\tilde{w}} \right\|_{\mathrm{TV}} \leq K e^{\kappa |y| \vee |\tilde{y}|} e^{-\lambda t}. \tag{4.14}$$

The constants above can be expressed in terms of the functions $a$ and $b$ following the lines of the proof.

# 5 To go further

With Joaquin Fontbona and Florent Malrieu, we have been interested in study the long time behavior of this simple PDMP with the idea to estimate the speed of convergence to equilibrium of one bacterium in a quite simple environment (only one nutriment and no poison). The process has been studied since for more general framework, namely with different assumptions on the dimension of the state space or on the velocity kernel [CRS15, BRZ17, Fét17], mainly using the Meyn-Tweedie method. This process has been also studied in parallel for completely others aims. Indeed PDMPs are really easy to simulate and in the case of the ergodic variant of the Telegraph process, also called the Zig-Zag process, the invariant measure is known. Then new Monte-Carlo algorithms have been recently

introduced using this class of process, [Mon14, BFR16, BR17, BRZ17]. The question of the speed of convergence to equilibrium is then also of great interest for algorithmic reasons.

But if we still keep in mind the biological application of this process, we can easily think about some extensions. For example, we can consider a population of bacteria with a collaborative behavior: each bacterium is attracted by the average behaviour of the population. The study of the long time behavior of this interacting particle system driven by a piecewise deterministic Markov behavior and its evolution when the size of the population growth will be central to understand the evolution of the system. We can easily imagine to obtain propagation of chaos (see [Szn91]) under some good assumptions, as for the Landau process in [GM03a], when the size goes to infinity to a nonlinear stochastic differential equation driven by a Poisson measure. The evolution in time is not easy to predict, because it is not obvious that the system will be ergodic. We may need additional assumptions than the usual ones on the Zig-Zag process or we will have to introduce some renormalized system.
There are few works on interacting particle systems driven by PDMP, we can mention the work of Fournier and LÃűcherbach on interacting neurons, [FL16], of Graham and Robert on multi-class transmissions in communication networks, [GR09], and the PhD thesis of Andreis [And17]. There are many applications in this domain.

# Chapter 3

# Markov Switching Processes, Spectrally Negative Lévy Processes
# Some actuarial and financial issues

This chapter is based on four articles with applications in finance and in actuarial science. The first one concern Ornstein-Ulhenbeck diffusions with Markov switching, the others are about risk measures in insurance and occupations times of spectrally negative Lévy processes.
The first section of this chapter focus on a collaboration with Jean-Baptiste Bardet, Université de Rouen, and Florent Malrieu, Université de Tours, Long time behavior of diffusions with Markov switching, published in ALEA, Latin American Journal of Probability and Mathematical Statistics, [BGM10]. The second section relies on a work with Jean-François Renaud, Université du Québec à Montréal, On the distribution of finite-time cumulative Parisian ruin, published in Insurance: Mathematics & Economics, [GR17] and on a collaboration with Zied Ben Salah, American University in Cairo, Manuel Morales, Université de Montréal and Hassan Omidi Firouzi, published in European Actuarial Journal, On the depletion problem for an insurance risk process: new non-ruin quantities in collective risk theory, [BSGMF15]. The third part presents the results on the occupation times of spectrally negative Lévy processes, Joint distribution of a spectrally negative Lévy process and its occupation time, with step option pricing in view, published in Advances in Applied Probability, [GR16].

# 1 Diffusions with Markov switching

The regime switching model has become a popular model in mathematical finance and actuarial science. We can mention the Black-Scholes model with Markov switching, (see for e.g. [DMKR94]), or more recenlty the Cox-Ingersoll-Ross model with Markov switching (see [ZTH16]). With J-B. Bardet and F. Malrieu, we have studied the case of Ornstein-Uhlenbeck diffusions with Markov switching. This first section of the chapter is independent of the others.

The aim of this section is to draw a complete picture of the ergodicity of Ornstein-Uhlenbeck diffusions with Markov switching by characterizing the tails of the invariant measure and giving quantitative convergence to equilibrium. In particular we make more precise the results of [GIY04, dSY05]. The so-called diffusion with Markov switching $Y = (Y_t)_{t\geq 0}$ is defined as follows.
The *switching process* $X = (X_t)_{t\geq 0}$ is a Markov process on the finite state space $E = \{1, \dots, d\}$ (with $d \geq 2$), of infinitesimal generator $A = (A(x, \tilde{x}))_{x,\tilde{x}\in E}$. We denote by $a(x)$ the jump rate at state $x \in E$ and $P = (P(x, \tilde{x}))_{x,\tilde{x}\in E}$ the transition matrix of the embedded chain. One has, for $x \neq \tilde{x}$ in $E$,

$$a(x) = -A(x, x) \quad \text{and} \quad P(x, \tilde{x}) = -\frac{A(x, \tilde{x})}{A(x, x)}.$$

We assume that $P$ is irreducible recurrent. The process $X$ is ergodic with a unique invariant probability measure denoted by $\mu$. See [Nor98] for details. Let $\mathcal{F}_t^X = \sigma(X_u, 0 \leq u \leq t)$.

Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion on $\mathbb{R}$ and $Y_0$ a real-valued random variable such that $B$, $Y_0$ and $X$ are independent. Conditionnally to $X$, the process $Y = (Y_t)_{t\geq 0}$ is the real-valued diffusion process defined by:

$$Y_t = Y_0 - \int_0^t \lambda(X_u) Y_u \, du + \int_0^t \sigma(X_u) \, dB_u, \tag{1.1}$$

where $\lambda$ and $\sigma$ are two functions from $E$ to $\mathbb{R}$ and $(0, \infty)$ respectively. When $\lambda$ and $\sigma$ are constants, $Y$ is just an Ornstein-Uhlenbeck process with attractive ($\lambda > 0$), neutral ($\lambda = 0$) or repulsive coefficient ($\lambda < 0$). One has to notice that Equation (1.1) has an "explicit" solution:

$$Y_t = Y_0 \exp\left(-\int_0^t \lambda(X_u) \, du\right) + \int_0^t \exp\left(-\int_u^t \lambda(X_v) \, dv\right) \sigma(X_u) \, dB_u. \tag{1.2}$$

**Remark 3.1.** *In others words, the full process $(X, Y)$ is the Markov process on $E \times \mathbb{R}$ associated to the infinitesimal generator $\mathcal{A}$ defined by:*

$$\mathcal{A}f(x, y) = \sum_{\tilde{x} \in E} A(x, \tilde{x})(f(\tilde{x}, y) - f(x, y)) + \frac{\sigma(x)^2}{2} \partial_{22}^2 f(x, y) - \lambda(x) \partial_2 f(x, y).$$

Previous works investigated the ergodicity of $Y$ and some integrability properties for the invariant measure. For example, in [BBG96], the multidimensional case is addressed together with the case of diffusion coefficients depending on $Y$. Stability results and sufficient conditions for the existence of moments are established under Lyapunov-type conditions. In [GIY04], it is proved that the Markov switching diffusion $Y$ is ergodic if and only if

$$\sum_{x \in E} \lambda(x) \mu(x) > 0, \tag{1.3}$$

that is if the process is attractive "in average". Let us denote by $\nu$ its invariant probability measure of $Y$.

## 1.1 Invariant measure of the diffusion with Markov switching

It is also shown in [GIY04] that $\nu$ admits a moment of order $p$ if, for any $x \in E$, $p\lambda(x) + a(x)$ is positive and the spectral radius of the matrix

$$M_p = \left(\frac{a(x)}{a(x) + p\lambda(x)} P(x, \tilde{x})\right)_{x, \tilde{x} \in E}$$

is smaller than 1. In the sequel $\rho(M)$ stands for the spectral radius of a matrix $M$.
In [dSY05], the result is more precise: a dichotomy is exhibited between heavy and light tails for $\nu$. Let us define

$$\underline{\lambda} = \min_{x \in E} \lambda(x) \quad \text{and} \quad \overline{\lambda} = \max_{x \in E} \lambda(x).$$

**Theorem 3.2** (de Saporta-Yao [dSY05])**.** *Under Assumption* (1.3)*, the following dichotomy holds:*

1. *if $\underline{\lambda} < 0$, then there exists $C > 0$ such that*

$$t^{\kappa}\nu((t,+\infty)) \xrightarrow[t\to+\infty]{} C,$$

   *where $\kappa$ is the unique $p \in (0, \min\{-a(x)/\lambda(x),\ \lambda(x) < 0\})$ such that the spectral radius of $M_p$ is equal to 1;*

2. *if $\underline{\lambda} \geq 0$, then $\nu$ has moments of all order.*

**Remark 3.3.** *Note that the constant $\kappa$ does not depend on the parameters $(\sigma(x))_{x\in E}$, and that Point 1. from previous theorem implies that, for $\underline{\lambda} < 0$, the $p^{th}$ moment of $\nu$ is finite if and only if $p < \kappa$.*

The main idea of the proofs in [GIY04] and [dSY05] is to study the discrete time Markov chain $(X_{\delta n}, Y_{\delta n})_{n\geq 0}$ for any $\delta > 0$ with renewal theory and then to let $\delta$ goes to 0.
In our work, we show that there are three (and not only two) different behaviors for the tails of $\nu$.
Let us gather below several useful notations.

**Notations 3.4.** *Let us define for the diffusion coefficients*

$$\underline{\sigma}^2 = \min_{x\in E} \sigma^2(x) \quad \textit{and} \quad \overline{\sigma}^2 = \max_{x\in E} \sigma^2(x).$$

*We denote by $A_p$ the matrix $A - p\Lambda$ where $\Lambda$ is the diagonal matrix with diagonal $(\lambda(1), \ldots, \lambda(d))$ and associate to $A_p$ the quantity*

$$\eta_p := -\max_{\gamma\in \mathrm{Spec}(A_p)} \mathrm{Re}\ \gamma. \tag{1.4}$$

*When $\underline{\lambda} \geq 0$, the set $E$ is the union of*

$$M = \{x \in E,\ \lambda(x) > 0\} \quad \textit{and} \quad N = \{x \in E,\ \lambda(x) = 0\}.$$

*Let us then define*

$$\beta(x) = \frac{\sigma(x)^2}{2a(x)} \quad \textit{and} \quad \overline{\beta} = \max_{x\in N} \beta(x),$$

*and, for any $v$ such that $v^2 < \overline{\beta}^{-1}$, the matrix*

$$P_v^{(N)} = \left(\frac{1}{1-\beta(x)v^2} P(x,x')\right)_{(x,x')\in N^2}.$$

With J-B. Bardet and F. Malrieu, we obtain the following result.

**Theorem 3.5.** *Let us define*

$$\kappa = \sup\{p \geq 0,\ \eta_p > 0\} \in (0, +\infty].$$

*Then $\eta_p$ is continuous, positive on the set $(0, \kappa)$ and negative on $(\kappa, +\infty)$. Under Assumption (1.3), the following trichotomy holds:*

1. *if $\underline{\lambda} < 0$ then $0 < \kappa \leq \min\{-a(x)/\lambda(x),\ \lambda(x) < 0\}$, and the $p^{th}$ moment of $\nu$ is finite if and only if $p < \kappa$.*
2. *if $\underline{\lambda} = 0$, then $\kappa$ is infinite and the domain of the Laplace transform of $\nu$ is $(-v_c, v_c)$ where*
$$v_c = \sup\Big\{v > 0,\ \rho(P_v^{(N)}) < 1\Big\};$$
3. *if $\underline{\lambda} > 0$, then $\kappa$ is infinite and $\nu$ has a Gaussian-like Laplace transform: for any $v \in \mathbb{R}$,*
$$\exp\left(\frac{\underline{\sigma}^2 v^2}{4\overline{\lambda}}\right) \leq \int e^{vy}\,\nu(dy) \leq \exp\left(\frac{\overline{\sigma}^2 v^2}{4\underline{\lambda}}\right).$$
*Moreover, its tail looks like the one of the Gaussian law with variance $\overline{\alpha}/2$ where $\overline{\alpha} = \max_{x\in E} \sigma(x)^2/\lambda(x)$ since $y \mapsto e^{\delta y^2}$ is $\nu$-integrable if and only if $\delta < 1/\overline{\alpha}$.*

**Remark 3.6.** *In the sequel we will respectively refer to Points 1. 2. and 3. as the polynomial, exponential-like and Gaussian-like cases.*

The first point of this theorem is indeed a reformulation of the first point of Theorem 3.2 by de Saporta and Yao. We provide a direct and simple proof of this result based on Itô formula and some basic results on finite Markov chains. The proof of Points 2. relies on precise estimates on the Laplace transform of $Y_t$ that can be derived from a discrete time model already studied in [GG96, HW09, AIR09]. The point 3. of Theorem 3.5 is quite trivial, except for the second part.
We give in the next section the intuition of the proof of Theorem 3.5, but for a complete proof we refer the reader to [BGM10]. Let us first notice from the explicit form (1.2) of $Y_t$ that, for any measure $\pi_0$ on $E \times \mathbb{R}$, the Laplace transform $L_t$ of $Y_t$ is

$$L_t(v) := \mathbb{E}_{\pi_0}\big(e^{vY_t}\big) = \mathbb{E}_{\pi_0}\left[\exp\left(vY_0 e^{-\int_0^t \lambda(X_s)\,ds} + \frac{v^2}{2}\int_0^t \sigma(X_s)^2 e^{-2\int_s^t \lambda(X_r)\,dr}\,ds\right)\right]. \quad (1.5)$$

## 1.2 Intuition and scheme of the proof

**The gaussian-like case: $\underline{\lambda} > 0$**

Let us assume that $Y_0 = 0$. Then we easily get from the explicit Laplace transform of $Y_t$ (1.5) that

$$L_t(v) \leq \mathbb{E}\left[\exp\left(\frac{v^2}{2}\int_0^t \overline{\sigma}^2 e^{-2\int_s^t \underline{\lambda}\,dr}\,ds\right)\right] \leq \exp\left(\left(1 - e^{-2\underline{\lambda}t}\right)\frac{\overline{\sigma}^2 v^2}{4\underline{\lambda}}\right),$$

which gives the upper bound as $t$ goes to infinity. The lower bound follows from a symmetric argument.
Moreover, using Itô formula to the function $f(y) = e^{\delta y^2}$, we have $y \mapsto e^{\delta y^2}$ is $\nu$-integrable if and only if $\delta < 1/\overline{\alpha}$, $\bar{\alpha}$ being introduced in Theorem 3.5. This result gives a meaning to the terminology of *gaussian-like case*.

**The exponential-like case:** $\underline{\lambda} = 0$

We notice that in a state $x$ with $\lambda(x) > 0$, the process $Y$ is attractive, whereas in a state $x$ with $\lambda(x) = 0$, the process $Y$ behaves like a Brownian motion. It is well known that the process stasy in a state $x$ during an exponential time with parameter $a(x)$. Besides if $B$ is a standard Brownian motion and $T$ an independent exponential random variable with parameter $a$, we have for $v < \sqrt{2a}/\sigma$,

$$\mathbb{E}\left[e^{v\sigma B_T}\right] = \frac{2a}{2a - v^2\sigma^2}.$$

Consequently, $\sigma B_T$ has a symmetric Laplace Law and then it is not surprising to lose the gaussian behavior when $\underline{\lambda} = 0$.
Then considering the following partition of the state space $E$

$$N = \{x \in E : \lambda(x) = 0\} \text{ and } M = \{x \in E : \lambda(x) > 0\},$$

we can decompose the process $Y$ according to the successive entrance times of each subset of the partition. Assume that $X_0 \in M$, we introduce by induction the sequence of random time $(T_n)_{n\geq 0}$ by $T_0 = 0$ and for $n \geq 0$

$$T_{2n+1} = \inf\{t > T_{2n} : X_t \in N\} \text{ and } T_{2n+2} = \inf\{t > T_{2n+1} : X_t \in M\}.$$

In Figure 3.1, we visualize the successive states of the process $Y$.

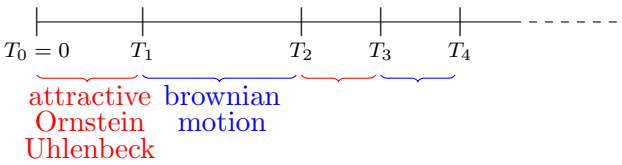



Figure 3.1: Successive states of $Y$.

Since $\lambda = 0$ between $T_{2n+1}$ and $T_{2n+2}$, from the explicit form (1.2) of $Y$, we have

$$Y_{T_{2n+2}} = Y_{T_{2n}} \exp\left(-\int_{T_{2n}}^{T_{2n+1}} \lambda(X_s)ds\right) + \int_{T_{2n}}^{T_{2n+1}} \exp\left(-\int_s^{T_{2n+2}} \lambda(X_r)dr\right)\sigma(X_s)dB_s + \int_{T_{2n+1}}^{T_{2n+2}} \sigma(X_s)dB_s.$$

Denoting $U_n = X_{T_{2n}}$ and $V_n = Y_{T_{2n}}$, we notice that $V_{n+1} = M_n(U_n)V_n + Q_n(U_n)$, where $(M_n(x), Q_n(x))$ are i.i.d. random variables, independent of $(U_n)$ with law given by

$$M_n(x) \stackrel{\mathcal{L}}{=} \exp\left(-\int_0^{T_1} \lambda(X_s^x)ds\right),$$
$$Q_n(x) \stackrel{\mathcal{L}}{=} \int_0^{T_1} \exp\left(-\int_s^{T_1} \lambda(X_r^x)dr\right)\sigma(X_s)dB_s + \int_{T_1}^{T_2} \sigma(X_s^x)dB_s,$$

and $X^x$ is the Markov process with infinitesimal generator $A$ starting at $x$. Consequently, the sequence $(V_n)_{n\geq 0}$ is an auto-regressive sequence. We then deduce from the exhaustive study on perpetuities of Alsmeyer et al. [AIR09] the domain of the Laplace transform of $V_n$ and by extension the domain of the Laplace transform of $Y$.

**The polynomial case:** $\underline{\lambda} < 0$

For $p > 0$, we introduce $A_p = A - p\Lambda$ with $\Lambda$ the diagonal matrice with diagonal $(\lambda(1), \ldots, \lambda(d))$ and $\eta_p := -\max_{\gamma\in \mathrm{Spec}(A_p)} \mathrm{Re}\ \gamma$.
We establish the equivalence between the positivity of $\eta_p$ and the existence of a $p^{\text{th}}$ moment for the invariant measure $\nu$ of $Y$. To this aim we first relate $\eta_p$ with exponential functionals of $\lambda$ along the trajectories of $X$.
Let us define, for any $p > 0$ and $t > 0$, the matrix $A_{(p,t)}$ by

$$A_{(p,t)}(x, \tilde{x}) = \mathbb{E}_x\left(\exp\left(-\int_0^t p\lambda(X_u)\,du\right)\mathbb{1}_{\{X_t=\tilde{x}\}}\right).$$

On the one hand, one remarks that

$$\mathbb{E}_\pi\left(\exp\left(-\int_0^t p\lambda(X_u)\,du\right)\right) = \pi A_{(p,t)}\mathbf{1}$$

where the coordinates of $\mathbf{1}$ are all equal to 1 and $\pi$ is a probability measure on $E$ seen as a row vector.
On the other hand, a simple application of the Feynman-Kac formula shows that $A_{(p,t)} = e^{tA_p}$. This fact relates the spectra of $A_p$ and $A_{(p,t)}$. In particular, the spectral radius $\rho(A_{(p,t)}) = e^{-\eta_p t}$ and, since all coefficients of $A_{(p,t)}$ are positive, we can apply the Perron-Frobenius Theorem to ensure that $-\eta_p$ is a simple eigenvalue of $A_p$, all other eigenvalues having a strictly smaller real part. Then, using the spectral theory of linear operators, we deduce the following result.

**Proposition 3.7.** *For any $p > 0$, there exist $0 < C_1(p) < C_2(p) < +\infty$ such that, for any initial probability measure $\pi$ on $E$, any $t > 0$,*

$$C_1(p)e^{-\eta_p t} \leq \mathbb{E}_\pi\left(\exp\left(-\int_0^t p\lambda(X_u)\,du\right)\right) \leq C_2(p)e^{-\eta_p t}.$$

We now describe the properties of $p \mapsto \eta_p$:

1. the function $p \mapsto \eta_p$ is smooth and concave on $\mathbb{R}_+$. Its derivative at $p = 0$ is equal to
$$\sum_{x \in E} \lambda(x)\mu(x) > 0,$$
and $\eta_p/p$ tends to $\underline{\lambda}$ as $p$ goes to infinity,
2. we have the following dichotomy:
   - if $\underline{\lambda} \geq 0$, then for all $p > 0$, $\eta_p > 0$,
   - if $\underline{\lambda} < 0$, there is $\kappa \in (0, \min\{-a(x)/\lambda(x), \lambda(x) < 0\})$ such that $\eta_p > 0$ for $p < \kappa$ and $\eta_p < 0$ for $p > \kappa$.

We now consider the moments of $Y_t$. Let us denote by $\alpha_p$ the function defined on $[0, T]$ by
$$\alpha_p(t) = \mathbb{E}\big(|Y_t|^p | \mathcal{F}_T^X\big).$$
Using Itô's formula and the independence between $B$ and $X$, we obtain
$$\alpha_p'(t) = -p\lambda(X_t)\alpha_p(t) + \frac{p(p-1)}{2}\sigma^2(X_t)\alpha_{p-2}(t),$$
which implies that for any $\varepsilon > 0$ there exists $c > 0$ such that
$$\alpha_p(t) \leq \alpha_p(0)e^{\int_0^t(-p\lambda(X_r)+\varepsilon)\,dr} + c\int_0^t e^{\int_u^t(-p\lambda(X_r)+\varepsilon)\,dr}\,du.$$
Taking the expectation of the previous expression, using Proposition 3.7 and taking $\varepsilon < \eta_p$, we deduce $\sup_{t>0}\mathbb{E}(|Y_t|^p)$ is finite.
If $p = \kappa$, one has
$$\alpha_\kappa'(t) = -\kappa\lambda(X_t)\alpha_\kappa(t) + \frac{\kappa(\kappa-1)}{2}\sigma^2(X_t)\alpha_{\kappa-2}(t).$$
Then
$$\begin{aligned}\alpha_\kappa(t) &= \int_0^t e^{-\kappa\int_s^t \lambda(X_u)\,du}\kappa(\kappa-1)\sigma(X_s)^2\alpha_{\kappa-2}(s)\,ds + \mathbb{E}(|Y_0|^\kappa)e^{-\kappa\int_0^t\lambda(X_u)\,du} \\ &\geq \kappa(\kappa-1)\underline{\sigma}^2\int_0^t e^{-\kappa\int_s^t\lambda(X_u)\,du}\alpha_{\kappa-2}(s)\,ds.\end{aligned}$$
Using the second inequality of Proposition 3.7 and the fact that $\eta_\kappa = 0$, we deduce that
$$\lim_{t\to\infty}\mathbb{E}(|Y_t|^\kappa) = +\infty,$$
and the $\kappa^{th}$ moment of $\nu$ is infinite. This is also true for the $p^{th}$ moment for any $p > \kappa$.

### 1.3 Speed of convergence to equilibrium

In order to control the convergence of the law of $Y_t$ to the invariant measure $\nu$, we give an explicit exponential bound for the Wasserstein distance of order $p$ for any $p < \kappa$.
The strategy is to couple two processes $(X, Y)$ and $(\tilde{X}, \tilde{Y})$ in such a way that the Wasserstein distance between $\mathcal{L}(Y_t)$ and $\mathcal{L}(\tilde{Y}_t)$ goes to zero as $t$ goes to infinity. This requires to couple the initial conditions and the dynamics (of both the Markov chains and the diffusion part). When $X_0$ and $\tilde{X}_0$ have the same law, the coupling is trivial: we choose $X = \tilde{X}$ and the same driving Brownian motion. Thanks to the upper bound of Proposition 3.7, we have

**Theorem 3.8.** *Let $p < \kappa$. Assume that $X_0$ and $\tilde{X}_0$ have the same law. Let $Y$ and $\tilde{Y}$ be solutions of (1.1) associated to $(X_t)$ and $(\tilde{X}_t)$ and assume that $Y_0$ and $\tilde{Y}_0$ have finite moment of order $p$. Then there exists $C(p)$ such that*

$$\mathcal{W}_p\Big(\mathcal{L}(Y_t), \mathcal{L}(\tilde{Y}_t)\Big)^p \leq C(p) e^{-\eta_p t} \mathcal{W}_p\Big(\mathcal{L}(Y_0), \mathcal{L}(\tilde{Y}_0)\Big)^p,$$

*where $\eta_p$ is given by (1.4) and $\mathcal{W}_p$ is the Wasserstein distance of order $p$.*

If $X_0$ and $\tilde{X}_0$ do not have the same law, one first has to make the Markov chains $X$ and $\tilde{X}$ stick together and then to use Theorem 3.8. This provides an explicit but rather intricate bound given in Theorem 5.1 in [BGM10].

## 2 Different notions of ruin in insurance

We now consider the notion of risk measures in insurance. Historically, in classical actuarial ruin theory, the central topic has been the analysis of the probability of ruin, which is a measure to assess the overall risk of an insurance company. Ruin occurs when the surplus process falls below a certain threshold level for the first time, e.g. the insurer's minimum capital requirement set by the regulatory body. Then, more sophisticated ruin-based risk measures using the timing and the severity of such a capital shortfall have been proposed, culminating with so-called Gerber-Shiu distributions; see [GS98, Kyp13].
However, as discussed in [dR93]: "*[...] sometimes the event ruin has a very small probability and the portfolio is just one out of many existing in the company. The company can have enough funds available to support (or ask for external support) some negative surplus for some time, in the hope that the portfolio will recover in the future, allowing the company to keep this business alive. This can be regarded as an investment, since the process will recover in the future.*" This is why, in recent years, new ruin models, incorporating this realistic feature that a company is not immediately liquidated when it defaults, have been proposed. Therefore, new risk measures and models have been introduced: Parisian ruin (see, e.g., [CP11, LCP13, LRZ14a, BPPR16]), random observations (see, e.g., [ACT11, ACT13]) and

Omega models (see, e.g., [AGS11, GSY12, AL13, LP17]). In those papers, new definitions of *ruin*, *bankruptcy* and/or *liquidation* are proposed and studied.
In this direction, the concept of cumulative Parisian ruin, based on the time spent in the red by the underlying surplus process, has been introduced. While Parisian ruin has been inspired by Parisian options, the idea of cumulative Parisian ruin comes from cumulative Parisian options, another type of occupation-time options. In the definition of Parisian ruin (the duration of) periods of financial distress are considered separately, while cumulative Parisian ruin considers the aggregation of those durations. As expected, this notion of ruin is closely related to the occupation-time process of the underlying surplus process in the model.
We can also consider others risk measures for an insurance company. Quantities such as the speed of depletion and drawdowns have been studied in finance in connection to the concept of market crash [ZH12]. Indeed, in finance one would be interested in knowing how fast and how frequent drawdowns of a certain size occur. These concepts are also meaningful from an insurance risk management point of view. Clearly knowing how your insurance reserve is affected by drawdowns and how fast and frequent these are could be useful to devise risk management tools. These quantities provide a measure of riskiness that is not linked to the ruin event but rather to the depletion features of the reserve. The ruin event is an object of concern over the long-run but a risk manager might also keep an eye on any series of particularly large drawdowns especially if they happen particularly fast. So concerning oneself with the ruin event overlooks other risky events that also have an impact on the solvency and financial planning of an insurance company. This problem is technically challenging due to the jump nature of insurance models.

## 2.1 Some models in insurance

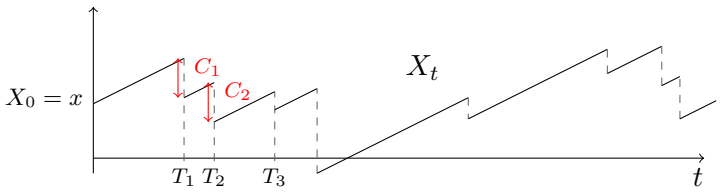
Let $X = \{X_t, t \geq 0\}$ be a surplus process. In what follows, the law of $X$ with initial capital $X_0 = x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$. We write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$.

### The Cramér-Lundberg model with Exponential Claims

The Cramér-Lundberg model is a classical (and simple) model in insurance and was introduced in [Lun03]. The risk process is a compound Poisson process starting at $x \geqslant 0$,

$$X_t = x + ct - \sum_{i=1}^{N_t} C_i;, \tag{2.6}$$

where $c$ represents the premium per unit of time, the number of claims is assumed to follow a Poisson process $(N_t)_{t\geqslant 0}$ which is independent of the positive and *iid* random variables $(C_n)_{n\geqslant 1}$ representing claim sizes.




**Lévy insurance risk processes**

Lévy insurance models are models where the surplus process $X = \{X_t, t \geq 0\}$ is assumed to be a spectrally negative Lévy process (SNLP), i.e. a Lévy process with no positive jumps. Hereby we exclude the case of $X$ having monotone paths. The Cramér-Lundberg process, the drifted Brownian motions are some examples of Lévy insurance risk processes.

As the Lévy process $X$ has no positive jumps, its Laplace transform exists: for $\theta, t \geq 0$,

$$\mathbb{E}\left[\mathrm{e}^{\theta X_t}\right] = \mathrm{e}^{t\psi(\theta)},$$

where

$$\psi(\theta) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_0^\infty \left(\mathrm{e}^{-\theta z} - 1 + \theta z \mathbb{1}_{(0,1]}(z)\right) \Pi(\mathrm{d}z),$$

for $\gamma \in \mathbb{R}$ and $\sigma \geq 0$, and where $\Pi$ is a $\sigma$-finite measure on $(0, \infty)$ such that

$$\int_0^\infty (1 \wedge z^2)\Pi(\mathrm{d}z) < \infty.$$

This measure is called the Lévy measure of $X$, while $(\gamma, \sigma, \Pi)$ is refered to as the Lévy triplet of $X$. Note that for convenience we define the Lévy measure in such a way that it is a measure on the positive half line instead of the negative half line. Further, note that $\mathbb{E}[X_1] = \psi'(0+)$.
One can define its right-inverse $\Phi\colon [0, \infty) \to [0, \infty)$ by $\Phi(q) = \sup\{\theta \geq 0 \mid \psi(\theta) = q\}$. Note that

$$\psi(\Phi(q)) = q, \quad q \geq 0,$$

and $\Phi(q) = 0$ if and only if $q = 0$ and $\psi'(0+) \geq 0$.
The process $X$ has jumps of bounded variation if $\int_0^1 z\Pi(\mathrm{d}z) < \infty$. In that case we denote by $\mathsf{c} := \gamma + \int_0^1 z\Pi(\mathrm{d}z) > 0$ the so-called drift of $X$ which can now be written as

$$X_t = \mathsf{c}t - S_t + \sigma B_t,$$

where $S = \{S_t, t \geq 0\}$ is a driftless subordinator (for example a Gamma process or a compound Poisson process with positive jumps).

The distribution of a SNLP is characterized by a family of functions, called scale functions. For $q \geq 0$, the $q$-scale function $W^{(q)}$ of the process $X$ is defined as the continuous function on $[0, \infty)$ with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\theta y} W^{(q)}(y)\mathrm{d}y = \frac{1}{\psi(\theta) - q}, \quad \text{for } \theta > \Phi(q). \tag{2.7}$$

This function is unique, positive and strictly increasing for $x \geq 0$ and is further continuous for $q \geq 0$. We extend $W^{(q)}$ to the whole real line by setting $W^{(q)}(x) = 0$ for $x < 0$. We write $W = W^{(0)}$ when $q = 0$. The initial value of $W^{(q)}$ is known to be

$$W^{(q)}(0) = \begin{cases} 1/\mathsf{c} & \text{when } \sigma = 0 \text{ and } \int_0^1 z\Pi(\mathrm{d}z) < \infty, \\ 0 & \text{otherwise,} \end{cases}$$

where we used the following definition: $W^{(q)}(0) = \lim_{x \downarrow 0} W^{(q)}(x)$. We also have that, when $\psi'(0+) > 0$,

$$\lim_{x \to \infty} W(x) = \frac{1}{\psi'(0+)}.$$

We also introduce the function

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y)\mathrm{d}y, \quad x \in \mathbb{R}. \tag{2.8}$$

Now, for any $a, b \in \mathbb{R}$, define the stopping times

$$\tau_a^- = \inf\{t > 0 \colon X_t < a\} \quad \text{and} \quad \tau_b^+ = \inf\{t > 0 \colon X_t > b\},$$

with the convention $\inf \emptyset = \infty$.
It is well known that, if $a \leq x \leq c$, then the solution to the two-sided exit problem for $X$ is given by

$$\mathbb{E}_x \left[ \mathrm{e}^{-q\tau_c^+}; \tau_c^+ < \tau_a^- \right] = \frac{W^{(q)}(x-a)}{W^{(q)}(c-a)}, \tag{2.9}$$

$$\mathbb{E}_x \left[ \mathrm{e}^{-q\tau_a^-}; \tau_a^- < \tau_c^+ \right] = Z^{(q)}(x-a) - \frac{Z^{(q)}(c-a)}{W^{(q)}(c-a)} W^{(q)}(x-a), \tag{2.10}$$

where, for a random variable $Y$ and an event $A$, we use the following notation

$$\mathbb{E}[Y; A] := \mathbb{E}[Y \mathbb{1}_A].$$

For more details on spectrally negative Lévy processes, the reader is referred to [Kyp06].

## 2.2 Some notions of ruin

Let $X = \{X_t, t \geq 0\}$ be a surplus process.

**The classical ruin time**

The time of *classical* ruin is defined as

$$\tau_0^- = \inf\{t > 0 \colon X_t < 0\},$$

yielding the classical ruin-based risk measure $\mathbb{P}_x\left(\tau_0^- < \infty\right)$, i.e. the probability of ruin. From a practical point of view, it is more interesting to consider the finite-time probability of ruin $\mathbb{P}_x\left(\tau_0^- \leq t\right)$, where $t > 0$ is a fixed deterministic time corresponding to the time horizon of interest for the risk managers.
In the last few years, in order to distinguish between default and bankruptcy, several concepts of *ruin* have been proposed. In particular, the idea of Parisian ruin has attracted a lot of attention; it considers the application of an implementation delay in the recognition of an insurer's capital insufficiency. More precisely, it assumes that Parisian ruin occurs if the excursion below the critical threshold level 0 is longer than a pre-determined time called the implementation delay, or the *clock*. Two types of Parisian ruin have been considered: with deterministic delays or with stochastic delays.

**The Parisian ruin time with deterministic delays**

In [CP11, LCP13], a Parisian ruin time (with a deterministic delay $r > 0$) is defined as

$$\tau_r = \inf\{t > 0 \colon t - g_t > r\},$$

where $g_t = \sup\{0 \leq s \leq t \colon X_s \geq 0\}$, and studied for Lévy insurance risk models. In other words, the company is said to be *Parisian ruined* the first time an excursion below zero lasts longer than the fixed implementation delay $r$. We will call this *deterministic Parisian ruin*, where *deterministic* refers to the implementation delay.

**The Parisian ruin time with exponential delays**

In [LRZ11, LRZ14a, BPPR16], exponentially distributed delays have been considered. A descriptive definition of Parisian ruin with exponential delay is given in [BPPR16], for spectrally negative Lévy processes. The idea is to mark the Poisson point process of excursions away from 0 by independent copies of a generic exponential random variable $\mathbf{e}_q$ with rate $q > 0$. Then, the Parisian ruin time is defined as

$$\kappa_q = \inf\left\{t > 0 \colon t - g_t > \mathbf{e}_q^{g_t}\right\},$$

where $g_t = \sup\{0 \leq s \leq t \colon X_s \geq 0\}$ and where each random variable $\mathbf{e}_q^{g_t}$ is exponentially distributed with rate $q > 0$. We will call this *exponential Parisian ruin*.

Note that for the two definitions of Parisian ruin mentioned above, there is a different clock for each excursion, either deterministic or stochastic.

**The cumulative Parisian ruin**

With J-F. Renaud, UQAM, we propose in [GR17] a new definition of actuarial ruin based on the occupation-time process of the surplus process $X$, namely the process

$$t \mapsto \int_0^t \mathbb{1}_{(-\infty,0)}(X_s)\mathrm{d}s.$$

We define the time of cumulative Parisian ruin (at level $r > 0$) by

$$\sigma_r = \inf\left\{t > 0 \colon \int_0^t \mathbb{1}_{(-\infty,0)}(X_s)\mathrm{d}s > r\right\}. \tag{2.11}$$

Note that for cumulative Parisian ruin, the parameter $r$ could be interpreted as the length of a clock started at the beginning of the first excursion, paused when the process returns above zero, and resumed at the beginning of the next excursion, and so on. In other words, it is the length of this single clock that is compared to the sum of all excursions below zero, as opposed to the previous definitions of Parisian ruin where a new clock is started at the beginning of each excursion.
We are interested in the following new ruin-based risk measure

$$\mathbb{P}_x\left(\sigma_r \leq t\right),$$

where $t > 0$ is a fixed deterministic time. Clearly, for fixed values $t, r > 0$, we have

$$\{\sigma_r > t\} = \left\{\int_0^t \mathbb{1}_{(-\infty,0)}(X_s)\mathrm{d}s \leq r\right\},$$

so, the finite-time probability of cumulative Parisian ruin is given by

$$\begin{aligned}\mathbb{P}_x\left(\sigma_r \leq t\right) &= 1 - \mathbb{P}_x\left(\int_0^t \mathbb{1}_{(-\infty,0)}(X_s)\mathrm{d}s \leq r\right)\\ &= 1 - \int_0^r \mathbb{P}_x\left(\int_0^t \mathbb{1}_{(-\infty,0)}(X_s)\mathrm{d}s \in \mathrm{d}u\right).\end{aligned} \tag{2.12}$$

As the distribution of cumulative Parisian ruin is closely related to the distribution of the occupation-time process associated with the underlying surplus process $X$ over a finite-time horizon, we will now study the latter for classical surplus processes.
Of course, it is very difficult to obtain analytical expressions for the probability of finite-time cumulative Parisian ruin for a general insurance risk process. However, there is a quite simple expression of its Laplace transform when the initial capital is zero in the case of spectrally negative Lévy processes (See e.g. Remark 4.1 in [LRZ14a]).

**Proposition 3.9.** *If $X$ is a SNLP such that $\mathbb{E}[X_1] = \psi'(0+) > 0$, then, for $p, q > 0$,*

$$\int_0^\infty \mathrm{e}^{-pt}\mathbb{E}\left[\mathrm{e}^{-q\int_0^t \mathbb{1}_{(-\infty,0)}(X_s)\mathrm{d}s}\right]\mathrm{d}t = \frac{\Phi(p+q)}{p+q}\frac{1}{\Phi(p)}. \tag{2.13}$$

Noticing that

$$\int_0^\infty \mathrm{e}^{-pt}\mathbb{E}\left[\mathrm{e}^{-q\int_0^t \mathbb{1}_{(-\infty,0)}(X_s)\mathrm{d}s}\right]\mathrm{d}t = \int_0^\infty\int_0^\infty \mathrm{e}^{-pt-qu}\mathbb{P}\left(\int_0^t \mathbb{1}_{(-\infty,0)}(X_s)\mathrm{d}s \in \mathrm{d}u\right)\mathrm{d}t,$$

we obtain the distribution of the occupation time by inverting this double Laplace transform. When the process starts at 0 and $\mathbb{E}[X_1] = \psi'(0+) > 0$, using [LRZ14b, Corollary 3], [Ren14, Corollary 2] and [LCP13, Lemma 2], we deduce from (2.13) that for any $\alpha > 0$, $t > \alpha$,

$$\begin{aligned}\mathbb{P}\left(\sigma_r \leq t\right) &= \mathbb{P}\left(\int_0^t \mathbb{1}_{(-\infty,0)}(X_s)\mathrm{d}s > \alpha\right)\\ &= \int_\alpha^t \int_0^\infty \frac{y}{t-r}\mathbb{P}\left(X_{t-r} \in \mathrm{d}y\right)\int_0^\infty \frac{z}{r}W'(z)\mathbb{P}\left(X_r \in \mathrm{d}z\right)\mathrm{d}r. \qquad (2.14)\end{aligned}$$

When the surplus process is continuous, it is easy, thanks to the strong Markov property, to extend this result on the distribution of the cumulative ruin time $\sigma_r$ to a surplus process starting at an arbitrary positive capital. Unfortunately, when the surplus process is not continuous we can not control the value of $X_{\tau_0^-}$, and then deduce the distribution of $\sigma_r$ when the surplus process starts at a positive capital is not straightforward.

**Links between the ruin times**

We easily observe that

$$\tau_0^- \leq \sigma_r \leq \tau_r \quad \text{and} \quad \tau_0^- \leq \kappa_q,$$

and, for all $x \geq 0$, the deterministic Parisian ruin time $\tau_r$ converges $\mathbb{P}_x$-almost surely, as $r \to 0$, to the time of classical ruin $\tau_0^-$.
Furthermore, we have this remarkable result :

**Proposition 3.10.** *[GR17, Proposition 3.4] If $X$ is an SNLP and $\mathbf{e}_q$ an independent exponential random variable with rate $q > 0$, then for all initial surplus $x \geq 0$, $\sigma_{\mathbf{e}_q} \overset{(d)}{=} \kappa_q$, i.e. for all $t \geq 0$,*

$$\mathbb{P}_x\big(\sigma_{\mathbf{e}_q} \leq t\big) = \mathbb{P}_x(\kappa_q \leq t).$$

## 2.3 Other risk measures

With Z. Ben-Salah, M. Morales and H.O. Firouzi in [BSGMF15], we study other risk measures, based on the large fluctuations of the surplus process. Considering a spectrally negative Lévy surplus process $X = \{X_t, t \geq 0\}$ (with no monotone paths), we introduce its running infimum and its running supremum respectively defined by

$$\underline{X}_t := \inf_{0\leqslant s\leqslant t} X_s \qquad \text{and} \qquad \overline{X}_t := \sup_{0\leqslant s\leqslant t} X_s\ .$$

Now we characterize the depletion problem for $X$. We first define the *drawdown* process $Y = (Y_t)_{t \geqslant 0}$, associated with a given risk process $X$, to be

$$Y_t := \overline{X}_t - X_t \ , \qquad t \geqslant 0 \ . \tag{2.15}$$

The first-passage time over a level $a > 0$ of the drawdown process $Y$ is then defined to be

$$\tau_a^D := \inf\{t \geqslant 0 : Y_t > a\} \ . \tag{2.16}$$

It is well-known that $\tau_a^D < \infty$ $\mathbb{P}$-almost surely (see [AKP04], Theorem 1). This new random time in (2.16) contains relevant information on potentially risky behavior of the reserve. Their distribution can be used to measure the likeliness of path-related events that might have a negative impact on the financial health of the reserve. The random time $\tau_a^D$ records the time at which a drawdown in the reserve is larger than a, previously agreed upon, critical level $a$. An interesting set of associated tale-telling random variables can be built upon the random time (2.16). First, we need to define a process that will be useful in constructing meaningful non-ruin quantities. The last time before $t$ that $X$ reaches its running supremum, denoted by $\overline{G}_t$, is defined as

$$\overline{G}_t := \sup\{s \leq t : X_s \text{ or } X_{s-} = \overline{X}_s\} \ . \tag{2.17}$$

Thus the time $\tau_a^D$ of the critical drawdown of size $a$ along with the following quantities characterize the depletion problem for $X$:

- the last time the reserve was at its maximum level prior to critical drawdown, $\overline{G}_{\tau_a^D}$;
- the *speed of depletion*, $\tau_a^D - \overline{G}_{\tau_a^D}$;
- the maximum reserve level attained before critical drawdown is observed, $\overline{X}_{\tau_a^D}$;
- the minimum reserve level prior to critical drawdown, $\underline{X}_{\tau_a^D}$;
- the largest drawdown observed before critical drawdown of size $a$, $Y_{\tau_a^D -}$;
- the overshoot of the critical drawdown over level $a$, $Y_{\tau_a^D} - a$.

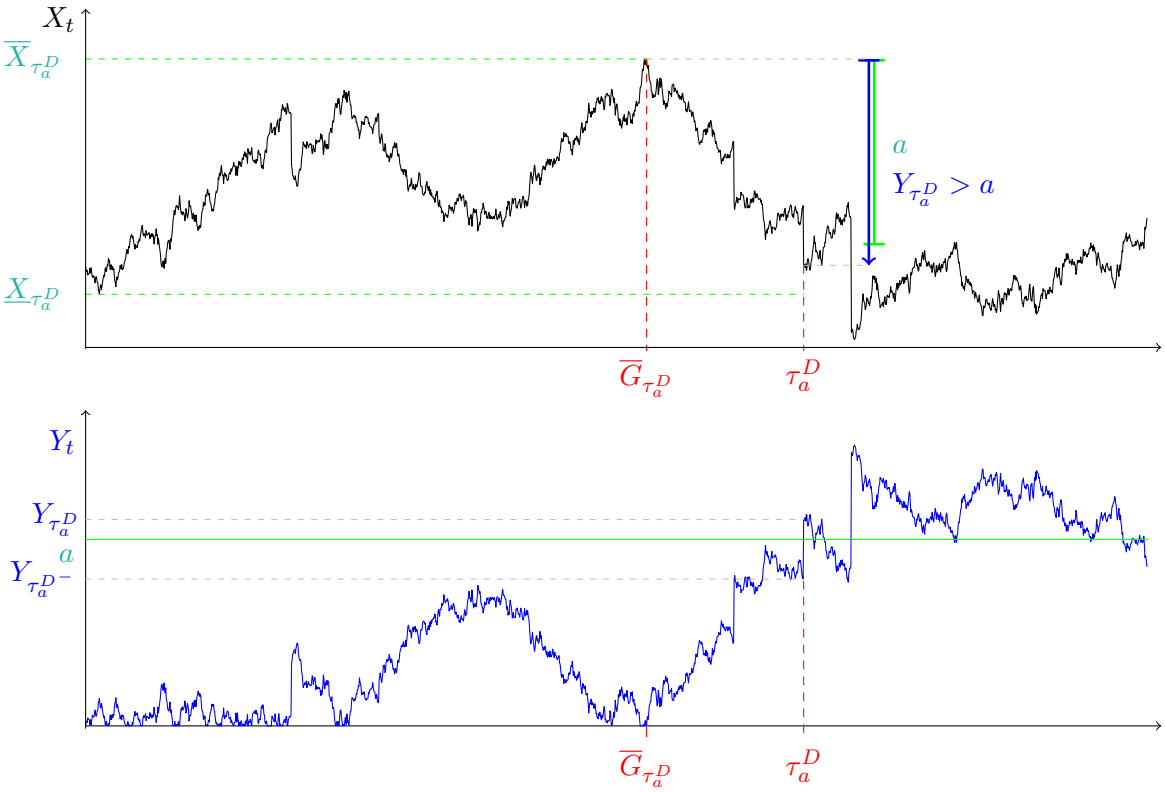



Figure 3.2: A path of $X_t = 10 + t + 2B_t - S_t$, its corresponding drawdown process $Y$, and their related depletion quantities, where $(B_t)_{t\geq 0}$ is a standard Brownian motion and $S$ is an independent compound Poisson process with Lévy measure $\nu(\mathrm{d}x) = e^{-2y}\mathrm{d}x$.

Clearly, these variables contain information on how the insurance deserve depletes over time. All of these quantities encapsulate relevant knowledge about the critical drawdown event. A risk manager would be potentially interested in gaining information regarding the distribution of the time of the critical drawdown of size $a$, i.e. $\mathbb{P}_x(\tau_a^D \leqslant t)$. Even more valuable information can be found in the distribution of the *speed of depletion*, this random variable indicates how fast critical drawdowns tend to occur.

The main results obtained on this question of depletion have been obtained by A. Mijatovic and M. Pistorius [MP12]. They give an explicit expression of the Laplace transform of $(\tau_a^D, \overline{G}_{\tau_a^D})$ in terms of the scale functions of the process $X$. In [BSGMF15], we study the probability to have ruin before time $\tau_a^D$ of the critical drawdown of size $a$ and we explicit the distribution of the drawdown $Y_{\tau_a^{D-}}$ observed juste before critical drawdown, of the overshoot over the critical drawdown $Y_{\tau_a^D} - a$. We also observe that the speed of depletion

$\tau_a^D - \overline{G}_{\tau_a^D}$ is independent of $\overline{G}_{\tau_a^D}$. Some classical insurance models are studied: the Cramér-Lundberg model with exponential claims, the Gamma risk process, the spectrally negative stable risk process, the Brownian motion with drift and at last the meromorphic Beta-risk process.

# 3 Occupation time of Lévy processes in finite time

The previous section highlights the link between Parisian ruin and occupation times of the surplus process $X = \{X_t, t \geq 0\}$. We assume that $X$ is a Lévy insurance process, i.e. a spectrally negative Lévy process with no monotone paths. In this section, we study the distribution of occupation times of a SNLP $X$.

One classical result of occupation time is the Paul Lévy's arcsine laws which gives the distribution of the occupation time of the positive/negative half-line for a standard Brownian motion. More precisely, if $\{B_t, t \geq 0\}$ is a standard Brownian motion, then, for $t > 0$,

$$\mathbb{P}\left(\int_0^t \mathbb{1}_{(-\infty,0)}(B_u) \leq s\right) = \frac{2}{\pi} \arcsin\left(\sqrt{\frac{s}{t}}\right) \mathbb{1}_{(0,t)}(s).$$

This result was then extended to a Brownian motion with drift by Akahori [Aka95] and Takács [Tak96].
In the last few years, several papers have looked at the distribution of functionals involving occupation times of a spectrally negative Lévy process (SNLP), in each case over an infinite time horizon. First, in [LRZ11], the Laplace transform of the occupation time of semi-infinite intervals for a SNLP has been derived. More precisely, the Laplace transform of

$$\int_0^\infty \mathbb{1}_{(-\infty,0)}(X_s)\mathrm{d}s \quad \text{and} \quad \int_0^{\tau_{-b}^-} \mathbb{1}_{(-\infty,0)}(X_s)\mathrm{d}s,$$

where $X = \{X_t, t \geq 0\}$ is a spectrally negative Lévy process and where $\tau_{-b}^- = \inf\{t > 0 \colon X_t < -b\}$ with $b > 0$, were expressed in terms of the Laplace exponent and the scale functions of the underlying process $X$. Then, in [LRZ14b], those results were significantly extended, first by considering more general quantities, i.e.

$$\left(\tau_0^-, \int_0^{\tau_0^-} \mathbb{1}_{(a,b)}(X_s)\mathrm{d}s\right) \quad \text{and} \quad \left(\tau_c^+, \int_0^{\tau_c^+} \mathbb{1}_{(a,b)}(X_s)\mathrm{d}s\right),$$

where

$$\tau_0^- = \inf\{t > 0 \colon X_t < 0\} \quad \text{and} \quad \tau_c^+ = \inf\{t > 0 \colon X_t > c\},$$

and where $0 \leq a \leq b \leq c$, and by obtaining considerably simpler expressions for the joint Laplace transforms. Note that [KPP14] and [Ren14] have also looked at the abovementioned quantities involving occupation times, but for a so-called refracted Lévy process, while similar quantities for diffusion processes were studied in [LZ13] and [Lac13].

With J-F. Renaud in [GR16], we study the joint distribution of a spectrally negative Lévy process and its occupation time when both are sampled at a fixed time. This is closer in spirit to Lévy's arcsine law and also much more useful for actuarial applications (e.g. for Parisian ruin) or financial applications (especially for the pricing of occupation time options).

## 3.1 Joint distributions of occupations times and the process sampled at a fixed time

We are interested in the following distribution: for fixed $q, t > 0$, and for all $x \in \mathbb{R}$,

$$\mathbb{E}_x \left[ \mathrm{e}^{-q \int_0^t \mathbb{1}_{(0,a)}(X_s)\mathrm{d}s}; X_t \in \mathrm{d}y \right]. \tag{3.18}$$

We will consider the Laplace transform, with respect to $t$, of the expectation in (3.18): for all $x \in \mathbb{R}$, set for $p > 0$

$$v(x, \mathrm{d}y) := \int_0^\infty \mathrm{e}^{-pt} \mathbb{E}_x \left[ \mathrm{e}^{-q \int_0^t \mathbb{1}_{(0,a)}(X_s)\mathrm{d}s}; X_t \in \mathrm{d}y \right] \mathrm{d}t.$$

We notice that it can be written

$$v(x, \mathrm{d}y) = \frac{1}{p} \mathbb{E}_x \left[ \mathrm{e}^{-q \int_0^{\mathbf{e}_p} \mathbb{1}_{(0,a)}(X_s)\mathrm{d}s}; X_{\mathbf{e}_p} \in \mathrm{d}y \right], \tag{3.19}$$

where $\mathbf{e}_p$ is an exponentially distributed random variable (independent of $X$) with mean $1/p$.
Recall that if we assume $X$ has paths of bounded variation (BV), then, for any $b \in \mathbb{R}$, we have $X_{\tau_b^-} < b$ almost surely; in other words, $X$ does not creep downward. The main idea in [GR16] is to decompose the quantity $v(., \mathrm{d}y)$ given by (3.19) according to the position of the hitting times of $(0, a)$ with respect to the exponential time $\mathbf{e}_p$ and to use the strong Markov property. Then $v(., \mathrm{d}y)$ can be expressed in terms of $v(0, \mathrm{d}y)$, $v(a, \mathrm{d}y)$ and of the $p-$potential measures of $X$ killed on exiting an interval. The proof consists in solving the expression and expressed all the coefficients in terms of scale functions, which is quite technical.
For the sake of compactness of the main result, we introduce the following functions. First, for $p, p + q \geq 0$ and $x \in \mathbb{R}$, set

$$\mathcal{W}_a^{(p,q)}(x) = W^{(p+q)}(x) - q \int_0^a W^{(p+q)}(x - z) W^{(p)}(z) \mathrm{d}z. \tag{3.20}$$

We notice that $\mathcal{W}_a^{(p,q)}(x) = W^{(p)}(x)$ for $x \leq a$. Secondly, for $p \geq 0$, $q \in \mathbb{R}$ with $p + q \geq 0$ and $x \in \mathbb{R}$, set

$$\mathcal{H}^{(p,q)}(x) = \mathrm{e}^{\Phi(p)x} \left[ 1 + q \int_0^x \mathrm{e}^{-\Phi(p)z} W^{(p+q)}(z) \mathrm{d}z \right]. \tag{3.21}$$

Note that the Laplace transform of $\mathcal{H}^{(p,q)}$ on $[0,\infty)$ is explicitly given by

$$\int_0^\infty \mathrm{e}^{-\theta x}\mathcal{H}^{(p,q)}(x)\mathrm{d}x = \frac{1}{\theta-\Phi(p)}\left(1+\frac{q}{\psi(\theta)-p-q}\right), \text{ for } \theta > \Phi(p+q). \tag{3.22}$$

We now give an explicit form of the Laplace transform of the joint distribution of the occupation time and the process sampled at a deterministic time.

**Theorem 3.11.** *[GR16, Theorem 1] Fix $a>0$, $q\geq 0$ and $x\in\mathbb{R}$. For $p>0$, $y\in\mathbb{R}$,*

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-pt}\mathbb{E}_x\left[\mathrm{e}^{-q\int_0^t \mathbb{1}_{(0,a)}(X_s)\mathrm{d}s}; X_t\in\mathrm{d}y\right]\mathrm{d}t \\
&\quad = \mathrm{e}^{-\Phi(p)a}\left(\frac{\mathcal{H}^{(p,q)}(x)-q\int_a^x W^{(p)}(x-z)\mathcal{H}^{(p,q)}(z)\mathrm{d}z}{\psi'(\Phi(p))+q\int_0^a \mathrm{e}^{-\Phi(p)z}\mathcal{H}^{(p,q)}(z)\mathrm{d}z}\right) \\
&\qquad \times\left\{\mathcal{H}^{(p,q)}(a-y)-q\int_0^{-y}\mathcal{H}^{(p,q)}(a-y-z)W^{(p)}(z)\mathrm{d}z\right\}\mathrm{d}y \\
&\qquad\quad -\left\{\mathcal{W}^{(p,q)}_{x-a}(x-y)-q\int_0^{-y}\mathcal{W}^{(p,q)}_{x-a}(x-y-z)W^{(p)}(z)\mathrm{d}z\right\}\mathrm{d}y.
\end{aligned}$$

The result of Theorem 3.11 can be extended to any finite and then semi-infinite interval, see [GR16, Corollary 1], i.e. the Laplace transform of

$$\mathbb{E}_x\left[\mathrm{e}^{-q\int_0^t \mathbb{1}_{(a,b)}(X_s)\mathrm{d}s}; X_t\in\mathrm{d}y\right].$$

with $t>0$ and $-\infty\leq a<b\leq+\infty$.
As explained in the previous section, this result has many applications, especially in finance and in insurance. We develop in the sequel two of those applications.

## 3.2 Applications in finance and in insurance

### Occupation time options

Let the risk-neutral price of an asset $S=\{S_t, t\geq 0\}$ be of the form:

$$S_t = S_0\mathrm{e}^{X_t},$$

where $X=\{X_t, t\geq 0\}$ is the log-return process. For example, in the Black-Scholes-Merton model, $X$ is a Brownian motion with drift. The time spent by $S$ in an interval $I$, or equivalently the time spent by $X$ in an interval $I'$, from time 0 to time $T$, is given by

$$\int_0^T \mathbb{1}_{\{S_t\in I\}}\mathrm{d}t = \int_0^T \mathbb{1}_{\{X_t\in I'\}}\mathrm{d}t.$$

Introduced by Linetsky [Lin99], (barrier) step options are exotic options linked to occupation times of the underlying asset price process. They are generalized barrier options: instead of being activated (or canceled) when the underlying asset price crosses a barrier, which is a problem from a risk management point of view, the payoff of occupation-time options will depend on the time spent above/below this barrier. Therefore, the change of value occurs more gradually. For instance, a (down-and-out call) step option admits the following payoff:

$$\mathrm{e}^{-\rho\int_0^T \mathbb{1}_{\{S_t\leq L\}}\mathrm{d}t}\left(S_T-K\right)_+ = \mathrm{e}^{-\rho\int_0^T \mathbb{1}_{\{X_t\leq \ln(L/S_0)\}}\mathrm{d}t}\left(S_0\mathrm{e}^{X_T}-K\right)_+,$$

where $\rho>0$ is called the *knock-out rate*. Therefore, its price can be written as

$$\begin{aligned} C(T) &:= \mathrm{e}^{-rT}\mathbb{E}\left[\mathrm{e}^{-\rho\int_0^T \mathbb{1}_{\{S_t\leq L\}}\mathrm{d}t}\left(S_T-K\right)_+\right] \\ &= \mathrm{e}^{-rT}\int_{\ln(K/S_0)}^{\infty}\left(S_0\mathrm{e}^{y}-K\right)\mathbb{E}\left[\mathrm{e}^{-\rho\int_0^T \mathbb{1}_{\{X_s\leq \ln(L/S_0)\}}\mathrm{d}s};X_T\in\mathrm{d}y\right] \end{aligned}$$

with $r$ the risk-free interest rate and its Laplace transform, with respect to the time of maturity $T$, can be written

$$\begin{aligned} &\int_0^\infty \mathrm{e}^{-pT}C(T)\mathrm{d}T \\ &\quad= \int_0^\infty \mathrm{e}^{-(p+r)T}\int_{\ln(K/S_0)}^{\infty}\left(S_0\mathrm{e}^{y}-K\right)\mathbb{E}\left[\mathrm{e}^{-\rho\int_0^T \mathbb{1}_{\{X_s\leq \ln(L/S_0)\}}\mathrm{d}s};X_T\in\mathrm{d}y\right]\mathrm{d}T \\ &\quad= \int_{\ln(K/S_0)}^{\infty}\left(S_0\mathrm{e}^{y}-K\right)\int_0^\infty \mathrm{e}^{-(p+r)T}\mathbb{E}\left[\mathrm{e}^{-\rho\int_0^T \mathbb{1}_{\{X_s\leq \ln(L/S_0)\}}\mathrm{d}s};X_T\in\mathrm{d}y\right]\mathrm{d}T. \end{aligned}$$

Thus, pricing step options boils down to identifying the distribution:

$$\mathbb{E}\left[\mathrm{e}^{-\rho\int_0^T \mathbb{1}_{(-\infty,b)}(X_s)\mathrm{d}s};X_T\in\mathrm{d}y\right],$$

for a given value $b\in\mathbb{R}$. Other occupation time options can also be priced using this distribution. For references on occupation-time option pricing, see e.g. [CCW10, Hug99]. In the specific cases of Lévy jump-diffusions, we deduce from Theorem 3.11 and [GR16, Corollary 1] an explicit expression of the price of a step option in terms of the characteristic of the Lévy process (see Section 4.2 in [GR16]).

**Cumulative ruin time for the Cramér-Lundberg model with exponential claims**

We consider in this section the Cramér-Lundberg insurance model with exponential claims. The surplus is given by

$$X_t = X_0 + ct - \sum_{i=1}^{N_t} C_i,$$

where $c > 0$, $N = \{N_t, t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ and the $C_i$'s form a sequence of independent and identically distributed random variables following an exponential distribution with rate $\alpha > 0$.
In this model, the Laplace exponent is given by $\psi(\theta) = c\theta - \frac{\lambda\theta}{\alpha+\theta}$ and for $p \geq 0$, the solutions of Lundberg's equation $\psi(\theta) - p = 0$ are given by

$$\Phi(p) = \frac{1}{2c}\Big(p + \lambda - c\alpha + \sqrt{\Delta_p}\Big), \quad \Theta(p) = \frac{1}{2c}\Big(p + \lambda - c\alpha - \sqrt{\Delta_p}\Big),$$

where $\Delta_p = (p + \lambda - c\alpha)^2 + 4c\alpha p$.
Clearly, for a fixed $t > 0$, the distribution

$$\mathbb{P}\left(\int_0^t \mathbb{1}_{(-\infty,0)}(X_u)\mathrm{d}u \in \mathrm{d}s\right)$$

has a mass $a_t$ at 0, which is the survival probability $\mathbb{P}\left(\tau_0^- > t\right)$. From Proposition 3.9, we obtain an expression for this distribution.

**Proposition 3.12.** *[GR17, Proposition 3.2] For a fixed $t > 0$, we have*

$$\mathbb{P}\left(\int_0^t \mathbb{1}_{(-\infty,0)}(X_u)\mathrm{d}u \in \mathrm{d}s\right) = a_t\delta_0\left(\mathrm{d}s\right) + a_{t-s}\left(\lambda - c\alpha(1 - a_s)\right)\mathbb{1}_{(0,t)}(s)\mathrm{d}s, \tag{3.23}$$

*with*

$$a_t = \left(1 - \frac{\lambda}{c\alpha}\right)_+ + \frac{2\lambda}{\pi}\mathrm{e}^{-(\lambda+c\alpha)t}\int_{-1}^1 \frac{\sqrt{1-u^2}}{\lambda + c\alpha + 2\sqrt{c\alpha\lambda}\,u}\mathrm{e}^{-2\sqrt{c\alpha\lambda}\,t\,u}\mathrm{d}u, \tag{3.24}$$

*and $z_+ := \max(z, 0)$.*

Note that there exist other equivalent expressions for $a_t = \mathbb{P}\left(\tau_0^- > t\right)$ in the Cramér-Lundberg model with exponential claims. See e.g. [Dre09], or [AA10, Proposition 1.3] and the references therein.
To obtain the corresponding distribution under $\mathbb{P}_x$ for $x > 0$, i.e. when $X_0 = x > 0$, we have to invert the double Laplace transform of occupation times given in Theorem 3.11 (since $X_{\tau_0^-} < 0$ a.s.). Again note that, for a fixed $t > 0$, the distribution

$$\mathbb{P}_x\left(\int_0^t \mathbb{1}_{(-\infty,0)}(X_u)\mathrm{d}u \in \mathrm{d}s\right)$$

has a mass $a_t^x$ at 0, which is the survival probability $\mathbb{P}_x\left(\tau_0^- > t\right)$. By a quite technical proof we finally obtain:

**Theorem 3.13.** *[GR17, Theorem 2]*
*For a fixed $t > 0$, we have*

$$\mathbb{P}_x\left(\int_0^t \mathbb{1}_{(-\infty,0)}(X_u)\mathrm{d}u \in \mathrm{d}s\right) = a_t^x\delta_0(\mathrm{d}s) + \left(a_{t-s}^x + k_{t-s}^x\right)\left(\lambda - c\alpha\left(1 - a_s^0\right)\right)\mathbb{1}_{(0,t)}(s)\mathrm{d}s, \tag{3.25}$$

*with*

$$\begin{aligned} a_t^x &= 1 - \lambda \mathrm{e}^{-\alpha x}\int_0^t \mathrm{e}^{-(\lambda+c\alpha)s}\left[I_0\left(2\sqrt{\lambda c\alpha}\sqrt{s(s+x/c)}\right) - \frac{s}{s+x/c}I_2\left(2\sqrt{\lambda c\alpha}\sqrt{s(s+x/c)}\right)\right]\mathrm{d}s, \\ k_t^x &= \mathrm{e}^{-\alpha x} - 1 \\ &\quad + x\alpha\lambda\mathrm{e}^{-\alpha x}\int_0^t s^{-1}\mathrm{e}^{-(\lambda+c\alpha)s}\left[I_0\left(2\sqrt{c\alpha\lambda}\sqrt{s(s+x/c)}\right) - I_2\left(2\sqrt{c\alpha\lambda}\sqrt{s(s+x/c)}\right)\right]\mathrm{d}s, \end{aligned} \tag{3.26}$$

*where $I_\nu$ represents the modified Bessel function of the first kind of order $\nu$, which can be written as: for $s \geq 0$,*

$$I_0(s) = \frac{1}{\pi}\int_{-1}^1 \mathrm{e}^{-su}\left(1-u^2\right)^{-1/2}\mathrm{d}u \quad \textit{and} \quad I_2(s) = \frac{s^2}{3\pi}\int_{-1}^1 \mathrm{e}^{-su}\left(1-u^2\right)^{3/2}\mathrm{d}u.$$

# 4 To go further

The cumulative Parisian ruin, introduced with J.-F. Renaud (UQAM) in [GR17], consists in declaring bankruptcy as soon as the occupation times of the capital $X$ below zero exceeds a predefined level $\alpha > 0$:

$$\tau_\alpha = \inf\left\{t \geq 0 : \int_0^t \mathbb{1}_{X_s<0} > \alpha\right\}.$$

This ruin time indeed belongs to a more general framework, one speaks about *Omega* models (see for example [GSY12, LRZ11]). Ruin times from *Omega* models are expressed in term of an integral of a functional of the capital. Recently Li and Palmowski, [LP18], have adapted the notion of scale functions for SNLP to the case of *Omega* models, work then extended to refracted Lévy processes by Li and Zhou, [LZ18]. One interesting and unsolved example of Omega models is a ruin time taking into account both the spent time of the process in the negative line and the amplitude below zero of the capital. But this ruin time does not satisfy the assumptions of [LP18]. Moreover, in [LP18], the distribution of the *Omega* ruin times are expressed in terms of a new class of scale functions, which are defined in an implicit way. A recent publication by Döring and Kyprianou, [DK16], gives a characterization of the finiteness of an integral on $\mathbb{R}^+$ of a functional of a Lévy process. An interesting question is to obtain explicit (and not implicit) results by using local times and by expressing the excursions of the process for some specific *Omega* models, as the one presented just above.

Another challenge in ruin theory is to expand the classical insurance framework which is based on the nice properties of the spectrally negative Lévy processes. This is quite restrictive but also allows very sharp results on ruin times. Consequently to enlarge the framework, we have to proceed by small steps, as what is done with the refracted Lévy process. We can also think to the PDMP introduced in [BCT08] to model the toxine level in a body or financial models taking into account investment rates (see e.g. [Pau08]). A general objective could be to understand the mechanisms which allow us to express the distribution of the ruin times in terms of the characteristics of the surplus process in order to introduce innovative models.

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## Titre : Modèles probabilistes en physique, biologie et actuariat.

**Mot clés :** EDS non linéaires, Systèmes de particles, PDMP, processus de Lévy

**Resumé :** Ce mémoire est un survol de mes travaux à propos de différents modèles probabilistes. Les questions de comportement en temps long, de dépendance en la condition initiale, de simulation et de caractérisation des fluctuations sont abordées ici. Tout d'abord, la sensibilité en la condition initiale des solutions des équations de Boltzmann et de Landau en théorie cinétique est examinée. Nous introduisons également une méthode innovante pour simuler la solution de l'équation de Landau. Nous considérons ensuite le comportement en temps long d'une bactérie flagellée dans son environnement, mouvement modélisé par un processus de Markov déterministe par morceaux. Enfin, de nouveaux instants de ruine en assurance et certains processus avec Markov switching sont étudiés.

## Title : Probabilistic models in physics, biology and actuarial science.

**Keywords :** Nonlinear SDE, Particle systems, PDMP, Lévy processes

**Abstract :** This document is an overview of my works on different probabilistic models for various applications, always with the aim of studying carefully the behavior and properties of the underlying process. First, the question of the dependence on the initial condition of the solutions of the Boltzmann and the Landau equations, in kinetic theory, is investigated. We also introduce an innovative method to simulate the solution of the Landau equation. We then consider the long time behavior of a flagellated bacterium in its environment, modelled by a piecewise deterministic Markov process. At last new ruin times in actuarial science and some Markov switching processes are introduced and studied in the last chapter.