Toward a deeper understanding of the relationship between interaction constraints and visual isomorphs

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Abstract

Interaction and manual manipulation have been shown in cognitive science literature to play a critical role in problem solving. Given different types of interactions or constraints on interactions, a problem can appear to have different degrees of difficulty. While this relationship between interaction and problem solving has been well studied in the cognitive science literatures, the visual analytics community has yet to exploit this understanding for analytical problem solving. In this paper, we hypothesize that constraints on interactions and constraints encoded in visual representations can lead to strategies of varying effectiveness during problem solving. To test our hypothesis, we conducted a user study in which participants were given different levels of interaction constraints when solving a simple mathematic game called number scrabble. Number scrabble is known to have an optimal visual problem isomorph, and the goal of this study is to learn if and how the participants could derive the isomorph and to analyze the strategies that the participants utilize in solving the problem. Our results indicate that constraints on interactions do affect problem solving, and that although the optimal visual isomorph is difficult to derive, certain interaction constraints can lead to a higher chance of deriving the isomorph.

Introduction

In his insightful keynote address to the EuroVis 2009 conference, Pat Hanrahan[1] discussed how visualization plays a role as a system of thought. One remarkable topic of his talk is the use of visual problem isomorphs to make complex problem solving seem simple and often trivial. The examples Dr Hanrahan provided demonstrated the idea that once the right visual representation of the problem isomorph is found, solving that problem can be as simple as looking at the visual representation and identifying the right answer immediately. Of particular interest to us is the example of using a "magic square" as a visual isomorph to the number scrabble game (which is discussed further in "The number scrabble problem" section). It is clear that by transforming the number scrabble game into a magic square, this relatively difficult game of finding and adding multiple numbers becomes as simple as playing a game of tic-tac-toe.

This example is compelling because the process of encouraging a user to discover a useful visual isomorph for a problem can be thought of as the primary goal of visualization. However, the obvious question is, how does someone find the right visual isomorph for

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a problem? Unfortunately, the answer to this question is not trivial or well understood. Building on work in cognitive science and diagrammatic reasoning[2], we argue that helping a user find a useful visual isomorph is not just a matter of presenting an appropriate visual representation. Rather, people can best discover visual solutions to problems through interaction with visual representations. Unfortunately, although visualization researchers understand how to design visualizations to represent data, they have not exploited the relationship between interaction and problem solving to the same extent as cognitive scientists.

The goal of the research presented in this paper is, therefore, to bridge the gap between the findings in the cognitive science community and the visualization community. Specifically, we acknowledge research in the cognitive science community which shows that interaction plays a critical role in problem solving[2]. However, given our emphasis on visualization, we do not simply seek to corroborate their existing findings. Instead, our interest lies at the intersection of these two fields where we look to understand how using interaction to solve problems can lead to the identification of potential visual isomorphs.

First we extend the notion that interaction generally facilitates problem solving. However, we further hypothesize that interactions with different constraints and amounts of encoded information will lead to different solutions to the number scrabble problem. In addition, we hypothesize that during the problem-solving process the different constraints on interaction will lead to different types of isomorphs, both visual and non-visual. Finally, we hypothesize that these different types of isomorphs have varying degrees of effectiveness in solving the problem, which can be measured quantitatively.

To test our hypotheses, we conducted a user study in which 117 participants were given different types of interaction constraints while developing strategies for the number scrabble game. We chose to use the number scrabble problem because it is self-contained and is known to have an optimal visual problem isomorph in the form of the magic square[3]. The participants' accuracy and time in playing the game against a computer were logged and tracked, and their strategizing session was video recorded. Based on the data obtained from the study, we find that (1) different constraints on interactions do affect the participants' performance while playing the game; (2) with more constraints, the participant has a higher chance to derive the optimal visual isomorph (the magic square); and, finally, while not all participants were able to derive the optimal visual isomorph, (3) using visual isomorphs in general leads to a better performance than using non-visual isomorphs.

We begin by reviewing related work on interaction and problem isomorphs in the context of problem solving. Next, we present our experiment exploring the effect of
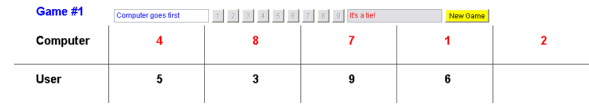
Figure 1. Number scrabble game interface. The first line provides information regarding which player goes first, which player won, and buttons for users to click on as the next move. The numbers in the grid illustrate the moves by both the computer and the user.

interaction constraints on deriving visual problem isomorphs. We then discuss the implications of our experimental results and limitations of the study.

Related work

Our conception of visualization as providing externalizations for problem solving draws on work in visualization theory as well as cognitive science. In particular, we study how visual representations can provide useful isomorphs of the information they visualize. Two problems or representations are isomorphic if they are informationally equivalent but present that information in different structures. As an example, we use the number scrabble problem and its isomorphic magic square representation.

The number scrabble problem

The original number scrabble[3] is a game played by two people with nine cards: ace through nine. The cards are placed in a row, face up. The players draw alternately, one at a time, selecting any one of the unselected cards. The objective of the game is for a player to get three cards which add up to 15 before his opponent does. If all nine cards have been drawn without either player having a combination that adds up to 15, the game is a draw.

The main reason we chose to use the number scrabble game is that there is a known visual isomorph of the problem called the "magic square" (Figure 1). As the magic square visually represents all possible combinations of three numbers that can be added up to 15 in a succinct manner, it can significantly help a player to perform well at the game. In other words, once this visual isomorph is identified, the number scrabble problem is turned into a much simpler tic-tac-toe game, which is played by two players who take turns marking the spaces in a 3×3 grid. The number scrabble game represents a large number of well-defined problems that show how visual isomorphs can make evident what was previously true but obscure[3].

Isomorphs and diagrammatic reasoning

Simon[3] defined problem isomorphs as problems whose solutions and moves can be placed in a one-to-one relation with the solutions and moves of the given problem.

The key to isomorphism is that even when two representations contain the same information, they can still provide very different sets of operations for accessing and inferring about that information, which can make a given problem easier or harder to solve[4]. According to Larkin and Simon[4], a given representation consists of both data structures and programs operating on them to make new inferences. Whether a representation is worth 10,000 words depends on what productions are available for searching the data structure, for recognizing relevant information, and for drawing inferences from that information[4]. In our example, the magic square and the number scrabble problem are isomorphs of the same problem in that they both contain all the information needed to play the game. However, in number scrabble, the numbers appear in a single sequence (structure) and the operations available to the player to access information about the game—such as whether your cards contain a winning combination—are mathematic. In the magic square case, that information is contained in a visual operation: seeing whether the cards form a line across the magic square grid. Such simple search operations are much easier to perform than mathematic calculations. As the brain processes visual search operations faster than mathematic ones, the visual isomorph is more efficient in this case.

The idea that visual representations make certain operations more efficient to perform is at the core of the theory of diagrammatic reasoning[4,5]. However, efficiency is not the only measure of interest in visualization; our goal is to make information not just accessible, but understandable. The distinction between these goals is highlighted by Carroll et al.[6], who had participants solve a design problem presented as one of two isomorphs: a spatial arrangement problem and a temporal scheduling problem. The spatial isomorph was easier and faster for participants to solve and led to fewer failures in understanding the problem. That is, in the temporal case there were several participants whose solutions did not follow the requirements of the task. Interestingly, when participants in both cases were provided with a simple graphical representation (a grid) in which to work on their solution, the temporal case was as easy to solve as the spatial one, but participants in the temporal case remained more likely to fail to understand the problem requirements. The authors took this to mean that appropriate graphical representations can make problems easier to solve, but not necessarily easier to understand.

Another way to interpret this is that there is more to designing a visual isomorph than making information easier to access. Much of the power of visual representations comes from how they set constraints on interpretation and reasoning. Constraints inherent in visual isomorphs can encode constraints on the information they represent, leading to a more direct preservation of information structure[7]. As Stenning and Oberlander[8] argue, these constraints inherent to visual representations help to meaningfully restrict the number and kinds of inferences that can be made about a problem, focusing processing power only on valid cases. In this way, visual isomorphs can not only make operations more efficient, but also model the constraints of a problem directly. This can affect the difficulty of solving a problem by reducing the cognitive load of remembering rules[9] or by encouraging different types of strategies[10].

Interaction and problem solving

In this section, we explore the relationship between interaction and problem solving based on previous work in the field of mathematic education and visualization.

Interacting with manipulatives in the field of mathematic education. In the field of mathematic education, manipulatives have long been in use as popular aids in learning and problem solving. Manipulative can be defined as any tangible object, tool, model, or mechanism that may be used to clearly demonstrate a depth of understanding, while problem solving refers to a specified mathematic topic or topics[11]. The role of interacting with manipulatives in learning and problem solving has received much attention in educational psychology research. Particularly, Dienes[12] studied children's learning of mathematic concepts from experiences with concrete materials under the assumption that interaction and the higher cognitive activities are closely connected. However, empirical evidences that directly support such a claim are spotty. Most of the studies focused on specific aspects of mathematic problem solving and the results of using manipulatives as aids are mixed. As pointed out by Beishuizen[13], manipulatives bring both positive and negative consequences as opposed to purely relying on mental strategies on addition and subtraction problems. There also seems to be a discrepancy between solution strategies on the manipulative level and the mental level.

On the other hand, Resnick[14] argues that interacting with manipulatives can help students to develop a deeper understanding of advanced concepts such as system dynamics. Chao et al.[15] further investigated whether structure or variability in physical manipulatives would have greater effectiveness in kindergartners' learning of number concepts. They found that interacting with each type of material had an impact on a different aspect of learning. More recently, the effect of interacting with digital manipulatives such as computer games was also examined. Bright et al.[16] have conducted a study to investigate whether there were differences in achievement by constraining resources available to game players in a mathematic game. Although interacting with the game was proven effective in improving students' achievement

of its mathematic content, the authors did not find significant differences in achievement between groups with varying available resources. More specifically, constraining resources in the digital manipulative did not lead to different level of improvement on participants' fraction ordering skills.

The effect of interaction on problem solving in the field of visualization. While we believe visual representations can aid problem solving significantly on their own, they gain even more power to model a problem when interaction is introduced. Interaction is increasingly seen as central to the process of reasoning with visualization.[17–19] Lending weight to the intuition that interaction improves reasoning, Hundhausen et al.[20] found that interacting with an algorithm visualization produces better understanding than viewing an equivalent animation.

We use the term "interaction" in the broad sense defined by Yi et al.[21] "the dialogue between the user and the system as the user explores the dataset to uncover insights." In this sense, the relationship between interaction and problem solving has been the subject of much research by cognitive scientists in the field of distributed cognition[21]. In particular, David Kirsh[2,22–25] has argued extensively that projection and interaction with external representations are fundamental to human reasoning. Kirsh points to the pervasive use of external representations and interaction with the world in everyday problem solving, and identifies several functions performed by interaction in the reasoning process[2]. Of these, most relevant to our work is *reformulation*, or the ability to restate ideas. Kirsh sees reformulation as a process that is frequently too complex to perform entirely in memory, and so is often managed with external tools. As reformulation is closely related to identifying different problem isomorphs, we argue that this process can also be made easier through certain types of interaction.

The effect of interaction on problem solving in visual mathematic representations. Sedig and Liang[26] and Sedig and Sumner[27] have conducted a series of research on how visual mathematic representations (VMRs) affect learning and cognitive processes. VMRs are defined as graphical representations that encode properties and relationships of mathematic concepts. Sedig and Sumner[27] identified 12 interactivity factors that affect the learning and cognitive processes of learners who use VMR-based mathematic cognitive tools. The factors include affordance, cognitive offloading, constraints, distance, epistemic appropriateness, etc. The identified factors are part of the endeavor to establish a descriptive framework to aid in the design and evaluation of VMRs.

One of the factors—constraints—that Sedig et al.[27] have identified is specifically related to our experiment.

As defined by them, the constraints factor deals with restrictions in the possible interactive operations that can be performed with the VMR. This factor refers to how interaction can focus, canalize, and direct a learner's cognitive processes and how interaction can guide thinking and reasoning. In this paper, we evaluate how interaction constraints affect problem solving in the context of the number scrabble game.

Hypotheses

Our research objective is to investigate the question of how constraints on interaction affect problem solving through the derivation of visual isomorphs. We propose that in developing a strategy for playing a game like number scrabble, participants will tend to derive an isomorph for the problem that is easier for them to use than the representation in the original game, and that the availability of different levels of interaction while strategizing will lead to different types of isomorphs. If this is the case, it can help to clarify the relationship between interaction with visual representations and reasoning. To what extent does the nature of a visual representation, and the type of interactions a user is allowed to perform upon it, affect the kind of strategy that a user develops for solving a problem?

We therefore designed a study based on the aforementioned number scrabble game because of its known optimal visual isomorph, the magic square. In our study, we developed five different interaction conditions, ranging from free form to very restrictive, and studied how strategizing under these conditions affects problem solving and the development of isomorphs. In particular, we propose three interrelated hypotheses concerning interaction, problem solving, and isomorphs:

1. **Interactions and problem solving**: We hypothesize that different types of interactions will affect the participants' performance in playing the number scrabble game. Specifically, we hypothesize that more constrained interactions can promote desired strategies, and will therefore lead to better problem solving.

2. **Interactions and isomorphs**: We hypothesize that the different constraints on interaction will affect the isomorphs generated by the participants. With higher constraints on interaction, a participant will be more likely to derive the optimal visual isomorph (the magic square).

3. **Isomorphs and problem solving**: Based on Larkin's[3] definition of isomorphism, we hypothesize that not all isomorphs developed by participants will be visual, but that visual isomorphs will be more effective for playing the number scrabble game.
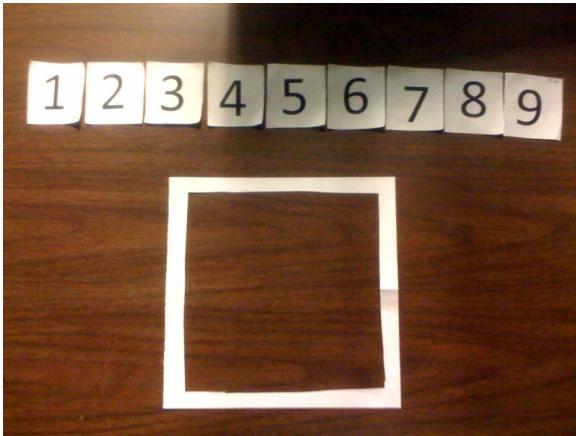
Figure 2. Cards and boundary.

Experiment design

The main factor of interaction constraint had five levels (no interaction, pen and paper, multiple sets of cards, single set of cards, and boundary). Details of each constraint and design rationale will be discussed in the "Interaction constraints" section. We used a between-subjects design with repeated measures. Each subject is randomly assigned to one of the five interaction constraint conditions which determine what interactions are available to them during their strategy session. Qualitative measures in our experiment are the types of isomorphs that our subject derived during their strategy session. Quantitative measures involved response time and scores on number scrabble games played against a computer, using the game interface shown in Figure 2. The computer was programmed to play the game optimally so that it never loses. While our subjects played the game against the computer, we recorded the number of games tied or lost and the time it took for them to figure out the next move for response time. We alternated who makes the first move between the subjects and the computer for every game played.

Participants

We recruited a total number of 117 participants (86 male, 31 female) from "Introduction to computer science" courses at our university. Participants' ages ranged from 18 to 40 years with median of 25 years. Students were primarily undergraduates, and 80% were enrolled in computing-related majors.

Task

The experiment began with investigators introducing the number scrabble game to the subjects based on a training script. The investigators were asked to play the game with the participants until they fully grasped the rules. Next,

the participants filled out a demographic form on age, gender, and experience with mathematic courses through a web interface. The rest of the experiment was divided into four major sessions: pre-test, strategizing, externalizing isomorph, and post-test.

1. **Pre-test**: During the pre-test session, the participants were asked to play the number scrabble game six times against the computer. To make sure that our participants did not start developing strategies during the pre-test, we enforced a maximum time limit of 18 minutes to finish all six pre-test games. Failing to meet the time limit resulted in a participant's data being dropped from analysis.
2. **Strategizing**: During the strategizing session, the subjects were given 20 minutes to interact with the materials we provided under the different interaction constraints and were told to look for a strategy that could help them play the game better.
3. **Externalizing isomorphs**: At the end of the strategizing session, all participants were given 2–3 minutes to make a "cheat sheet" out of the strategy they had developed so that they could refer to it during the post-test session when they play number scrabble again. This cheat sheet was a single sheet of paper onto which participants were told they could write anything they felt would help them play the game (in the case of the pen and paper condition, this was a separate sheet from that which they wrote on during the strategizing session.) This gave us a record of the isomorph used by participants in forming a strategy and reduced the cognitive load on participants during the post-test session. We gave them a very short amount of time to make their "cheat sheet" so that they could not continue elaborating on it after the end of the strategizing session.
4. **Post-test**: During the post-test session, participants were asked to play the number scrabble game six more times against the computer while consulting their "cheat sheet." To be consistent with the pre-test, and also to make sure that the participants did not refine their isomorphs during the post-test session, 18 minutes was set as the upper limit for playing all six games. As in the pre-test session, failing to meet the time limit resulted in a participant's data being dropped from the analysis.

After the post-test session, participants were asked to fill out a questionnaire regarding how they arrived at their strategy and their experience during the strategizing session. The investigators collected all the participants' "cheat sheets" for further analysis of the isomorphs they had derived during the experiment. In addition, the strategizing sessions were video recorded, which allowed us to examine how the interaction constraints affected

our participants' behavior during the process of searching for an isomorph.

Interaction constraints

We went through multiple rounds of a refining process to design the interaction constraint conditions used in our study. Our goal was to design constraints that ranged from placing no limit on the interaction to restricting the interaction a great deal.

- Constraint #1 (no interaction): The participants were asked to think about the problem in their head during the strategizing session to develop a strategy to help them play the game better. The participants were not allowed to interact with any materials.
- Constraint #2 (pen and paper): The participants were provided with a pen and paper to work out their strategy for the number scrabble problem.
- Constraint #3 (multiple sets of cards): The participants assigned to this constraint were provided with multiple sets of cards, with each set consisting of the numbers one through nine. Each card is square in shape and made from paper with the numbers printed on them. Within the strategizing session, the participants were encouraged to organize the cards freely.
- Constraint #4 (single set of cards): The participants were further limited to interact with only one set of cards labeled with the numbers one through nine.
- Constraint #5 (boundary): This is the most restrictive case. Participants were presented with nine cards and a square space large enough to fit only the cards in a grid, and were told to confine their interactions to that space. Figure 3 shows this condition.

In our study, the term constraint refers to not only confinement of interaction space, but also the amount of resource a user can have to externalize his or her mental states. Our conditions are designed so that "no interaction" serves as a control group where participants have no external aids on solving the number scrabble problem. The condition "pen and paper" places no limit on how participants could interact with the nine numbers in the sense that users could duplicate each number as many times as possible and move them around on a piece of paper. In addition, users could also group numbers and annotate the grouping. Then, based on both the original description of the number scrabble problem and the optimal visual isomorph, we derived the other three interaction constraints from "multiple sets of cards" to "boundary" by adding more constraints on interaction each time, all of which encode some information about the optimal
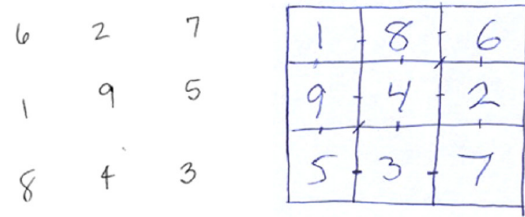
(a) Partial magic square examples
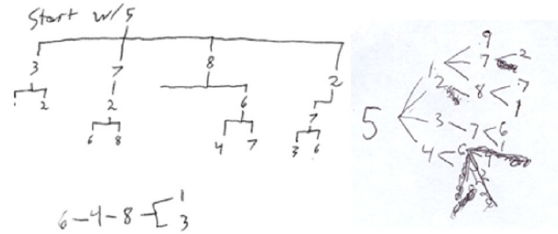
(b) Other visual isomorph examples

(c) Permuted isomorph examples

Figure 3. Isomorph examples.

visual isomorph of the problem. In the condition of multiple sets of cards, duplicating and moving cards around are still allowed but annotation is no longer available compared with the pen and paper condition. Next, in the "single set of cards" condition, users could not duplicate cards but still had the freedom to form different patterns using the nine cards. Finally, the boundary further limited the way in which a user could possibly organize the nine cards.

Results

When analyzing the experimental data, we were concerned with the impact of outliers due to random responses. Therefore, we trimmed out the data of four participants whose response times were unusually fast during the pre-test. In addition, 11 of our participants reached the 18-minute time limit and another two participants accidentally skipped one or two games during either pre- or post-test, and thus their data were dropped automatically because their missing data made it impossible to compare fairly the pre-test and post-test scores. As a result, we have valid data from 100 participants with 20 subjects under each interaction constraint.
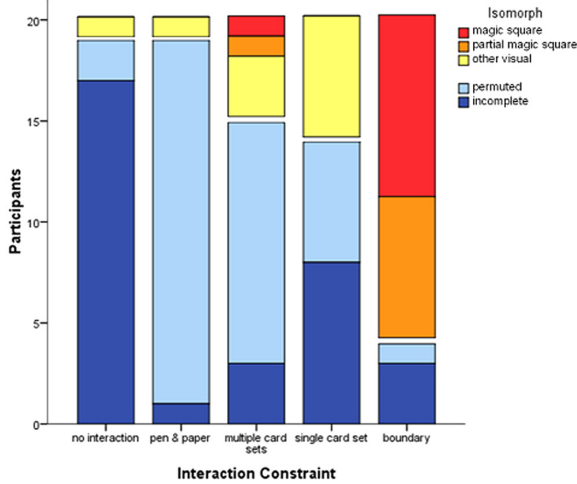
Figure 4. Distribution of isomorphs developed under five different interaction constraints. The gaps divide visual isomorphs (1,2 and 3) from non-visual isomorphs (4 and 5).

Isomorph versus interaction constraint

Based on the strategies recorded on their cheat sheets, our participants developed a wide range of problem isomorphs during the experiment. Some of these were visual while the others were either mathematic or purely descriptive. We classified these isomorphs into five different categories:

1. **Magic square (visual)**: The magic square isomorph.
2. **Partial magic square (visual)**: Same layout as the magic square isomorph with different ordering or numbers.
3. **Other visual isomorph**: Visual isomorph but numbers are not organized in a 3×3 matrix manner.
4. **Permuted isomorph**: All possible combinations of three numbers adding to 15.
5. **Incomplete isomorph**: Strategies that do not involve all nine numbers.

Note that categories 1–3 are visual isomorphs of the number scrabble problem but 4 and 5 are not. In addition, examples of different types of isomorphs are shown in Figure 4

The distribution of different isomorphs developed by our subjects within each interaction constraint is shown in Figure 5. This distribution supports our hypothesis in the sense that as the interactions become increasingly constrained (from pen and paper to boundary), more participants developed visual isomorphs of the number scrabble problem. More importantly, 9 of 20 subjects under the most restrictive constraint (boundary) discovered the optimal visual isomorph (the magic square), while another six subjects developed partial magic square isomorphs. In contrast, only 1 of
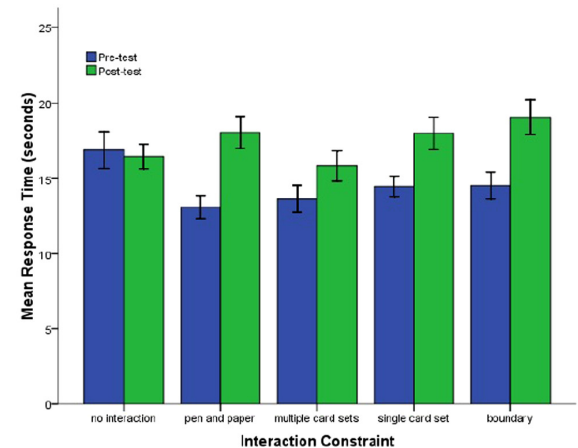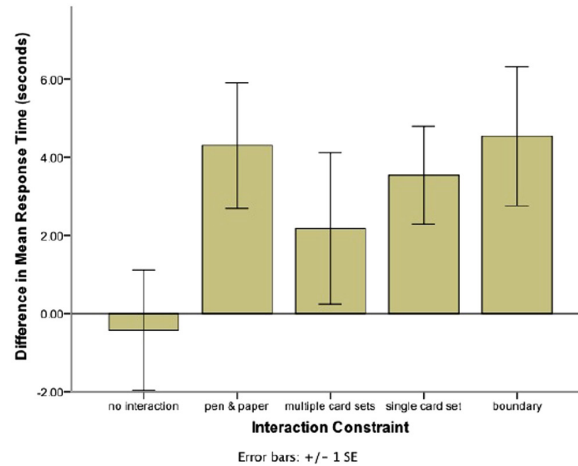
Figure 5. Difference in mean response time (top); mean response (bottom).

20 participants in either the no-interaction condition or the pen-and-paper condition discovered any visual solution. A Pearson's chi-squared test of independence finds a highly significant interaction between interaction constraint and isomorph, $\chi^2(16, n = 100) = 116.9$, $p < 0.001$. As 15 cells have an expected count of less than five, we performed a Fisher's exact test, which also yielded a probability of < 0.001.

The effect of interaction constraints on response time and score

Results regarding time and score were analyzed statistically using an analysis of variance (ANOVA) followed by Tukey's honestly significant difference test for pair-wise comparisons. The factor in our experiment was interaction constraint (five levels) and the dependent variables were difference in response time and improved score.

Difference in response time was derived from the time it took to decide which card to choose next at each move during a game. Response time per game was defined as the average time it took the participants to choose the next card during each game,

$$T = \sum ResponseTime \,/\, n$$

with n being the number of cards chosen following the opponent's move during a specific game. As both the pre-test and post-test sessions comprised six games, the difference in response time was thus defined as

$$IT = \sum_{i=1}^{6} T(i, posttest) - \sum_{i=1}^{6} T(i, pretest)$$

In a similar vein, improved score was derived from whether the subjects tied with or lost to the computer during each game, with tying counting as 1 point and losing as 0 points. Thus, improved score was defined as

$$IS = \sum_{i=1}^{6} S(i, posttest) - \sum_{i=1}^{6} S(i, pretest)$$

Response time. We expected participants to choose the next card faster during the post-test as the interaction constraints increased, as we hypothesized that they would be more likely to derive a better visual isomorph similar to the "magic square." However, we did not observe a significant main effect of difference in response time ($F(4,95) = 1.54$, $p = 0.097$). Figure 6 (top) shows the difference in response time under different interaction constraints. However, interesting yet surprising findings emerged once we considered response time during the pre-test and post-test sessions separately. Figure 6 (bottom) shows the mean response time during both pre- and post-tests under the five interaction constraints. It should be noted that participants in the no-interaction condition had an unusually slow average response time in the pre-test, which makes comparisons between that condition and the others problematic. In general, however, we found that most of our participants spent more time deciding which card to choose next during the post-test session, and participants under the most confined constraints took the longest time, which ran counter to our expectations. We discuss possible reasons for this in the discussion session.

Score. If we consider mean scores on the pre-test and the post-test separately, it is clear that in general our participants scored higher after the strategizing session under all five interaction constraints ($F(1,1190) = 57.7$, $\eta_p^2 = 0.046$, $p < 0.001$). More importantly, the subjects in the more constrained interaction groups tended to score higher than those in the less restrictive interaction groups.

For improved score (Figure 7, top), we observed a significant main effect of interaction constraint type ($F(4,95) = 6.5$, $\eta_p^2 = 0.215$, $p < 0.001$). Post hoc tests showed that the improved scores were significantly different between numerous pairs of interaction constraints. Note that the mean scores have improved during the post-test among all conditions (Figure 7, bottom). To
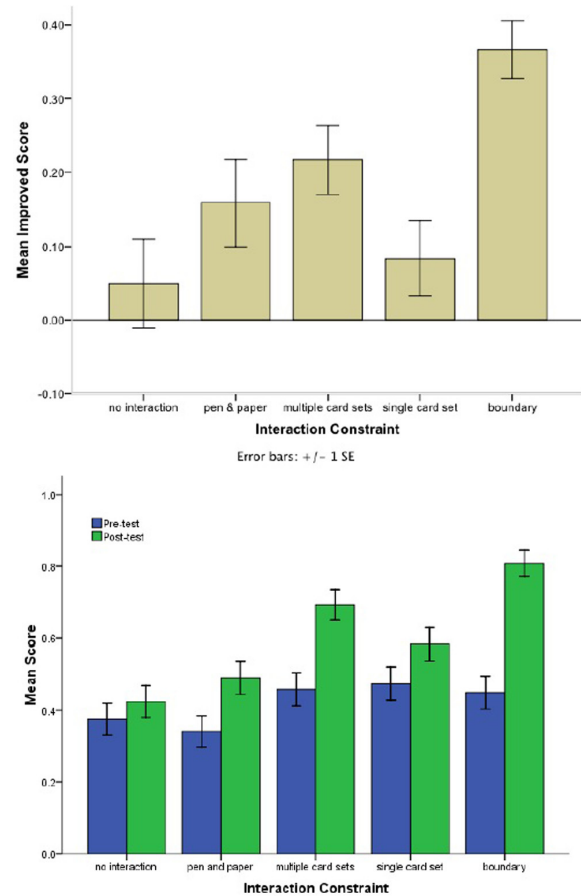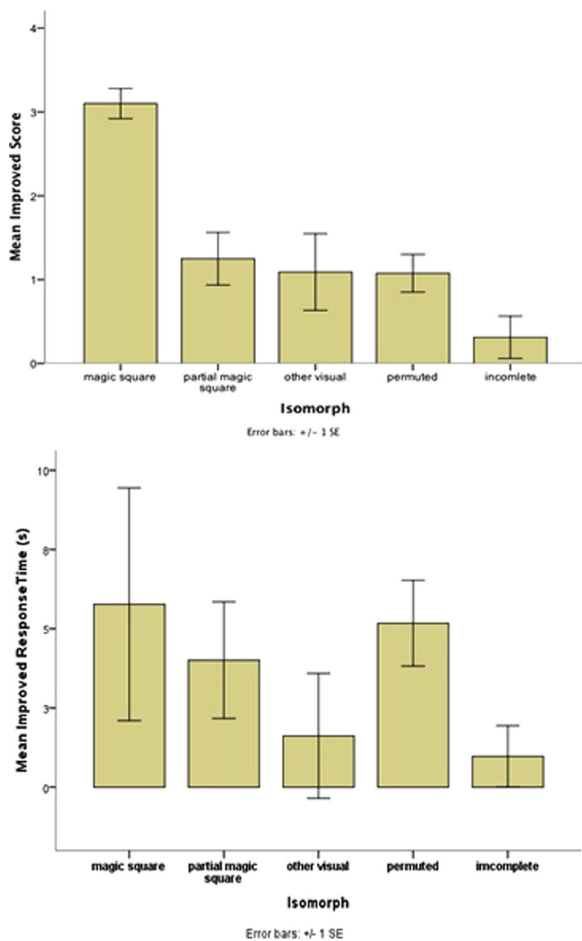

Figure 6. Mean improved score (top), mean score of both pre and post-test across five conditions (bottom).

elaborate, the improved score for participants assigned to interaction constraint #5 (boundary) was significantly larger than that for participants assigned to interaction constraint #1 (no interaction), $p = 0.001$, constraint #2 (pen and paper) with $p < 0.01$, and constraint #4 (one set of cards) with $p < 0.01$. Although the result of other pair-wise comparisons were not significant, we saw a clear trend (Figure 7, top) that as the interaction constraints become more restrictive, the improvement of score increases, except in the case of constraint #4. We further analyze this unexpected "dip" in the discussion section.

The effect of isomorphs on response time and score

Overall, the main effect of types of derived isomorph is significant ($F(4,95) = 8.495$, $\eta_p^2 = 0.263$, $p < 0.001$) on improved score (Figure 8). Post hoc tests showed that the improved scores for participants who derived the magic square isomorph is significantly higher than for participants who derived partial magic squares at $p < 0.05$, and significantly higher than those of all other participants at $p < 0.01$. The result supports our hypothesis

Figure 7. Mean improved score vs. Isomorph(top); mean improved response time (bottom).

that the optimal solution does lead to much better performance in terms of accuracy. Although the other pairs were not significantly different on mean improved score, we saw a trend that as the isomorphs were further from the optimal magic square, the mean improved score decreased. We further performed a linear contrast between visual isomorphs (1, 2, 3) and non-visual isomorphs (4, 5) on improved score. The result showed that the mean improved score for participants using visual isomorphs was significantly larger than for those using non-visual isomorphs ($t(95) = 3.822$, $p < 0.001$).

Unexpectedly, we did not observe a significant result of isomorph type in terms of difference in response time (Figure 8, bottom). In fact, the results showed that users who developed the magic square solution on average took longer in the post-test to decide the next move. We provided further explanation in the "Why is response time not a good measure?" section.

Discussion

We start our discussion by addressing the key questions based on our hypotheses.

Do more confined interaction constraints yield a better chance of deriving a visual isomorph?

Yes, based on Figure 5 and the chi-squared analysis ("Isomorph versus interaction constraint" section), we observed that as the interaction constraints were increasingly restricted, larger number of visual isomorphs were developed. In addition, the strictest interaction constraints led to the highest number of the optimal visual isomorphs discovered. Nine out of 20 participants under constraint #5 (boundary) discovered the magic square isomorph during the strategizing session and seven participants out of the remaining 11 discovered a partial magic square isomorph. Based on further analysis of feedback about the interaction constraints, most participants under this condition found constraint #5 very helpful in their discovery of the visual isomorphs. Many of them left comments such as, "It helped me visualize the problem and make competitive moves." Similarly, most subjects under interaction constraints #3 (multiple sets of cards) and #4 (one set of cards) felt that being able to manipulate the cards freely was helpful, while subjects under constraint #1 (no interaction) commented that they wished to have materials that they could externalize their thoughts on. Thus, both statistics and user feedbacks support the hypothesis that interaction constraints significantly affect the types of isomorphs that users are able to derive by altering the way in which participants approach the same problem. In other words, the manipulation of the isomorphs could be embodied in the interaction.

Does a more advanced visual isomorph outperform a non-visual isomorph in terms of score?

Yes. We consider an isomorph as more advanced if it is more similar to the optimal visual isomorph (the magic square). Thus, our results summarized in the "The effect of isomorphs on response time and score" section confirm that visual isomorphs lead to a greater increase in score than non-visual isomorphs. What is more, within the group of visual isomorphs, the optimal visual isomorph outperforms the other two significantly. Comparing the effect of isomorphs (Figure 8) with interactions constraints (Figure 7), the type of visual isomorphs present a much larger effect on the improved scores. The contrast reflects the relationship among interaction constraints, strategies (isomorphs), and performance. On the one hand, interaction constraints could affect the problem-solving strategies—in other words, the final problem isomorphs that a user is able to develop. On the other hand, the problem isomorphs with a different efficiency directly affect the user's performance
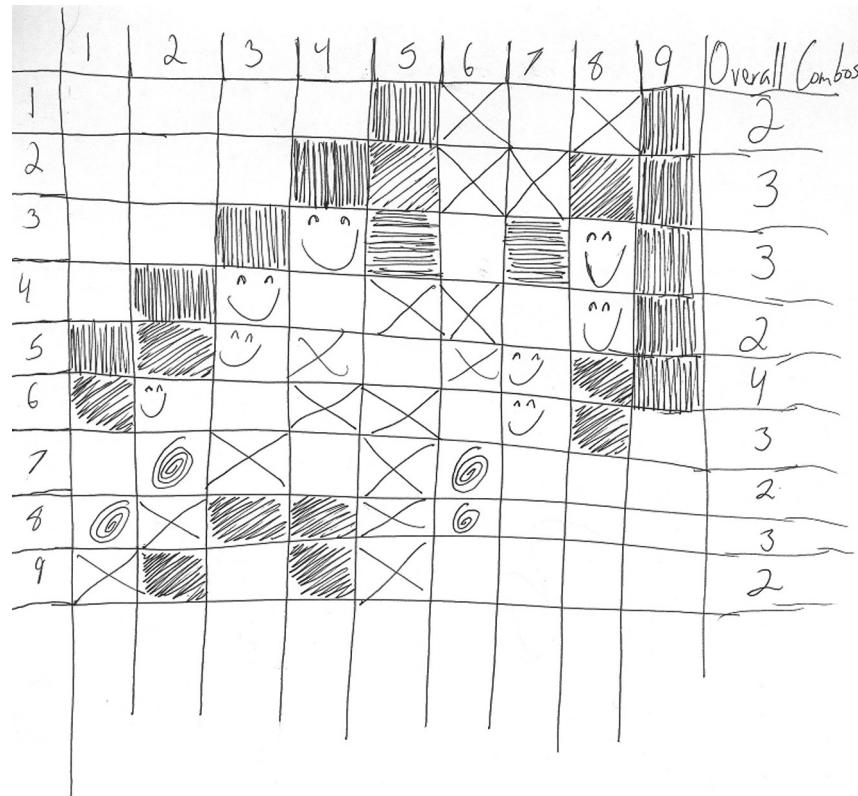
Figure 8. A matrix-like visual isomorph.

on the number scrabble game. Therefore, we observed a more salient effect of the problem isomorphs than the interaction constraints on the improved scores.

Does more confined interaction constraint always yield larger improvements on score?

The short answer is: not always. As seen in Figure 7, the general trend showed that as the interaction constraints became more restricted, the improved score tended to rise, with the exception of constraint #4 (one set of cards). The low improved score in this condition can be explained by considering Figure 5, which shows that none of the participants under this condition derived a magic square (red) or partial magic square (orange) isomorph. Without more efficient visual isomorphs, it made sense that the subjects did not do much better in their post-test than in their pre-test. However, when we designed the five interaction constraints, we considered one set of cards as a highly restrictive constraint, and thus we expected better scores and more derivation of the optimal isomorph. Based on the comments they left, many participants in this condition felt limited by only being able to interact with one set of cards and wished they were given paper to write down the combinations of numbers they found to offload the burden of having to memorize them.

After the experiment, when we presented the magic square isomorph to participants, most in this condition thought they were close to discovering the optimal isomorph at some point during the experiment. But without the extra boundary to further constrain their interaction, it was hard for them to find the bridge between one set of cards and the magic square. This finding highlights the fact that more restrictive interaction constraints are not necessarily helpful unless they meaningfully encode information about the problem. The single set of cards constrained interaction, but without the boundary this constraint did not by itself tell participants anything about the nature of the problem.

Why is response time not a good measure?

Unexpectedly, we did not observe a significant result of isomorph type in terms of both post-test response time and difference in response time (Figure 8, bottom). In fact, response times in the post-test were generally longer than in the pre-test, and the participants who discovered the optimal isomorph tended to take an especially long time responding during the post-test. We contacted them afterwards about why they made decisions more slowly during the post-test and found out that instead of playing defensively using the magic square, they spent more time

thinking about how to beat the computer. Thus, we can infer that the bar set by this particular group of participants was higher than just "not to lose." Overall, it may have been the case that participants in the post-test took a longer time because they were consulting their cheat sheets or otherwise thinking harder about their strategy, as we encouraged them to do in the strategizing session.

Another reason that we did not observe a significant result of different types of isomorphs on difference in response time is that the search time for each of the visual isomorphs our subjects derived to decide the next card might vary drastically. For example, searching through a partial magic square might yield a much faster decision than searching through a 9×9 matrix if the winning combinations are covered by the partial magic square, but this might not necessarily be the case when the partial magic square does not contain the needed combinations of three numbers adding to 15. Overall, as there are many other factors involved in the difference in response time (such as search time and self-expectation of performance), we did not observe a strong causal relationship between the types of isomorph and differences in response time.

A note on the variety of visual isomorphs

In the section "Isomorph versus interaction constraint," we roughly categorized all the isomorphs that our subjects developed during the study into five categories, including three visual and two non-visual isomorph types. In this section we focus mainly on the visual isomorphs discovered by the participants. It is interesting to see that eight participants across interaction constraints #3 (multiple set of cards) and #5 (boundary) developed a partial magic square isomorph, and that 11 participants discovered other forms of visual isomorph across interaction constraints #1, 2, 3, and 4. Within the partial magic square isomorph, there are many variations. Figure 4(a) illustrates a few of them, and we can see that the variations are caused mainly by ordering. There are even more variations under the "Other visual isomorph" category. One type of variation was a decision tree, such as the examples in Figure 4(b); additionally, a few participants built a 9×9 matrix (Figure 9). To see how this isomorph can be used in playing the number scrabble game, refer to Appendix 1.

In Figure 5, we can see a strong contrast between the types of visual isomorphs that the participants generated. Most participants under interaction constraint #5 (boundary) developed magic square-like visual isomorphs during the strategizing session, while there was a relatively larger number of participants under both constraints #3 and #4 who discovered more creative visual isomorphs (such as different forms of decision trees and node-link diagrams). Thus, there seemed to be a trade-off between

Figure 9. 3*3 magic square.

interaction constraint and the creativity of the resulting visual isomorph.

Visual isomorphs and visualization design

Our results show that changing interaction constraints can influence the visual isomorphs discovered by users in a problem-solving situation. The situation we presented to our participants is not directly analogous to that faced by users and designers of a visualization. Nonetheless, we argue that it is valuable to view visualization design from the perspective of encouraging efficient visual isomorphs. In this section, we discuss the relationship between the visual isomorphs studied in our experiment and visual representations of information.

What is a visual isomorph?

In this work, we consider isomorphic visual representations to be analogous to the problem isomorphs described by Simon and colleagues (see Kotovsky and Fallside[28]). Simon et al.[23] define two problem spaces as isomorphic if they have:

- the same number of move operators;
- the same starting position; and
- the same goal.

So, for example, while the magic square and number scrabble look very different, they both have a single move operator, start in the same game state, and end in the same state. Of course, to say that they are the "same" states requires a certain level of abstraction, which introduces some uncertainty into the definition. As we shall see, the same applies to visual isomorphs.

While these criteria make the term "isomorph" more concrete in relation to problem spaces, this definition must be adapted for application to visualization. In this work, we consider two visual representations to be

isomorphic if they are informationally equivalent, which means that the information in one can also be inferred from the other. This is not the same thing as two problem spaces being equivalent, but we argue that there is overlap between the concepts. The criteria by Simon are another way of saying that two problems can be expressed as similar finite state machines. However, the more free-form nature of operations that could be performed on a visual representation does not easily map onto operations in a state machine. Still, the idea that two visual isomorphs must be able to express similar states and move between states with equal freedom can serve as the basis for a more formal definition. Adapting key concepts from Simon's definition of problem isomorph, we define visual isomorph as:

> Two visual representations that present the same dataset, can represent the same set of variables and relationships from that dataset, in a way that information in one representation is also inferable from the other, and vice versa.

An example would be a simple bar chart and a simple line graph, which seem to be obviously isomorphic when presenting the same variables and relationships. After all, a user of Microsoft Excel can freely swap between the two. It could nonetheless be argued that a line graph expresses trend information whereas a bar chart does not. Indeed, Zacks and Tversky[29] provide evidence that perceptions of data can be significantly affected by this difference. That is the beauty of visual isomorphs—while representing the same information, they still make the trending either implicit or explicit. Such differences between implicit and directly expressed information are also found in Simon's problem isomorphs. Both magic square and number scrabble can express the same problem states, but by using a spatial layout, a magic square board makes the moves toward the goal state more obvious than in the number scrabble game. Similarly, a line graph makes the operation of detecting the trend much easier than a bar chart, but both contain the same information. To disambiguate, when we say that two visualizations are isomorphic, we are always referring to two specific visual representations that are being used to represent the same dataset. While it may be possible, using this definition, to argue that a certain visualization technique is *always* equivalent to another, regardless of implementation, making such an argument is outside the scope of this discussion.

Two visualizations cannot be isomorphs unless they make visible all the same variables and relations. A trivial example can be found in cases where certain kinds of data cannot be expressed at all in one visualization. For example, a dataset containing only numerical variables cannot be expressed in a treemap, which requires at least one categorical variable to display at all. In such a case, a treemap can never be isomorphic to a scatterplot. Even when showing the same dataset, two visualizations may not be capable of expressing all the same information.

Previous work in the field of information visualization has provided examples of visual isomorphs and how they influence users on performing common visualization tasks. Ghoniem et al.[30] performed a comparison of the readability of graphs using node-link and matrix-based representations. More specifically, in the experiments, two visualizations displayed the same undirected graph containing the same number of vertices and edges. The two visual representations can be considered isomorphic in this experiment because information such as connections and a common neighbor between two nodes can be inferred from both visualizations. Having evaluated the two visualizations on seven generic tasks, the authors demonstrated that matrix-based visualization performs better than node-link diagrams on most tasks, with the exception of path-finding tasks, which were consistently in favor of node-link diagrams. Stasko et al.[31] evaluated the effectiveness of two space-filling visualizations on file/directory search and analysis tasks. The first visualization implemented the treemap display technique developed by Shneiderman[32] with a rectangular layout, while the second visualization utilized a similar space-filling technique, but with a circular or radial layout. Both visualizations are designed to present hierarchical information structures. During the evaluation, both visualizations presented the same datasets (computer file hierarchies), and thus are considered isomorphic because information in one visualization, such as the size of a specific file or the location of the file in the hierarchies, can be inferred from the other visualization. The evaluation results confirmed that the two isomorphic visualizations affect the task performance differently. Each visualization afforded different search strategies, which in turn influenced the performance. In general, performance trends favored the visualization with the circular layout with respect to correct task performance, especially on initial use. Another interesting finding is regarding the learning cost of two visual representations: treemap bore a higher learning cost, but the cost was partially recouped over time. This evaluation presented similar findings to our experiment in that different isomorphs promote diverse strategies, and the strategies affect users' performance differently.

Why visual isomorphs are important

Having a definition of visual isomorphism clarifies the theory underlying the current research. However, this definition, combined with the findings of our study, can have practical implications for visualization research and

design as well. The fact that two visualizations can contain equal information without being equally effective is an important one to keep in mind when evaluating and creating visualizations. By demonstrating that two visualizations are isomorphic, a researcher can more clearly analyze how and why they differ.

A clear application for this is in evaluation design. The criteria that lead to our definition are in many ways the same criteria used intuitively by a researcher when making two visualization designs comparable in an evaluation study. When doing this, it is important to make sure that visualizations make equal information visible. In this way, researchers ensure that the representation choice itself is being measured, rather than any confounding variables. A formal definition of information equivalence would help to standardize this process and make evaluation results comparable across experiments. We hope that our definition can serve to encourage discussion in the research community about what makes two visualizations a fair comparison and how to design comparison studies more rigorously.

In addition to showing how two visualizations are basically similar, our work can also help to illuminate where visualizations differ. Our study, like Simon's work with problem isomorphs, found that some visual isomorphs are better than others when it comes to problem solving. Once it is established that two visual representations are informationally equivalent, we can see more clearly how they differ in terms of what information they make implicit and explicit. Visual isomorphism can therefore serve as a useful framework for discussing how visualizations make different parts of an information state more or less apparent. For example, a treemap and a node-link diagram can be designed to be isomorphic, but the treemap will still emphasize the part–whole relationships in a dataset while the node-link diagram makes fewer visual distinctions between levels of the tree. Such differences in emphasis can be used to explain research findings such as those of Zacks and Tversky[30] as well as the effects of visual metaphors found by Ziemkiewicz and Kosara[33].

Furthermore, we found that interaction constraints can be used to guide a participant to a more optimal visual isomorph. This could be generally applied to visualization design by "encoding" the constraints of a task or dataset into a system's interaction design. More directly, guiding the user with constraints could be implemented as a design philosophy in visualization frameworks intended for broad usage, such as Tableau. Such systems allow a user to interactively build a visual representation of their data. In Tableau, a user can choose an option called "Show Me," which automatically selects an optimal visualization method for a given data type[34]. Interaction constraints suggest another method. For interactive visualization interfaces, the creation of a visual

problem isomorph is a collective effort from both visualization designers and users. If a user indicates that he or she wants to find trends in his or her data, the available visual objects could be restricted to lines, guiding the user toward a more optimal solution. Although visualization designers try to provide the best possible representations based on data type, users could still have the freedom to put their own spin on the final representation through interacting with the interface. This may be less efficient than Show Me, but, as our study suggests, it could help the user to actually learn the rules of good design through interaction. Such a design could even try to hit the "sweet spot" between creativity and optimal performance found in our study, guiding a user to good solutions but leaving open the possibility of solutions that the designer did not consider. A good example of supporting users to find optimal representations while promoting creativity is the newly launched website "visual.ly." Visual.ly encourages users to create their own visualization elements and layout instead of using existing ones. Visual.ly also provides recommendations on what are the better representations based on the data, so users could build upon the suggested visualizations.

Implications and future work

Our findings suggest that there is a clear connection between the nature of interactions available in a visual representation and the types of strategies users tend to develop when working with the representation. While we have demonstrated this in the context of a specific problem-solving scenario, we argue that our results have significant implications for the more general area of interaction with visual representations with which visual analytics concerns itself.

In particular, this research suggests that degree of constraint is an important dimension to consider when designing interactions for visual analytics systems, although this is not a common way of talking about interaction design in visualization. In cases where a task has an optimal solution path—for example, when there is a standardized procedure that analysts are expected to follow—highly constrained interaction is likely to be a good way to guide a user toward this procedure without the need for extensive training. In situations where the designer needs to encourage creative solutions to a problem, some middle ground between constrained and unconstrained interaction is likely to be more helpful. One strong implication of our findings, however, is that complete freedom of interaction may make problem solving more difficult; encoding some degree of boundaries into the interaction will likely help users to understand the task in a more intuitive fashion.

In particular, when considering the core goals of visual analytics of identifying the expected and discovering the

unexpected[35], our findings would suggest that constraints in the user's interactions would have an impact. Specifically, the results of our study imply that highly constrained interactions can impede the discovery of the unexpected, but can also potentially guide the users to consistently identify the expected findings. In contrast, complete freedom in an interactive visual analytics system has the potential to encourage open-ended explorations that could lead to unexpected discoveries, but users of such systems are likely to find different results in their analysis each time. Although our findings do not directly inform the design decision of the degree of constraints in a user's interactions in a visual analytics system, we do believe that the relationship between the degree of constraint in interactivity and creativity should be carefully considered when designing visual analytics systems.

As demonstrated by our results, the optimal visual isomorph indeed makes the number scrabble problem easier to solve. But, as mentioned in the section "Isomorphs and diagrammatic reasoning," efficiency is not the only measure of interest in visualization; our goal is to make information not just accessible, but also understandable. In this context, it is worth mentioning that we had one participant who discovered the magic square visual isomorph but failed to realize that the nature of the game is just like tic-tac-toe given the optimal isomorph. While one incident does not warrant enough evidence to confirm or counter any existing theory, it is an interesting phenomenon to consider.

As the problem we considered has a known and clearly defined optimal visual isomorph, our designed interaction constraints were geared toward this isomorph. Realizing the limitations of our task, we certainly hope that this proof of concept could be generalized to more complex problems. The obvious next step is to examine how to design interaction constraints for problems that might not have known optimal visual isomorphs.

Conclusion

We have demonstrated that constraining user interactions indeed affects problem solving through exploring the relationship between interaction constraints, visual isomorphs, and problem-solving performance as measured by response time and score. Our results showed that more confined constraints lead to better visual isomorphs, and better visual isomorphs result in large improvements in scores on the number scrabble game. Our hypothesis is further confirmed by a significant effect of interaction constraints on improved score. Overall, our results indicate that the manipulation of isomorphs can be embodied in user interaction by imposing different constraints, and that certain interaction constraints can lead to a higher chance of deriving

a better visual isomorph for a problem. With better visual isomorphs yielding a higher performance, our results demonstrate that we indeed can improve the effectiveness of problem-solving activities by embodying information in user interaction.

Declaration of conflicting interests

The authors declare that they do not have any conflicts of interest.

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Appendix 1

The matrix is designed to allow the user to quickly identify three numbers that can add up to 15. For instance, if player 1 (the user) chooses the number 5, the user will then focus on the row in the matrix that starts with the number 5. If player 2 (the computer) chooses the number 3, the user will be able to see that the cell that intersects row 5 with column 3 contains a smiley face. To quickly identify the number that will complement 5 and 3 for a sum of 15, the user will then follow the same row (row 5) until she finds the other smiley face. In this case, the user will find the second smiley face under the column 7.

In addition, Figure 9 also suggests strategies for choosing the best numbers in playing the number scrabble game. The right-most column, denoted as "Overall Combos," is a count of the number of different types of glyphs in each row. For instance, in row 5, there are four distinct types of glyphs (vertical stripes, diagonal stripes, smiley face, and a cross), which represents the number of combinations of 15 that includes the number 5. The user of this matrix will then choose the number 5 as the optimal starting move, followed by 2, 4, 6, or 8 as the second-best alternative. Note that in Figure 9, the user made a mistake and flipped the values for Overall Combos between the rows 3 and 4. These strategies correspond well to the magic-square isomorph in that the number 5 is always at the center of the square and the even numbers (2, 4, 6, and 8) are at the corners of the square.

Figure 9 is a particularly interesting example of how our participants derived creative isomorphs. Here we explain how this isomorph can be used to assist with playing the number scrabble game.