Comp 141 Probabilistic Robotics Homework 1: Kalman Filter

1 Kalman Filter: Prediction

A balloon drone has encountered a glitch in its program and needs to reboot its on-board computer. While rebooting, the drone is helpless and cannot issue motor commands. To help the drone, you'll need some understanding of the Kalman filter algorithm.

The drone operates in a 1-D world where x_t is the position at time t, while $\dot{x}_t$ and $\ddot{x}_t$ are the velocity and acceleration. For simplicity, assume that $\triangle t = 1$.

Due to random wind fluctuations, at each new time step, your acceleration is set randomly accordingly to the distribution $\mathcal{N}(\mu_{wind}, \sigma^2_{wind})$, where $\mu_{wind} = 0.0$ and $\sigma^2_{wind} = 1.0$.

Question 1.1: The minimal state vector for the Kalman Filter is the state vector for the Kalman Filter is $\begin{pmatrix} x_t & \dot{x}_t \end{pmatrix}$

Question 1.2: The state transition model:

$$\begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{t-1} \\ \dot{x}_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where matrix A is

$$\begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$

=

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and matrix B is 0 since there's no control signal

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

state transition probability function $p(x_t|u_t, x_{t-1}) = det(2\pi R_t^{-1})^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1}(x_t - A_t\right.$
where R is

$$\begin{bmatrix} \frac{1}{4}\Delta t^4 & \frac{1}{2}\Delta t^3 \\ \frac{1}{2}\Delta t^3 & \Delta t^2 \end{bmatrix} \sigma^2_{wind}$$

$=$

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

Question 1.3: assuming the initial time step is t=0,

$$\mu_{t1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \sigma_{t1} = \begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 1.0 \end{bmatrix}$$

$$\mu_{t2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \sigma_{t2} = \begin{bmatrix} 2.5 & 2.0 \\ 2.0 & 2.0 \end{bmatrix}$$

$$\mu_{t3} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \sigma_{t3} = \begin{bmatrix} 8.75 & 4.5 \\ 4.5 & 3.0 \end{bmatrix}$$

$$\mu_{t4} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \sigma_{t4} = \begin{bmatrix} 21.0 & 8.0 \\ 8.0 & 4.0 \end{bmatrix}$$

$$\mu_{t5} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \sigma_{t5} = \begin{bmatrix} 41.25 & 12.5 \\ 12.5 & 5.0 \end{bmatrix}$$

Question 1.4: the uncertainty ellipse of the joint posterior over x and $\dot{x}$ in a diagram where x is the horizontal and $\dot{x}$ is the vertical axis for t=1 to t=5

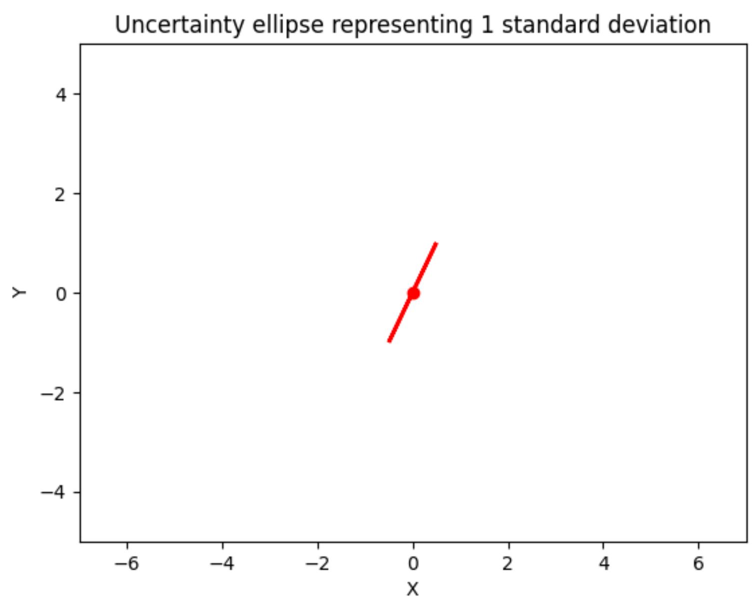

Figure 1: Uncertainty Ellipse at t=1

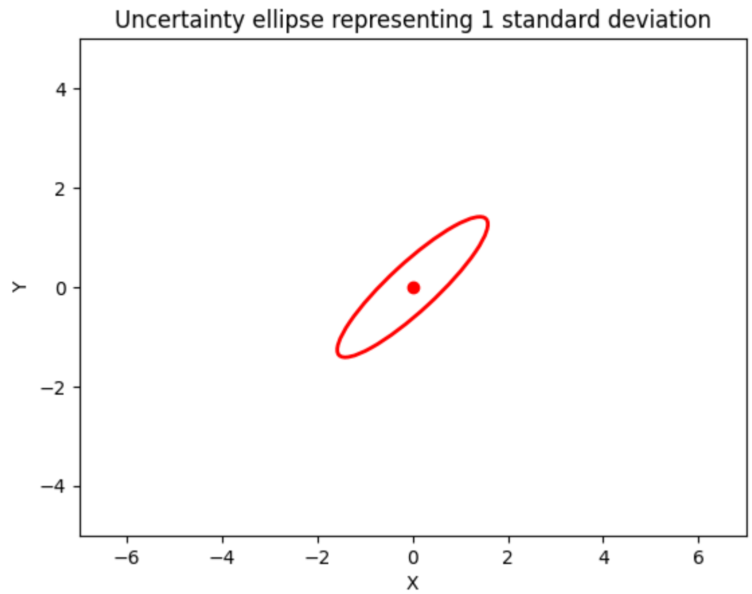

Figure 2: Uncertainty Ellipse at t=2

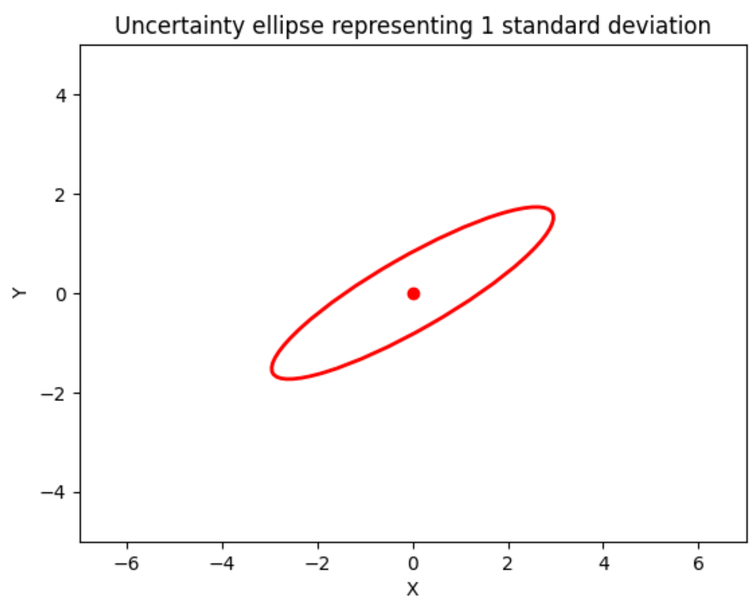

Figure 3: Uncertainty Ellipse at t=3

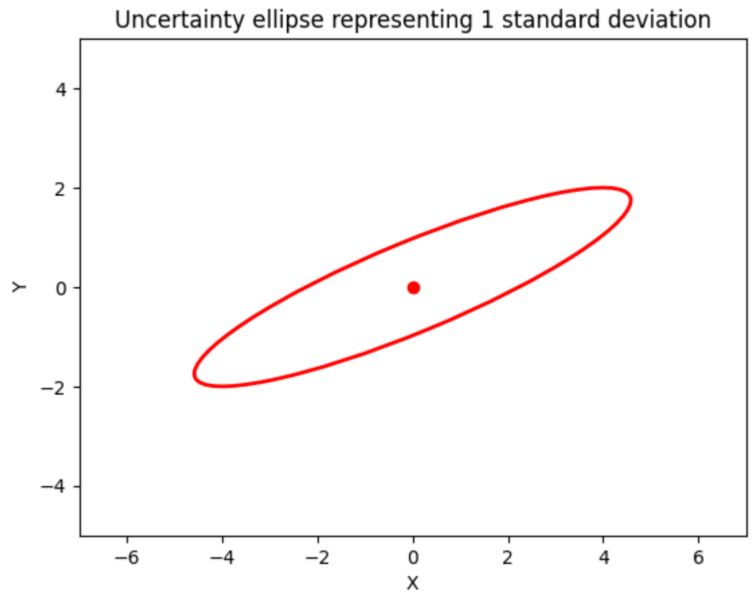

Figure 4: Uncertainty Ellipse at t=4

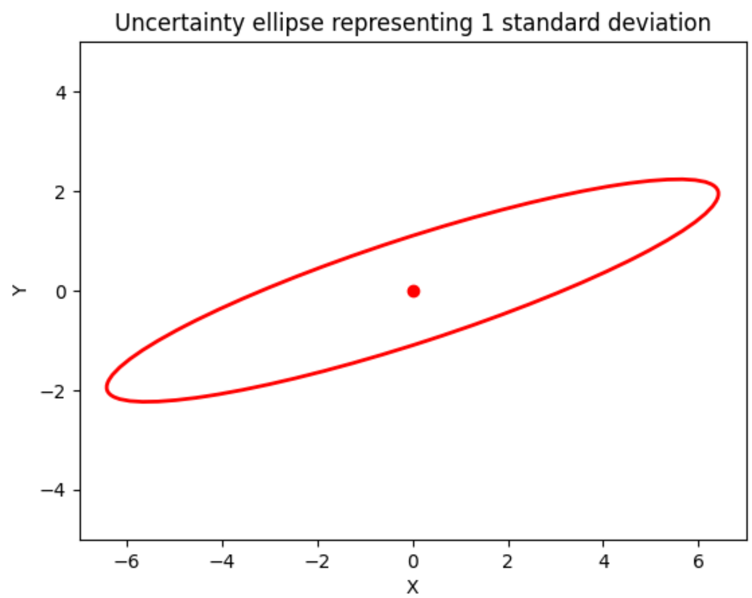

Figure 5: Uncertainty Ellipse at t=5

2 Kalman Filter: Measurement

Prediction alone will result in greater and greater uncertainty as time goes on. Fortunately, your drone has a GPS sensor, which in expectation, measures the true position. However, the measurement is corrupted by Gaussian noise with covariance $\sigma^2_{gps} = 8.0$.

Question 2.1: the measurement model:

$$C = \begin{bmatrix} 1.0 & 0.0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 8.0 \end{bmatrix}$$

Question 2.2: Suppose at time $t = 5$, we obtain $z = 10$. Then:

$$\bar{\sigma_{t5}} = \begin{bmatrix} 41.25 & 12.5 \\ 12.5 & 5. \end{bmatrix}$$

$$\bar{\mu_{t5}} = \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}$$

$$\sigma_{t5} = \begin{bmatrix} 6.70050761 & 2.03045685 \\ 2.03045685 & 1.82741117 \end{bmatrix}$$

$$\mu_{t5} = \begin{bmatrix} 8.37563452 \\ 2.53807107 \end{bmatrix}$$

The simulated positions of the drone from t=1 to t=5 are:

$$x_{t1} = 0.0$$

$$x_{t2} = -0.27283527$$

$$x_{t3} = 0.49668245$$

$$x_{t4} = 3.00267159$$

$$x_{t5} = 6.22960415$$

The measurement at t=5 is:

$$z_{t5} = 8.58188962$$

with a 37.75% error compared to ground truth

Question 2.3: at time $t = 20$, the expected error in terms of distance from the true position for each sensor fail probability is:
for $p_{gps-fail} = 0.1$, expected distance from true position = 1.83
for $p_{gps-fail} = 0.5$, expected distance from true position = 2.82
for $p_{gps-fail} = 0.9$, expected distance from true position = 16.57

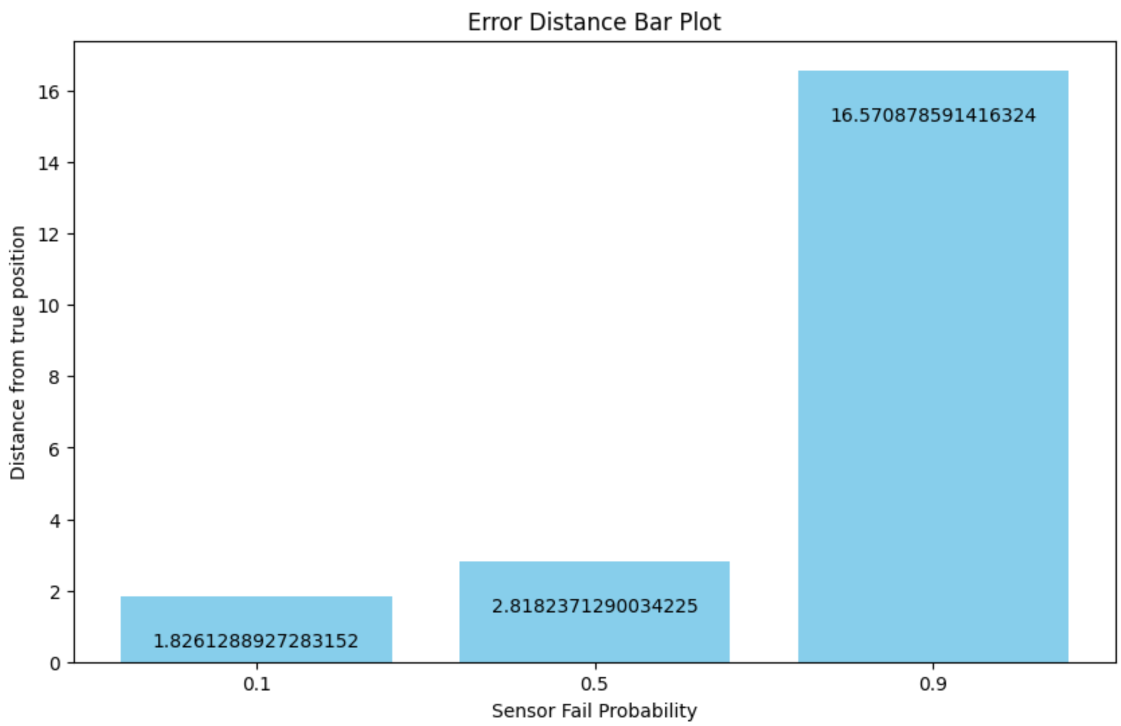
Error Distance Bar Plot
Distance from true position
Sensor Fail Probability
0.1
0.5
0.9
0
2
4
6
8
10
12
14
16
1.8261288927283152
2.8182371290034225
16.570878591416324

Figure 6:

3 Kalman Filter: Movement

Question 3.1 The drone is now fully operational and can not only take measurements, but also issue motor commands in the form of acceleration commands to its propeller. For example, a command of 1.0 will increase the drone's velocity by 1.0. Revisit Question 1.3 to provide the matrix B. If at time $t-1$, the drone's position and velocity are 5.0 and 1.0, compute the mean estimate for the state at time t given a motor command of 1.0. Your answer should be based on the constants provided but also include a random variable due to the wind effects. State the distribution of that random variable.

4 Extra Credit

Now, formulate both the prediction and measurement steps in the 2-D case. Construct a plot showing the true position and the position tracked by the Kalman filter over the first 30 time steps.

What to turn in: A PDF document with the answers to the questions, along with the code implementation and a README file that describes what to run in order to get the results in your PDF. You can use a language of your choice.