Modal Logic

Supplemental module

A formal logic for possibility and necessity claims.

The Problem

Translate the sentences:

1. If it is cloudy then it must rain.
2. If it is cloudy then it might rain.

- It seems like these two sentences have the *same content*.
- The difference is how it's said.
- But in predicate logic, both are just normal conditionals.

Modals

Express that something *can*, *must*, *or cannot* happen.

- Possibility: Possibly, Can, Could, Might, May
- Necessity: Necessarily, Must, Will
- Impossibility: Impossibly, Cannot

Modal Operators

- $\Diamond$ p : Possibly, p.
- $\Box$ p : Necessarily, p.
- $\Box$ p $:= \sim \Diamond \sim$ p

Translations

C: it is cloudy
R: it rains

It must rain.

$\Box$ R

If it is cloudy then it might rain.

C $\rightarrow \Diamond$ R

If it is cloudy then it must rain.

C $\rightarrow \Box$ R

Semantics

What do our modal operators mean?

Possible Worlds

A possible world represents an alternative scenario.

For example, I can imagine myself being an inch taller, or two inches shorter, etc.

We think of the set of possible worlds as “all the different ways things could be”.

Accessibility Relation

This relation provides a notion of "relative possibility".

Currently it is possible for me to be in Chicago on June 10, 2021.

But on June 11, 2021, if I'm still in California, then clearly that isn't possible anymore.

Worlds are possible *relative to* other worlds.

Accessibility Relation

If $w1$ and $w2$ are possible worlds, then we write R$w1w2$ to say that in $w1$, $w2$ is possible.

Alternatively: $w1$ *accesses* $w2$ (hence the name).

We use possible worlds and the accessibility relation to interpret the modal operators.

Modal Operator Semantics

Let's say we're in world *w*.

◇ p is true just in case there is some other world *w*2 such that R *w w2* and p is true at *w*2.

"Possibly, p" is true if there's some way the world could be where p is true.

□ p is true just in case every world *w*' that *w* accesses is such that p is true in *w*'.

"Necessarily, p" is true if every other way the world could be makes p true.