What are the quantifier negation rules?

Week 8. Deeper dive

On the Week 5 PP, did you put

S = "has a subway"
F = "has a football team"

4.4

Ax~(Sx∧Fx) **Submit**

No place which has a subway has a football team.

I've been saying for a while that if we have negated quantifiers, there are usually two equivalent ways of translating the statement. Today, we make this claim precise with quantifier negation rules.

or

4.4

~Ex(Sx∧Fx) **Submit**

No place which has a subway has a football team. ✓

?

The basic idea

$$\neg\forall x\, \phi \leftrightarrow \exists x\, \neg\phi$$

$$\neg\exists x\, \phi \leftrightarrow \forall x\, \neg\phi$$

"Not everything is blue" iff "something is non-blue"

"Nothing is blue" iff "everything is non-blue"

Quantifier negation rules for proofs

$\neg \forall x\, \phi$
$\exists x \neg \phi$: QN

$\neg \exists x\, \phi$
$\forall x \neg \phi$: QN

$\exists x \neg \phi$
$\neg \forall x \phi$: QN

$\forall x \neg \phi$
$\neg \exists x \phi$: QN

The slogan here is ‘flip’ the quantifier when you move the negation in or out.

Quantifier negation rules for proofs

$\neg\forall x\, \phi$
$\exists x \neg\phi$: QN

$\neg\exists x\, \phi$
$\forall x \neg\phi$: QN

$\exists x \neg\phi$
$\neg\forall x \phi$: QN

$\forall x \neg\phi$
$\neg\exists x \phi$: QN

In proofs, we often use these one after the other because existentials are easier to introduce but universals are easier to eliminate!

Show that $\forall x(Px \to Qx), \neg Qa \vdash \exists z \neg Pz$

1.	$\forall x(Px \to Qx)$	:assumption
2.	$\neg Qa$	:assumption
3.	$Pa \to Qa$	:E$\forall$ 1
4.	Pa	:assumption
5.	Qa	:E$\to$3,4
6.	!?	:E~2,5
7.	$\sim Pa$	:I~
8.	$Ex{\sim}Px$	:I$\exists$ 7

1.	$\forall x(Px \to Qx)$	:assumption
2.	$\neg Qa$	:assumption
3.	$Pa \to Qa$	:E$\forall$ 1
4.	$\forall xPx$	:assumption
5.	Pa	:E$\forall$ 4
6.	Qa	:E $\to$3,5
7.	!?	:E~2,6
8.	$\sim AxPx$	:I~
9.	$Ex{\sim}Px$	:QN

two equally good proofs!

Show that :

$\forall x(Px \rightarrow \exists yRxy), \forall x\forall y(Rxy \rightarrow Sxy), \neg\exists x\exists ySxy \vdash \neg\exists xPx$

(using base rules, shortcuts, and QN)

1. $\forall x(Px \rightarrow \exists yRxy)$: assumption
2. $\forall x\forall y(Rxy \rightarrow Sxy)$:assumption
3. $\neg\exists x\exists ySxy$:assumption
4. Pa-> $\exists yRay$:E$\forall$ 1
5. $\forall y(Ray \rightarrow Say)$:E$\forall$ 2
6. $(Rab \rightarrow Sab)$:E$\forall$ 5
7. $\forall x{\sim}\exists ySxy$:QN 3
8. ${\sim}\exists ySay$:E$\forall$ 7
9. $\forall y{\sim}Say$:QN 8
10. ${\sim}Sab$:E$\forall$ 9

using the QN rules successively to simplify

Show that :

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9. $\forall y{\sim}Say$:QN 8
10. ${\sim}Sab$:E$\forall$ 9
11. ${\sim}Rab$:MT 6,10

Recall MT:
From wffs of the forms
$(\phi \rightarrow \psi)$ and ${\sim}\psi$, infer ${\sim}\phi$

Show that :

$\forall x(Px \rightarrow \exists yRxy), \forall x\forall y(Rxy \rightarrow Sxy), \neg\exists x\exists ySxy \vdash \neg\exists xPx$

(using base rules, shortcuts, and QN)

1. $\forall x(Px \rightarrow \exists yRxy)$: assumption
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9. $\forall y{\sim}Say$:QN 8
10. ${\sim}Sab$:E$\forall$ 9
11. ${\sim}Rab$:MT 6,10
12. $\forall y{\sim}Ray$:I$\forall$ 11
13. ${\sim}\exists yRay$:QN 12

Show that :

$\forall x(Px \to \exists yRxy), \forall x\forall y(Rxy \to Sxy), \neg\exists x\exists ySxy \vdash \neg\exists xPx$

(using base rules, shortcuts, and QN)

1. $\forall x(Px \to \exists yRxy)$: assumption
2. $\forall x\forall y(Rxy \to Sxy)$:assumption
3. $\neg\exists x\exists ySxy$:assumption
4. Pa-> $\exists yRay$:E$\forall$ 1
5. $\forall y(Ray \to Say)$:E$\forall$ 2
6. $(Rab \to Sab)$:E$\forall$ 5
7. $\forall x{\sim}\exists ySxy$:QN 3
8. ${\sim}\exists ySay$:E$\forall$ 7
9. $\forall y{\sim}Say$:QN 8
10. ${\sim}Sab$:E$\forall$ 9
11. ${\sim}Rab$:MT 6,10
12. $\forall y{\sim}Ray$:I$\forall$ 11
13. ${\sim}\exists yRay$:QN 12
14. ${\sim}Pa$:MT 4,13

Show that :
$\forall x(Px \rightarrow \exists yRxy), \forall x\forall y(Rxy \rightarrow Sxy), \neg\exists x\exists ySxy \vdash \neg\exists xPx$
(using base rules, shortcuts, and QN)

1. $\forall x(Px \rightarrow \exists yRxy)$:	assumption	11. ~Rab	:MT 6,10
2. $\forall x\forall y(Rxy \rightarrow Sxy)$	:assumption	12. ∀y~Ray	:I∀ 11
3. $\neg\exists x\exists ySxy$	:assumption	13. ~∃yRay	:QN 12
4. Pa-> ∃yRay	:E∀ 1	14. ~Pa	:MT 4,13
5. $\forall y(Ray \rightarrow Say)$	:E∀ 2	15. ∀y~Py	:I∀ 14
6. $(Rab \rightarrow Sab)$	:E∀ 5	16. ~∃yPy	:QN 15
7. ∀x~∃ySxy	:QN 3		
8. ~∃ySay	:E∀ 7		
9. ∀y~Say	:QN 8		
10. ~Sab	:E∀ 9		