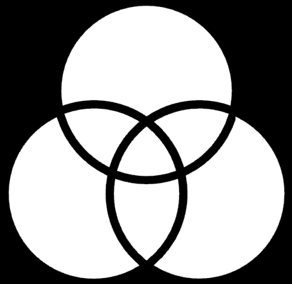
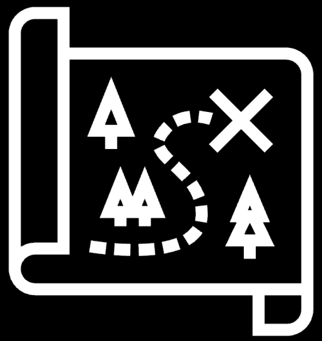
Arrow models

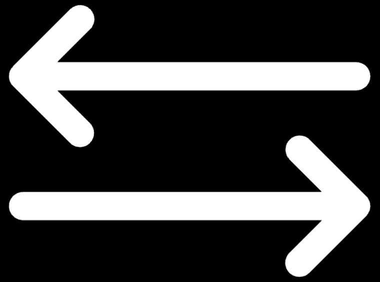
Week 6 . Deeper dive.



Venn diagrams

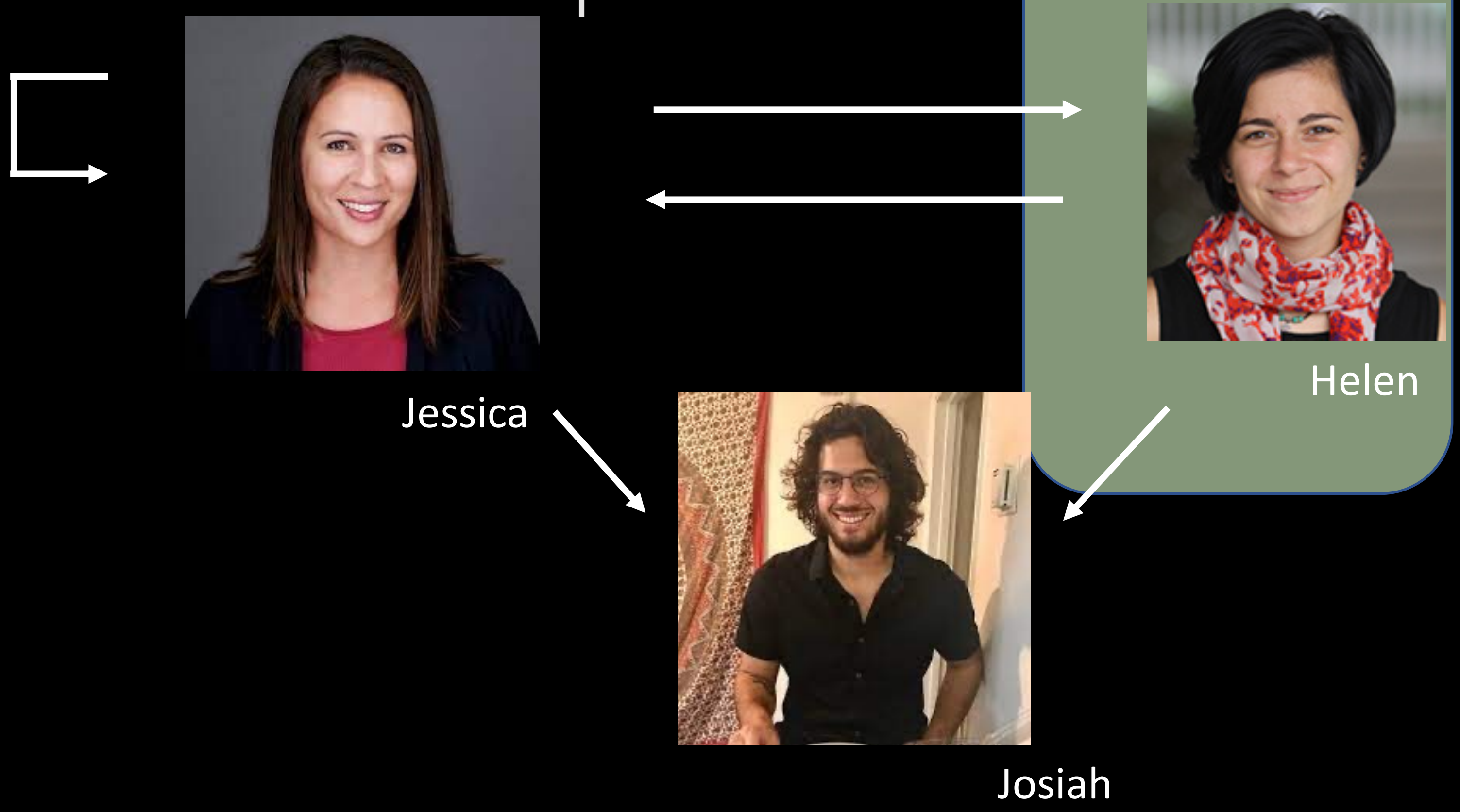


Map models



Arrow models

Let's start with an example:



where "Rxy" means "x relies upon y",
"Nx" means "is in the instructor",
"h" means "Helen",
"j" means "Jessica",
"o" means "Josiah"

The **domain** is the unodered collection of objects in the model:
|M| = {Helen, Jessica, Josiah}

What is...

1. I(h) = Helen
2. I(j) = Jessica
3. I(o) = Josiah

For a name, the interpretation function provides the object denoted by the name. This is asking "What does "h" pick out?"

Jessica

Helen

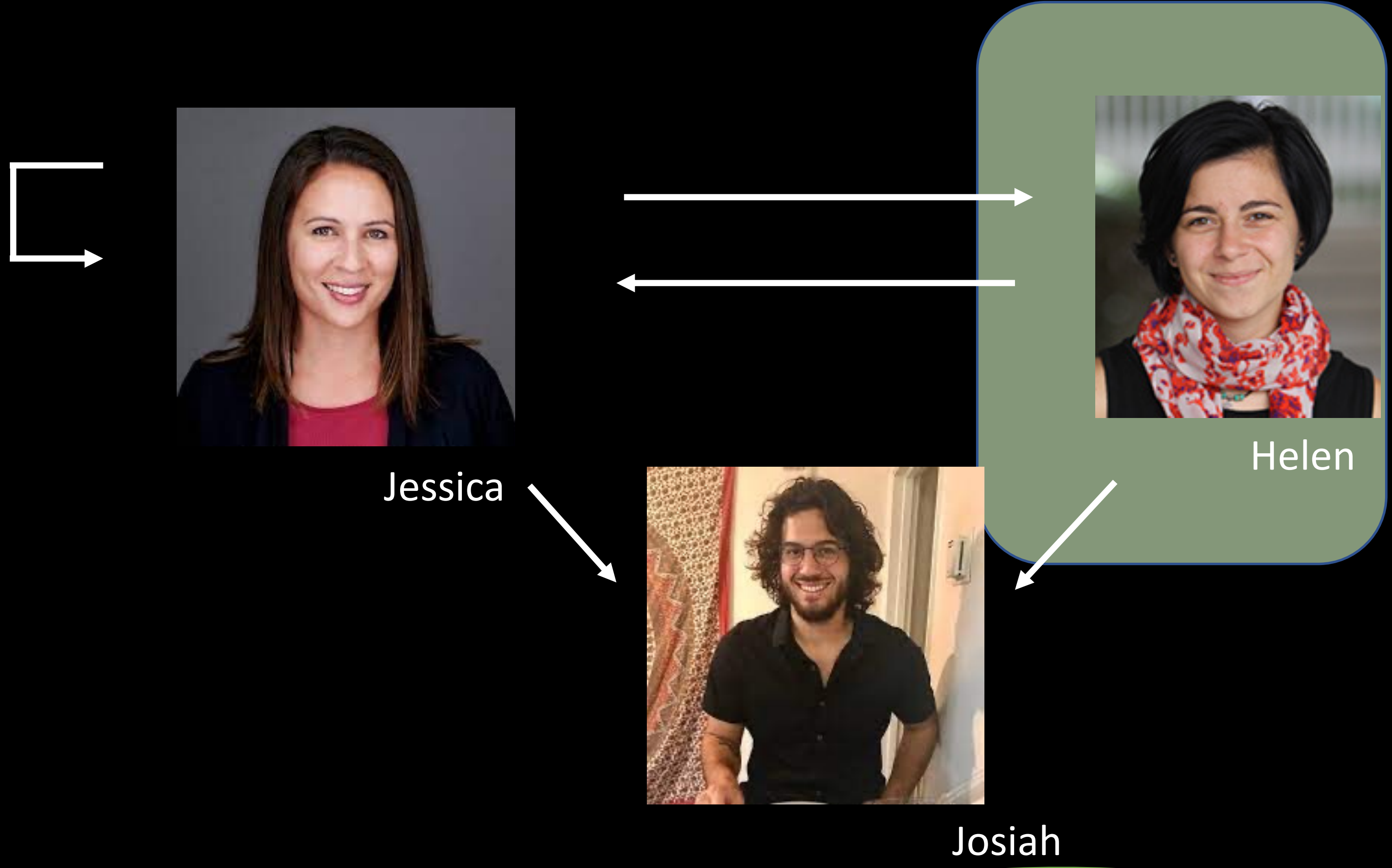
Josiah

where "Rxy" means "x relies upon y",
"Nx" means "is in the instructor",
"h" means "Helen",
"j" means "Jessica",
"o" means "Josiah"

For a predicate, the interpretation function provides the extensions. This is asking "What objects does G hold of?"

What is…

$I(N) = \{h\}$

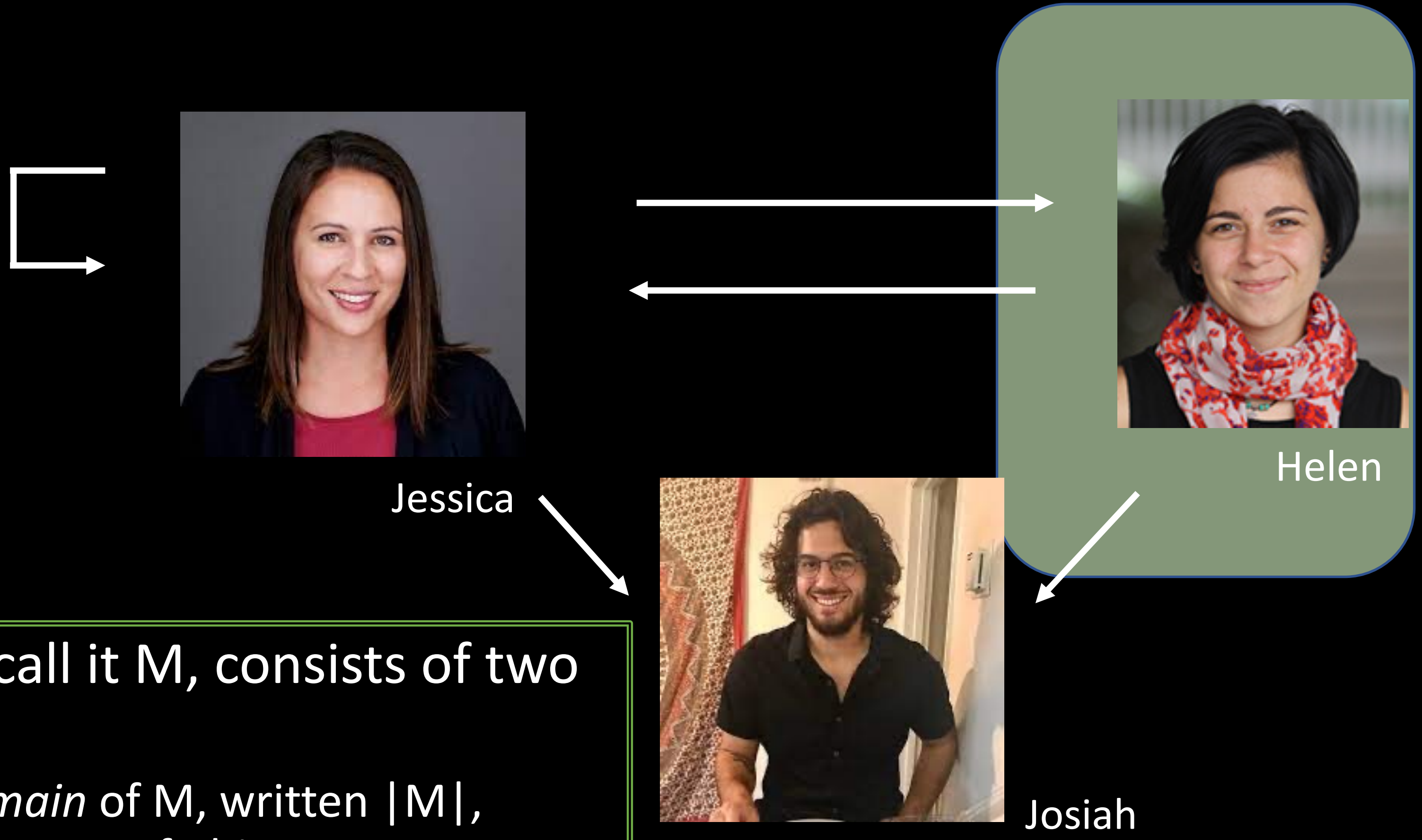


where “Rxy” means ”x relies upon y”,
“Nx” means “is in the instructor”,
“h” means “Helen”,
“j” means “Jessica”,
“o” means “Josiah”

What is…

I(R) = {<j,j>,<j,h>,<h,j>,<h,o>,<j,o>}

This is asking “What objects does R hold of?” But we need an ordered set of objects now since the order matters for R. We use angled brackets for ordered sets: “< >” We collect together our ordered sets in an ordered set with “{ }”



where "Rxy" means "x relies upon y",
"Nx" means "is in the instructor",
"h" means "Helen",
"j" means "Jessica",
"o" means "Josiah"

- A **model**, call it M, consists of two parts:
  - The *domain* of M, written |M|, which is a set of objects
  - The *interpretation function* of M, written I (or sometimes I_M), which assigns an interpretation to each of the names and predicates in the form of objects and extensions, respectively.

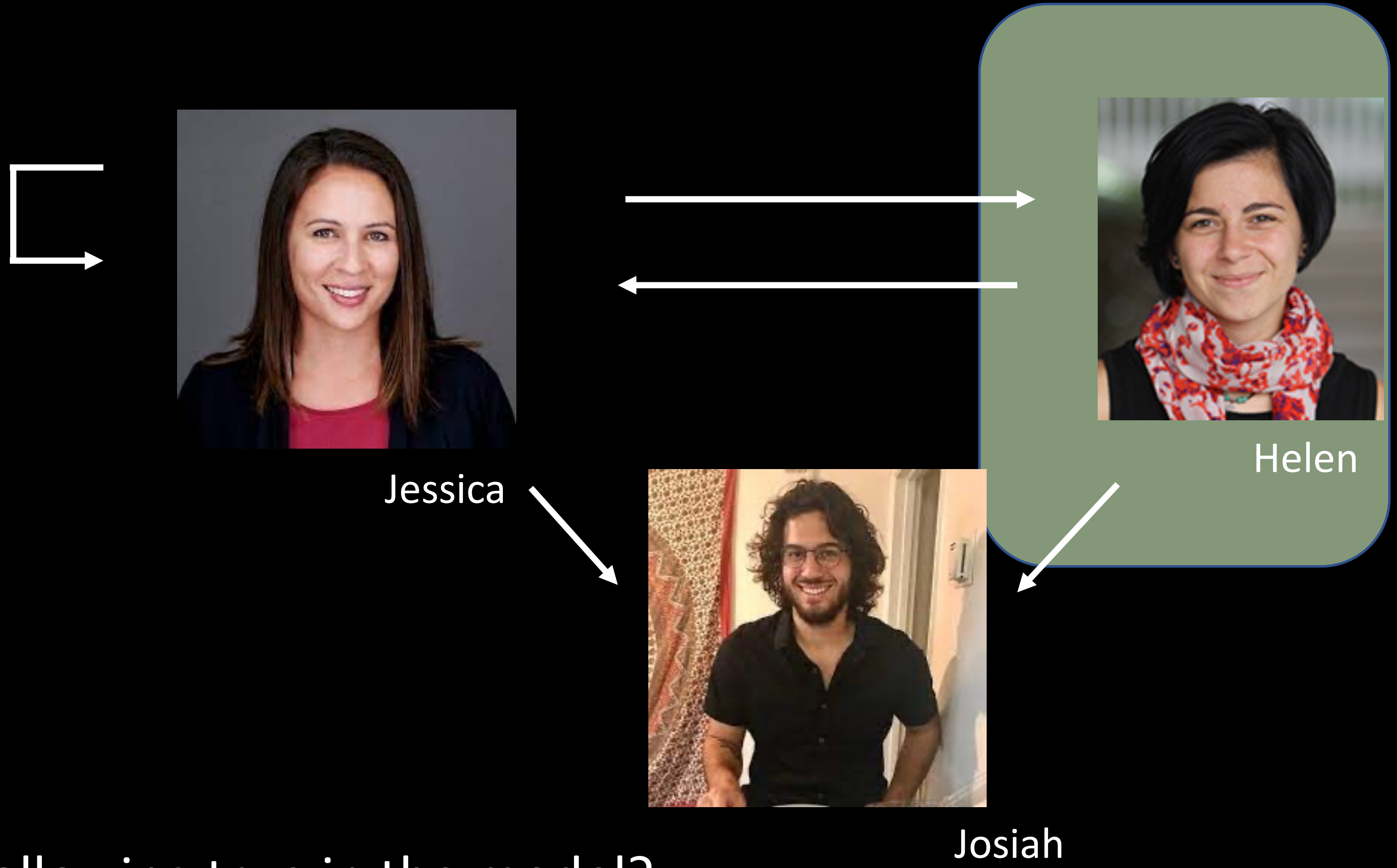
- |M| = {Helen, Jessica, Josiah}
- I(h) = Helen, I(j) = Jessica, I(o) = Josiah, I(G) = {h}, I(R) = {<j,j>,<j,h>,<h,j>,<h,o>,<j,o>}

where “Rxy” means ”x relies upon y”,
“Nx” means “is in the instructor”,
“h” means “Helen”,
“j” means “Jessica”,
“o” means “Josiah”

What about truth in a model?

Recall: Truth in models…

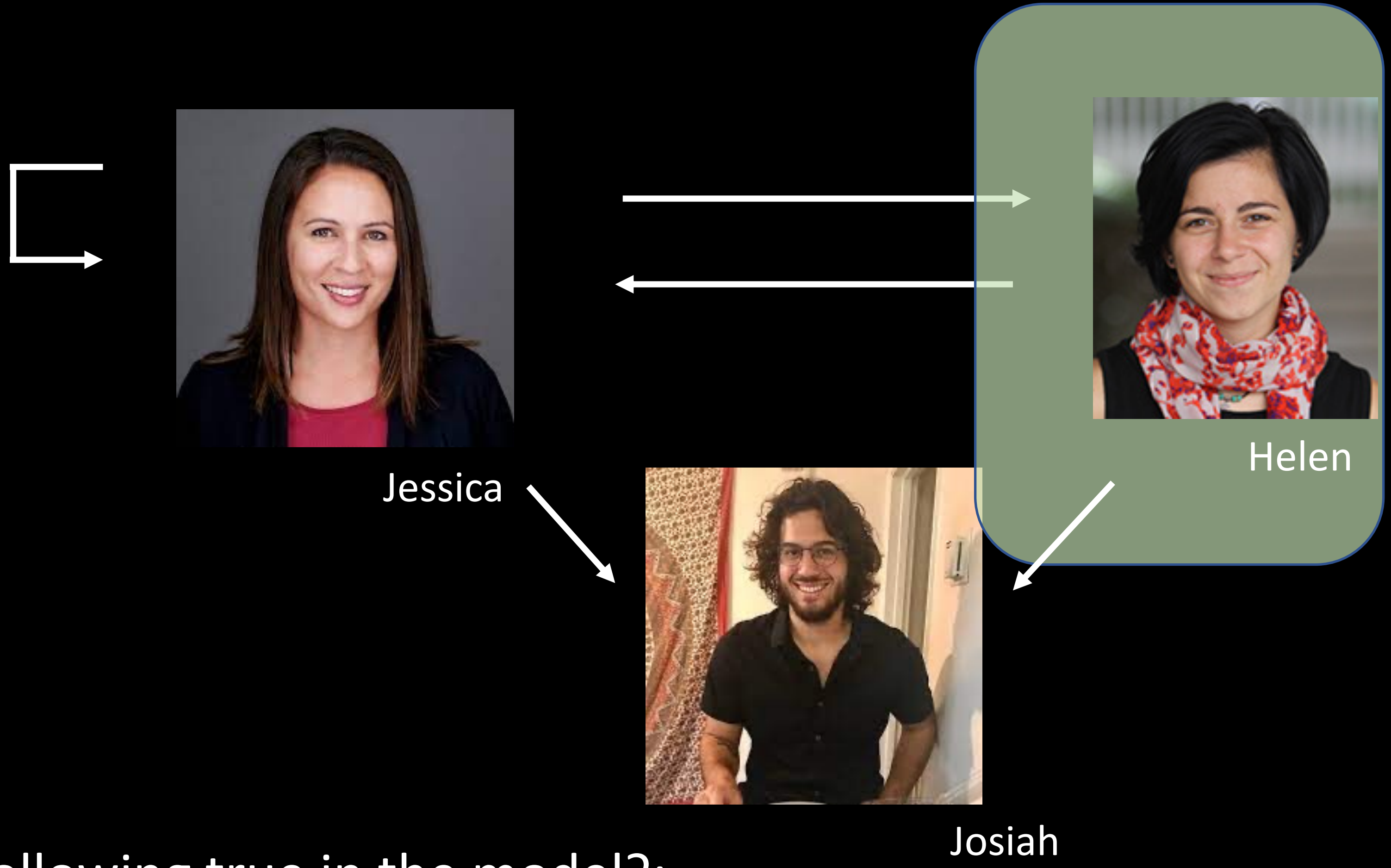
Given a property S and a name h, the proposition Sh is true in a model M if and only if: I(h) ∈ I(S)



where “Rxy” means ”x relies upon y”,
“Nx” means “is in the instructor”,
“h” means “Helen”,
“j” means “Jessica”,
“o” means “Josiah”

Are the following true in the model?:

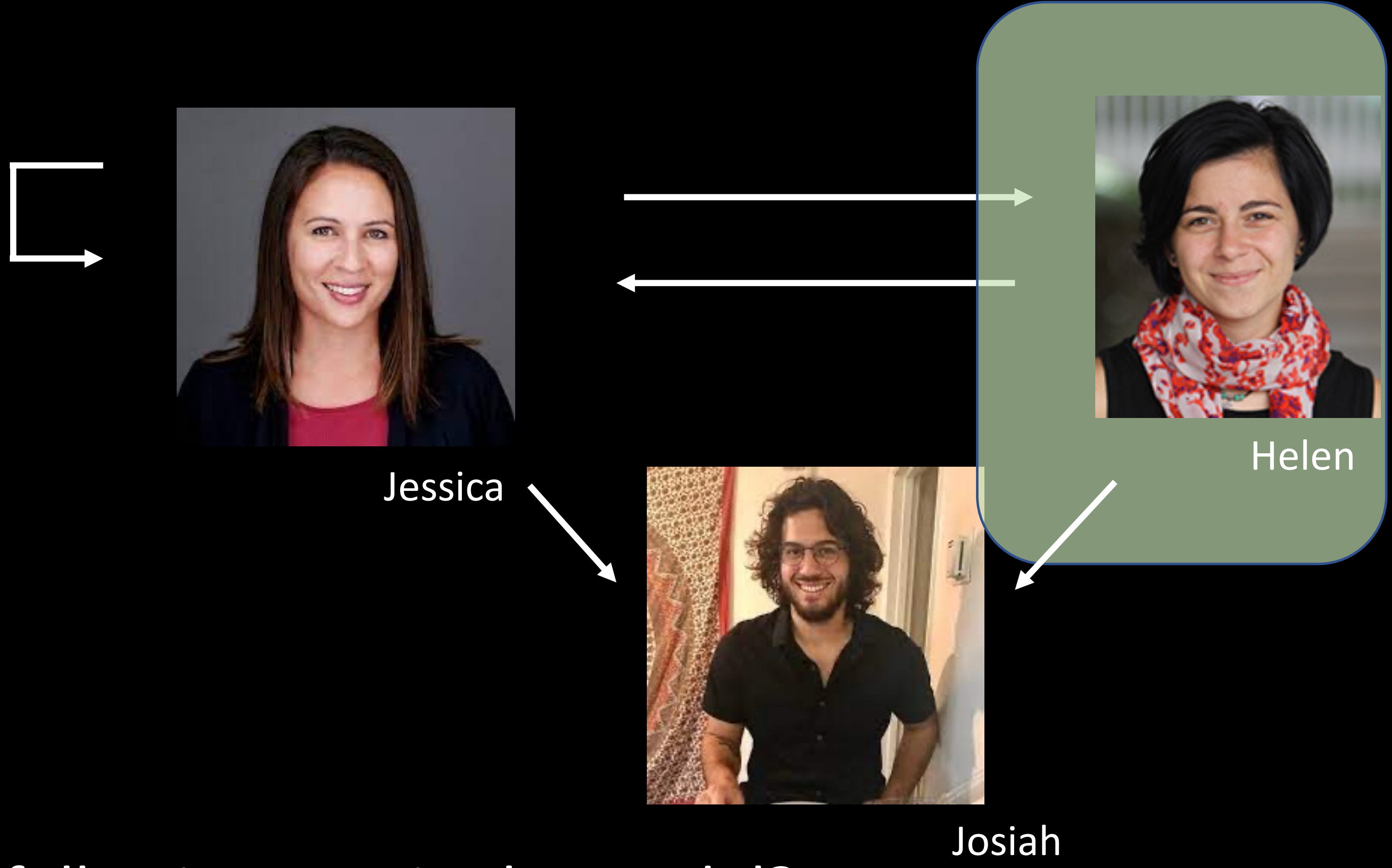
1. Rjo ✓
2. Roj ✗
3. Rho ✓
4. Rjj ✓
5. Roo ✗



where “Rxy” means ”x relies upon y”,
“Nx” means “is in the instructor”,
“h” means “Helen”,
“j” means “Jessica”,
“o” means “Josiah”

Are the following true in the model?:

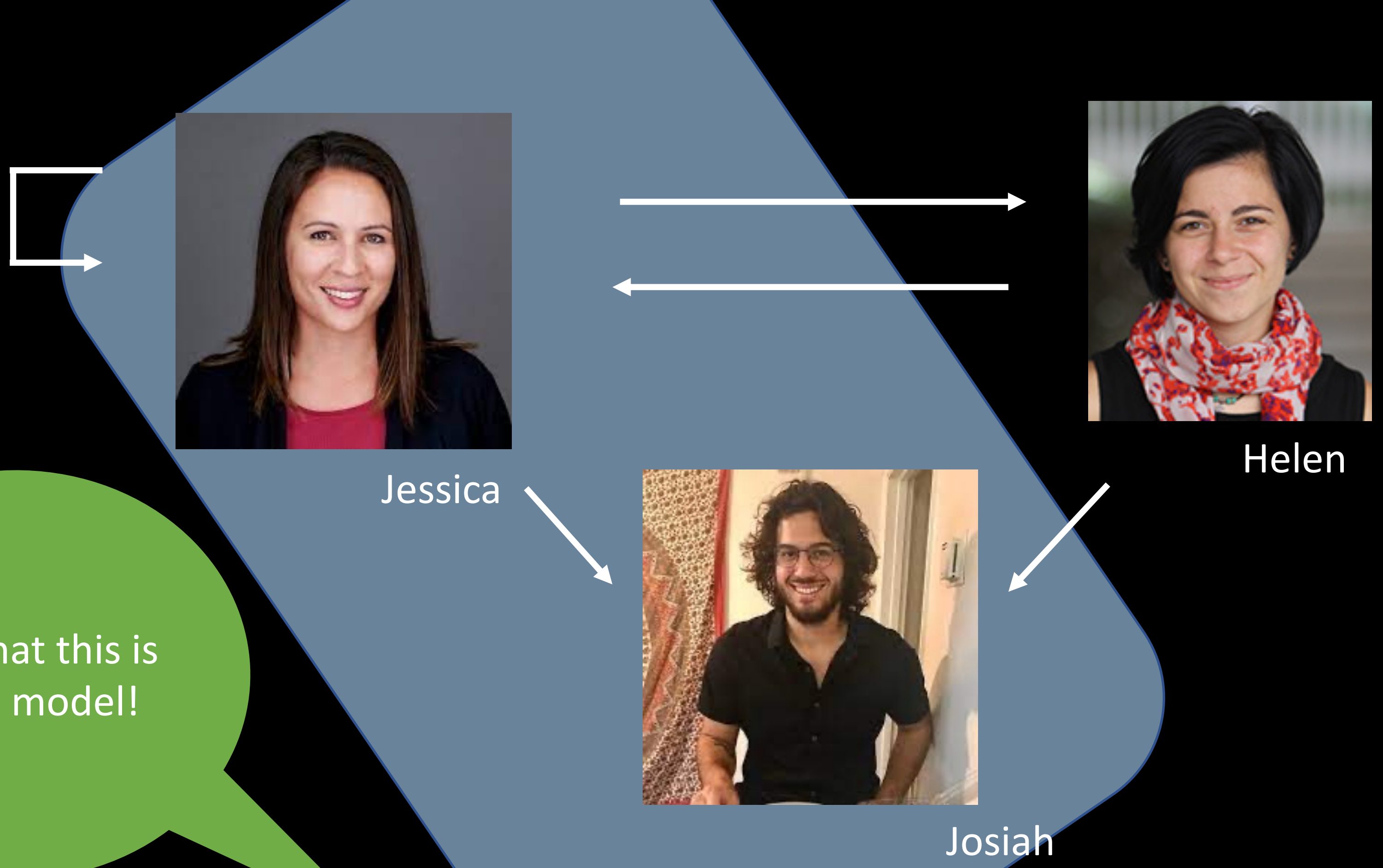
1. $\exists x\ Rxx$
2. $\exists x\ \neg Rxx$
3. $\forall x\ Rxx$
4. $\forall x\ \neg\ Rxx$
5. $\neg\forall x\ Rxx$



where "Rxy" means "x relies upon y",
"Nx" means "is in the instructor",
"h" means "Helen",
"j" means "Jessica",
"o" means "Josiah"

Are the following true in the model?:

1. $\forall x\,(Rxx \rightarrow x = o)$ ✗
2. $\forall x\,\forall y\,(((Rxy \wedge Ryx) \rightarrow ((x = j \wedge y = h) \vee (x = h \wedge y = j)))$ ✓



where "Rxy" "x relies upon y",
"Tx" means "x is a TA"
"h" means "Helen",
"j" means "Jessica",
"o" means "Josiah"

Note that this is a new model!

Are the following true in the model?:

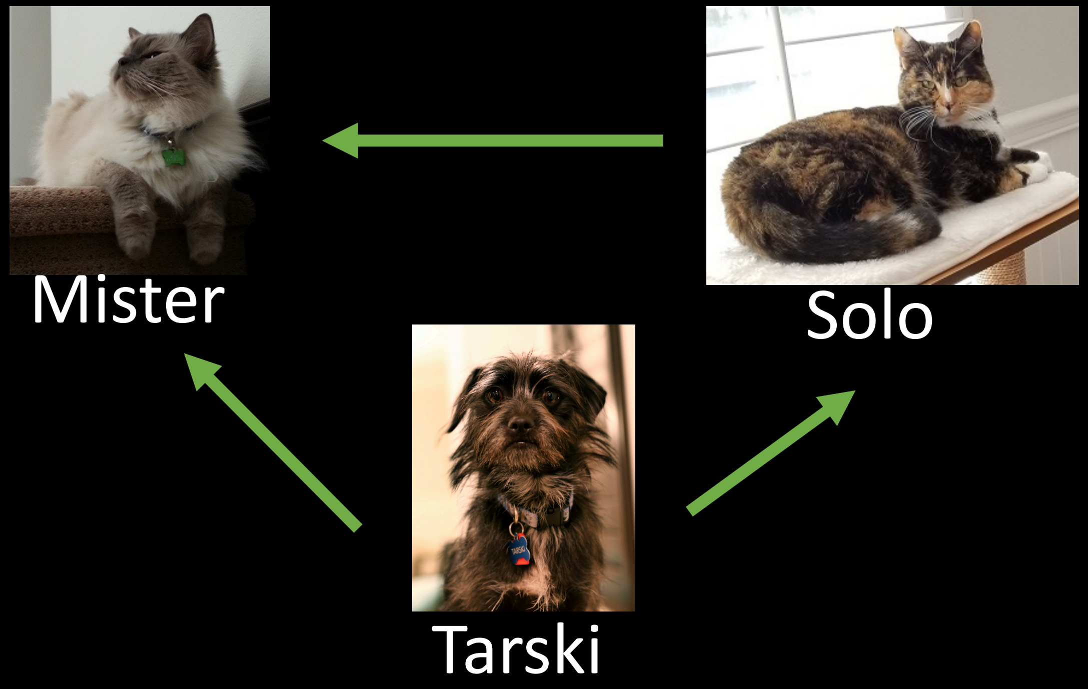
1. $\forall x\,(Bx \land x = 0)$ ✗
2. $\exists x\,(Bx \land x = 0)$ ✓

Model example

m	Mister	Dx	x is a dog
s	Solo	Fx	x is a feline (cat)
t	Tarski	Cxy	x chases y (there's an arrow from x to y)

The domain: $|M| = \{\text{Mister, Solo, Tarski}\}$

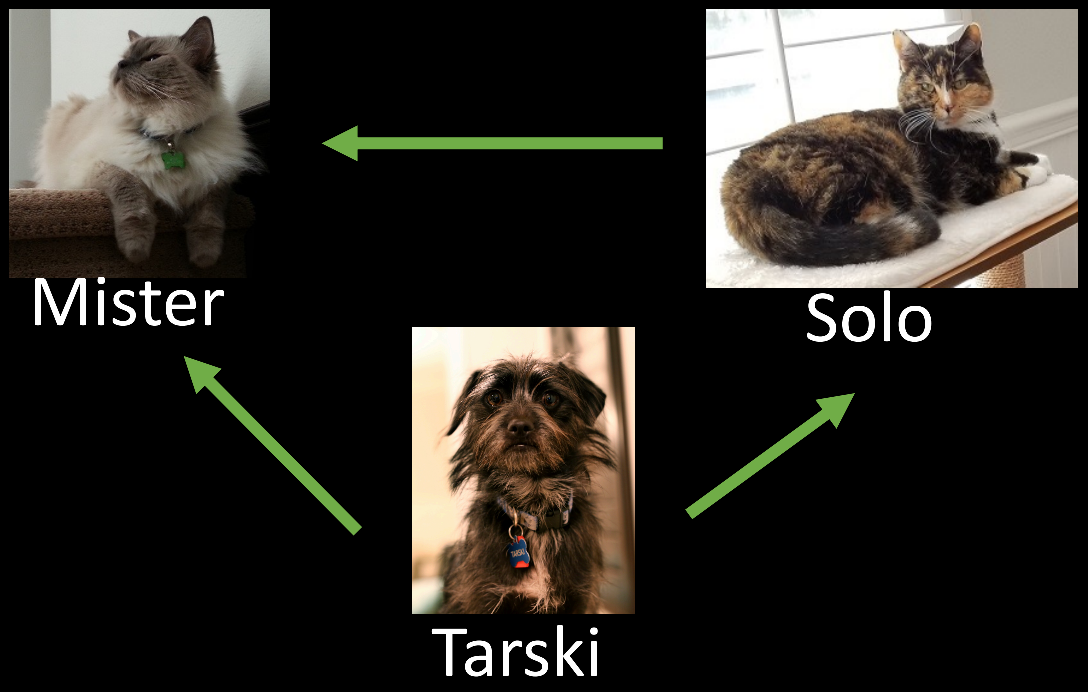
- $I(m) = \text{Mister}$, $I(s) = \text{Solo}$, $I(t) = \text{Tarski}$
- $I(D) = \{\text{Tarski}\}$
- $I(F) = \{\text{Mister, Solo}\}$
- $I(C) = \{\langle \text{Tarski, Mister} \rangle, \langle \text{Tarski, Solo} \rangle, \langle \text{Solo, Mister} \rangle\}$



m	Mister	Dx	x is a dog
s	Solo	Fx	x is a feline (cat)
t	Tarski	Cxy	x chases y (there's an arrow from x to y)

Is… true in M?

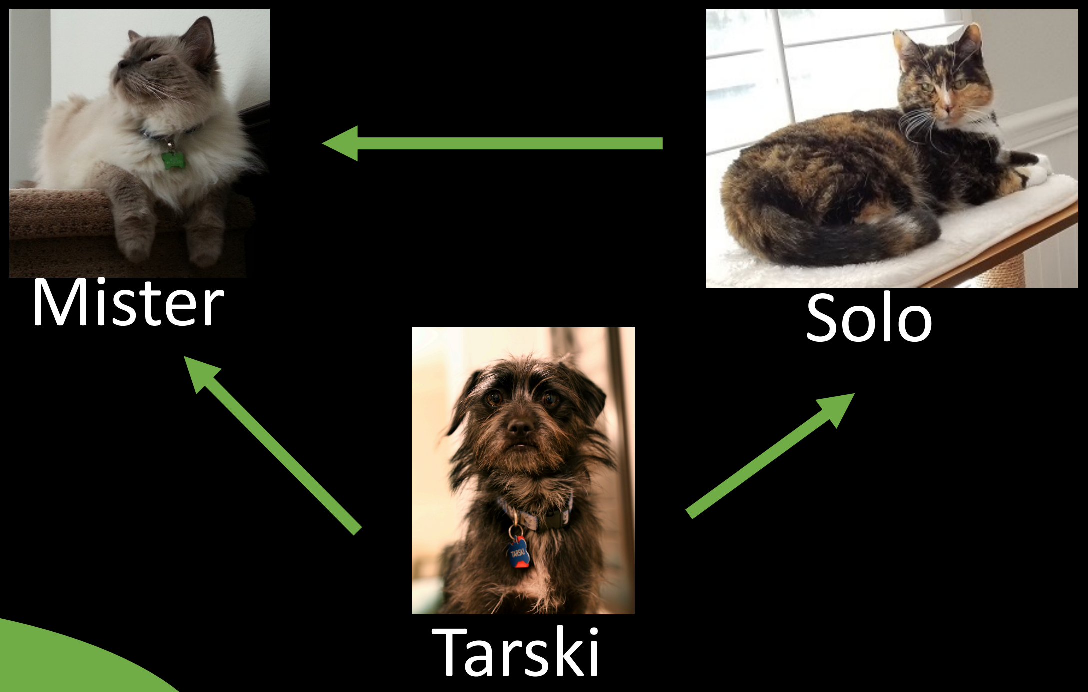
- Csm
- Cmt
- Ctm
- (Dt∧Cst)
- $\exists x\ (Fx \land Dx)$
- $\exists x\ Fx\ \land \exists y\ Dy$
- $\forall x\ (Fx \rightarrow \exists y Cxy)$



m	Mister	Dx	x is a dog
s	Solo	Fx	x is a feline (cat)
t	Tarski	Cxy	x chases y (there's an arrow from x to y)

The domain: $|M|$ = {Mister, Solo, Tarski}

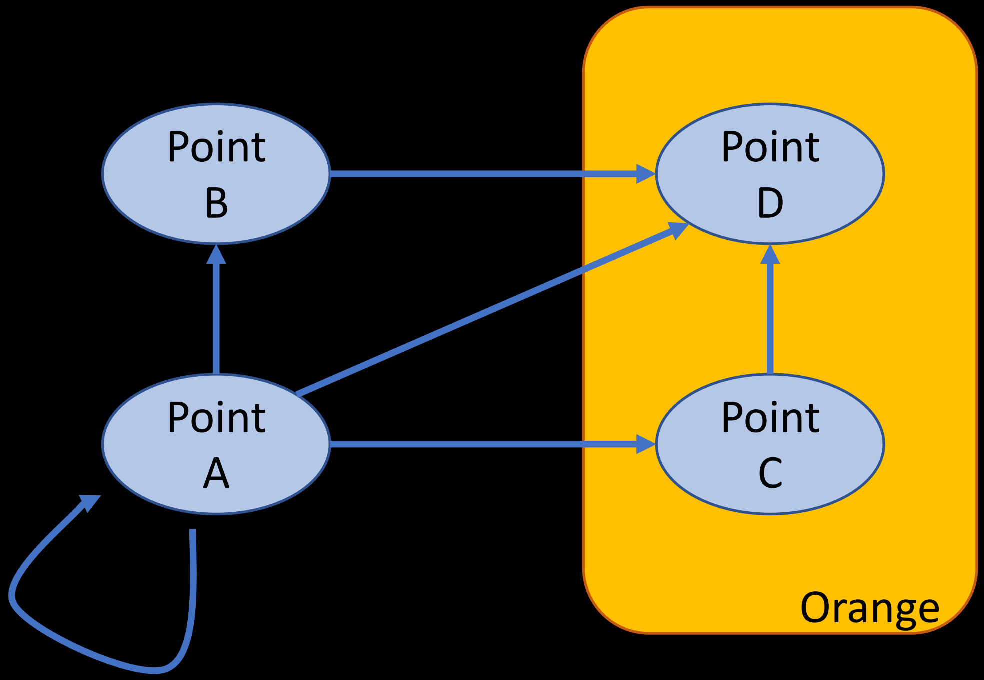
- I(m) = Mister, I(s) = Solo, I(t) = Tarski
- I(D) = {Tarski}
- I(F) = {Mister, Solo}
- I(C) = {⟨Tarski, Mister⟩, ⟨Tarski, Solo⟩, ⟨Solo, Mister⟩}



Note: these are actually both saying the same thing^!

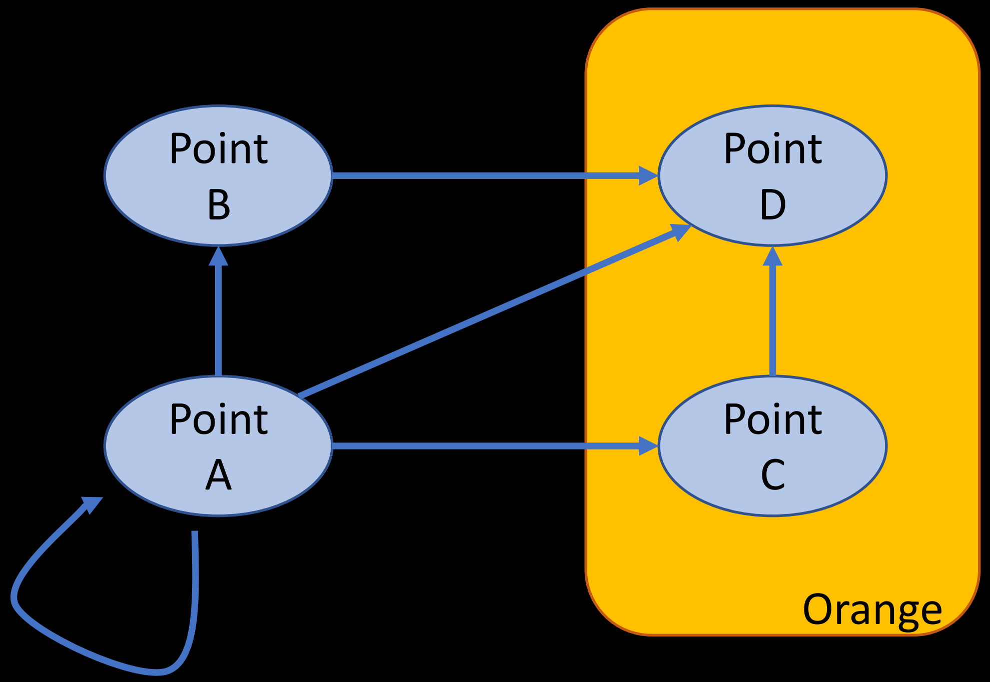
Arrow models

One special type of model (which is very similar to the ones we've just discussed!) is called an **arrow model**



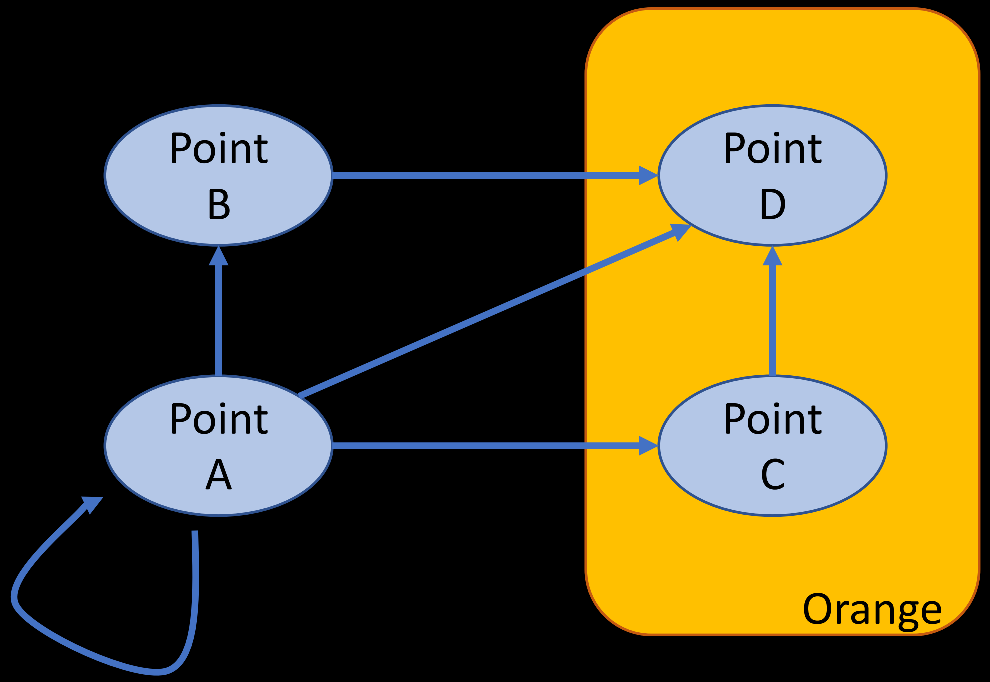
Arrow models, continued

- In an *arrow model*, we have a domain of *points*
- These points are connected with one or more types of arrows
- And sometimes we group some of the points, like the ‘Orange’ box here

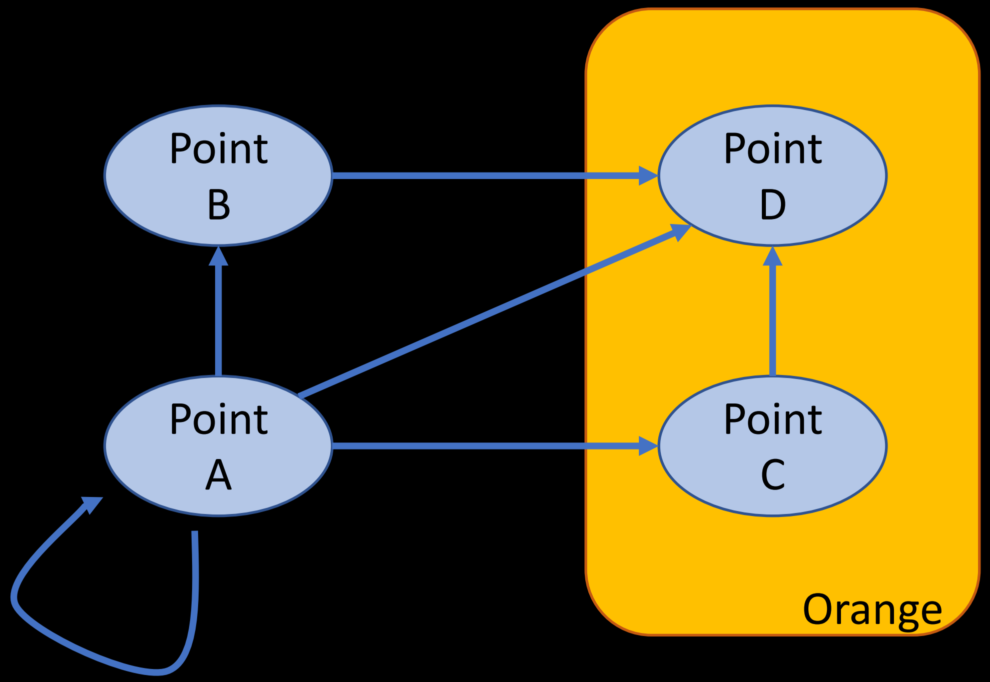


Example of an arrow model

- Let M be the arrow model represented by this diagram, with the name ‘a’ standing for ‘Point A’, etc.
- Let ‘Rxy’ mean ‘there is an arrow from x to y’
- Let ‘Ox’ mean ‘x belongs to the orange group’



- Write out the interpretation I(O)
    - I(O)={c,d}
- Decide the truth of the following in M:
    - (Rbd&Od) ✓
    - ∃zRcz ✓
    - ∀xRxd ✗
    - ∀w(Ow v Rbw) ✗
    - ∀x∃yRyx ✓



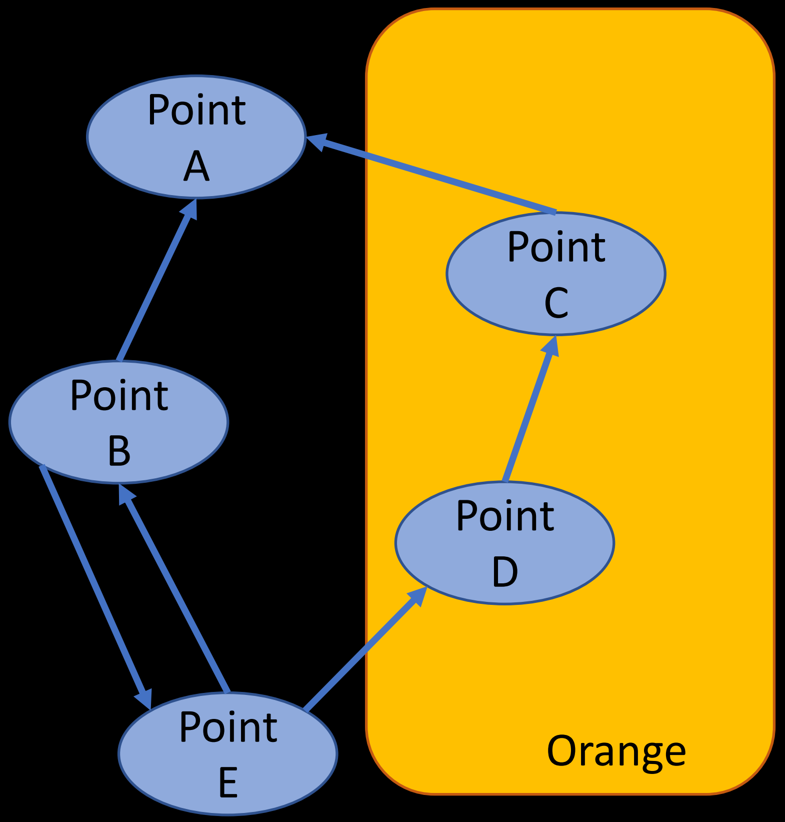
More examples

(1) $\forall x Rxx$ ✗

(2) $\forall x(Ox \rightarrow \exists y Ryx)$ ✓

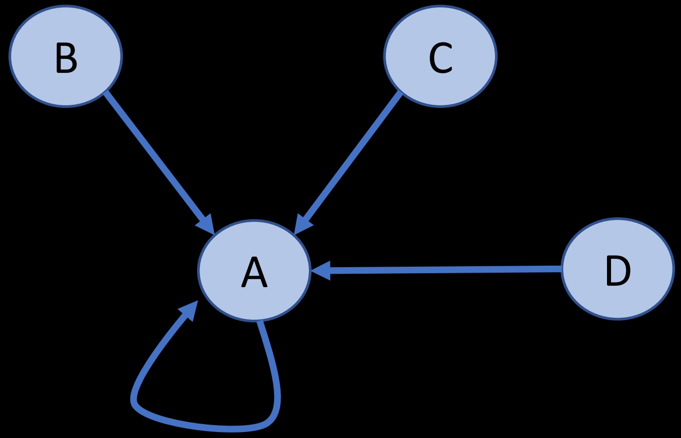
(3) $\exists z \forall y \neg Rzy$ ✓

(4) $\exists x \exists y(Rxy \wedge Ryx)$ ✓



Generating models

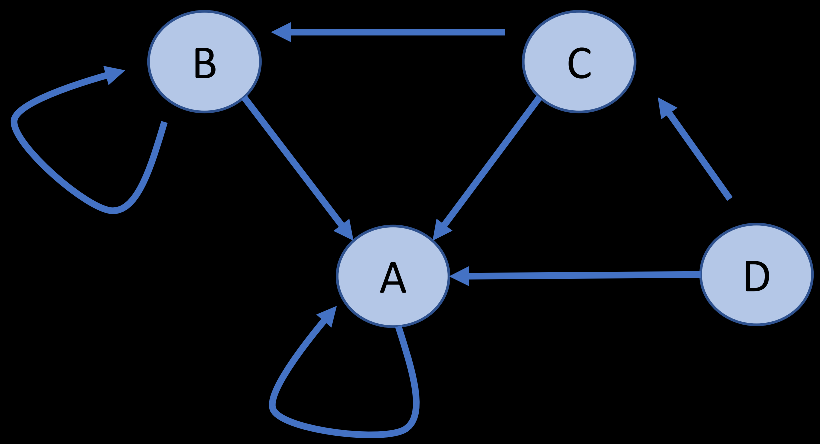
- Draw an arrow model M with at least 4 points where ∃z∀yRyz is true
  - This says 'there is a point z such that every point has an arrow to z', or more simply 'there is a point that everything points to'



Generating models

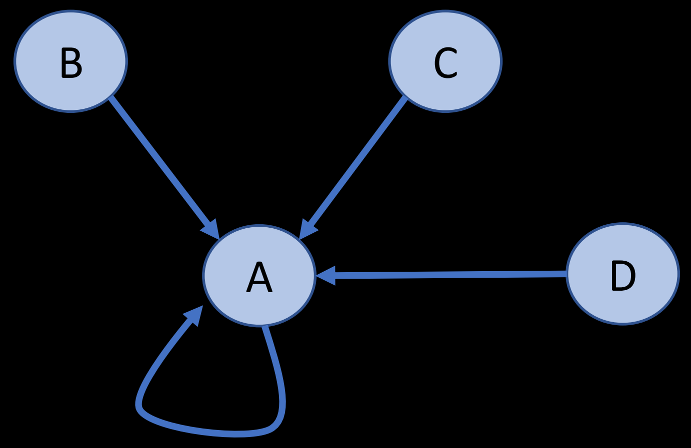
- Draw an arrow model M with at least 4 points where $\exists z \forall y Ryz$ is true
  - This says 'there is a point z such that every point has an arrow to z', or more simply 'there is a point that everything points to'

Does this count too? Sure! Think of the previous model as the minimal possible model to make the statement true (with at least 4 points).



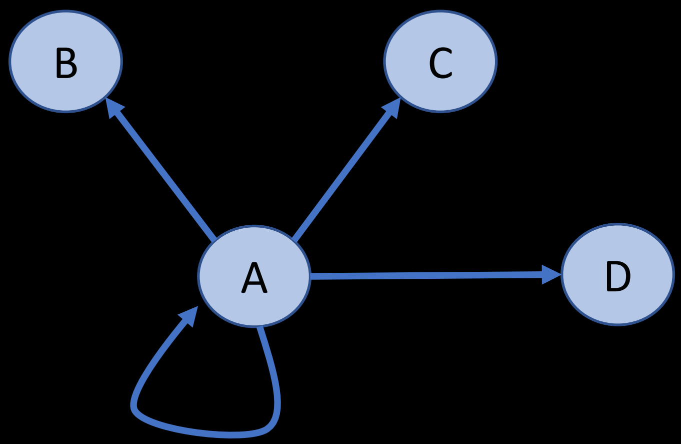
Generating models

- Draw an arrow model M with at least 5 points where
  - This says 'there is a point z such that every point has an arrow to z', or more simply 'there is a point that everything points to'



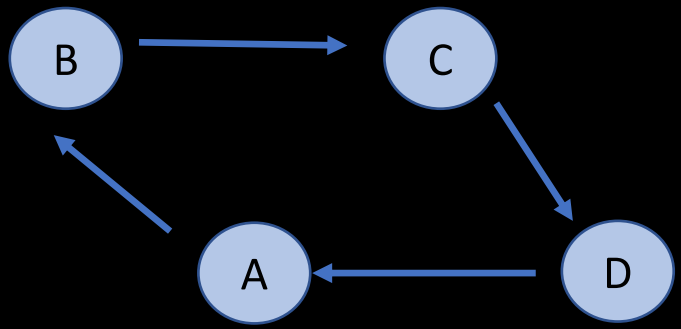
Generating models

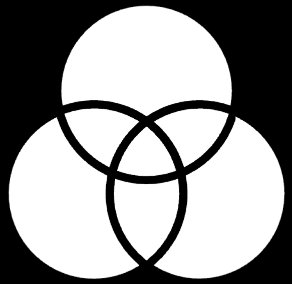
- Draw an arrow model M with at least 4 points where ∃y∀zRyz is true
  - This says 'there is a point z that points to all the other points.'



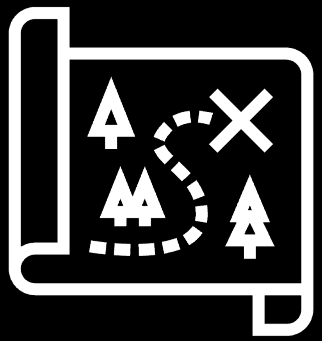
Generating models

- Draw an arrow model M with at least four points where ∃z∀yRyz is false *and* ∀y∃zRyz is true.
- This says 'It's false that there is a point that everything points to' And 'It's true that every point has an arrow to some point.'

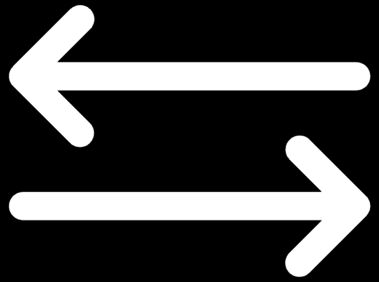




Venn diagrams



Map models



Arrow models

Check in: But what does this have to do with validity?

Validity via models: a valid argument is an argument in which the conclusion is true *in every model* in which the premises are true.

Instead, we'll often use models as *counterexamples!* Remember an argument was invalid if we found a row on which the premises were true but the conclusion wasn't. We often want to give these kinds of counterexamples to show an argument is invalid!

Invalidity via models: an invalid argument is an argument in which *there is some model where the premises are true and the conclusion is false*.

In other words, we just need to come up with one model that makes the premises true and the conclusion false.

- This is what we call a ***countermodel***!

Ex: Show that $Sa \nvDash Pa$

- E.g. 'Alice is a student. Therefore, Alice is the pope.'

This is obviously invalid…but how do we show this with models?

We need to construct a countermodel which makes Sa true and Pa false.

Show that Sa ⊭ Pa

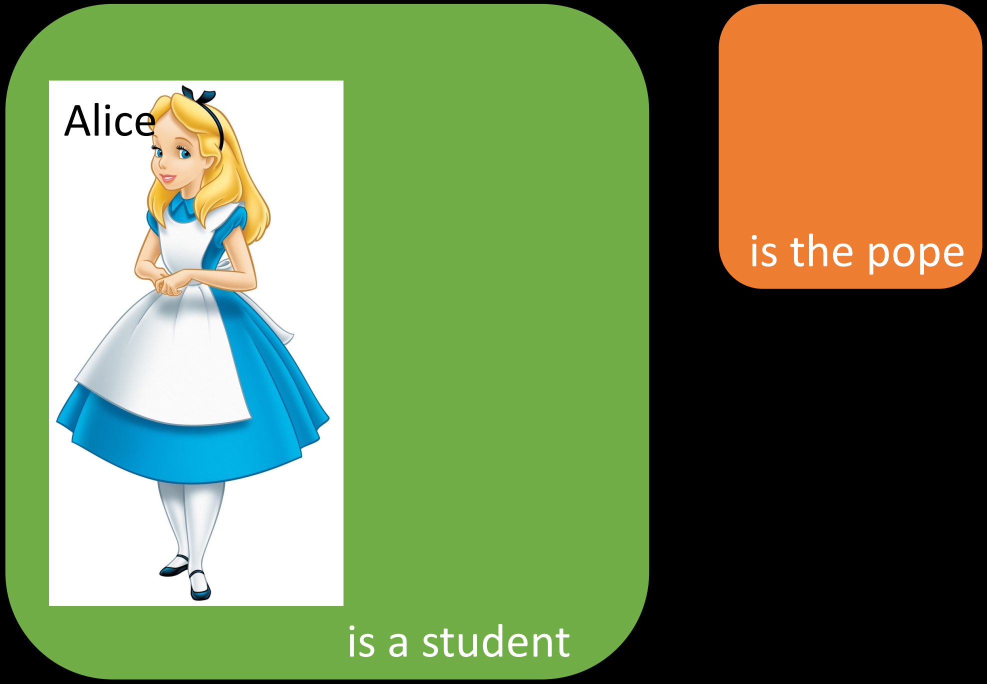
- Let's call our countermodel M
- We want Sa to be true in M. This means $I(a) \in I(S)$
- We want Pa to be false in M. So $I(a) \notin I(P)$

That's it! We just need to fill in the details and meet these constraints.

Show that $Sa \nvDash Pa$

- Let the domain(M) = {Alice}
- Let I(a) = Alice
- So let I(S) = {Alice}
- So let I(P) = { }
  - (i.e. P doesn't hold of anything!)

We've fully-specified a countermodel M



Countermodels

We still don’t have a procedure (like a truth table) for constructing a countermodel, but we do get a lot of guidance from the premises and conclusion

Another countermodel

Ex: Show that $\forall x(Px \rightarrow Qx) \nvDash \exists xQx$

- E.g. 'All unicorns are magical beings. Therefore, there is a magical being.'

There are no names mentioned in the argument, so we don't have a lot of guidance on our domain. Let's start out by letting domain(M) = {Coco}

- We might have to revise this, but we need to start somewhere.
- So we have: I(c) = Coco and |M| = {Coco}

- If we want $\exists xQx$ to be false in M, that means that there should be nothing that is Q. In other words, this tells us that I(Q) = { }

Show that $\forall x(Px \rightarrow Qx) \nvDash \exists xQx$

- We also want $\forall x(Px \rightarrow Qx)$ to be true in M
- This means for every name n, we need $(Pn \rightarrow Qn)$ to be true in N
  - We only have one name 'c', so we want $(Pc \rightarrow Qc)$ to be true in N
  - We know this happens if Pc is false or if Qc is true
  - But Qc can't be true because we choose $I(Q) = \{\}$
  - So let's make Pc false. Let $I(P) = \{\}$

Putting it together…

Show that $\forall x(Px \to Qx) \nvDash \exists xQx$

- Domain(M) = {Coco}
- I(c) = Coco
- I(P) = { }
- I(Q) = { }

M is a countermodel to the above argument!

Arrow countermodels

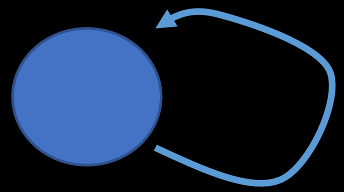
Ex: Give an arrow model which shows that
$\exists x \exists y Rxy, \forall x \forall y (Rxy \rightarrow Ryx) \nvDash \exists x Rxx$

- The first premise expresses ‘there is an arrow’
  - Recall: without explicitly stating $x \neq y$ we’re assuming it’s possible for x and y to refer to the same thing.
- The second premise says ‘if there is an arrow from x to y, there is an arrow going back from y to x’
- The conclusion says ‘there is some point with an arrow pointing to itself’

Ex: Give an arrow model which shows that
$\exists x \exists y Rxy, \forall x \forall y(Rxy \to Ryx) \nvDash \exists x Rxx$

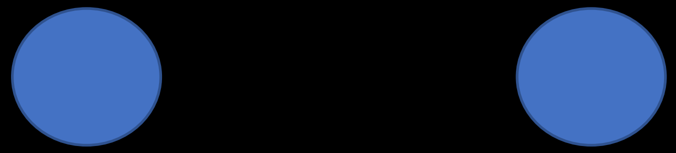
- The conclusion says 'there is some point with an arrow pointing to itself'

- We want the conclusion to be false, so we want to avoid any points like this in our model:



Ex: Give an arrow model which shows that
$\exists x\exists yRxy, \forall x\forall y(Rxy \rightarrow Ryx) \nvDash \exists xRxx$

- The first premise says there has to be at least one arrow, so if we want to avoid the conclusion, we need at least two points



Ex: Give an arrow model which shows that
$\exists x \exists y Rxy, \forall x \forall y(Rxy \rightarrow Ryx) \nvDash \exists x Rxx$

- The first premise says we need an arrow, so we add one.
- The second one says that whenever we have an arrow, we need another arrow coming back

- This model succeeds in making the premises true and the conclusion false, so we have our desired countermodel

Validity via models, recap

- Showing that an argument is *valid* via models is hard because you need to consider all models (all possible ways the world could be).
- Showing that an argument is semantically *invalid* is easier, because you only need to find one countermodel (one possible way the world could be that makes the argument invalid).
- To do so, you look at the premises and conclusion for guidance in constructing the countermodel