What are the quantifier negation rules?

Week 8. Deeper dive

On the Week 5 PP, did you put

S = "has a subway"
F = "has a football team"

4.4

Ax~(Sx∧Fx) **Submit**

No place which has a subway has a football team.

or

4.4

~Ex(Sx∧Fx) **Submit**

No place which has a subway has a football team. ✓

?

I've been saying for a while that if we have negated quantifiers, there are usually two equivalent ways of translating the statement. Today, we make this claim precise with quantifier negation rules.

Recall also (from 5.3):

But wait…were there repeats?

Equivalences:

∃xSx : There exists someone who is a student.

∃x ¬Sx: There exists someone who is not a student.

¬ ∃xSx There doesn't exist anyone who is a student.

¬ ∀x ¬ Sx: Not everyone is a non-student

¬ ∀xSx: Not everyone is a student.

∀x ¬Sx: Everyone isn't a student (alternatively, "no one is a student").

etc..

The basic idea

$$\boxed{\neg\forall x\, \phi \leftrightarrow \exists x\, \neg\phi}$$

$$\boxed{\neg\exists x\, \phi \leftrightarrow \forall x\, \neg\phi}$$

"Not everything is blue" iff "something is non-blue"

"Nothing is blue" iff "everything is non-blue"

Quantifier negation rules for proofs

$\neg\forall x\,\phi$
$\exists x\neg\phi$: QN

$\neg\exists x\,\phi$
$\forall x\neg\phi$: QN

$\exists x\neg\phi$
$\neg\forall x\phi$: QN

$\forall x\neg\phi$
$\neg\exists x\phi$: QN

The slogan here is ‘flip’ the quantifier when you move the negation in or out.

Quantifier negation rules for proofs

$\neg\forall x\, \phi$
$\exists x \neg\phi$: QN

$\neg\exists x\, \phi$
$\forall x \neg\phi$: QN

$\exists x \neg\phi$
$\neg\forall x\phi$: QN

$\forall x \neg\phi$
$\neg\exists x\phi$: QN

In proofs, we often use these one after the other because existentials are easier to introduce but universals are easier to eliminate!

Show that $\forall x(Px \rightarrow Qx), \neg Qa \vdash \exists z \neg Pz$

1.	$\forall x(Px \rightarrow Qx)$	:assumption
2.	$\neg Qa$	:assumption
3.	$Pa \rightarrow Qa$	:E$\forall$ 1
4.	$\quad$ Pa	:assumption
5.	$\quad$ Qa	:E$\rightarrow$3,4
6.	$\quad$!?	:E~2,5
7.	~Pa	:I~
8.	Ex~Px	:I$\exists$ 7

1.	$\forall x(Px \rightarrow Qx)$	:assumption
2.	$\neg Qa$	:assumption
3.	$Pa \rightarrow Qa$	:E$\forall$ 1
4.	$\quad \forall x Px$	:assumption
5.	$\quad$ Pa	:E$\forall$ 4
6.	$\quad$ Qa	:E $\rightarrow$3,5
7.	$\quad$!?	:E~2,6
8.	~AxPx	:I~
9.	Ex~Px	:QN

two equally good proofs!

Show that :

∀x(Px → ∃yRxy), ∀x∀y(Rxy → Sxy), ¬∃x∃ySxy ⊢ ¬∃xPx

(using base rules, shortcuts, and QN)

1. ∀x(Px → ∃yRxy):	assumption
2. ∀x∀y(Rxy → Sxy)	:assumption
3. ¬∃x∃ySxy	:assumption
4. Pa-> ∃yRay	:E∀ 1
5. ∀y(Ray → Say)	:E∀ 2
6. (Rab → Sab)	:E∀ 5
7. ∀x~∃ySxy	:QN 3
8. ~∃ySay	:E∀ 7
9. ∀y~Say	:QN 8
10.~Sab	:E∀ 9

using the QN rules successively to simplify

Show that :

$\forall x(Px \rightarrow \exists yRxy), \forall x\forall y(Rxy \rightarrow Sxy), \neg\exists x\exists ySxy \vdash \neg\exists xPx$

(using base rules, shortcuts, and QN)

1. $\forall x(Px \rightarrow \exists yRxy)$: assumption
2. $\forall x\forall y(Rxy \rightarrow Sxy)$:assumption
3. $\neg\exists x\exists ySxy$:assumption
4. $Pa\text{->} \exists yRay$:E∀ 1
5. $\forall y(Ray \rightarrow Say)$:E∀ 2
6. $(Rab \rightarrow Sab)$:E∀ 5
7. $\forall x{\sim}\exists ySxy$:QN 3
8. ${\sim}\exists ySay$:E∀ 7
9. $\forall y{\sim}Say$:QN 8
10. ${\sim}Sab$:E∀ 9
11. ${\sim}Rab$:MT 6,10

Recall MT:
From wffs of the forms
$(\phi \rightarrow \psi)$ and ${\sim}\psi$, infer ${\sim}\phi$

Show that :
∀x(Px → ∃yRxy), ∀x∀y(Rxy → Sxy), ¬∃x∃ySxy ⊢ ¬∃xPx
(using base rules, shortcuts, and QN)

1. ∀x(Px → ∃yRxy): assumption
2. ∀x∀y(Rxy → Sxy) :assumption
3. ¬∃x∃ySxy :assumption
4. Pa-> ∃yRay :E∀ 1
5. ∀y(Ray → Say) :E∀ 2
6. (Rab → Sab) :E∀ 5
7. ∀x~∃ySxy :QN 3
8. ~∃ySay :E∀ 7
9. ∀y~Say :QN 8
10. ~Sab :E∀ 9
11. ~Rab :MT 6,10
12. ∀y~Ray :I∀ 11
13. ~∃yRay :QN 12

Show that :

∀x(Px → ∃yRxy), ∀x∀y(Rxy → Sxy), ¬∃x∃ySxy ⊢ ¬∃xPx

(using base rules, shortcuts, and QN)

1. ∀x(Px → ∃yRxy): assumption
2. ∀x∀y(Rxy → Sxy) :assumption
3. ¬∃x∃ySxy :assumption
4. Pa-> ∃yRay :E∀ 1
5. ∀y(Ray → Say) :E∀ 2
6. (Rab → Sab) :E∀ 5
7. ∀x~∃ySxy :QN 3
8. ~∃ySay :E∀ 7
9. ∀y~Say :QN 8
10. ~Sab :E∀ 9
11. ~Rab :MT 6,10
12. ∀y~Ray :I∀ 11
13. ~∃yRay :QN 12
14. ~Pa :MT 4,13

Show that :
$\forall x(Px \rightarrow \exists yRxy), \forall x\forall y(Rxy \rightarrow Sxy), \neg\exists x\exists ySxy \vdash \neg\exists xPx$
(using base rules, shortcuts, and QN)

1. $\forall x(Px \rightarrow \exists yRxy)$:	assumption	11. $\sim Rab$	:MT 6,10
2. $\forall x\forall y(Rxy \rightarrow Sxy)$	:assumption	12. $\forall y\sim Ray$	:I∀ 11
3. $\neg\exists x\exists ySxy$	:assumption	13. $\sim\exists yRay$	:QN 12
4. Pa-> $\exists yRay$	:E∀ 1	14. $\sim Pa$	:MT 4,13
5. $\forall y(Ray \rightarrow Say)$	:E∀ 2	15. $\forall y\sim Py$	:I∀ 14
6. $(Rab \rightarrow Sab)$	:E∀ 5	16. $\sim\exists yPy$	:QN 15
7. $\forall x\sim\exists ySxy$	:QN 3		
8. $\sim\exists ySay$	:E∀ 7		
9. $\forall y\sim Say$	:QN 8		
10. $\sim Sab$	:E∀ 9		