Quantified logic, p.2

Week 5 . Topic introduction.

Recall…

What's next: Quantified logic

The vocabulary of QL builds on the vocabulary of SL, which is what gives QL a lot more expressive power!

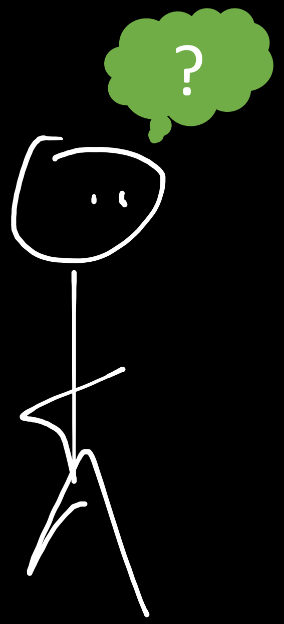
We will have:

- ✓ Connectives: ∧, v, →, ¬, ↔
- ✓ Parentheses: (,)
- ✓ Predicate/relation terms: capital letters like A, B …
- ? A special, logical relation: =
- ✓ Constants: lower case letters like a, b …
- ? Variables: x, y, z…
- ? Quantifiers: ∀, ∃

New pieces of vocab are indicated in purple!

We will learn these by using them in translations!

Variables



- Remember: we can put constants in the blanks of predicates:
  - “Fabio is a student” (Sf)
  - “Jessica is a student” (Sj)
- These sentences are about specific people.

- However, sometimes we want to make claims about non-specific people, e.g.:
  - “Everyone is a student”
  - “No one is a student”

Variables

- When we want to talk about a non-specific individual, we want to use a placeholder and say "every individual can fill this role"

- So "everyone is a student" is like: Sx where every individual can play the role of x.

- 'x' here is a **variable** because it stands in for a number of possible values.

Variables

- However, just saying ‘Sx’ doesn’t yield a definite meaning.
  - Sx says “x is a student” where x signals a role that can be played by individuals. Sx itself doesn’t say whether the role of x can be played by: all individuals? Some individuals? No individuals?

- When this happens (when there is a variable in a formula but it’s unclear who can play the role it signals), we say that the variable is a free variable.
- The logical formula Sx, even though it’s well formed, is not a proper sentence. It’s called an open formula.

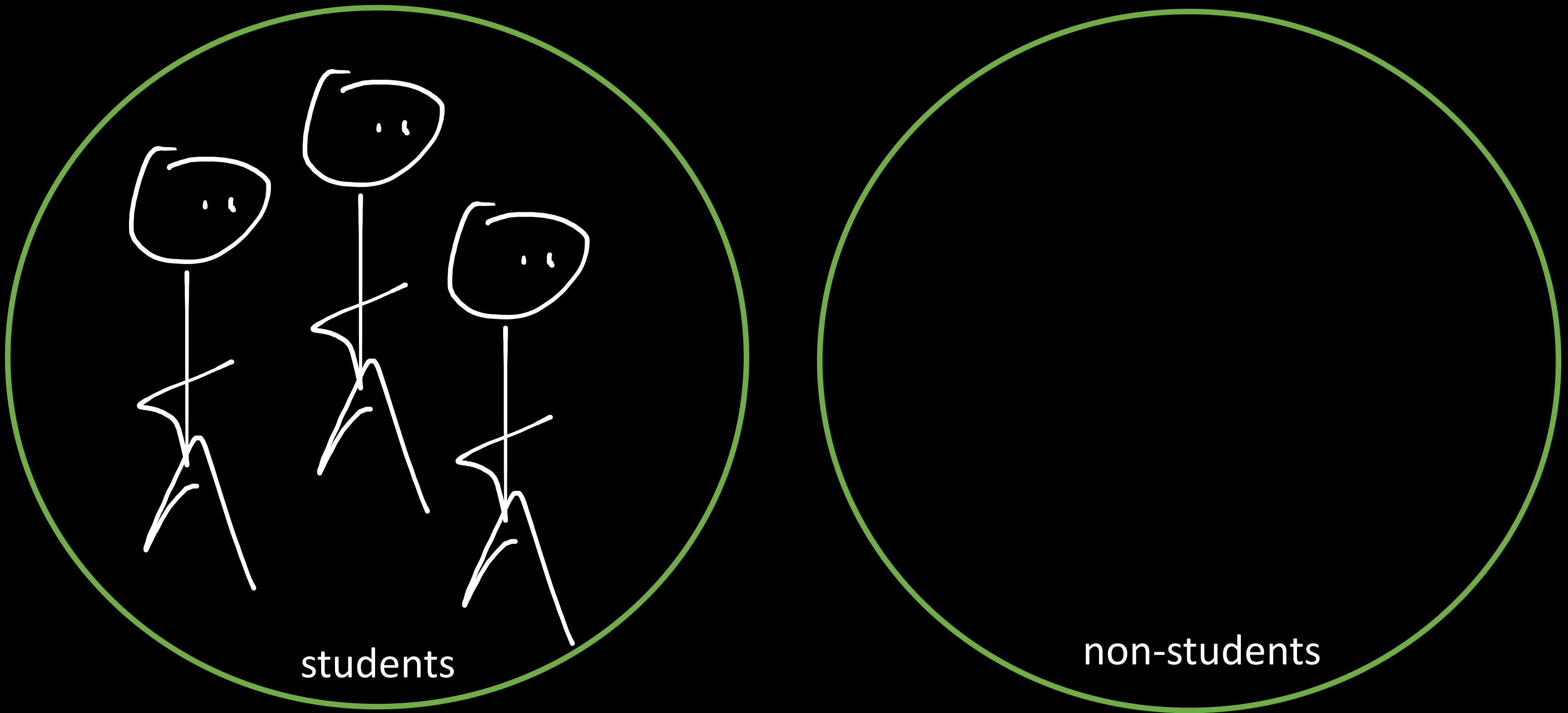
We’ll come back to these definitions once we’ve gone over a bit more notation.

Quantifiers: Or, why quantified logic is called quantified logic…

- We can specify which individuals can play the role of x by adding quantifiers to the open formula.
- We will learn 2 quantifiers:
  - ∀: the universal quantifier; read as “for all”.
    - All individuals in the **domain** can play the role of the variable.
  - ∃: the existential quantifier; read as “exists”.
    - There exists at least one individual in the domain that can play the role of the variable.

To say "everyone is a student", we write ∀xSx.

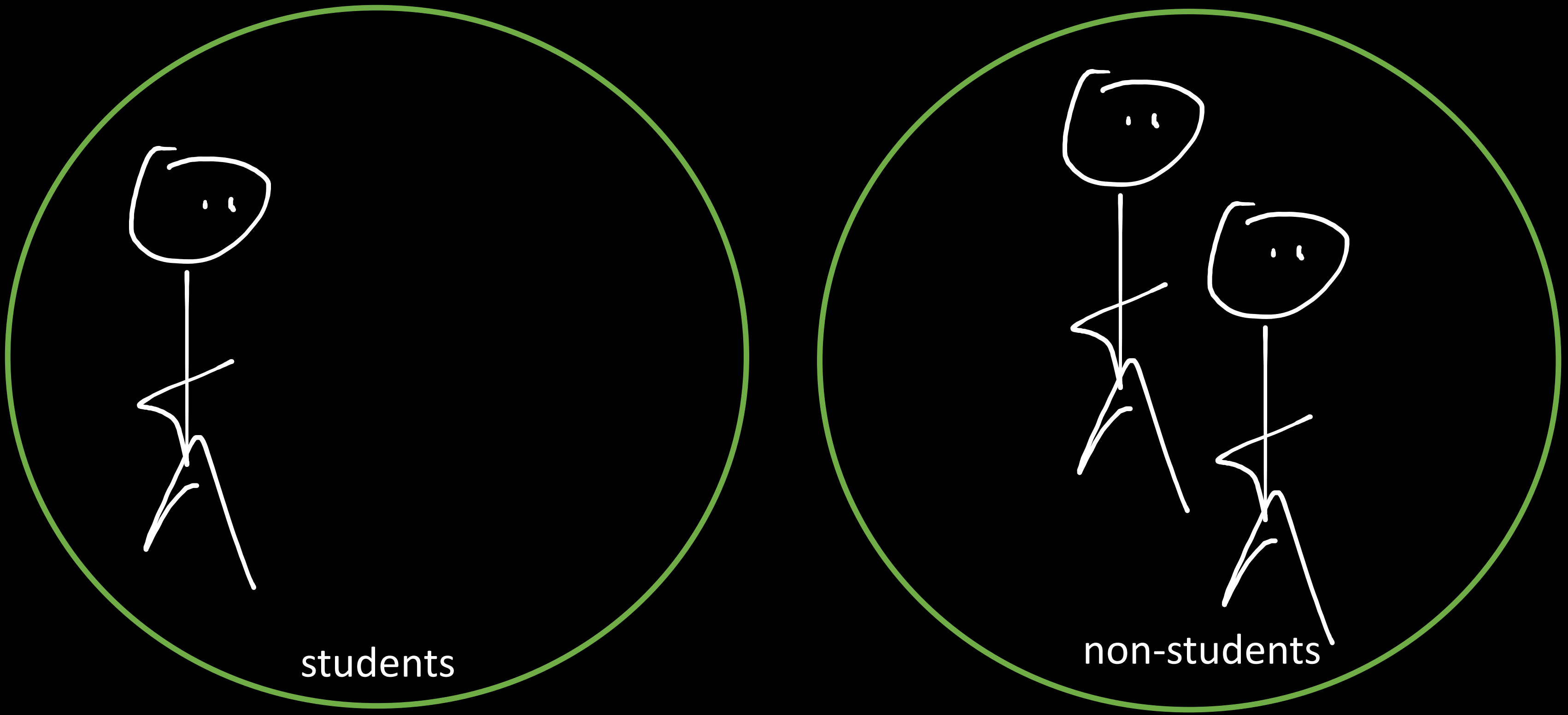
- ∀x says that every individual can fill the role of x, and Sx says that the role of x is that it's a student.



To say "there exists a student", written as ∃xSx.

- ∃x says that there exists at least one individual that can fill the role of x.

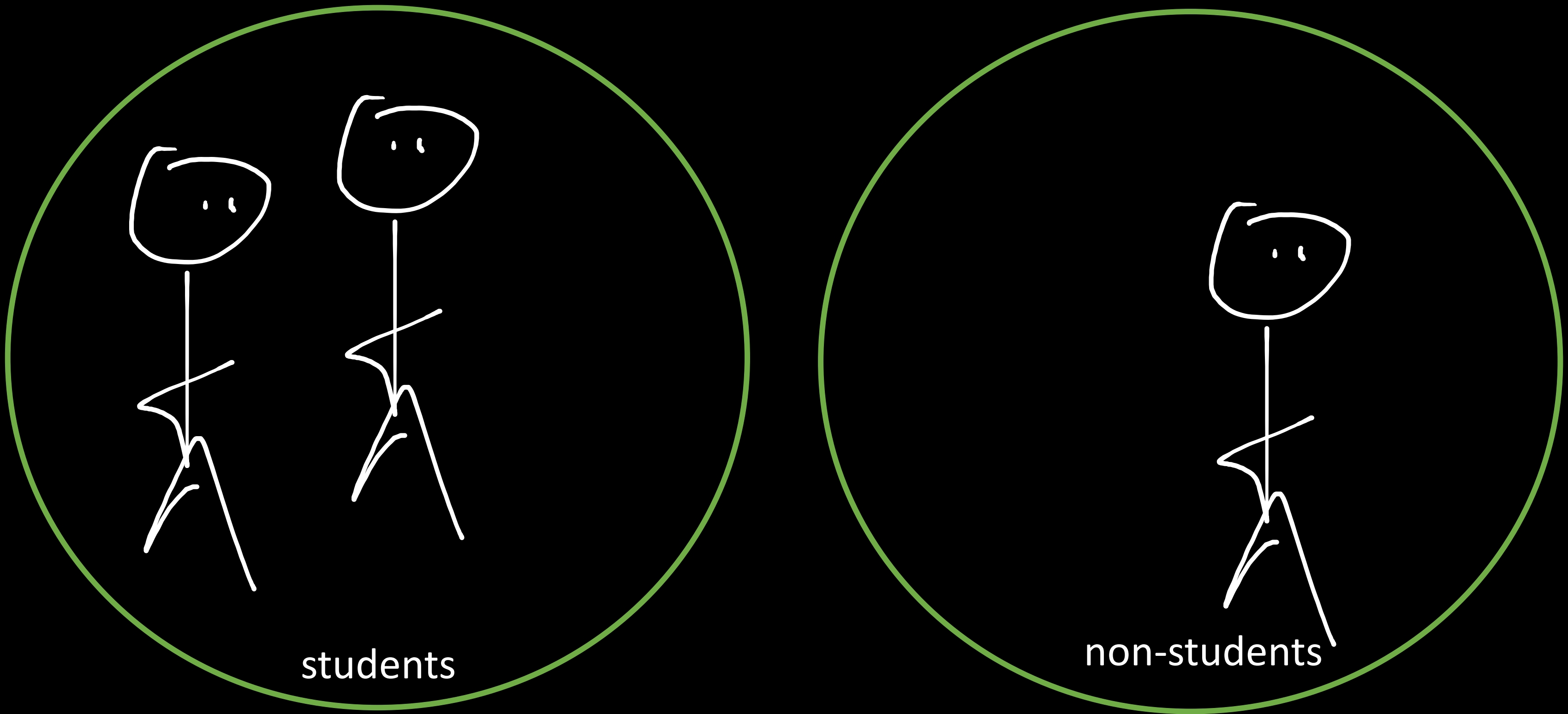
One example:



To say "there exists a student", written as $\exists x S x$.

- $\exists x$ says that there exists at least one individual that can fill the role of x.

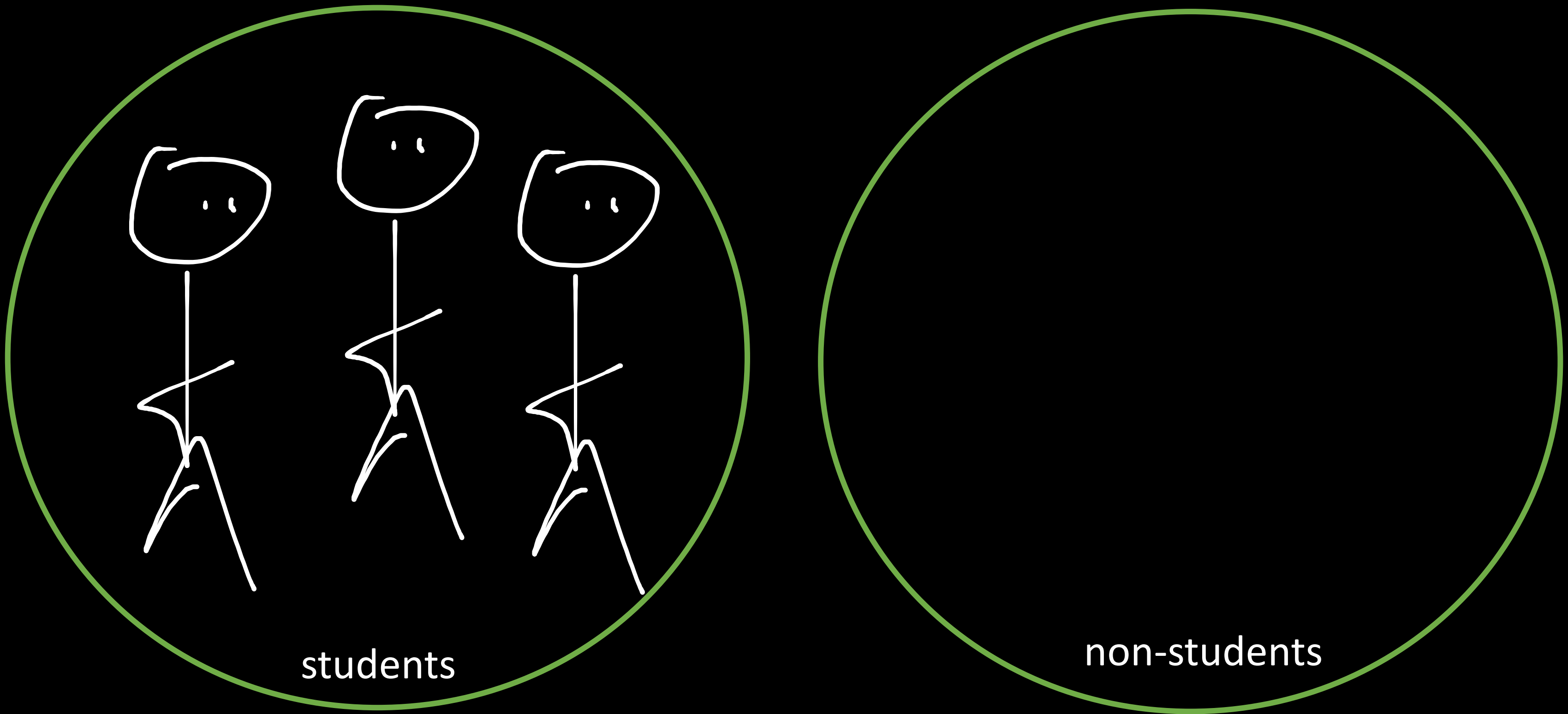
Another example:



To say "there exists a student", written as $\exists x Sx$.

- $\exists x$ says that there exists at least one individual that can fill the role of x.

Even this works:



Some definitions

- bound variable: An occurrence of a variable in a formula which is in the scope of a quantifier followed by the same variable
  - When there is a quantifier that specifies which individual(s) can play the role of a variable, the variable is called a bounded variable.

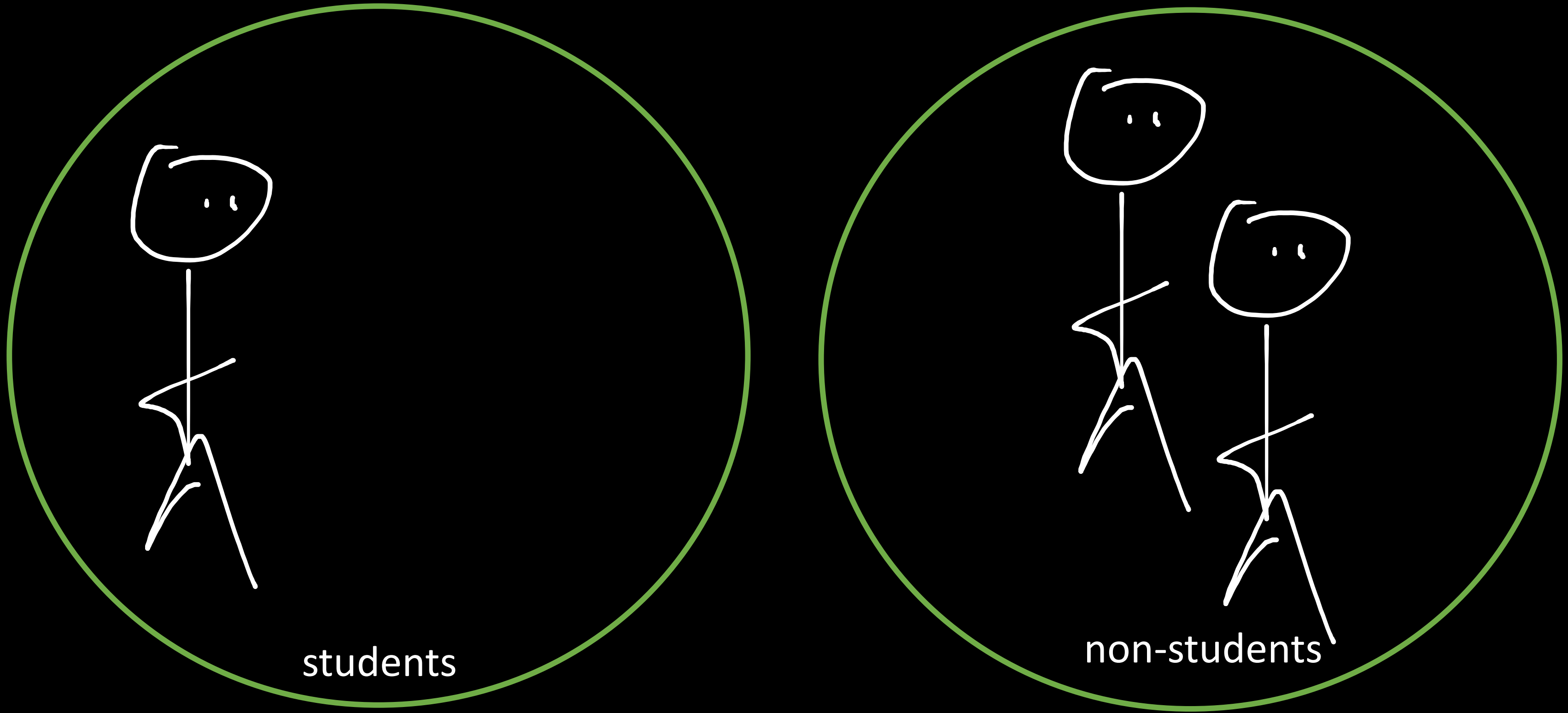
- free variable: An occurrence of a variable in a formula which is not a bound variable.

Some definitions

- open formula: A formula which results from removing one or more initial quantifiers from a quantified wff.

- When a formula has **no free variable** (it has only constants only or all variables are bounded), the formula is a **closed formula**, aka a **sentence**.

Consider the sentence "someone is a student"



Quantifiers are interdefinable! these are equivalent so they are both correct!

We could mean:

"there exists someone who is a student": $\exists x Sx$

"it's not the case that everyone is a non-student": $\neg \forall x \neg Sx$

The details…

- Each quantifier *bounds* one variable in the wff that immediately follows it.
  - ∀xPx: ∀ bounds the x in Px.
  - ∀x(Px & Qx): ∀ bounds the x in both Px and Qx, because the wff that follows ∀x is (Px & Qx).
  - ∀xPx & Qx: here, ∀ *only bounds the x in P*x, because Px is the wff that follows ∀x. Qx here is a separate wff where x appears as a free variable.
- All variables in the wff that immediately follows a quantifier are considered to be **in the scope of** that quantifier.
  - Quantifiers only bound variables within their scope.

Well formed formula in QL

- A **well-formed formula of PL (wff)** is constructed using the following rules:
    1. If P is an n-place predicate and m_1, …, m_n are names, then $Pm_1 \ldots m_n$ is a wff.
    2. If ϕ and χ are wffs, then any combination of them using our sentential connectives (e.g. &) is a wff (i.e., all the SL rules still hold).
    3. If ϕ is a wff and m is a name occurring in it, then $\forall x\phi^*$ and $\exists x\phi^*$ are wffs, where ϕ^* is the result of replacing the name m with the variable x in ϕ.

Identify major operator

- Here we have to talk about the major "operator", because it's not just connectives anymore.
- The idea is the same: the connective **or quantifier** that is added *last* is the *major operator.*
  - main operator of $\exists x Px$ is $\exists$
  - main operator of $\neg\exists x Px$ is $\neg$
  - main operator of $\neg \exists x Px \wedge \forall y Qy$ is $\wedge$.
- The major operator (like the main connective) tells you what this sentence is primarily about.
  - E.g., $\forall x(Px \wedge Qx)$ is about "everything" being such and so
  - $\forall x Px \wedge Qx$ is about two things both being true.

Let S be the 1-place predicate “is a student”.
Here are some wffs we can make with the quantifier ∃.

- ∃xSx : There exists someone who is a student.
- ∃x ¬Sx: There exists someone who is not a student.
- ¬ ∃xSx There doesn’t exist anyone who is a student.
- ¬ ∃x ¬ Sx : There doesn’t exist anyone who is not a student.
- ∃xSx ∧ ∃y¬Sy: There exists someone who is a student, and there exists someone who isn’t a student
- ∃x(Sx v ¬Sx) : There exists someone who either is a student or isn’t a student:

Let S be the 1-place predicate "is a student" and T be the 1-place predicate "is tall".
Here are some wffs we can make with the quantifier ∀:

- $\forall x Sx$: Everyone is a student.
- $\forall x \neg Sx$: Everyone isn't a student (alternatively, "no one is a student").
- $\neg \forall x Sx$: Not everyone is a student.
- $\neg \forall x \neg Sx$: Not everyone is a non-student
- $\forall x(Sx \rightarrow Tx)$: Everyone who is a student is tall
- $\neg \forall x(Sx \land Tx)$: Not everyone is both a student and tall.

But wait…were there repeats?

Equivalences:

$\exists xSx$: There exists someone who is a student.

$\exists x \neg Sx$: There exists someone who is not a student.

$\neg \exists xSx$ There doesn't exist anyone who is a student.

$\neg \forall x \neg Sx$: Not everyone is a non-student

$\neg \forall xSx$: Not everyone is a student.

$\forall x \neg Sx$: Everyone isn't a student (alternatively, "no one is a student").

etc..

Trickier translations:

Let S be the 1-place predicate "is a student" and let *a* be the constant "Alice".

- Sa: Alice is a student.
- ∃x(Sa&Sx) : Alice and someone are both students.
  - Note 1: this sentence is equivalent to Sa&∃xSx. Because Sa does not have a variable in it, it's not affected by ∃x.
  - Note 2: if there is only one student and that student is Alice, the above sentence would still be true. This is because Sa is true if Alice is a student, and ∃xSx is true if Alice is a student, so the conjunction is true.

Trickier translations:

- Consider: “everyone who’s younger than a student is also a student”.
  - We need 2 variables, one stands in for “a student”, and we are claiming something about the other – that if it’s older than the first then it’s a student.
  - Without quantifiers we can start with “if x is a student and y is younger than x then y is also a student”, which is (Sx & Yyx) → Sy
  - Then, we bound these variables by noticing that this relationship applies to “everyone” and “every student”, we get ∀x ∀y [(Sx & Yyx) → Sy]

Trickier translations:

- Consider: "there's someone who likes someone"
  - (a more natural expression might be "at least someone here understands the meaning of love")
  - We translate this as: $\exists x \; \exists y \; Lxy$.
- Note: when the quantifiers are the same type, order doesn't matter. It's equivalent to $\exists y \; \exists x \; Lxy$. Writing it differently will change how we use it in proofs but not its truth value.
- Note 2: we haven't required that x and y be distinct…

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Recall: identity

But wait…what if something has multiple names?

UCI goes by i: "University of California, Irvine" or say, a: "University of the Anteaters."

When this happens, we use a special relation "=" and write "i=a."

This identity relation is a lot more powerful than it looks. We'll revisit it again and again in the future.

Non-identity between variables

- In $\exists x\ \exists y\ Lxy$, we haven't required that x and y be distinct individuals. That is, this sentence is true even when there's only one person and they like themself.
- If we want to talk about liking other people, we have to specify that x and y are not the same individual: $\exists x\ \exists y(x \neq y\ \&\ Lxy)$
  - We write $x \neq y$ to mean $\neg(x=y)$.
- Similarly, if we want to say that every student is taught by someone else, then we need to specify $\forall x\ (Sx \rightarrow \exists y\ (x \neq y\ \&\ Tyx))$

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