Homework 2

Helen Ngo

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Problem 2.1 In Equation (2.1), set $\delta = 0.03$ and let

$$\epsilon(M, N, \delta) = \sqrt{\frac{1}{2N} \ln \frac{2M}{\delta}}.$$

(a) For $M = 1$, how many examples do we need to make $\epsilon \leq 0.05$?

(b) For $M = 100$, how many examples do we need to make $\epsilon \leq 0.05$?

(c) For $M = 10,000$, how many examples do we need to make $\epsilon \leq 0.05$?

Solution:

(a) Let

$$\epsilon(M, N, \delta) = \sqrt{\frac{1}{2N} \ln \frac{2M}{\delta}}, \tag{1}$$

where $\epsilon \leq 0.05$, $M = 1$ and $\delta = 0.03$. Then

$$\begin{aligned}
0.05 \geq \epsilon(M, N, \delta) &= \sqrt{\frac{1}{2N} \ln \frac{2}{0.03}} \\
0.0025 &\geq \frac{1}{2N} \ln \frac{2}{0.03} \\
0.0025 \left(\ln \frac{2}{0.03} \right)^{-1} &\geq \frac{1}{2N} \\
400 \ln \frac{2}{0.03} &\leq 2N \\
200 \ln \frac{2}{0.03} &\leq N \\
839.941 &\leq N \\
N &\geq 840.
\end{aligned}$$

We need at least $N \geq 840$ examples to make $\epsilon \leq 0.05$ for $M = 1$.

(b) Substituting $\epsilon \leq 0.05$, $M = 100$ and $\delta = 0.03$ into equation (1), we can solve for N,

$$\begin{aligned}
0.05 &\geq \sqrt{\frac{1}{2N} \ln \frac{200}{0.03}} \\
0.0025 &\geq \frac{1}{2N} \ln \frac{200}{0.03} \\
0.0025 \left(\ln \frac{200}{0.03} \right)^{-1} &\geq \frac{1}{2N} \\
400 \ln \frac{200}{0.03} &\leq 2N \\
200 \ln \frac{200}{0.03} &\leq N \\
1761.975 &\leq N \\
N &\geq 1762.
\end{aligned}$$

We need at least $N \geq 1762$ examples to make $\epsilon \leq 0.05$ for $M = 100$.

(c) Notice that between $M = 1$ and $M = 100$, only one number changes in the result before the answer. Thus, for $M = 10,000$,

$$N \geq 200 \ln \frac{20,000}{0.03} \approx 2682.009 = 2683.$$

We need at least $N \geq 2683$ examples to make $\epsilon \leq 0.05$ for $M = 10,000$.

Problem 2.11 Suppose $m_H(N) = N + 1$, so $d_{vc} = 1$. You have 100 training examples. Use the generalization bound to give a bound for E_{out} with confidence 90%. Repeat for $N = 10,000$.

Solution: The generalization bound states, "For any tolerance $\delta > 0$

$$E_{\text{out}} \leq E_{\text{in}} + \sqrt{\frac{8}{N} \ln \frac{4m_H(2N)}{8}} \tag{2}$$

with probability $\geq 1 - \delta$." Substitute $N = 100$ into equation (2),

$$\begin{aligned} E_{\text{out}} &\leq E_{\text{in}} + \sqrt{\frac{8}{N} \ln \frac{4m_H(2N)}{8}} \\ &= E_{\text{in}} + \sqrt{\frac{8}{100} \ln \frac{4(2 \times 100 + 1)}{8}} \\ &\approx E_{\text{in}} + \sqrt{0.3688} \\ &\approx E_{\text{in}} + 0.6073. \end{aligned}$$

The bound for $E_{\text{out}} \leq E_{\text{in}} + 0.6073$. Similarly, for $N = 10,000$, the bound is

$$E_{\text{out}} \leq E_{\text{in}} + \sqrt{\frac{8}{10000} \ln \frac{4(2 \times 10000 + 1)}{8}} \approx E_{\text{in}} + 0.0858.$$

Problem 2.12 For an H with $d_{vc} = 10$, what sample size do you need to have a 95% confidence that your generalization error is at most 0.05?

Solution: Using equation (2.13), we need

$$N \geq \frac{8}{0.05^2} \ln \left(\frac{4(2N)^{10} + 4}{0.05} \right).$$

Trying an initial guess of $N_0 = 1,000$ in the RHS, we get

$$N \geq \frac{8}{0.05^2} \ln \left(\frac{4(2 \times 1000)^{10} + 4}{0.05} \right) \approx 257,251.$$

Then we try the new value $N = 257,251$ in the righthand side and continue the iterative process until $N_i/N_{i-1} \approx 1$. The iterative process converges to an estimate of $N \approx 451,652$.

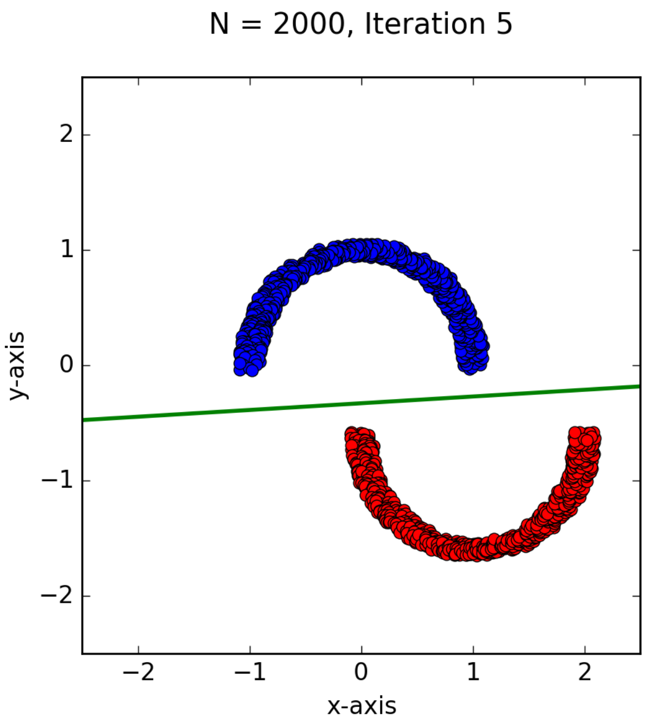
Problem 3.1 Consider the double semi-circle "toy" learning task. This task is linearly separable when $sep \geq 0$, and not so for $sep < 0$. Set $rad = 10, thk = 5$ and $sep = 5$. Then, generate $2,000$ examples uniformly, which means you will have approximately $1,000$ examples for each class.

(a) Run the PLA starting from $\mathbf{v} = \mathbf{0}$ until it converges. Plot the data and the final hypothesis.

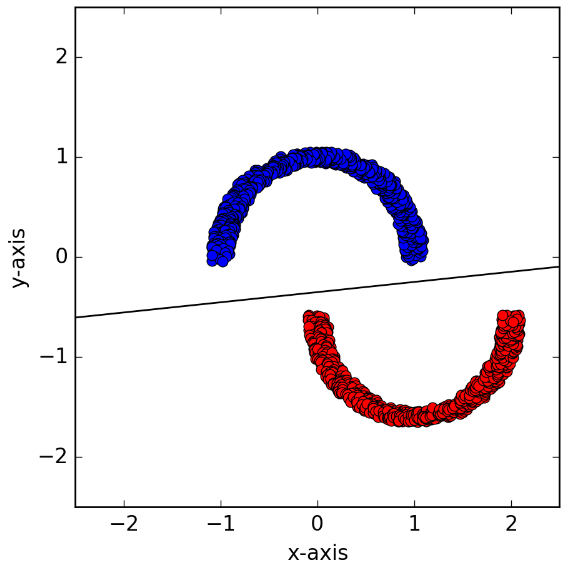
(b) Repeat part (a) using the linear regression (for classification) to obtain $\mathbf{w}$. Explain your observations.

Solution: The python code is attached.

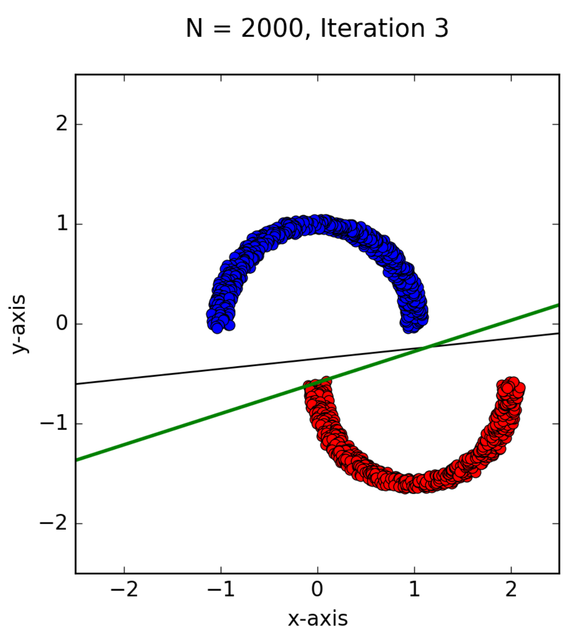
(a) Adjusting the Perceptron from Assignment 1, so that the points are generated by the function *make_semi_circles*, we can run the PLA as before. The plot below shows the data and the final hypothesis as produced by PLA.



(b) The *linear_regression* function added to the Perceptron code so that using the same point, a linear classification line was produced. The classification line, produced using the same data as part (a), is shown in the plot below.



While the separating lines above are similar, linear regression classification will always give us the line with the lowest least square error while the PLA produces a classifying line with no error, but have a higher least square error. This can be better observed for another set of data, the linear regression line and perceptron line are shown below for it.



Problem 3.2 For the double-semi-circle task, vary *sep* in the range $\{0.2, 0.4, ..., 5\}$.

Solution: As the separation *sep* between the semicircles increase, the number of iterations it takes for the PLA to find a solution increases. The overall trend trend of the relationship follows an exponential trend instead of a linear trend. A plot of the *sep* versus the number of iterations is shown below with the code in the main function of the attached python code.