1. a) $X = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

$-[(x_1 - x_2) - x_3] = -x_1 + (x_2 + x_3)$

$\begin{matrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{matrix} \quad \begin{matrix} 0 \\ 0 \\ -1 \\ -2 \\ 0 \end{matrix} \qquad \begin{bmatrix} -1 \\ -2 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

$\Rightarrow \quad x_4 = -x_1 + x_2 + x_3$

$0 = -x_1 + x_2 + x_3 - x_4$

↳ linear dependence bc $x_4$ can be expressed as a linear combination of the 1st three $x_4 = -x_1 + x_2 + x_3$

meaning set of all 4 cols isn't independent

⇒ so 3 cols is the largest set of linearly indep. columns in $X$ (i.e. $\{x_1, x_2, x_3\}$).

1. b) Not unique.

we can also show $x_3 = x_1 - x_2 + x_4$

$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ✓.

1. c) rank of $X = 3$

since this is the no. of cols in any largest set of linearly independence cols of $X$.

d) $X^T X = \begin{bmatrix} 1 & 2 & 0 & -2 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 2 & -4 \\ 3 & 2 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ -4 & 0 & 1 & 5 \end{bmatrix}$

$x_4 = -x_1 + x_2 + x_3$

$\begin{bmatrix} -9 \\ -3 \\ -2 \\ +4 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -4 \\ 0 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -9 \\ -3 \\ -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \\ 3 \\ 1 \end{bmatrix}$

linearly dependent

shows $\{x_1, x_2, x_3\}$ independent ⇒ rank=3.

2. a) Yes, since the elements in the 2nd row don't match, meaning the 2 columns are not related ⇒ no work around makes them equal zero vector

2. b) Yes.

$a_1 - a_2 + a_3 = 0 \Rightarrow a_2 = a_1 + a_3$

$a_1 + a_2 - a_3 = 0 \Rightarrow 2a_1 = 0 \Rightarrow a_1 = 0$

$a_1 - a_2 = 0 \rightarrow a_2 = 0.$ which forces $a_3 = 0$.

Therefore, $a_1 = a_2 = a_3 = 0$ and columns of $X$ are linearly independent.

2. c) $X = \begin{bmatrix} 3 & 8 & 6 \\ 2 & 1 & 2 \\ 9 & 11 & 12 \end{bmatrix} \Rightarrow$

$3a_1 + 8a_2 + 6a_3 = 0$

$2a_1 + a_2 + 2a_3 = 0 \Rightarrow a_2 = -2a_1 - 2a_3$

$9a_1 + 11a_2 + 12a_3 = 0$

$\Rightarrow 3a_1 + 8(-2a_1 - 2a_3) + 6a_3 = 0$

$-16$

$-13a_1 - 10a_3 = 0$

$9a_1 + 11(-2a_1 - 2a_3) + 12a_3 = 0$

$-22$

$-13a_1 - 10a_3 = 0$

$\rightarrow \quad 13a_1 - 10a_3 = 0$

$13a_1 - 10a_3 = 0$

$0 = 0$

meanining infinitely many solutions where columns are linearly dependent.

2. d) rank of $X = \begin{bmatrix} 2 & 4 \\ 8 & 12 \\ 4 & 8 \\ 7 & 9 \end{bmatrix}$

$2a_1 + 4a_2 = 0 \; / 2$

$8a_1 + 12a_2 = 0 \; / 4$

$4a_1 + 8a_2 = 0 \; / 4$

$7a_1 + 9a_2 = 0$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 2 \\ 7 & 9 \end{bmatrix}$

Either way, rank is at most 2 bc $\exists$ 2 columns. The columns aren't multiples of one another so they're independent and therefore, rank$(X) = 2$.

3. a) $f(w) = w^T(8x) + x^T w$

$= 9w^T x \qquad \Rightarrow \quad w^T x = x^T w$ (known)

$\nabla_w f = 9x$

3. b) $f(w) = (2w - 4x)^T (2w - x)$

$= 4w^T w - 2w^T x - 8x^T w + 4x^T x$

$= 4w^T w - 10 w^T x + 4x^T x$

note: gradient of $w^T w = 2w$
gradient of $x^T w = x$.

$\nabla_w f = 4(2w) - 10x$

$\nabla_w f = 8w - 10x$

3. c) $f(w) = x^T \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} w$

$\nabla_w f = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} x = A^T x$ where $A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$

3. d) $f(w) = w^T \begin{bmatrix} 2 & 2 \\ -2 & 1 \end{bmatrix} w$

$\nabla_w f = \left( \begin{bmatrix} 2 & 2 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix} \right) w = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} w \quad \Rightarrow \quad \nabla_w f = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} w$

3. e) $f(w) = w^T \begin{bmatrix} 1 & 8 \\ 8 & 8 \end{bmatrix} w$

let $C = \begin{bmatrix} 1 & 8 \\ 8 & 8 \end{bmatrix} \Rightarrow$ symmetric $(C = C^T)$

$\begin{bmatrix} 1 & 8 \\ 8 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 8 & 8 \end{bmatrix}$

$\Rightarrow$ function is in quadratic form: $f(w) = w^T C w$

$\nabla_w f = 2Cw$

$= 2 \begin{bmatrix} 1 & 8 \\ 8 & 8 \end{bmatrix} w$

$\nabla_w f = \begin{bmatrix} 2 & 16 \\ 16 & 16 \end{bmatrix} w$

4. a) code...

b) 1. get the 9 features from the new face image. Put them into a row vector $v^T$.

2. $s = v^T w$ where $w$ = stuff calculated from part (a) w/ LSE.

3. If $s > 0$, then classify as smiling. Else, $s \leq 0$, classify as non-smiling.

c) code....

d) Since the features are normalized to have zero mean and unit variance, we should first look @ the magnitude of the weights. We should rank the features by the absolute value of their weight and drop the ones w/ the smallest magnitudes. From here, we should test if the error still remains below a certain threshold.

5. code.

# PROBLEM SET 2

You may also find the code here: https://github.com/helenxtian/math-for-ml.

4. a)

```
import numpy as np

##### Part a - main #####
# Load in training data and labels
# File available on Canvas

face_data_dict = np.load("face_emotion_data.npz")
face_features = face_data_dict["X"]
face_labels = face_data_dict["y"]
n, p = face_features.shape

# Solve the least-squares solution. weights is the
    array of weight coefficients
# TODO: find weights
XT_X = np.dot(face_features.T, face_features)
             # X^T X
XT_y = np.dot(face_features.T, face_labels)
             # X^T y
weights = np.dot(np.linalg.inv(XT_X), XT_y)  # (X^T
    X)^{-1} X^T y

print(f" P a r t 4a .     Foundweights     :\n{weights}")
```

```
helentian@Helens-MacBook-Air pset2_code % ...
 P a r t 4a .     Foundweights     :
[[ 0.94366942]
 [ 0.21373778]
 [ 0.26641775]
 [-0.39221373]
 [-0.00538552]
 [-0.01764687]
 [-0.16632809]
 [-0.0822838 ]
 [-0.16644364]]
```

c)

```
##### Part b  - function #####
def lstsq_cv_err(features: np.ndarray, labels: np.
    ndarray, subset_count: int = 8) -> float:
    """Estimate the error of a least-squares
       classifier using cross-validation. Use
       subset_count different train/test splits with
        each subset acting as the holdout set once.
```

```
    Parameters:
        features (np.ndarray): dataset features as a
             2D array with shape (sample_count ,
            feature_count)
        labels (np.ndarray): dataset class labels
            (+1/-1) as a 1D array with length (
            sample_count)
        subset_count (int): number of subsets to
            divide the dataset into
            Note: assumes that subset_count divides
                the dataset evenly

    Returns:
    cls_err (float): estimated classification error
        rate of least-squares method"""

    sample_count, feature_count = features.shape
    subset_size = sample_count // subset_count

    # Reshape arrays for easier subset-level
        manipulation
    reshaped_feat = features.reshape(subset_count,
        subset_size, feature_count)
    reshaped_lbls = labels.reshape(subset_count,
        subset_size)

    subset_idcs = np.arange(subset_count)
    train_set_size = (subset_count - 1) *
        subset_size
    subset_err_counts = np.zeros(subset_count)

    for i in range(subset_count):
        # TODO: select relevant dataset,
        # fit and evaluate a linear model,
        # then store errors in subset_err_counts[i]
        # Hint: you could extract the training
            subset with train_subset_idcs =
            subset_idcs[subset_idcs != i]

        test_feat = reshaped_feat[i]
        test_label = reshaped_lbls[i]

        train_subset_idcs = subset_idcs[subset_idcs
            != i]
        train_feat = reshaped_feat[train_subset_idcs
            ].reshape(train_set_size, feature_count)
        train_label = reshaped_lbls[
            train_subset_idcs].reshape(train_set_size
            )
```

```
        XT_X = np.dot(train_feat.T, train_feat)
        XT_y = np.dot(train_feat.T, train_label)
        w = np.dot(np.linalg.inv(XT_X), XT_y)

        y_pred = np.sign(np.dot(test_feat, w))
        subset_err_counts[i] = np.sum(y_pred !=
            test_label)

    # Average over the entire dataset to find the
        classification error
    cls_err = np.sum(subset_err_counts) / (
        subset_count * subset_size)
    return cls_err
```

```
##### Part b - main #####
# Run on the dataset with all features included
full_feat_cv_err = lstsq_cv_err(face_features,
    face_labels)
print(f"Part 4b. Error estimate: {
    full_feat_cv_err*100:.3f}%")
```

```
helentian@Helens-MacBook-Air pset2_code % ...
 Part 4b. Error estimate: 4.688%
```

e)

```
##### Part e - function #####
def drop_features_heuristic(features, labels,
    max_cv_err=0.06):
    chosen_features = list(range(features.shape[1]))
    best_err = lstsq_cv_err(features, labels)

    while len(chosen_features) > 1:
        reducedX = features[:, chosen_features]
        XT_X = reducedX.T @ reducedX
        XT_y = reducedX.T @ labels
        w = np.linalg.inv(XT_X) @ XT_y

        # absolute value weights, sorted from least
            to most
        min_idx = np.argmin(np.abs(w))
        feat_to_drop = chosen_features[min_idx]

        trial_features = [f for f in chosen_features
             if f != feat_to_drop]
        trialX = features[:, trial_features]
        trial_err = lstsq_cv_err(trialX, labels)
```

```
        if trial_err <= max_cv_err:
            chosen_features = trial_features
            best_err = trial_err
            print(f" P a r t 4e .  dropping   feature {
                feat_to_drop}, error: {trial_err
                *100:.2f}%")
        else:
            print(f" P a r t 4e .  dropping   feature {
                feat_to_drop}, error {trial_err
                *100:.2f}% > {max_cv_err*100:.2f}%")
            break

    return chosen_features, best_err
```

```
##### Part e - main #####
chosen_features, final_err =
    drop_features_heuristic(face_features,
    face_labels, max_cv_err=0.06)
print(" P a r t 4e .    Selectfeatures    :",
    chosen_features)
print(f" P a r t 4e .
               CVerrorwithselectfeatures          :   {
    final_err*100:.2f}%")
```

```
helentian@Helens-MacBook-Air pset2_code % ...
 P a r t 4e .  dropping   feature 4, error: 4.69%
 P a r t 4e .  dropping   feature 5, error: 4.69%
 P a r t 4e .  dropping   feature 7, error: 4.69%
 P a r t 4e .  dropping   feature 8, error 6.25% > 6.00%
 P a r t 4e .    Selectfeatures    : [0, 1, 2, 3, 6, 8]
 P a r t 4e .          CVerrorwithselectfeatures          :
      4  .69%
```

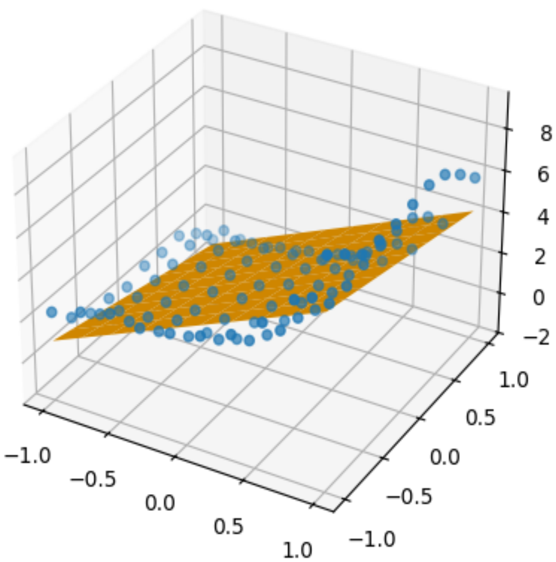
5.

```
import numpy as np
import matplotlib.pyplot as plt

# File available on Canvas
data = np.load('polydata_a24.npz')
x1 = np.ravel(data['x1'])
x2 = np.ravel(data['x2'])
y = data['y']

N = x1.size
p = np.zeros((3,N))

for d in [1 ,2 ,3]:
```

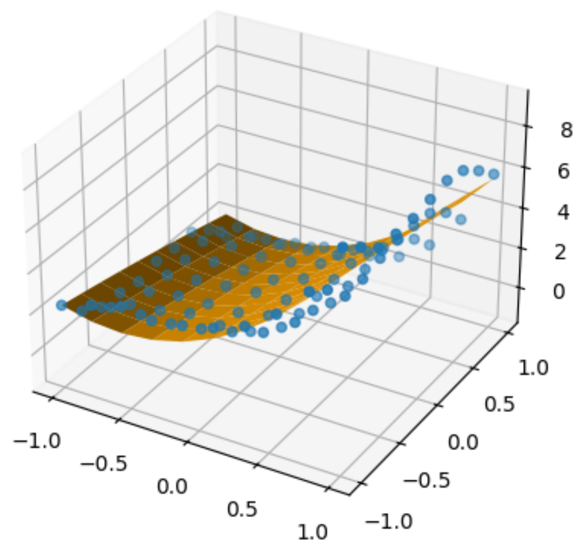
```
    # Generate the X matrix for this d
    # Note that here d is the degree of the polynomial,
       not the dimension of a vector
    # Find the least-squares weight matrix w_d
    # Evaluate the best-fit polynomial at each point (x1
       ,x2) # and store the result in the corresponding
        column of p
    # Report the relative error of the polynomial fit

    X = np.zeros((N, 2 * d + 1))
    for i in range(N):
        X[i, 0] = 1.0
        for j in range(1, d + 1):
            X[i, j] = x1[i] ** j
            X[i, d + j] = x2[i] ** j

    # (X^T X)^-1 X^T y
    XT_X = np.dot(X.T, X)
    XT_y = np.dot(X.T, y)
    w_d = np.dot(np.linalg.inv(XT_X), XT_y)

    y_hat = np.dot(X, w_d)
    p[d - 1, :] = y_hat

    rel_err = np.linalg.norm(y - y_hat) / np.linalg.
       norm(y)
    print(f"d={d}:      relativeerror      =   {rel_err
       *100:.3f}%")

# Plot the degree 1 surface
Z1 = p[0,:].reshape(data['x1'].shape)
ax = plt.axes(projection='3d')
ax.scatter(x1,x2,y)
ax.plot_surface(data['x1'],data['x2'],Z1,color='orange')
plt.show()

# Plot the degree 2 surface
Z2 = p[1,:].reshape(data['x1'].shape)
ax = plt.axes(projection='3d')
ax.scatter(x1,x2,y)
ax.plot_surface(data['x1'],data['x2'],Z2,color='orange')
plt.show()

# Plot the degree 3 surface
Z3 = p[2,:].reshape(data['x1'].shape)
ax = plt.axes(projection='3d')
ax.scatter(x1,x2,y)
ax.plot_surface(data['x1'],data['x2'],Z3,color='orange')
plt.show()
```

```
helentian@Helens-MacBook-Air pset2_code % ...
d=1:      relativeerror      =  32  .174%
d=2:      relativeerror      =  18  .979%
d=3:      relativeerror      =  2  .912%
```

d=1




d=2




d=3