# Math 449 Directed Study Report

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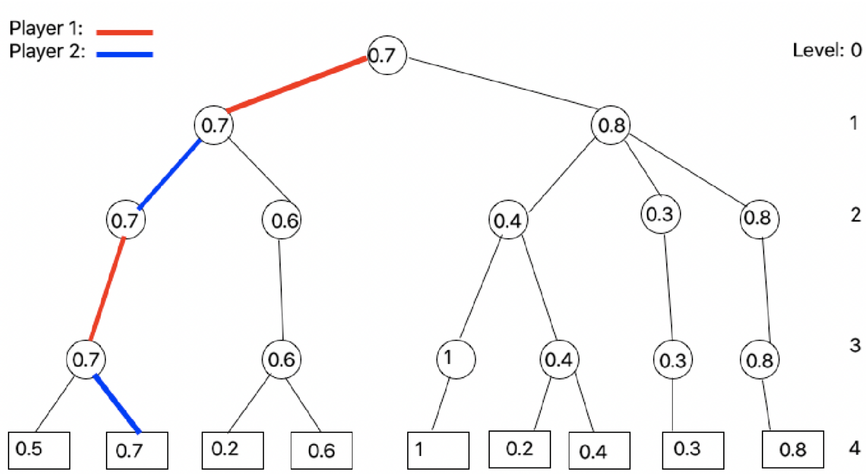
## 1 Introduction

In this directed study supervised by Professor Omer Angel, we discussed Martin and Stasinski's [2] paper about minimax functions on Galton-Watson tree, in which they examined the convergence behaviour and discussed the question of endogeny. We also studied some other previous papers such as the paper by Ali Khan, Devroye and Neininger [4] in which they considered minimax function on regular trees and obtained a distributional limit under a suitable rescaling.. This report will summarize the game setting, the main theorems and proofs in these papers, our attempt of some further analysis and several open questions.

## 2 Game Setting and Example

Consider a two player game on a finite random tree of depth m, where the root starts at level 0. Each node has one or more children. Each leaf node has a corresponding value that are drawn independently from a common distribution. The games starts with a token placed at the root, and two players move the token to a child node alternatively. When the token reaches a leaf node, the game terminates, and the outcome of the game is the value on that leaf node. Player 1 tries to minimize the outcome, while player 2 tried to maximize the outcome.

Here is an example of the minimax game on a tree of depth 4, where player 1 (red) tries to minimize the outcome, player two (blue) tries to maximize the outcome, and player 1 starts.



Martin and Stasinski's [2] paper considered the case where the tree is given by a Galton-Watson branching process truncated at level 2n. In the paper by Ali Khan, Devroye and Neininger [4], they considered the case of d-regular tree where $d \geq 2$. Given that the tree is finite and each player always take the optimal move, the outcome of the game is deterministic.

# 3 Results in the Papers

Consider a Galton-Watson tree truncated at $2n$ level. Assume the offspring distribution has mass function $p_i$ on having $i$ children, $i \geq 1$. Let $G(x) = \sum_{k=0}^{\infty} p_k x^k$ be the probability generating function of the offspring distribution. Let $R(x) = 1 - G(x)$, and $f(x) = R(R(x))$.

Assume the value of the leaf nodes are drawn independently from uniform distribution on the interval [0,1]. Then assign values to each node of the tree by the minimax process described before. Denote the randome variable of the root value of a tree of depth 2n to be $W_{2n}$. Then we have the following recursion:

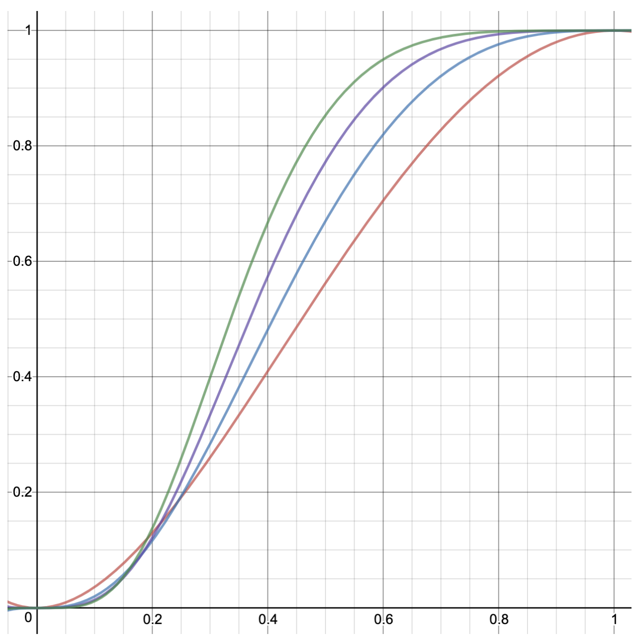
$$W_{2n} \stackrel{d}{=} \min_{1 \leq i \leq M} \max_{1 \leq j \leq M_i} W_{2n-2}^{(i,j)}, \tag{1}$$

where $M$ and each $M_i$ are independently drawn from the offspring distribution, and $W_{2n-2}^{(i,j)}$ are copies of random variable $W_{2n-2}$, the value of the nodes at level 2. Then we have the generating function:

$$\mathbb{P}(W_{2n} \leq x) = f(\mathbb{P}(W_{2n-2} \leq x)). \tag{2}$$

One result from Pearl's [3] paper is that in the case of a d-regular tree, $W_{2n} \xrightarrow{q} w$ as $n \to \infty$ where q is the unique fixed point on the interval [0,1] of the function $f$ defined as

$$f(x) = (1 - (1 - x)^d)^d. \tag{3}$$




In Ali Khan, Devroye and Neininger's [4] paper, in the case of a d-regular tree, letting $\zeta = f'(q)$, then

$$\zeta^n (W_{2n} - q) \xrightarrow{d} W \text{ as } n \to \infty \tag{4}$$

where W has a continuous distribution functiom $F_w$ with $0 \leq F_w \leq 1$, $F_w(0) = q$ and that

$$F_w(x) = f(F_W(x/\zeta)) \tag{5}$$

where f is the f defined in equation (3).

Martin and Stasinski's [2] paper considered the non-regular tree, discusses the behaviour of the $W_{2n}$ for large n and the function f, in particular its fixed points. They proved that when f is the identity function, the distribution of the root converges to uniform distribution on the interval [0,1] for all n. Otherwise, the distribution converges to a discrete distribution that has atoms percisly at elements of Q, where Q denotes the set of fixed point of f which are unstable from at least one side.

Ali Khan, Devroye and Neininger's [4] paper also discussed the question of endogeny, in other words, when the tree is very deep, whether the players need to know the boundary values or a very large part of the tree structure to make a reasonable first move, or is the information of the first several level of the trees sufficient for the player to have a nearly optimal strategy. Considering a 2-level periodic tree, the recursive tree process is said to be endogenous if the value associated to the root is measurable with respect to the structure of the tree. They concluded that for a stationary recursive tree process with Bernoulli(1-x) marginals for the values at even levels, the process is endogenous if and only if $f'(x) \leq 1$ where x is a fix point of f. As a corollary, if f is indentity, then for any $\mu$, the recursive tree process with marginals $\mu$ for the values at even levels is endogenous.

Holroyd and Martin's paper [1] considered two- player combinatorial games whose position can be discribed in directed Galton- Watson trees. Started with a token at the root, two player take turns to move the token forward along an edge to reach a new vertex in the tree. They pointed out that there are three most general rules of games: The normal game, the mesere game and the escape game. In the normal game, a player loses the game if they cannot move, such as when the token is already at a leaf of the tree, and the other player win. In the misere game, a player wins if they can not move. In the escape game, there is a stopper and a escaper. The stopper wins if either player is not able to move; the escaper wins when the game is a draw and goes on forever.

| Game | N | P | Draw |
|---|---|---|---|
| Normal | Player 1 win | Player 2 win | Draw |
| Misere | Player 2 win | Player 1 win | Draw |
| Escape(Player 1 is stopper) | Player 1 win | Player 1 win | Player 2 win |
| Escape(Player 1 is escaper) | Player 2 win | Player 2 win | Player 1 win |

Clearly, in any game, either Player 1 as a winning strategy, or player 2 has a winning strategy, or it is a draw. Here N-position is when the next player wins in a normal game (next player reaches a leaf), and P is when the previous player wins in a normal game. We discussed in class that there are more than three combinations of rules, or more precisely 27 combinations of rules. We could have a game where player 1 wins in all three scenario but it will not be very interesting. Some other combinations can be understood buy analyzing these three games, for example, having Player 1 win in N and Draw games and Player 2 win in P games can be understood by using the normal games: the probability of player 1 win in this case is the sum of probability of Player 1 wins and the probability of a draw in the normal game.

The paper also discussed several examples. When the offspring distribution be Poisson with mean $\lambda$, The normal and misere games have continuous phase transitions at $\lambda = \exp$ and $\lambda = 2.103$, and The escape game has a discontinuous phase transition at $\lambda = 3.319$.

## 4 Further Analysis

We attempted to analysis the uniqueness of the distribution function $F_w(x)$ in equation (5) by observing its Taylor series expansion around 0. Consider the following equation:

$$F(x\zeta) = f(F(x))$$

where $f(x) = (1-(1-x)^d)^d$ is the function for d-regular tree. Taking the derivative of both side,

$$F'(x\zeta)\zeta = f'(F(x))F'(x)$$

When x = 0,

$$F'(0)\zeta = f'(F(0))F'(0) = f'(q)F'(0)$$

After taking more derivatives of the function, we learned that the coefficient of the derivative of the composition function can be solved by Faa di Bruno's fomular recursively. For example,

$$(f \circ g)''' = f'''(g(x))g'(x)^3 + 3g'(x)g''(x)f''(g(x)) + g'''(x)f'(g(x))$$

The second term correspond to the partition of 1 + 2 = 3, because there are one factor of $g''(x)$ and one factor of $g'(x)$. The coefficient of the second term is 3, because there are exactly three partition of a set with three elements that break it into one part of size 2 and one parts of size 1. This term has the factor $f''(g(x)$ because 3 is partitioned into two parts.

In order to compute the coefficient, we found out that number of partitions of a set of size n corresponding to the integer partition can be expressed in closed form:

$$\frac{n!}{m_1!m_2!m_3!...1!^{m_1}2!^{m_2}3!^{m_3}}$$

where $m_i$ is the number of $i$ in the partion of n, meaning $n = \sum m_i * i$. We then implement the following function in python in order to solve further derivatives quickly.

```
def partition(number):
        answer = set()
        answer.add((number, ))
        for x in range(1, number):
            for y in partition(number - x):
                answer.add(tuple(sorted((x, ) + y)))
        return list(answer)

def partition_number(number):
    return len(partition(number))

def faa_di_bruno_coeff(order):
    n = sum(order)
    count = np.zeros(n+1)
    for i in range(n+1):
        count[i] = order.count(i)
    answer = math.factorial(n)
    for i in range(n+1):
        answer = answer/(math.factorial(count[i]) * math.factorial(i)**(count[i]))
    return int(answer)

def taylor_Fx(derivatives, k):
    answer = var('q')
    for i in range(k):
```

```
        answer = answer + var('x')**(i+1)*derivatives[i]/(math.factorial(i+1))
    return answer

def taylor_Fxoverf1(derivatives, k):
    answer = var('q')
    for i in range(k):
        answer = answer + var('x')**(i+1)*derivatives[i]/(math.factorial(i+1)*f1**(i+1))
    return answer

def fx(x,d):
    return 1-(1-x**d)**d

d = 6
derivatives_F = [var('f1')*var('F1')]
derivatives_F_0 = [1]

for i in range(2,d):
    terms = 0
    orders = partition(i)
    for order in orders:
        if len(order) is not 1:
            term = 1
            term = term * var('f'+str(len(order)))
            for power in order:
                #term = term * var('F'+str(power))
                term = term * derivatives_F_0[power-1]
            term = term * faa_di_bruno_coeff(order)
            terms = terms + term
    terms = terms / (var('f1')**i-var('f1'))
    derivatives_F_0.append(terms)
```

This function is able to output the Taylor series coefficient for $f(F(x))$, but unfortunately we were not able to simplify the expression much further to have more meaningful result.

# 5 Remaining Questions

Due to limited course time, there are some questions that we did not understand or wish to work on further.

1. What can we say about the uniqueness of $F_w(x)$ mentioned before? Could we further simplify the Taylor series, maybe using numeral derivatives, to have a nicer expression of $F$?
2. What can we say about the form of $f$ and its fixed points in the interval [0,1]? Can $f$ have arbitrarily many fixed points or infinite many fixed points?
3. What about other transformation of the game, for example, when there is positive probability of choosing second biggest value instead of the maximum value from children nodes? What could we say about the root distribution, convergence and the question of endogeny?

# References

[1] J. B. Martin A. E. Holroyd. Galton-watson games. 2018.

[2] R. Stasinski J. B. Martin. Minimax functions on galton-watson trees. 2018.

[3] J. Pearl. Asymptotic properties of minimax trees and game-searching procedures. 1980.

[4] L. Devroye T. Ali Khan and R. Neininger. A limit law for the root value of minimax trees. 2005.