Домашнє завдання № 2

1.3.21. $(2x-3y+1)dx+(9x+y-1)dy=0$

Робимо заміну $(x,y)\to(\xi,\eta)$

$$\begin{cases} x=\xi+x_0 & dx=d\xi \\ y=\eta+y_0 & dy=d\eta \end{cases}$$

$(2(\xi+x_0)-3(\eta+y_0)+1)d\xi+(9(\xi+x_0)+(\eta+y_0)-1)d\eta=0$

$(2\xi-3\eta+2x_0-3y_0+1)d\xi+(9\xi+\eta+9x_0+y_0-1)d\eta=0$

Знайдемо x_0, y_0:

$$\begin{cases} 2x_0-3y_0+1=0 \\ 9x_0+y_0-1=0 \mid \cdot 3 \end{cases} \quad \begin{cases} 2x_0-3y_0=-1 \\ 27x_0+3y_0=3 \end{cases}$$

$x_0=\frac{2}{29} \quad y_0=\frac{11}{29}$

Робимо заміну $(x,y)\to(\xi,\eta)$

$$\begin{cases} x=\xi+\frac{2}{29} & dx=d\xi \\ y=\eta+\frac{11}{29} & dy=d\eta \end{cases}$$

$(2\xi-3\eta+\cancel{\frac{4}{29}}-\cancel{\frac{33}{29}}+\cancel{1})d\xi+(9\xi+\eta+\cancel{\frac{18}{29}}+\cancel{\frac{11}{29}}-\cancel{1})d\eta=0$

$(2\xi-3\eta)d\xi+(9\xi+\eta)d\eta=0$ - однор. р-ня

$\frac{\eta}{\xi}=z(\xi) \quad \eta=\xi\cdot z \quad d\eta=zd\xi+\xi dz$

$(2\xi-3\xi z)d\xi+(9\xi+\xi z)(zd\xi+\xi dz)=0 \mid :\xi$

$(2-3z)d\xi+(9+z)(zd\xi+\xi dz)=0$

$(2-3z)d\xi+(9+z)zd\xi+(9+z)\xi dz=0$

$(z^2+6z+2)d\xi+(9+z)\xi dz=0 \mid :(z^2+6z+2) \mid :\xi$

$\int\frac{d\xi}{\xi}+\int\frac{9+z}{z^2+6z+2}=0$

$$\ln|\xi| + \frac{1}{2}\ln|z^2+6z+2| + \frac{3\sqrt{7}}{7}\ln\left|\frac{z+3-\sqrt{7}}{z+\sqrt{7}+3}\right| = C$$

$$\ln\left|x+\frac{2}{28}\right| + \frac{1}{2}\ln\left|\left(\frac{y-\frac{11}{29}}{x-\frac{2}{29}}\right)^2 + 6\left(\frac{y-\frac{11}{29}}{x-\frac{2}{29}}\right)+2\right| +$$

$$+\frac{3\sqrt{7}}{7}\ln\left|\frac{x+3-\sqrt{7}}{x+\sqrt{7}-3}\right| + C$$

Перевірка:

$$\xi = 0 \longrightarrow x - \frac{2}{29} = 0$$

$$\left(2\cdot\frac{2}{29} - 3y + 1\right)\cdot 0 + \left(9\,\frac{2}{29} + y - 1\right)dy = 0$$

не тотоже

$$z^2 + 6z + 2 = 0$$

$$z_1 = \frac{-6+\sqrt{28}}{2};\quad z_2 = \frac{-6-\sqrt{28}}{2}$$

1.3.22. $2(x\sqrt{y} + 2y)dx - x\,dy = 0$

$$\frac{dy}{dx} = 2\sqrt{y} + \frac{4y}{x};$$

$$\sqrt{y} = z \qquad \frac{dy}{2\sqrt{y}} = dz$$

$$dy = 2z\,dz$$

$$\frac{2z\,dz}{dx} = 2z + \frac{4z^2}{x} \quad |:2z$$

$$\frac{dz}{dx} = \frac{4z^2}{x} + 1$$

Заміна: $\frac{z}{x} = u,\ z = ux$

$$dz = x\,du + u\,dx$$

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$$\frac{x\,du + u\,dx}{dx} = 1 + 2u$$

$$\frac{x\,du}{dx} + u = 1 + 2u$$

$$\frac{x\,du}{dx} = 1 + u$$

$$\int \frac{du}{1+u} = \int \frac{dx}{x}$$

$$\ln|1+u| = \ln|x| + C$$

$$\ln\left|1 + \frac{z}{x}\right| = \ln|x| + C$$

$$\ln\left|1 + \frac{\sqrt{y}}{x}\right| = \ln|x| + C$$

Перевірка:

$y = -x$

$2x^2dx - 2x^2dx = 0$

$0 = 0$

$x = 0$ - розв.

$y = 0$ - розв.

1.3.23. $y(x^2y^2+1)dx + (x^2y^2-1)x\,dy = 0$

Заміна: $x^2y^2 - 1 = u$

$$y = \frac{\sqrt{u+1}}{x}$$

$$\frac{dy}{dx} = \frac{-\sqrt{u+1}}{x^2} + \frac{\frac{du}{dx}}{2x\sqrt{u+1}}$$

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$$\frac{\sqrt{u+1}\,(u+2)}{x} - \frac{u\sqrt{u+1}}{x} + \frac{u\,du}{\sqrt{u+1}\,dx} = 0$$

$$\frac{\sqrt{u+1}}{x}(u+2-u) + \frac{u\,du}{\sqrt{u+1}\,dx} = 0$$

$$\frac{2\sqrt{u+1}}{x} = -\frac{u\,du}{\sqrt{u+1}\,dx}$$

$$\int\frac{2dx}{x} = -\int\frac{u\,du}{u+1}$$

$$2\ln|x| = -x^2y^2 - 1 + \ln|x^2+y^2| + C$$

Перевірка:

$x = 0$:

$y(0+1)\cdot 0 + 0\cdot dy = 0$

$0 = 0$ - розв.

$\sqrt{u+1} = 0$ не розв.

1.3.24. $y' = \frac{2x-y-4}{2y-x+5}$

Заміна: $(x,y) \rightsquigarrow (\xi,\eta)$

$$\begin{cases} x = \xi + x_0 & dx = d\xi \\ y = \eta + y_0 & dy = d\eta \end{cases}$$

$$y' = \frac{dy}{dx} = \frac{d\eta}{d\xi}$$

$$\frac{d\eta}{d\xi} = \frac{2(\xi+x_0)-(\eta+y_0)-4}{2(\eta+y_0)-(\xi+x_0)+5}$$

$$\frac{d\eta}{d\xi} = \frac{2\xi-\eta+2x_0-y_0-4}{2\eta-\xi+2y_0-x_0+5}$$

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Знайдемо x_0, y_0:

$$\begin{cases} 2x_0 - y_0 - 4 = 0 & \cdot 2 \\ -x_0 + 2y_0 + 5 = 0 \end{cases} \qquad \begin{cases} 4x_0 - 2y_0 = 8 \\ -x_0 + 2y_0 = -5 \end{cases}$$

$$x_0 = 1$$
$$y_0 = -2$$

Заміна: $(x, y) \rightsquigarrow (\xi, \eta)$

$$\begin{cases} x = \xi + 1 \\ y = \eta - 2 \end{cases} \qquad \begin{matrix} dx = d\xi \\ dy = d\eta \end{matrix}$$

$$\frac{d\eta}{d\xi} = \frac{2\xi - \eta}{2\eta - \xi}$$

$$\frac{\eta}{\xi} = z(\xi), \quad \eta = \xi z, \quad d\eta = z\,d\xi + \xi\,dz$$

$$\frac{z\,d\xi + \xi\,dz}{d\xi} = \frac{2\xi - \xi z}{2\xi z - \xi} \quad | : \xi$$

$$\frac{z\,d\xi + \xi\,dz}{d\xi} = \frac{2 - z}{2z - 1}$$

$$z + \xi\frac{dz}{d\xi} = \frac{2 - z}{2z - 1}$$

$$\xi\frac{dz}{d\xi} = + \frac{2 - 2z^2}{2z - 1} = + \frac{2 - 2z^2}{2z - 1}$$

$$\int + \frac{2z - 1}{2 - 2z^2}\,dz = \int \frac{d\xi}{\xi}$$

$$-\frac{1}{2}\ln\left|1 - \left(\frac{y+2}{x-1}\right)^2\right| + \frac{1}{4}\ln\left|\frac{\frac{y+2}{x-1} - 1}{\frac{y+2}{x-1} + 1}\right| = \ln|x - 1| + C$$

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Перевірка:

$$\frac{\cdots}{2z-1}=0 \qquad z_1 = 1 \quad z_2 = -1$$

1.3.25. $x^3(y'-x)=y^2$

$$x^3y'-x^4=y^2$$

$$y'=\frac{y^2+x^4}{x^3}$$

$$y=z^{\alpha},\quad y'=z'\alpha\cdot z^{\alpha-1}$$

$$z'\alpha\cdot z^{\alpha-1}=\frac{z^{2\alpha}+x^4}{x^3}\ \Big|:\alpha z^{\alpha-1}$$

$$z'=\frac{z^{2\alpha}}{x^3\cdot\alpha z^{\alpha-1}}+\frac{x}{\alpha\cdot z^{\alpha-1}}$$

$$\begin{cases}2\alpha-(3+\alpha-1)=0\\1-(\alpha-1)=0\end{cases}\quad\begin{cases}2\alpha-3-\alpha+1=0\\1-\alpha+1=0\end{cases}\quad\begin{cases}\alpha=2\\\alpha=2\end{cases}$$

$$y=z^2,\quad y'=z'\cdot2\cdot z$$

$$2z'z=\frac{z^4}{x^3}+x\ \Big|:2z$$

$$z'=\frac{z^3}{2x^3}+\frac{x}{2z}$$

$$z=xu,\quad z'=xu'+u$$

$$xu'+u=\frac{\cancel{x^3}\cdot u^3}{2\cancel{x^3}}+\frac{\cancel{x}}{2\cancel{x}u}$$

$$xu'+u=\frac{u^3}{2}+\frac{1}{2u};\quad xu'=\frac{u^3}{2}+\frac{1}{2u}-u=\frac{u^4+1-2u^2}{2u}$$

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$$x\frac{du}{dx} = \frac{u^4-2u^2+1}{2u} \quad \Big| \cdot dx \quad \Big| : x$$

$$du = \frac{u^4-2u^2+1}{2u\,x}\,dx \quad \Big| \cdot 2u \quad \Big| \, u^4-2u^2+1$$

$$\int \frac{2u}{u^4-2u^2+1}\,du = \int \frac{dx}{x}$$

$$-\frac{1}{u^2-1} = \ln|x| + C$$

$$-\frac{1}{\left(\frac{2}{x}\right)^2-1} = \ln|x| + C$$