Домашнє завдання 7
студентки групи ПМО-21
Кравець Ольги

№1148.

$$\frac{dx}{y+z} = \frac{dy}{x+z} = \frac{dz}{x+y}$$

1. $$\frac{dx-dy}{y+z-x-z} = \frac{dy-dz}{x+z-x-y}$$

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

$$\ln|x-y| = \ln|y-z| + \ln C_1$$

$$x-y = C_1(y-z)$$

2. $$\frac{dx+dy+dz}{2(x+y+z)} = -\frac{dx-dy}{x-y}$$

$$\frac{d(x+y+z)}{2(x+y+z)} = -\frac{d(x-y)}{x-y}$$

$$\frac{1}{2}\ln|x+y+z| = -\ln|x-y| + \ln C_2$$

$$(x+y+z)(x-y)^2 = C_2$$

В: $\boxed{x-y = C_1(y-z),\ (x+y+z)(x-y)^2 = C_2}$

№ 1152.

$$\frac{dx}{z} = \frac{dy}{xz} = \frac{dz}{y}$$

1. $\frac{dx}{z} = \frac{dy}{xz} \;/\cdot z$

$$\frac{dx}{1} = \frac{dy}{x} \Rightarrow \frac{x^2}{2} = y + C_1 \Rightarrow x^2 - 2y = C_1$$

$$y = \frac{x^2 - C_1}{2}$$

2. $\frac{dx}{z} = \frac{2dz}{x^2 - C_1}$

$$(x^2 - C_1)\,dx = 2z\,dz$$

$$\frac{x^3}{3} - \underbrace{C_1}_{x^2-2y} x = z^2 + C_2$$

$$\frac{x^3}{3} - x^3 + 2xy - z^2 = C_2$$

$$-\frac{2}{3}x^3 + 2xy - z^2 = C_2 \;/\cdot 3$$

$$-2x^3 + 6xy - 3z^2 = C_2$$

В: $\boxed{\begin{array}{l} x^2 - 2y = C_1 \\ -2x^3 - 3z^2 + 6xy = C_2 \end{array}}$

№ 1155.

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy\sqrt{z^2+1}}$$

1. $\frac{dx}{xz} = \frac{dy}{yz} \Rightarrow \frac{dx}{x} = \frac{dy}{y} \Rightarrow x = C_1 y$

2. $\frac{dy}{\cancel{y}z} = \frac{dz}{C_1 y^2 \sqrt{z^2+1}}$; $C_1 y dy = \frac{z dz}{\sqrt{z^2+1}}$

$$C_1 \cdot \frac{y^2}{2} = \frac{1}{2} \cdot \sqrt{z^2+1} \cdot 2 + C_2$$

$$\frac{C_1 y^2}{2} = \sqrt{z^2+1} + C_2$$

$$xy = 2\sqrt{z^2+1} + 2C_2$$

$$\frac{x}{y} = C_1$$

$$\frac{xy}{2} - \sqrt{z^2+1} = C_2$$

В: $\boxed{\frac{x}{y} = C_1, \quad \frac{xy}{2} - \sqrt{z^2+1} = C_2}$

№1156.

$$\frac{dx}{x+y^2+z^2} = \frac{dy}{y} = \frac{dz}{z}$$

1. $\frac{dy}{y} = \frac{dz}{z} \Rightarrow c_1 = \frac{y}{z}$; $y = c_1 z$

2. $\frac{dx}{x+(c_1^2+1)z^2} = \frac{dz}{z}$

$$z\,dx = x + (c_1^2+1)z^2\,dz$$

$$\frac{dx}{dz} = \frac{x+(c_1^2+1)z^2}{z} = \frac{x}{z} + (c_1^2+1)z$$

$$x = (c_1^2+1)z^2 + c_2 z = \left(\frac{y^2}{z^2}+1\right)z^2 + c_2 z = y^2 + z^2 + c_2 z$$

$$c_2 = \frac{x}{z} - \frac{y^2}{z} - z$$

B: $\boxed{\frac{y}{z} = c_1, \quad \frac{x}{z} - \frac{y^2}{z} - z = c_2}$

№1158.

$$-\frac{dx}{x^2} = \frac{dy}{xy-2z^2} = \frac{dz}{xz}$$

1. $-\frac{dx}{x^2} = \frac{dy}{xy-2z^2}$

$$-xy\,dx + 2z^2dx = x^2dy$$
$$x^2dy + xy\,dx = 2z^2dx$$
$$x(x\,dy + y\,dx) = 2z^2dx$$
$$x\,d(xy) = 2z^2dx$$

2. $-\frac{dx}{x^2} = \frac{dz}{xz}$

$$d(xy) = 2z^2\frac{dx}{x}$$

$$d(x\cdot y) = 2z^2x\left(-\frac{dz}{z}\right) = -2z\,dz$$
$$xy + z^2 = C_1$$

$$\ln|x| + \ln|z| = \ln C_2 \Rightarrow xz = C_2$$

В: $\boxed{xy + z^2 = C_1, \quad xz = C_2}$

№1159.

$$\frac{dx}{x(z-y)} = \frac{dy}{y(y-x)} = \frac{dz}{y^2-xz}$$

1. $$\frac{dx+dz}{x(z-y)+y^2-xz} = \frac{dy}{y(y-x)}$$

$$\frac{d(x+z)}{x(z-y)+y^2-xz} = \frac{dy}{y(y-x)}$$

$$x(z-y)+y^2-xz\,dy = y(y-x)\cdot d(x+z)$$

$$d(x-y+z)=0$$

$$x-y+z=C_1 \Rightarrow z=C_1-x+y$$

2. $$\frac{dx}{x(C_1-x)} = \frac{dy}{y(y-x)}$$

$$dy + \frac{y}{C_1-x} = \frac{y^2}{x(C_1-x)}$$

$$y = \frac{x-C_1}{\ln|x|+C_2} \Rightarrow y(\ln|x|+C_2) = x-C_1 \Rightarrow$$

$$\Rightarrow y\ln x + yC_2 = x-C_1 \Rightarrow$$

$$\Rightarrow yC_2 = x-C_1-y\ln x$$

$$\Rightarrow C_2 = \frac{x-C_1-y\ln x}{y}$$

$$C_2 = 1-\frac{z}{y}-\ln|x|$$

В: $\boxed{x-y+z=C_1, \quad 1-\frac{z}{y}-\ln|x| = C_2}$

№1162.

$$\begin{cases} \dot{x} = xy \\ \dot{y} = x^2+y^2 \end{cases} \Rightarrow \begin{cases} \frac{dx}{dt} = xy \\ \frac{dy}{dt} = x^2+y^2 \end{cases}$$

$$\begin{cases} \varphi_1 = x\ln y - x^2y \\ \varphi_2 = \frac{y^2}{x^2} - 2\ln x \end{cases}; \quad ① \ \frac{d}{dt}(x\ln y - x^2y) = 0 \Rightarrow$$

$$\Rightarrow \varphi_1 = x\ln y + \frac{x}{y}\cdot y - 2xy\,x - x^2y = xy\ln y + \frac{x}{y}(x^2+y^2) - 2(xy)^2 -$$

$$-x^2(x^2+y^2) = xy\ln y + \frac{x^3}{y} + xy - 2x^2y^2 - x^4 - x^2y^2 \neq 0$$

$$② \ \frac{d}{dt}\left(\frac{y^2}{x^2} - 2\ln x\right) = 0 \Rightarrow$$

$$\Rightarrow \varphi_2 = \frac{2y}{x^2}\cdot y - \frac{2y^2}{x^3}x - 2\cdot\frac{1}{x}x = \frac{2y}{x^2}(x^2+y^2) - \frac{2y^2}{x^3}xy - \frac{2}{x}xy =$$

$$= 2y + \frac{2y^3}{x^2} - \frac{2y^3}{x^2} - 2y = 0$$

В: $\boxed{\varphi_1 \text{ не } \in, \ \varphi_2 \in.}$

№1(a)

$$\begin{cases}\dot{x} = -x \\ \dot{y} = 2y\end{cases} \qquad A = \begin{pmatrix}-1 & 0 \\ 0 & 2\end{pmatrix}$$

$$\begin{vmatrix}-1-\lambda & 0 \\ 0 & 2-\lambda\end{vmatrix} = \lambda^2 - \lambda - 2 = 0$$

$\lambda_1 = 2$

$\lambda_2 = -1$

Розв.: $\begin{cases}x(t) = c_1 e^{-t} \\ y(t) = c_2 e^{2t}\end{cases}$

$$\frac{dx}{dt} = -x \rightsquigarrow dt = \frac{dx}{-x}$$

$$\frac{dy}{dt} = 2y \rightsquigarrow dt = \frac{dy}{2y}$$

$$\rightsquigarrow \frac{dy}{2y} = \frac{dx}{-x}, \quad \ln|2y| = \ln|x| + \ln|c|$$

$$2y = -cx; \quad y = -\frac{cx}{2}; \quad \frac{-2y}{x} = c$$

перший інтеграл

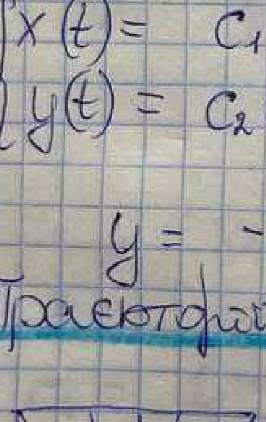
$$\lambda_1 = 2: \begin{pmatrix}-3 & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = 0 \Rightarrow \begin{matrix}\alpha = 0 \\ \beta = 1\end{matrix}$$

$$\lambda_2 = -1: \begin{pmatrix}0 & 0 \\ 0 & 3\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = 0 \Rightarrow \begin{matrix}\alpha = 1 \\ \beta = 0\end{matrix}$$

$$\begin{cases}x(t) = c_1 e^{-t} \\ y(t) = c_2 e^{2t}\end{cases}$$

$y = -\frac{cx}{2}$ – криві рівняння першого інт.

Траєкторії:



(e^{-t}, e^{2t}) – праміжь

№1 (б)

$$\begin{cases} \dot{x} = x - 2y \\ \dot{y} = 4x - 3y \end{cases} \qquad A = \begin{pmatrix} 1 & -2 \\ 4 & -3 \end{pmatrix}, \quad \lambda^2 + 2\lambda + 5 = 0$$

$D = -16 < 0$

$\lambda_1 = -1 - 2i$

$\lambda_2 = -1 + 2i$

Власні знач: $\lambda_1 = -1 - 2i$, $\lambda_2 = \overline{\lambda_1} = -1 + 2i$

Власні вектори:

$\lambda_1 = -1 - 2i$: $\begin{pmatrix} 2-2i & -2 \\ 4 & -2i-2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \begin{matrix} \alpha = 1 \\ \beta = 1-i \end{matrix}$ $\quad h_1 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$

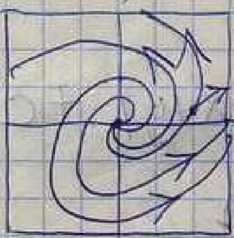
ФСР:

$$\vec{z}(t) = e^{(2i-1)t}\begin{pmatrix} 1 \\ 1-i \end{pmatrix} = e^{2it} \cdot e^{-t}\begin{pmatrix} 1 \\ 1-i \end{pmatrix} = e^{t}\left(\frac{i\sin 2t}{e^t} + \frac{\cos 2t}{e^t}\right)\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

$\vec{z}(t) = u(t) + iv(x)$

Розв. $\begin{cases} x(t) = e^{-t}(c_1 \sin 2t + c_2 \cos 2t) \\ y(t) = e^{-t}((c_1 + c_2)\sin 2t + (c_2 - c_1)\cos 2t) \end{cases}$

Траєкторії:



$$\frac{\dot{y}}{\dot{x}} = \frac{dy/dt}{dx/dt} = \frac{dy}{dx} = y'$$

$y' = \frac{4x-3y}{x-2y}$ — диф. р-ня, $v = \frac{y}{x}$, $y' = xv' + v$

$y = xv$; $x = \frac{v}{y}$

$$v\,dx + x\,dv = \frac{(4-3v)\,dx}{1-2v}$$

$$x\,dv = \left(\frac{4-3v}{1-2v} - v\right)dx; \quad \int \frac{1}{2(v^2-2v+2)} - \frac{v}{v^2-2v+2}\,dv = \int \frac{1}{x}\,dx$$

$$-\frac{\ln(v^2+2v+2)}{2} - \frac{\operatorname{arctg}(v-1)}{2} = \ln(x) + C$$

$$-\frac{\ln\left(\frac{y^2}{x^2} - \frac{2y}{x} + 2\right)}{2} - \frac{\operatorname{arctg}\left(\frac{y}{x} - 1\right)}{2} = C$$

$$\frac{\ln(y^2 - 2xy + 2x^2)}{2} - \frac{\operatorname{arctg}\left(\frac{y-x}{x}\right)}{2} = C$$ — перший інтеграл

№1 (в)

$$\begin{cases} \dot{x} = x-y \\ \dot{y} = -x+y \end{cases} \quad A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad (\lambda-2)\lambda = 0$$

$$\lambda_1 = 2,\ \lambda_2 = 0$$

Власні знач.: $\lambda_1 = 2,\ \lambda_2 = 0$

$\lambda_1 = 2$: $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \Rightarrow \begin{matrix} \alpha = -1 \\ \beta = 1 \end{matrix} \Rightarrow h_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda_2 = 0$: $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \Rightarrow \begin{matrix} \alpha = 1 \\ \beta = 1 \end{matrix} \Rightarrow h_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Відв.: $\begin{cases} x(t) = -c_1 e^{2t} + c_2 \\ y(t) = c_1 e^{2t} + c_2 \end{cases}$

$\frac{dx}{dt} = x-y \rightsquigarrow dt = \frac{dx}{x-y}$

$\frac{dy}{dt} = -x+y \rightsquigarrow dt = \frac{dy}{-x+y}$

$\rightsquigarrow \frac{dx}{x-y} = \frac{dy}{-x+y} \Rightarrow$

$$\Rightarrow \ln|x-y| = -\ln|-x+y| + c$$

$$x-y = -c(-x+y)$$

$$-c(-x+y) = x-y \Rightarrow -c = \frac{x-y}{-x+y} \Rightarrow c = \frac{-x+y}{x-y} \text{ — перший інтеграл}$$

$$\begin{cases} \dot{x} = x-y \\ \dot{y} = -x+y \end{cases} \rightsquigarrow \begin{cases} x(t) = -c_1 e^{2t} + c_2 \\ y(t) = c_1 e^{2t} + c_2 \end{cases} \rightsquigarrow U(x,y) = \frac{-x+y}{x-y}$$

перший інт.

$$y = cx - cy + x$$

$y = c(x-y) + x$ — рівн. першого інтеграла

Траєкторії: