Домашнє завдання 4.
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№421.

$x^2 \cdot y'' = y'^2$

Заміна: $u(x) = y'(x)$

$u'(x) = y''(x)$

$x^2 \cdot u' = u^2$

$x^2 \cdot \frac{du}{dx} = u^2 \Big/ \frac{dx}{u^2 \cdot x^2}$

$\frac{du}{u^2} = \frac{dx}{x^2}$

$\int \frac{du}{u^2} = \int \frac{dx}{x^2}$

$-\frac{1}{u} = -\frac{1}{x} + C_1$

$-\frac{1}{u} = \frac{-1 + xC_1}{x}$

$y'(x) = u = \frac{x}{1 + C_1 x}$

$dy = \frac{1}{C_1}\left(\cancel{\frac{1 + C_1 x}{1 + C_1 x}} dx - \frac{1}{1 + C_1 x} dx \right)$

$y = \frac{x}{C_1} - \frac{\ln|1 + C_1 x|}{C_1^2} + C_2$

В: $\boxed{y = \frac{xC_1 - \ln|1 + C_1 x| + C_1^2 \cdot C_2}{C_1^2}}$

№123.

$$y^3 y'' = 1$$

Структура рівняння: $F(y, y', y'') = 0$

Заміна: $y' = v(y)$

$$y'' = v' y' = v v'$$

$$y^3 \cdot v v' = 1 \quad \Big| \cdot \frac{1}{y^3}$$

$$v v' = \frac{1}{y^3}$$

$$v \cdot \frac{dv}{dy} = \frac{1}{y^3} \quad \Big| \cdot dy$$

$$\int v\, dv = \int \frac{dy}{y^3}$$

$$\frac{v^2}{2} = C_1 - \frac{1}{2y^2}$$

$$v^2 = C_1 - \frac{1}{y^2}$$

$$v = \pm\sqrt{C_1 - \frac{1}{y^2}}$$

$$y' = \pm\sqrt{C_1 - \frac{1}{y^2}}$$

$$\frac{dy}{dx} = \pm\sqrt{C_1 - \frac{1}{y^2}} \quad \Big| \cdot \frac{dx}{\pm\sqrt{C_1 - \frac{1}{y^2}}}$$

$$\pm\frac{dy}{\sqrt{C_1 - \frac{1}{y^2}}} = dx$$

$$\boxed{\pm\frac{dy}{\sqrt{\frac{C_1 y^2 - 1}{y^2}}} = dx}$$

① $C_1 = 0$

$$\int \frac{dy}{\sqrt{\frac{1}{y^2}}} = \int dx; \quad \frac{y^2}{2} + C_2 = x$$

$$\frac{y^2 + 2C_2}{2} = x$$

$$y^2 + 2C_2 = 2x; \quad y^2 = 2x - 2C_2; \quad \boxed{y = \pm\sqrt{2x - 2C_2}}$$

② $C_1 > 0$, $C_1 = A_1^2$

$$\int \frac{dy}{\sqrt{\frac{A_1 y^2 - 1}{y^2}}} = \int dx$$

$$\frac{1}{A_1}\sqrt{A_1 y^2 - 1} = x + A_2$$

$$\sqrt{A_1 y^2 - 1} = A_1(x + A_2)$$

$$A_1 y^2 - 1 = \left(A_1(x + A_2)\right)^2$$

$$A_1 y^2 = (A_1 x + A_1 A_2)^2 + 1$$

$$y^2 = \frac{(A_1 x + A_1 A_2)^2 + 1}{A_1}$$

$$y = \pm\sqrt{\frac{(A_1 x + A_1 A_2)^2 + 1}{A_1}}$$

$$\boxed{y = \pm\sqrt{\frac{A_1^2 x^2 + 2x A_1^2 A_2 + A_1^2 A_2^2 + 1}{A_1}}}$$

③ $C_1 < 0$, $C_1 = -A_1^2$

$$\int \frac{dy}{\sqrt{\frac{A_1 y^2 - 1}{y^2}}} = \int dx$$

$$-\frac{\sqrt{-A_1 y^2 - 1}}{x} = x + A_2$$

$$-\sqrt{-A_1 y^2 - 1} = x^2 + x A_2$$

$$A_1 y^2 - 1 = (x^2 + x A_2)^2$$

$$A_1 y^2 = (x^2 + x A_2)^2 + 1$$

$$y^2 = \frac{(x^2 + x A_2)^2 + 1}{A_1}$$

$$\boxed{y = \pm\sqrt{\frac{(x^2 + x A_2)^2 + 1}{A_1}}}$$

№426.

$yy''+1=y'^2$

Структура р-ня $F(y, y', y'')=0$

<u>Замена:</u> $y'=v(y)$

$y''=v'y'=vv'$

$yvv'+1=v^2$

$yv\cdot\frac{dv}{dy}=v^2-1$

$\int\frac{v\,dv}{v^2-1}=\int\frac{dy}{y}$

$\frac{1}{2}\int\frac{d(v^2-1)}{v^2-1}=\frac{1}{2}\ln|v^2-1|+C_1$

$\frac{1}{2}\ln|v^2-1|=\ln y+\ln c_1$

$\frac{1}{2}\ln|v^2-1|=\ln y\cdot c_1$

$v^2-1=yc_1$

$v=\pm\sqrt{cy^2+1}$

$\frac{dy}{dx}=\pm\sqrt{cy^2+1}$

$\pm\frac{dy}{\sqrt{cy^2+1}}=dx$

① $C_1=0$

$\pm\int dy=\int dx$; $y=x+C_2$

$\boxed{y=x\pm C_2}$

② $C_1<0$, $C_1=-A_1^2$

$\pm\int\frac{dy}{\sqrt{c y^2+1}}=\pm\frac{1}{C}\arcsin Cy$

$\pm\frac{1}{C}\arcsin cy=x+A_1$

$cy=\pm\sin(cy+cA_1)$

$\boxed{y=\pm\frac{\sin(cy+cA_1)}{c}}$

③ $C_1>0$, $C_1=A_1^2$

$\pm\int\frac{dy}{\sqrt{c^2y^2+1}}=\pm\frac{1}{c}\operatorname{arsh}\frac{y}{c}$

$\pm\frac{1}{c}\operatorname{arcsh} cy=x+A_1$

$cy=\pm\operatorname{sh}(cy+cA_1)$

$\boxed{y=\pm\frac{\operatorname{sh}(cy+cA_1)}{c}}$

№427.

$$y''(e^x+1)+y'=0$$

<u>Замена:</u> $y'(x)=u(x)$

$u'(x)=y''(x)$

$$u'(e^x+1)+u=0$$

$$\frac{du}{dx}(e^x+1)=-u \Big/ \cdot\frac{dx}{u(e^x+1)}$$

$$\int\frac{du}{u}=-\int\frac{dx}{e^x+1}=\int\frac{de^x}{e^x+1}-\int\frac{de^x}{e^x}=\ln(e^x+1)-x$$

$$\ln u+\ln c_1=\ln(e^x+1)-x$$

$$\ln(u\cdot c_1)=\ln(e^x+1)-x$$

$$u=c_1\cdot\frac{e^x+1}{e^x}$$

$$y'=c_1\cdot\frac{e^x+1}{e^x}$$

$$y=\int c_1\cdot\frac{e^x+1}{e^x}dx=$$

$$=c_1\int(1+e^{-x})dx=$$

$$=c_1(x-e^{-x})+c_2$$

$$\boxed{y=c_1(x-e^{-x})+c_2}$$

№436.

$$y''^2=y'^2+1$$

Структура ф-ия $F(y,y',y'')=0$

<u>Замена:</u> $y'=v(y)$

$$y''=v'y'=vv'$$

$$(vv')^2=v^2+1$$

$$vv'=\pm\sqrt{v^2+1}$$

$$\frac{dv}{dy}=\pm\frac{\sqrt{v^2+1}}{v}$$

$$\pm\int dy=\int\frac{v\,dv}{\sqrt{v^2+1}}=\sqrt{v^2+1}+C_1$$

$$y=\pm\sqrt{v^2+1}+C_1$$

$$y+C_1=\pm\sqrt{v^2+1}$$

$$(y+C_1)^2=v^2+1$$

$$v^2=(y+C_1)^2-1$$

$$v=\sqrt{(y+C_1)^2-1}$$

$$\frac{dy}{dx}=\pm\sqrt{(y+C_1)^2-1}$$

$$dx=\frac{\pm dy}{\sqrt{(y+C_1)^2-1}}$$

① $C_1=0$

$$x=\pm\ln\left|y+C_1+\sqrt{y^2+2C_1y+C_1^2-1}\right|+C_2$$

$$x=\pm\ln\left|y+C_1+\sqrt{(y+C_1)^2-1}\right|+C_2$$

$$\int dx=\int\pm\frac{dy}{y^2-1}=\ln\left(tg\left(\frac{asec(y)}{2}\right)\right)+C_2$$

$$\boxed{x=\ln\left(tg\left(\frac{asec(y)}{2}\right)\right)+C_2}$$

② $C_1>0$, $C_1=A_1^2$

$$x=\int\frac{\pm dy}{\sqrt{(y+A_1)^2-1}}$$

$$\boxed{x=\ln\left|y+A_1+\sqrt{y^2+2A_1y+A_1^2-1}\right|+C_2}$$

③ $C_1<0$, $C_1=-A_1^2$

$$x=\int\frac{\pm dy}{\sqrt{(y-A_1)^2-1}}$$

$$\boxed{x=\ln\left|y-A_1+\sqrt{y^2-2A_1y+A^2-1}\right|+C_2}$$

№464.

$yy'' = y'^2 + 15y^2\sqrt{x}$

$y' = yz$; $z = \frac{y'}{y}$

$y'' = y(z' + z^2) = yz' + yz^2$

$y \cdot y(z^2 + z') = (yz)^2 + 15y^2\sqrt{x}$

$y^2(z^2 + z') = y^2z^2 + 15y^2\sqrt{x}$

$\cancel{y^2z^2} + y^2z' = \cancel{y^2z^2} + 15y^2\sqrt{x}$

$\underline{y=0}$ – $\underline{\text{розв'язок}}$

$\cancel{y^2}z' = 15\cancel{y^2}\sqrt{x}$

$z' = 15\sqrt{x}$

$\int dz = \int 15\sqrt{x}\,dx$

$z = \frac{\cancel{15}^{5} \cdot 2x^{3/2}}{\cancel{3}} + C_1$

$z = 10x^{3/2} + C_1 \rightsquigarrow \frac{y'}{y} = 10x^{3/2} + C_1 \rightsquigarrow$

$\rightsquigarrow \frac{dy}{y} = (10x^{3/2} + C_1)\,dx$

$\ln y = \cancel{10}^{2} \frac{2x^{5/2}}{\cancel{5}} + C_1x + C_2$

$\ln y = 4x^{\frac{5}{2}} + C_1x + C_2$

$e^{\ln y} = e^{\ln(4x^{5/2} + C_1x + C_2)}$

$y = e^{4x^{5/2} + C_1x + C_2}$

В: $\boxed{\begin{array}{l} y = C_2 e^{4x^{5/2} + C_1x} \ ; \ C_1, C_2 \in \mathbb{R} \\ y = 0 \end{array}}$

№465.

$(x^2+1)(y'^2 - yy'') = xyy'$

$y' = yz$

$z = \frac{y'}{y}$

$y'' = y(z' + z^2)$

$(x^2+1)\left((yz)^2 - y\cdot y(z'+z^2)\right) = xy\cdot yz$

$(x^2+1)\left(\cancel{y^2z^2} - y^2z' - \cancel{y^2z^2}\right) = xy^2z$

$(x^2+1)(-y^2z') = xy^2z$

$-x^2\cancel{y^2}z' - \cancel{y^2}z' = x\cancel{y^2}z \; \left(-\frac{1}{y^2}\right)$; $\underline{y=0}$ - розв.

$x^2z' + z' = -xz$; $z'(x^2+1) = -xz$

$\frac{dz}{dx}(x^2+1) = -xz$

$\frac{dz}{z} = -\frac{xdx}{x^2+1} = -\frac{1}{2}\cdot\frac{dx^2}{x^2+1}$

$\ln z = -\frac{1}{2}\ln|x^2+1| + \ln C_1$

$z = \frac{C_1}{\sqrt{x^2+1}}$

$\frac{y'}{y} = \frac{C_1}{\sqrt{x^2+1}}$

$\frac{dy}{y} = \frac{C_1\,dx}{\sqrt{x^2+1}}$

$\ln y = C_1\cdot\ln|x + \sqrt{x^2+1}| + \ln C_2$

$y = C_2\left(x + \sqrt{x^2+1}\right)^{C_1}$

В: $y = 0$

$y = C_2\left(x + \sqrt{x^2+1}\right)^{C_1}$, $C_1, C_2 \in \mathbb{R}$

№501.

$yy'' = 2xy'^2$

$y(2) = 2$

$y'(2) = 0{,}5$

$y' = yz \; ; \; z = \frac{y'}{y}$

$y'' = y(z' + z^2)$

$y \cdot y(z' + z^2) = 2x(yz)^2$

$y^2(z' + z^2) = 2xy^2z^2$

$y^2z' + y^2z^2 = 2xy^2z^2 \Big/ \cdot \frac{1}{y^2} \; ; \; y = 0$ - розв.

$z' + z^2 = 2xz^2$

$z' = 2xz^2 - z^2$

$z' = z^2(2x-1)$

$\frac{dz}{dx} = z^2(2x-1)\,dx \Big/ \cdot \frac{dx}{z^2}$

$\int \frac{dz}{z^2} = \int (2x-1)\,dx$

$-\frac{1}{z} = x^2 - x + C_1$

$-1 = z(x^2 - x + C_1)$

$\frac{y'}{y} = z = \frac{1}{C_1 + x - x^2}$

$y' = \frac{y}{C_1 + x - x^2}$

$\frac{1}{2} = \frac{2}{C_1 + 2 - 4}$

$4 = C_1 - 2$

$C_1 = 6$

$\frac{dy}{y} = \frac{1}{C_1 + x - x^2}\,dx = \frac{1}{6 + x - x^2}\,dx = -(x+2)(x-3)\,dx = (x+2)(3-x)\,dx$

$\ln y - \ln C_2 = \int \frac{dx}{(x+2)(3-x)} \ominus$

$\frac{A}{x+2} + \frac{B}{3-x} = \frac{1}{(x+2)(3-x)}$

$A = \frac{1}{\cancel{(x+2)}(3-x)}\Big|_{x=-2} = \frac{1}{5}$

$B = \frac{1}{(x+2)\cancel{(3-x)}}\Big|_{x=3} = \frac{1}{5}$

$\ominus \int \frac{1}{5(x+2)} + \frac{1}{5(3-x)}\,dx =$

$= \frac{1}{5}\ln|x+2| - \frac{1}{5}\ln|3-x| + C_2$

$\ln\left(\frac{y}{C_2}\right) = \frac{1}{5}\ln|x+2| - \frac{1}{5}\ln|3-x| \Big/ \cdot 5$

$5\ln\left(\frac{y}{C_2}\right) = \ln|x+2| - \ln|3-x|$

$y = C_2 \sqrt[5]{\frac{x+2}{3-x}}$

$x = 2$

$y = 2$

$2 = C_2 \sqrt[5]{\frac{2+2}{3-2}}$

$2 = C_2 \sqrt[5]{4}$

$C_2 = \frac{2}{\sqrt[5]{4}} = \sqrt[5]{2^3} =$

$= \sqrt[5]{8}$

$C_2 = \sqrt[5]{8}$

В:

$\boxed{y = \sqrt[5]{8\frac{2+x}{3-x}}}$

Математична модель кривої погоні

$$\begin{cases} xy'' = \frac{1}{a}\sqrt{1+y'^2} \\ y(H)=0, \quad y'(H)=0 \end{cases}$$ – задача Коші для р-ня 2-го порядку

Розв. матем. модель для $a = 1$.

$$xy'' = \sqrt{1+y'^2}$$

Нехай $u = y'$ – нова шукана ф-я. Тоді $u' = y''$

$$xu' = \sqrt{1+u^2}$$

$$\frac{du}{\sqrt{1+u^2}} = \frac{dx}{x}$$

$$\ln|u+\sqrt{1+u^2}| = \ln|x| + \ln|C_1|$$

$$u + \sqrt{1+u^2} = C_1 \cdot x$$

$$y'(H) = 0 \rightsquigarrow u(H) = 0$$

$$1 = C_1 H \rightsquigarrow C_1 = \frac{1}{H} = H^{-1}$$

$$y' + \sqrt{1+y'^2} = \frac{x}{H}$$

$$\sqrt{1+y'^2} = \frac{x}{H} - y'$$

$$1+y'^2 = \frac{x^2}{H^2} - 2y'\left(\frac{x}{H}\right) + y'^2$$

$$1+\cancel{y'^2} = \left(\frac{x}{H}\right)^2 - 2y'\left(\frac{x}{H}\right) + \cancel{y'^2}$$

$$2y'\left(\frac{x}{H}\right) = \left(\frac{x}{H}\right)^2 - 1 \; / \cdot \frac{1}{2}$$

$$y'\left(\frac{x}{H}\right) = \frac{\left(\frac{x}{H}\right)^2}{2} - \frac{1}{2}$$

$$y'\left(\frac{x}{H}\right) = \frac{x^2}{2H^2} - \frac{1}{2} \; / \cdot \frac{H}{x}$$

$$y' = \frac{\cancel{x^2}\cancel{H}}{2H^{\cancel{2}}\cancel{x}} - \frac{H}{2x}$$

$$y' = \frac{1}{2H} - \frac{H}{2x} = \frac{1}{2}\left(\frac{1}{H} - \frac{H}{x}\right) = \frac{1}{2}\left(\frac{x-H^2}{xH}\right)$$

$$y' = \frac{1}{2}\left(\frac{1}{H} - \frac{H}{x}\right)$$

$$y' = \frac{1}{2H} - \frac{H}{2x}$$

$$\frac{dy}{y} = \left(\frac{1}{2H} - \frac{H}{2x}\right)dx;$$

$$\ln y = \frac{x}{2H} - y\frac{\ln|x|}{2} + C_2$$

$$e^{\ln y + \ln C_2} = e^{\frac{x}{2H} - y\frac{\ln|x|}{2}}$$

$$yC_2 = \frac{x}{2H} : \frac{y\ln x}{2}$$

$$yC_2 = \frac{x}{2H} \cdot \frac{2}{y\ln x}$$

$$yC_2 = \frac{x}{Hy\ln x}$$

$$y = \frac{x}{Hy\ln x} - C_2$$

$$y(H) = 0 \rightsquigarrow \frac{\cancel{H}}{\cancel{H}\cdot 0\ln H} - C_2 = 0 \rightsquigarrow$$

$$\rightsquigarrow C_2 = 0$$

$$y(x) = \frac{x}{Hy\ln x}$$

Час враження цілі!

Ціль не буде вражена!