Варіант 2

Екзаменаційна робота (Семестр 2)
з математичного аналізу
студента групи ПМО-11
Барського Андрія

$$①\ \int \frac{dx}{(1+x^2)\sqrt[3]{1+arctg\,x}} = \int \frac{d(arctg\,x)}{\sqrt[3]{1+arctg\,x}} = \int \frac{d(arctg\,x+1)}{(1+arctg\,x)^{\frac{1}{3}}} =$$

$$= (1+arctg\,x)^{-\frac{1}{3}}\, d(1+arctg\,x) = \frac{3(1+arctg\,x)^{\frac{2}{3}}}{2} + C$$

$$②\ \int \frac{3x+2021}{(x+3)(x-1)(x+2)}\,dx = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{x+2} \triangleq$$

$$A = \frac{3x+2021}{\cancel{(x+3)}(x-1)(x+2)}\Big|_{x=-3} = \frac{-9+2021}{-4\cdot(-1)} = \frac{2012}{4} = 503$$

$$B = \frac{3x+2021}{(x+3)\cancel{(x-1)}(x+2)}\Big|_{x=1} = \frac{3+2021}{4\cdot 3} = \frac{2024}{12} = \frac{506}{3}$$

$$C = \frac{3x+2021}{(x+3)(x-1)\cancel{(x+2)}}\Big|_{x=-2} = \frac{-6+2021}{1\cdot(-3)} = \frac{2015}{-3}$$

$$\triangleq \int \frac{503\,dx}{x+3} + \int \frac{\frac{506}{3}}{x-1}\,dx + \int \frac{\frac{2015}{-3}}{x+2}\,dx = 503\int \frac{d(x+3)}{x+3} +$$

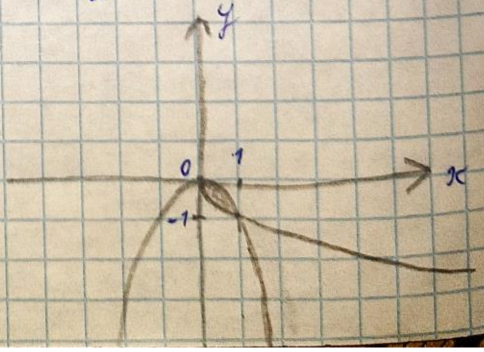
$$+ \frac{506}{3}\int \frac{d(x-1)}{x-1} - \frac{2015}{3}\int \frac{d(x+2)}{x+2} = 503\ln|x+3| +$$

$$+ \frac{506}{3}\ln|x-1| - \frac{2015}{3}\ln|x+2| + C.$$

③ a) $y = -x^2$, $y^2 = x$ $\rightarrow$ $\sqrt{x} = |y|$

$\begin{cases} y = \sqrt{x} \text{ — не уд. фігуру} \\ y = -\sqrt{x} \end{cases}$

$\begin{cases} y = -x^2 \\ y = -\sqrt{x} \end{cases}$



$-x^2 = -\sqrt{x}$

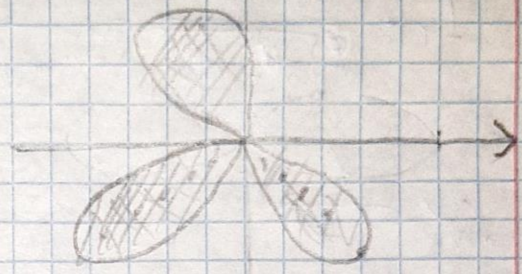
1 $-x^2 + \sqrt{x} = 0$

$x = 0 \quad x = 1$

$$S = \int_0^1 (-x^2 + \sqrt{x})\,dx = -\int_0^1 x^2\,dx + \int_0^1 \sqrt{x}\,dx =$$

$$= -\left(\frac{x^3}{3}\right)\Big|_0^1 + \left(\frac{2x\sqrt{x}}{3}\right)\Big|_0^1 = -\frac{1}{3} + \frac{2}{3} = \boxed{\frac{1}{3}}$$

δ) $r = 5\sin 2\varphi$



$r \geq 0$

$5\sin 2\varphi \geq 0$

$\sin 2\varphi \geq 0$

$0 + 2\pi n \leq 2\varphi \leq \pi + 2\pi n, \; n \in \mathbb{R}$

$\frac{2\pi n}{2} \leq \varphi \leq \frac{\pi}{2} + \frac{2\pi n}{2}$

$n = 0 \quad 0 \leq \varphi \leq \frac{\pi}{2}$

φ	0
r	0

$n = 1 \quad \pi \leq \varphi \leq \frac{3\pi}{2}$

$$S = 3 \cdot S_1 = 3 \cdot \frac{1}{2} \int_0^{\pi} (5\sin 2\varphi)^2\,d\varphi = 15 \cdot \frac{3}{2}$$

④ $$\int_0^{+\infty}\frac{\sqrt[3]{x}\,dx}{\sqrt[6]{x^7+x^{10}}}=\underbrace{\int_0^{1}\frac{\sqrt[3]{x}\,dx}{\sqrt[6]{x^7+x^{10}}}}_{I_1}+\underbrace{\int_1^{+\infty}\frac{\sqrt[3]{x}\,dx}{\sqrt[6]{x^7+x^{10}}}}_{I_2}$$

$$I_1:\ \frac{\sqrt[3]{x}}{\sqrt[6]{x^7+x^{10}}}\underset{x\to 0}{\sim}\frac{\sqrt[3]{x}}{\sqrt[6]{x^7(1+\underbrace{x^3}_{0})}}\underset{x\to 0}{\sim}\frac{x^{\frac{1}{3}}}{x^{\frac{7}{6}}}\underset{x\to 0}{\sim}$$

$$\underset{x\to 0}{\sim}\frac{1}{x^{\frac{5}{6}}}$$, $\frac{5}{6}<1$, отже I_1 збіжний

$$I_2:\ \frac{\sqrt[3]{x}}{\sqrt[6]{x^7+x^{10}}}\underset{x\to +\infty}{\sim}\frac{x^{\frac{1}{3}}}{\sqrt[6]{x^{10}(\underbrace{\frac{1}{x^3}}_{0}+1)}}\underset{x\to +\infty}{\sim}\frac{x^{\frac{1}{3}}}{x^{\frac{10}{6}}}\underset{x\to +\infty}{\sim}$$

$$\underset{x\to +\infty}{\sim}\frac{1}{x^{\frac{4}{3}}}$$, $\frac{4}{3}>1$, отже I_2 збіжний

Відповідь: ряди I_1 та I_2 збіжні, отже і весь ряд збіжний.

(5) $\sum\limits_{n=1}^{\infty} \dfrac{2^n (n+2)!}{(n+1)^n}$

За ознакою д'Аламбера:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q$$

$$\lim_{n\to\infty} \frac{2\cdot 2^n \cancel{(n+3)!}^{(n+1)}}{(n+2)\underbrace{(n+2)^n}_{(n+1)}} \cdot \frac{(n+1)^n}{2^n \cancel{(n+2)!}} = 2\lim_{n\to\infty} \frac{\cancel{n+1}}{(n+2)\cancel{(n+1)}} =$$

$$= 2\lim_{n\to\infty} \frac{1}{n+2} = \left[\frac{1}{\infty}\right] = 0$$

$0 <$, отже ряд збіжний.

(6) $\sum\limits_{n=1}^{\infty} \dfrac{\cos(\sqrt{n}\,x)}{\sqrt[3]{n^4} + \sin^2 x} \le \dfrac{1}{n^{\frac{4}{3}} + 2} = C_n$

$\sum\limits_{n=1}^{\infty} C_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{\frac{4}{3}} + 2}$ $\left(\alpha = \frac{4}{3} > 1\right)$, отже функціональний ряд збіжний рівномірно.

(7) $\sum\limits_{n=2}^{\infty} \dfrac{2^{n-1}}{n-1}$, за ф-ю Коші-Адамара

$$R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}} = \frac{1}{\frac{1}{2}\lim\limits_{n\to\infty} \frac{\sqrt[n]{2^n}}{\sqrt[n]{n-1}}} = \frac{2}{\lim\limits_{n\to\infty} \frac{2}{\sqrt[n]{n-1}}} =$$

$\sqrt[n]{n-1} \to 1$

$$= \frac{2}{2} = 1$$

Відповідь: $R = 1$, інтервал $(1; 3)$.

(8) $f(x, y) = \dfrac{2xy}{x^2+y^2}$ $O(0;0)$

$$\lim_{x\to 0}\left(\lim_{y\to 0} \frac{2xy}{x^2+y^2}\right) = \lim_{x\to 0}\left(\frac{0}{x^2}\right) = 0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{2xy}{x^2+y^2}\right) = \lim_{y\to 0}\left(\frac{0}{y^2}\right) = 0$$

$$\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{2xy}{x^2+y^2}\right) =$$

$(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right)$ $\lim\limits_{n\to\infty} f(x_n, y_n) = \dfrac{\frac{2}{n}}{\frac{1}{n^2}}$

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9) $u = x^2 - 2xy + 4y^2 + 6z^2 + 6yz - 6z$

1) $\begin{cases} u'_x = 2\,[illegible] - 2y \\ u'_y = -2x + 8y + 6z \\ y'_z = 12z + 6y \end{cases}$