Домашня робота (з. 15)

№3.1

$$y = \frac{2x}{1-x^2}$$

$$y' = \frac{(2x)'(1-x^2) - (1-x^2)'\,2x}{(1-x^2)^2} = \frac{2(1-x^2) - (-2x)2x}{(1-x^2)^2} =$$

$$= \frac{2-2x^2+4x^2}{(1-x^2)^2} = \boxed{\frac{2+2x^2}{(1-x^2)^2}}$$

№3.9

$$y = \sqrt[3]{x^2} - \frac{2}{\sqrt{x}} = x^{2/3} - \frac{2}{\sqrt{x}}$$

$$y' = \frac{2}{3}x^{-\frac{1}{3}} - \left(-2\frac{\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}\right) = \boxed{\frac{2}{3\sqrt[3]{x}} + \frac{1}{x\sqrt{x}}}$$

№3.10

$$y = x\sqrt{1+x^2}$$

$$y' = (x)'\sqrt{1+x^2} + (\sqrt{1+x^2})'\,x = \sqrt{1+x^2} + \frac{2x}{2\sqrt{1+x^2}}\cdot x =$$

$$= \sqrt{1+x^2} + \frac{x^2}{\sqrt{1+x^2}} = \frac{1+x^2+x^2}{\sqrt{1+x^2}} = \boxed{\frac{1+2x^2}{\sqrt{1+x^2}}}$$

№3.18

$$y = \cos 2x - 2\sin x$$

$$y' = \boxed{-2\sin 2x - 2\cos x}$$

№3.24

$$y = \frac{\cos x}{2\sin^2 x}$$

$$y' = \frac{(\cos x)' 2\sin^2 x - (2\sin^2 x)' \cos x}{(2\sin^2 x)^2} =$$

$$= \frac{-\sin x \cdot 2\sin^2 x - (2\cos^2 x)\cos x}{(2\sin^2 x)^2} =$$

$$= \frac{-2\sin^3 x - \cos x \cdot 2 \cdot \underbrace{2\sin x\cos x}_{\sin 2x}}{(2\sin^2 x)^2} = \frac{-2\sin^3 x - 2\cos x\sin 2x}{(2\sin^2 x)^2} =$$

$$= \frac{-2\sin^3 x - 2\cos x\sin 2x}{4\sin^4 x} = \frac{2(-\sin^3 x - \cos x\sin 2x)}{2 \cdot 2\sin^4 x} =$$

$$= \frac{-\sin^3 x - \cos x\sin 2x}{2\sin^4 x} = \frac{(-\sin x)(\sin^2 x + \cos x \cdot 2\cos x)}{2\sin^4 x^3} =$$

$$= \frac{-(\sin^2 x + 2\cos^2 x)}{2\sin^3 x} = -\frac{(\overbrace{\sin^2 x + \cos^2 x}^{=1} + \cos^2 x)}{2\sin^3 x} =$$

$$= \frac{-(1 + \cos^2 x)}{2\sin^3 x} = \boxed{-\frac{1+\cos^2 x}{2\sin^3 x}}$$

№ 3.27

$$y = tg\frac{x}{2} - ctg\frac{x}{2}$$

$$y' = \frac{1}{\cos^2\frac{x}{2}}\left(\frac{x}{2}\right)' - \left(-\frac{1}{\sin^2\frac{x}{2}}\left(\frac{x}{2}\right)'\right) = \frac{1}{\cos^2\frac{x}{2}} \cdot \frac{1}{2} + \frac{1}{\sin^2\frac{x}{2}} \cdot \frac{1}{2} =$$

$$= \frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}{2\cos^2\frac{x}{2}\sin^2\frac{x}{2}} = \boxed{\frac{1}{2\cos^2\frac{x}{2}\sin^2\frac{x}{2}}}$$

№ 3.31

$$y = e^{-x^2}$$

$$y' = e^{-x^2}(-x^2)' = e^{-x^2}(-2x) = \boxed{-\frac{2x}{e^{x^2}}}$$

№ 3.48

$$y = \ln(x+\sqrt{x^2+1})$$

$$y' = \frac{1}{x+\sqrt{x^2+1}}\left(x+\sqrt{x^2+1}\right)' = \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{1}{2\sqrt{x^2+1}}(x^2+1)'\right) =$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{2x}{2\sqrt{x^2+1}}\right) = \frac{1}{x+\sqrt{x^2+1}}\cdot\left(1+\frac{x}{\sqrt{x^2+1}}\right) =$$

$$= \frac{1}{x+\sqrt{x^2+1}}\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}+x}{(x+\sqrt{x^2+1})(\sqrt{x^2+1})} = \boxed{\frac{1}{\sqrt{x^2+1}}}$$

№3.60

$$y = \arcsin\frac{x}{2}$$

$$y' = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\left(\frac{x}{2}\right)' = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,\frac{1}{2} = \frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}} =$$

$$= \frac{1}{2\cdot\sqrt{1-\frac{x^2}{4}}} = \frac{1}{2\sqrt{\frac{4-x^2}{4}}} = \frac{1}{2\cdot\frac{\sqrt{4-x^2}}{2}} =$$

$$= \boxed{\frac{1}{\sqrt{4-x^2}}}$$

№3.62

$$y = \operatorname{arctg}\frac{x^2}{a}$$

$$y' = \frac{1}{1+\left(\frac{x^2}{a}\right)^2}\left(\frac{x^2}{a}\right)' = \frac{2x}{a}\cdot\frac{1}{1+\frac{x^4}{a^2}} = \frac{2x}{a}\cdot\frac{1}{\frac{a^2+x^4}{a^2}} =$$

$$= \frac{2x}{a\cdot\frac{a^2+x^4}{a^2}} = \frac{2x}{\frac{a^2+x^4}{a}} = \boxed{\frac{2xa}{a^2+x^4}}$$

№3.63

$$y = \frac{1}{\sqrt{2}}\operatorname{arcctg}\frac{\sqrt{2}}{x}$$

$$y' = \frac{\sqrt{2}}{2}\cdot\left(-\frac{1}{1+\left(\frac{\sqrt{2}}{x}\right)^2}\cdot\left(\sqrt{2}\cdot\frac{1}{x^2}\right)\right) = \frac{\sqrt{2}}{2}\left(\frac{1}{\frac{x^2+2}{\cancel{x^2}}}\cdot\frac{\sqrt{2}}{\cancel{x^2}}\right) =$$

$$= \frac{\sqrt{2}}{2}\left(\frac{1}{x^2+2}\sqrt{2}\right) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{x^2+2} = \frac{2}{2(x^2+2)} = \boxed{\frac{1}{x^2+2}}$$

№3.64

$y = \arccos\frac{1}{x}$

$y' = -\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\cdot\left(-\frac{1}{x^2}\right) = \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\cdot\frac{1}{x^2} = \frac{1}{x^2\sqrt{1-\left(\frac{1}{x}\right)^2}} =$

$= \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}} = \frac{1}{x^2\frac{\sqrt{x^2-1}}{|x|}} = \boxed{\frac{|x|}{x^2\sqrt{x^2-1}}}$

№3.77

$y = x + x^x + x^{x^x}$

$y' = 1 + x^x + x^x\ln x + x^{x^x}\left(x^{x-1} + \ln x\cdot x^x + \ln^2 x\cdot x^x\right) =$

$= 1 + x^x + x^x\ln x + x^{x^x+x-1} + x^{x^x+x}\ln x + x^{x^x+x}\ln^2 x =$

$= 1 + x^x(1+\ln x) + x^{x^x+x-1} + x^{x^x+x}\ln x + x^{x^x+x}\ln^2 x =$

$= \boxed{1 + x^x(1+\ln x) + x^{x^x+x-1} + x^{x^x+x}\ln x\,(1+\ln x)}$

№3.87

$y = \frac{x^2}{1+x^2}$ $\quad (y^{-1})' - ?$

$(y^{-1})' = \frac{1}{\frac{(x^2)'(1+x^2) - (1+x^2)'x^2}{(1+x^2)^2}} = \frac{1}{\frac{2x(1+x^2) - 2x\cdot x^2}{(1+x^2)^2}} =$

$= \frac{1}{\frac{2x+\cancel{2x^3}-\cancel{2x^3}}{(1+x^2)^2}} = \frac{1}{\frac{2x}{(1+x^2)^2}} = \frac{(1+x^2)^2}{2x} =$

$= \boxed{\frac{x^4+2x^2+1}{2x}}$

№3.95

$\begin{cases} x = e^{2t}\cos^2 t \\ y = e^{2t}\sin^2 t \end{cases}$ $\quad y'_x - ?$

$y'_x = \frac{(e^{2t}\sin^2 t)'_t}{(e^{2t}\cos^2 t)'_t} = \frac{\cancel{2}\cancel{e^{2t}}\cos^2 t}{\cancel{2}\cancel{e^{2t}}(-\sin^2 t)} = \frac{\cos^2 t}{-\sin^2 t} = \boxed{-\mathrm{ctg}^2 t}$

№ 3.99

$y'_x - ?$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2xa - 2x^2}{a^3} + \frac{2yb - 2y^2}{b^3}(y') = 0$

$\frac{2xab^3 - 2x^2b^3 + 2ya^3b - 2a^3y^2}{a^3b^3}(y') = 0$

$\frac{2yb - 2y^2}{b^3}y' = -\frac{2xa - 2x^2}{a^3}$

$$\boxed{y' = \frac{-\frac{2xa - 2x^2}{a^3}}{\frac{2yb - 2y^2}{b^3}}}$$

$y' = \frac{(-2xa - 2x^2)b^3}{(2yb - 2y^2)a^3} = \frac{\cancel{2}(-xa - x^2)b^3}{\cancel{2}(yb - y^2)a^3} =$

$= \frac{(-xa - x^2)b^3}{(yb - y^2)a^3}$

$1 + \left(\frac{y}{b}\right)'\left(\frac{y}{b}\right)' + \left(\frac{y}{b}\right)\left(\frac{y}{b}\right)'' = 0$

$\frac{y}{b} \cdot \left(\frac{y}{b}\right)'' = -1 - \left(\left(\frac{y}{b}\right)'\right)^2$

$\left(\frac{y}{b}\right)'' = \frac{-1 - \left(\left(\frac{y}{b}\right)'\right)^2}{\frac{y}{b}}$

$\left(\frac{y}{b}\right)'' = \frac{-1 - \left(\left(\frac{y}{b}\right)'\right)^2 \cdot b}{y}$

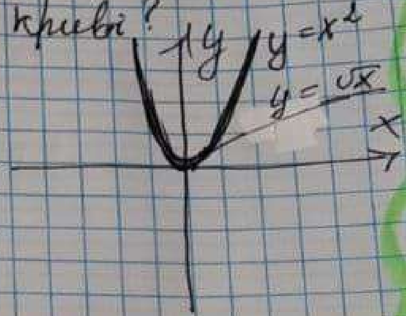
$$\left(\frac{y}{b}\right)'' = \frac{\dfrac{\dfrac{-1-2xa-2x^2}{a^3}}{\dfrac{2yb-2y^2}{b^3}}}{y}$$

$$\boxed{\left(\frac{y}{b}\right)'' = \frac{(-1-2xa-2x^2)b^3}{(2yb-2y^2)\,a^3 y}}$$

№ 3.106.

$y = x^2$

$x = y^2$ — під якими ∠ перетин. криві?



$\begin{cases} y = x^2 \\ x = y^2 \end{cases}$ $M_1(0;0)$ і $M_2(1;1)$

$\begin{cases} y = x^2 \\ y^2 = x \end{cases}$ $\begin{cases} y = x^2 \\ y = \sqrt{x} \end{cases}$

1) M_1: $y = x^2$ $y' = 2x$

$y_1'(0) = 2 \cdot 0 = 0$

$tg\varphi_1 = 0$

$k = 1$

$tg\alpha_1 = 1$

$$tg\lambda_1 = |tg(\varphi_1 - \alpha_1)| = \left|\frac{tg\varphi_1 - tg\alpha_1}{1 + tg\varphi_1\, tg\alpha_1}\right| = \left|\frac{0-1}{1+0\cdot 1}\right| = |-1| = 1$$

$\lambda_1 = 45°$

2) M_2: $y = \sqrt{x}$ $y' = \frac{1}{2\sqrt{x}}$

$y'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$; $tg\varphi_2 = \frac{1}{2}$; $k = \frac{1}{2}$; $tg\alpha_2 = \frac{1}{2}$

$$tg\lambda_2 = |tg(\varphi_2 - \alpha_2)| = \left|\frac{tg\varphi_2 - tg\alpha_2}{1 + tg\varphi_2\, tg\alpha_2}\right| = \left|\frac{\frac{1}{2} - \frac{1}{2}}{1 + \frac{1}{2}\cdot\frac{1}{2}}\right| = 0 \qquad \lambda_2 = 180°$$

№3109

$x = 2t - t^2$
$y = 3t - t^2$

р-ня дот. і норм — ?

а) $t_0 = 0$

$$\begin{cases} x = 2t - t^2 \\ y = 3t - t^2 \end{cases}$$

$x_0 = x(t_0) = 2 \cdot 0 - 0 = 0$

$y_0 = y(t_0) = 3 \cdot 0 - 0 = 0$

$$f'(x) = \frac{y'(t)}{x'(t)} = \frac{2-2t}{3-2t}$$

$$f'(x_0) = f(x_0)\Big|_{t_0=0} = \frac{2-2\cdot 0}{3-2\cdot 0} = \frac{2}{3}$$

$$y = \frac{2}{3}(x-0) + 0 = \frac{2}{3}x$$

$\boxed{y = \frac{2}{3}x \text{ — р-ня дотичної}}$

$$y = 0 - \frac{1}{\frac{2}{3}}(x-0) = -\frac{1}{\frac{2}{3}}x = -\frac{3}{2}x$$

$\boxed{y = -\frac{3}{2}x \text{ — р-ня нормалі}}$

б) $t_0 = 1$

$x_0 = x(t_0) = 2 \cdot 1 - 1^2 = 1$

$y_0 = y(t_0) = 3 \cdot 1 - 1 = 2$

$$f'(x) = \frac{y'(t)}{x'(t)} = \frac{2-2t}{3-2t}$$

$$f'(x_0) = f'(x_0)\Big|_{t_0=0} = \frac{2-2\cdot 1}{3-2\cdot 1} = 0$$

$y = 0(x-1)+2 = 2.$

$\boxed{y=2 \text{ р-не дотичної}}$

$y = 0(x-1)+2=2$

$\boxed{y=2 \text{ - р-не нормалі}}$