Заняття 4

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

№1.222

$$\int \sin^6 x\,dx = \int (\sin^2 x)^3 dx = \int \left(\frac{1}{2}(1-\cos 2x)\right)^3 dx =$$

$$= \frac{1}{8}\int (1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x)\,dx = \frac{1}{8}x - \frac{3}{16}\sin 2x + \frac{3}{8}\int \frac{1+\cos 4x}{2}\,dx$$

$$-\frac{1}{8}\int \cos^2 2x \cos 2x\,dx = \frac{1}{8}x - \frac{3}{16}\sin 2x + \frac{3}{16}x + \frac{3}{64}\sin 4x - \frac{1}{16}\int (1-\sin^2 2x)\,d(\sin 2x) =$$

$$= \frac{5}{16}x - \frac{3}{16}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{16}\sin 2x + \frac{1}{16}\frac{\sin^3 2x}{3} + C$$

№1.226

$$\int \sin^5 x \cdot \cos^5 x\,dx = \int \sin^5 x \cdot \cos^4 x\,(\cos x\,dx) = \int \sin^5 x\,(\cos^2 x)^2 d(\sin x) =$$

$$= \int \sin^5 x\,(1-\sin^2 x)^2 d(\sin x) = \left\{\begin{matrix}\text{Заміна:}\\ \sin x = t\end{matrix}\right\} = \int t^5 (1-t^2)^2 dt =$$

$$= \int t^5 (1 - 2t^2 + t^4)\,dt = \int t^5 - 2t^7 + t^9\,dt = \frac{t^6}{6} - \frac{2t^8}{8} + \frac{t^{10}}{10} =$$

$$= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{4} + \frac{\sin^{10} x}{10} + C$$

№1.230

$$\int \frac{dx}{\cos^3 x} = \int \frac{\sin^2 x + \cos^2 x}{\cos^3 x}dx = \int \frac{\sin^2 x}{\cos^3 x} + \frac{\cos^2 x}{\cos^3 x}dx = \int \frac{\sin^2 x}{\cos^3 x}dx + \int \frac{1}{\cos x}dx =$$

$$= \int \frac{\sin x}{\cos^3 x}(\sin x\,dx) + \int \frac{1}{\cos x}dx = \int \frac{\sin x}{\cos^3 x}\,d(\cos x) + \int \frac{1}{\cos x}\cdot\frac{\cos x}{\cos x}dx =$$

$$= \sin x \cdot \frac{1}{2\cos^2 x} - \int \frac{1}{2\cos^2 x}\cdot \cos x\,dx + \int \frac{\cos x}{\cos^2 x}dx =$$

$$= \sin x \cdot \frac{1}{2\cos^2 x} - \frac{1}{2}\int \frac{\cos x}{1-\sin^2 x}dx + \int \frac{\cos x}{1-\sin^2 x}dx =$$

{Заміна: $t = \sin x$} {Заміна: $t = \sin x$}

$$= \sin x \cdot \frac{1}{2\cos^2 x} - \frac{1}{2}\int \frac{1}{1-t^2}dt + \int \frac{1}{1-t^2}dt =$$

$$= \sin x \cdot \frac{1}{2\cos^2 x} - \frac{1}{2}\cdot\frac{1}{2\cdot 1}\ln\left|\frac{t-1}{t+1}\right| - \frac{1}{2\cdot 1}\ln\left|\frac{t-1}{t+1}\right| =$$

$$= \sin x\,\frac{1}{2\cos^2 x} - \frac{1}{4}\ln\left|\frac{\sin x - 1}{\sin x + 1}\right| - \frac{1}{2}\ln\left|\frac{\sin x - 1}{\sin x + 1}\right| + C =$$

$$= \sin x \cdot \frac{1}{2\cos^2 x} - \frac{1}{4}\ln\left|\frac{\sin x - 1}{\sin x + 1}\right| + C$$

№1.258

$$\int \frac{dx}{(2+\cos x)\sin x} = \left\{ \begin{matrix}\text{універс.}\\ \text{тригон.}\\ \text{підстан.}\end{matrix} \right\} = \int \frac{1}{2\sin x+\cos x\sin x}\,dx = \left\{ \begin{matrix} \sin x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} \\ \cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}} \\ tg\frac{x}{2} = t \end{matrix} \right\}$$

$$= \int \frac{1}{2\cdot\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}\cdot\frac{2t}{1+t^2}}\,dt =$$

$$= \int \frac{1}{\frac{4t}{1+t^2}\cdot\frac{2t(1-t^2)}{(1+t^2)^2}}\,dt = \int \frac{1}{\frac{4t(1+t^2)+2t-t^3}{(1+t^2)^2}}\,dt =$$

$$= \int \frac{1}{\frac{4t+4t^3+2t-2t^3}{(1+t^2)^2}}\,dt = \int \frac{1}{\frac{6t+2t^3}{(1+t^2)^2}}\,dt = \int \frac{(1+t^2)^2}{6t+2t^3}\cdot\frac{1}{d(1+t^2)\cdot\frac{1}{2}} =$$

$$= \int \frac{(1+t^2)^{\cancel{2}}}{6t+2t^3}\cdot\frac{2}{\cancel{1+t^2}}\,dt = \int \frac{1+t^2}{\cancel{2}(3t+t^3)}\,\cancel{2}\,dt = \int \frac{1+t^2}{3t+t^3}\,dt =$$

$$= \left\{ \begin{matrix} u = 3t+t^3 \\ du = 3+3t^2 \end{matrix} \right\} = \int \frac{1}{3u}\,du = \frac{1}{3}\int\frac{1}{u}\,du = \frac{1}{3}\ln|u| =$$

$$= \frac{1}{3}\ln|3t+t^3| = \frac{1}{3}\ln\left|3tg\frac{x}{2}+tg^3\frac{x}{2}\right| + C$$

№1.241

$$\boxed{I_n = \int \sin^n x\,dx \quad (n>2)}$$

$$\int \sin^n x\,dx = \int \sin^{n-1}x\,\sin x\,dx = \int \sin^{n-1} d(-\cos x) = \left\{ \begin{matrix} u = \sin^{n-1}x \\ v = -\cos x \end{matrix} \right\} \Rightarrow \begin{matrix} du = (n-1)\sin^{n-2}x \cdot \cos x\,dx \\ dv = \sin x\,dx \end{matrix}$$

$$= -\cos x\,\sin^{n-1}x - \int(-\cos x)(n-1)\sin^{n-2}x\cos x\,dx = \{\cos^2 x = 1-\sin^2 x\}$$

$$= -\cos x\cdot\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,(1-\sin x)\,dx =$$

$$= -\cos x\cdot\sin^{n-1}x + (n-1)\int \sin^{n-2}x - \sin^n x\,dx =$$

$$= -\cos x\cdot\sin^{n-1}x + (n-1)\left(\int \sin^{n-2}x\,dx - \int \sin^n x\,dx\right) =$$

$$= -\cos x\cdot\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

$$\int \sin^n x\,dx = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

$$\int \sin^n x\,dx + (n-1)\int \sin^n x\,dx = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx$$

$$n\int \sin^n x\,dx = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx \quad \Big|\cdot\frac{1}{n}$$

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

$\boxed{N1.242}$

$I_n = \int \cos^n x\,dx \quad (n>2)$

$$\int \cos^n x\,dx = \int \cos^{n-1}x\cos x\,dx = \left\{\begin{matrix} u=\cos^{n-1}x & dv=\cos x\,dx \\ du=(n-1)\cos^{n-2}x(-\sin x)dx & v=\sin x \end{matrix}\right\}$$

$$= \cos^{n-1}x\cdot\sin x - \int \sin x\cdot(n-1)\cos^{n-2}x(-\sin x)dx =$$

$$= \cos^{n-1}x\cdot\sin x + \int (n-1)\underbrace{\sin^2 x}_{\sin^2 x = 1-\cos^2 x}\cdot\cos^{n-2}dx =$$

$$= \cos^{n-1}x\cdot\sin x + \int (n-1)\cos^{n-2}x(1-\cos^2 x)dx =$$

$$= \cos^{n-1}x\cdot\sin x + (n-1)\int(\cos^{n-2}x - \cos^n x\,dx) =$$

$$= \cos^{n-1}x\cdot\sin x + (n-1)\int\cos^{n-2}x\,dx - (n-1)\int\cos^n x\,dx$$

$$\underline{\int\cos^n x\,dx} = \cos^{n-1}x\cdot\sin x + (n-1)\int\cos^{n-2}x\,dx - \underline{(n-1)\int\cos^n x\,dx}$$

$$1\int\cos^n x\,dx + (n-1)\int\cos^n x\,dx = \cos^{n-1}x\cdot\sin x + (n-1)\int\cos^{n-2}x\,dx$$

$$n\int\cos^n x\,dx = \cos^{n-1}x\cdot\sin x + (n-1)\int\cos^{n-2}x\,dx \;\Big/\cdot\frac{1}{n}$$

$$\int\cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\cdot\sin x + \frac{n-1}{n}\int\cos^{n-2}x\,dx$$

$\boxed{N1.244}$

$I_n = \int \frac{dx}{\cos^n x}, \quad (n>2)$

$$\int\frac{dx}{\cos^n x} = \int\frac{1}{\cos^n x}dx = \int\sec^n x\,dx = \int\sec^{n-2}x\,\sec^2 x\,dx =$$

$$= \left\{\begin{matrix} u=\sec^{n-2}x & \Rightarrow & \frac{du}{dx} = (n-2)\sec^{(n-3)}x\cdot\sec x\cdot tg\,x) \Rightarrow \\ & & \Rightarrow du = (n-2)\sec^{n-2}x\,tg\,x) \\ dv=\sec^2 x\,dx & \Rightarrow & v = tg\,x \end{matrix}\right\}$$

$$= \sec^{n-2}\cdot tg\,x - \int tg\,x(n-2)\sec^{n-2}x\,tg\,x\,dx = \sec^{n-2}tg\,x -$$

$$-(n-2)\int tg^2 x\cdot\sec^{n-2}x\,dx = \left\{ 1+tg^2\alpha = \frac{1}{\cos^2\alpha};\; tg^2\alpha = \frac{1}{\cos^2\alpha} - 1 = \sec^2\alpha - 1 \right\}$$

$$= \sec^{n-2} tgx - (n-2)\int (\sec^2 x - 1)\sec^{n-2} x\,dx =$$

$$= \sec^{n-2} tgx - (n-2)\int \sec^2 x \cdot \sec^{n-2} x - \sec^{n-2} x\,dx =$$

$$= \sec^{n-2} tgx - (n-2)\underbrace{\int \sec^n x\,dx}_{(n-2)I_n} - \underbrace{(n-2)\int \sec^{n-2} x\,dx}_{-(n-2)I_{n-2}} =$$

$$I_n = \sec^{n-2} tgx - (n-2)I_n + (n-2)I_{n-2}$$

$$1I_n + (n-2)I_n = \sec^{n-2}x\, tgx + (n-2)I_{n-2}$$

$$(n-1)I_n = \sec^{n-2}x\, tgx + (n-2)I_{n-2} \quad / \cdot \frac{1}{n-1}$$

$$I_n = \frac{\sec^{n-2}x \cdot tgx}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$\int \sec^n x\,dx = \frac{\sec^{n-2}x \cdot tgx}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

$$\int \frac{1}{\cos^n x}dx = \frac{\frac{1}{\cos^{n-2}x}\cdot tgx}{n-1} + \frac{n-2}{n-1}\int \frac{1}{\cos^{n-2}x}dx$$

$$\int \frac{1}{\cos^n x}dx = \frac{tgx}{(n-1)\cos^{n-2}x} + \frac{n-2}{n-1}\int \frac{1}{\cos^{n-2}x}dx$$

$\boxed{N 1.243}$

$$\boxed{I_n = \int \frac{dx}{\sin^n x} \quad (n>2)}$$

$$\int \frac{dx}{\sin^n x} = \int \frac{1}{\sin^n x}dx = \int \csc^n x\,dx = \int \csc^{n-2} x \cdot \csc^2 x\,dx =$$

$u = \csc^{n-2} x$ $\qquad$ $dv = \csc^2 x\,dx$

$\frac{du}{dx} = (n-2)\csc^{n-3}x \cdot (-ctgx)\csc x =$ $\qquad$ $v = -ctg\,x$

$= -(n-2)\csc^{n-2}x\, ctgx$ $\qquad$ $\frac{d}{dx}(ctgx) = -\csc x$

$du = -(n-2)\csc^{n-2}x\, ctg\,x\,dx$

$$= \csc^{n-2}x(-\mathrm{ctg}\,x) - \int -\mathrm{ctg}\,x \cdot (-(n-2)\csc^{n-2}x\,\mathrm{ctg}\,x)\,dx =$$

$$= -\mathrm{ctg}\,x \cdot \csc^{n-2}x - (n-2)\int \csc^{n-2}x \cdot \mathrm{ctg}^2 x\,dx = \left\{ \begin{array}{l} \mathrm{ctg}^2\alpha + 1 = \frac{1}{\sin^2\alpha} = \csc^2\alpha \\ \mathrm{ctg}^2 x = \csc^2 x - 1 \end{array} \right\}$$

$$= -\mathrm{ctg}\,x \cdot \csc^{n-2}x - (n-2)\int \csc^{n-2}x(\csc^2 x - 1)\,dx =$$

$$= -\mathrm{ctg}\,x \cdot \csc^{n-2}x - (n-2)\left(\int \csc^n x\,dx - \int \csc^{n-2}x\,dx\right) =$$

$$= -\mathrm{ctg}\,x \cdot \csc^{n-2}x - (n-2)\int \csc^n x\,dx + (n-2)\int \csc^{n-2}x\,dx$$

$$\int \csc^n x\,dx = -\mathrm{ctg}\,x \cdot \csc^{n-2}x \underline{- (n-2)\int \csc^n x\,dx} + (n-2)\int \csc^{n-2}x\,dx$$

$$1\int \csc^n x\,dx + (n-2)\int \csc^n x\,dx = -\mathrm{ctg}\,x \cdot \csc^{n-2}x + (n-2)\int \csc^{n-2}x\,dx$$

$$\int \csc^n x\,dx\,(1 + n - 2) = -\mathrm{ctg}\,x \cdot \csc^{n-2}x + (n-2)\int \csc^{n-2}x\,dx$$

$$(n-1)\int \csc^n x\,dx = -\mathrm{ctg}\,x \cdot \csc^{n-2}x + (n-2)\int \csc^{n-2}x\,dx \;\Big|\; \frac{1}{n-1}$$

$$\int \csc^n x\,dx = -\frac{1}{n-1} \cdot \mathrm{ctg}\,x \cdot \csc^{n-2}x + \frac{n-2}{n-1}\int \csc^{n-2}x\,dx$$

$$\int \frac{dx}{\sin^n x} = -\frac{1}{n-1}\mathrm{ctg}\,x \cdot \frac{1}{\sin^{n-2}x} + \frac{n-2}{n-1}\int \frac{1}{\sin^{n-2}x}\,dx$$

№1.228

$$\int \frac{\cos^4 x}{\sin^3 x}\,dx = \int \frac{\cos^3 x}{\sin^3 x} \cdot \cos x\,dx = \int \frac{\cos^3 x}{\sin^3 x}\,d(-\sin x) = -\int \frac{\cos^3 x}{\sin^3 x}\,d(\sin x) =$$

$$= -\int \frac{\cos^2 x \cdot \cos x}{\sin^3 x}\,d(\sin x) = -\int \frac{(1 - \sin^2 x)\cos x}{\sin^3 x}\,d(\sin x) =$$

$$= -\int \frac{\cos x - \sin^2 x \cdot \cos x}{\sin^3 x}\,d(\sin x) = -\left(\int \frac{\cos x}{\sin^3 x}\,d(\sin x) - \int \frac{\sin^2 x \cdot \cos x}{\sin^3 x}\,d(\sin x)\right) =$$

$$= -\left(\int \frac{\cos x}{\sin^3 x}\,d(\sin x) - \int \frac{\cos x}{\sin x}\,d(\sin x)\right) = -\left(\int \frac{1}{t^3}\,dt - \int \frac{1}{t}\,dt\right) =$$

заміна: $\sin x = t$ заміна: $\sin x = t$

$$= -\left(-\frac{1}{2t^2} - \ln|t|\right) = \frac{1}{2t^2} + \ln|t| = \frac{1}{2\sin^2 x} + \ln|\sin x| + C$$

$\boxed{№1.246}$

$$\int \cos x \cdot \cos 2x \cdot \cos 3x \, dx = \int (\cos x \cdot \cos 3x) \cos 2x \, dx =$$

$$= \frac{1}{2} \int (\cos(2x) + \cos 4x) \cos 2x \, dx = \frac{1}{2} \int (\cos 2x + \cos 4x) \cos 2x \, dx =$$

$$= \frac{1}{2} \int (\cos^2 2x + \cos 4x \cdot \cos 2x) \, dx = \frac{1}{2} \int \cos^2 2x + \cos 2x + \cos 6x \, dx =$$

$$= \frac{1}{4} \int ((1 + \cos 4x) + \cos 2x + \cos 6x) \, dx =$$

$$= \frac{1}{4} \left(\int 1 dx + \int \cos 4x \, dx + \int \cos 2x \, dx + \int \cos 6x \, dx \right) =$$

$$= \frac{1}{4} \left(x + \frac{1}{4} (\sin 4x) + \frac{1}{2} (\sin 2x) + \frac{1}{6} (\sin 6x) \right) =$$

$$= \frac{x}{4} + \frac{1}{16} \sin 4x + \frac{1}{8} \sin 2x + \frac{1}{24} \sin 6x + C$$