Заняття 12

$\boxed{N 3216}$

$$U = \frac{x}{\sqrt{x^2+y^2}}$$

$$\nabla U = U(x,y)$$

$$U'_x = \frac{\partial U}{\partial x} = \left(\frac{x}{\sqrt{x^2+y^2}}\right)'_x = \frac{1}{x^2+y^2} - \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$U'_y = \frac{\partial U}{\partial y} = \left(\frac{x}{\sqrt{x^2+y^2}}\right)'_y = -\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}$$

$$U''_{x^2} = \frac{\partial^2 U}{\partial x^2} = \left(\frac{1}{x^2+y^2} - \frac{x^2}{(x^2+y^2)^{3/2}}\right)'_x = \frac{x\left(\frac{3x^2}{x^2+y^2} - 3\right)}{(x^2+y^2)^{3/2}}$$

$$U''_{y^2} = \frac{\partial^2 U}{\partial y^2} = \left(\frac{-xy}{(x^2+y^2)^{\frac{3}{2}}}\right)'_y = \frac{x\left(\frac{3y^2}{x^2+y^2} - 1\right)}{(x^2+y^2)^{\frac{3}{2}}}$$

$$U''_{xy} = U''_{yx} = \frac{\partial^2 U}{\partial x \partial y} = \left(\frac{1}{x^2+y^2} - \frac{x^2}{(x^2+y^2)^{3/2}}\right)'_y =$$

$$= \frac{y\left(\frac{3x^2}{x^2+y^2} - 1\right)}{(x^2+y^2)^{3/2}}$$

$\boxed{N 3221}$

$$U = \ln(x+y^2)$$

$$\nabla U = U(x,y)$$

$$U'_x = \frac{\partial U}{\partial x} = \left(\ln(x+y^2)\right)'_x = \frac{1}{x+y^2}$$

$$U'_y = \left(\ln(x+y^2)\right)'_y = \frac{1 \cdot 2y}{x+y^2} = \frac{2y}{x+y^2}$$

$$U''_{x^2} = \left(\frac{1}{x+y^2}\right)'_x = -\frac{1}{(x+y^2)^2}$$

$$U''_{y^2} = \left(\frac{2y}{x+y^2}\right)'_y = \frac{2\left(-\frac{2y^2}{x+y^2}+1\right)}{x+y^2}$$

$$U''_{xy} = U''_{yx} = \left(\frac{1}{x+y^2}\right)'_y = \frac{-2y}{(x+y^2)^2}$$

№3222

$U = \arctg\frac{y}{x}$

$$\nabla\ U'_x = \left(\arctg\frac{y}{x}\right)'_x = -\frac{1}{\left(1+\frac{y^2}{x^2}\right)}\cdot\frac{y}{x^2} = \frac{-y}{x^2\left(1+\frac{y^2}{x^2}\right)} = \frac{-y}{x^2+y^2}$$

$$U'_y = \left(\arctg\frac{y}{x}\right)'_y = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{1}{x\left(1+\frac{y^2}{x^2}\right)} = \frac{x}{x^2+y^2}$$

$$U''_{x^2} = \left(\frac{-y}{x^2\left(1+\frac{y^2}{x^2}\right)}\right)'_x = \left(\frac{-y}{x^2+y^2}\right)'_x = \frac{2xy}{(x^2+y^2)^2}$$

$$U''_{y^2} = \left(\frac{x}{x^2+y^2}\right)'_y = -\frac{2yx}{(x^2+y^2)'}$$

$$U''_{xy} = U''_{yx} = \left(\frac{-y}{x^2+y^2}\right)'_y = \frac{2y^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2}$$

№3236

$U = \frac{x}{y}$; du — ? d^2u — ?

$$\nabla\ du = \underline{U'_x dx} + \underline{U'_y dy} + \underbrace{U'_z}_{0} dz =$$

$$= \frac{1}{y}dx - \frac{x}{y^2}dy$$

$$d^2u = U''_{x^2}dx^2 + U''_{y^2}dy^2 + 2U''_{xy}dxdy =$$

$$= 0 + \frac{2x}{y^3}dy^2 + 2\left(-\frac{1}{y^2}\right)dxdy = \frac{2x}{y^3}dy^2 - \frac{2}{y^2}dxdy \quad \Delta$$

№3239

$U = e^{xy}$; du — ? d^2u — ?

$\nabla\ u'_x = (e^{xy})'_x = ye^{xy}$

$u'_y = (e^{xy})'_y = xe^{xy}$

$du = ye^{xy}dx + xe^{xy}dy$

$u''_{x^2} = (ye^{xy})'_x = y\cdot e^{xy}\cdot y + e^{xy}\cdot 0 = y^2e^{xy}$

$u''_{y^2} = (xe^{xy})'_y = x\cdot e^{xy}\cdot x + e^{xy}\cdot 0 = x^2e^{xy}$

$u''_{xy} = (ye^{xy})'_y = e^{xy}\cdot(xy+1)$

$d^2u = y^2e^{xy}dx^2 + x^2e^{xy}dy^2 + 2e^{xy}(xy+1)$

$\boxed{№3258}$

$\frac{d^6u}{dx^3dy^3}$ - ?; $u = x^3\sin y + y^3\sin x$

$\nabla\ \frac{\partial u}{\partial y} = x^3\cos y + 3y^2\sin x$

$\frac{\partial^2 u}{\partial y^2} = x^3(-\sin y) + 6y\sin x$

$\frac{\partial^3 u}{\partial y^3} = -x^3\cdot\cos y + 6\cdot\sin x$

$\frac{d^4u}{\cancel{dy}^3} = -3x^2\cdot\cos y + 6\cdot\cos x$

$\frac{d^5u}{dx^2dy^3} = -6x\cdot\cos y + 6\cdot(-\sin x)$

$\frac{d^6u}{dx^3dy^3} = -6\cos y - 6\cdot\cos x$

$\boxed{N\ 3273}$

$d^3u - ?$

$u = \cos x \cdot \operatorname{ch} y$

$\nabla \; du = (\cos x \cdot \operatorname{ch} y)'_x dx + (\cos x \cdot \operatorname{ch} y)'_y dy$

$du = \underbrace{-\operatorname{ch} y \cdot \sin x}_{u'_x} dx + \underbrace{\operatorname{sh} y \cdot \cos x}_{u'_y} dy$

$d^2u = u''_{x^2} dx^2 + u''_{y^2} dy^2 + u''_{xy}$

$d^2u = (-\operatorname{ch} y \cdot \sin x)'_x dx^2 + (\operatorname{sh} y \cdot \cos x)'_y dy^2 + 2(\sin x \cdot (-\operatorname{ch} y))'_y$

$d^2u = \underbrace{-\operatorname{ch} y \cdot \cos x}_{=u''_{x^2}} dx^2 + \underbrace{\cos x \cdot \operatorname{ch} y}_{u''_{y^2}} dy^2 + 2(-\sin x \cdot \operatorname{sh} y)_{dxdy} =$

$= -\operatorname{ch} y \cdot \cos x\, dx^2 + \cos x \cdot \operatorname{ch} y\, dy^2 + 2\underbrace{(\sin x \cdot \operatorname{sh} y)}_{u''_{xy}} dx dy$

$d^3u = (-\operatorname{ch} y \cdot \cos x)'_x dx^3 + (\cos x \cdot \operatorname{ch} y)'_y dy^3 + (((-\operatorname{ch} y \cdot \cos x)'_x +$

$+ (\operatorname{sh} y \cdot \cos x)'_y) dx^2 dy) + (((-\operatorname{ch} y \cdot \sin x) + (\cos x \cdot \operatorname{ch} y)\, dx dy^2))'_{xy} =$

$= \underbrace{\sin x \cdot \operatorname{ch} y}_{u'''_{x^3}} dx^3 + \underbrace{\cos x \cdot \operatorname{sh} y}_{u'''_{y^3}} dy^3 - 3 \cos x \operatorname{sh} y\, dx^2 dy -$

$- 3 \sin x \operatorname{ch} y\, dx\, dy^2$

$\boxed{N\ 3285}$

$u = f(\underline{x}, \underline{xy}, \underline{xyz})$ $\qquad du - ? \quad d^2u - ?$

$\nabla \; u'_x = f'_1 \cdot 1 + f'_2 \cdot y \cdot 1 + f'_3 yz \cdot 1$

$u'_y = f'_1 \cdot 0 + f'_2 \cdot x \cdot 1 + f'_3 x \cdot z \cdot 1 = f'_2 x + f'_3 xz$

$u'_z = f'_1 \cdot 0 + f'_2 \cdot 0 + f'_3 x \cdot y = f'_3 \cdot xy$

$$U''_{x^2} = f''_{11}\cdot 1 + f''_{12}y + f''_{13}yz + y\cdot(f''_{21}\cdot 1 + f''_{22}y + f''_{23}yz)$$
$$+ yz(f''_{31}\cdot 1 + f''_{32}\cdot y + f''_{33}yz) \Rightarrow$$

$$U''_{x^2} = f''_{11} + 2y f''_{12} + 2yz\cdot f''_{13} + y^2\cdot f''_{22} + 2y^2 z\cdot f''_{23} +$$
$$+ y^2 z^2 f''_{33}$$

$$U''_{y^2} = x\cdot(f''_{21}\cdot 0 + f''_{22}x + f''_{23}xz) + xz(f''_{31}\cdot 0 + f''_{32}\cdot x +$$
$$+ f''_{33}xz) = x^2\cdot f''_{22} + 2x^2 z\cdot f''_{23} + x^2 z^2 f''_{33}$$

$$U''_{z^2} = xy\ (f''_{31}\cdot 0 + f''_{32}\cdot 0 + f''_{33}xy) = x^2y^2\cdot f''_{33}$$

$$U''_{xy} = (\underbrace{f'_2\cdot x + f'_3\cdot xz}_{u'_y})'_x = (f''_{21}\cdot 1 + f''_{22}y + f''_{23}yz)x +$$
$$+ f'_2\cdot 1 + z((f''_{31}\cdot 1 + f''_{32}y + f''_{33}yz)x + f'_3\cdot 1)$$

$$U'_{xz} = y\underbrace{(f'_3 x)}_{u'_z}{}'_x = y(f''_{31} + f''_{32}\cdot y + f''_{33}yz) + y\cdot f'_3\cdot 1 =$$
$$= y(f''_{31} + f''_{32}\cdot y + f''_{33}yz + f'_3)$$

$$U''_{yz} = x\underbrace{(f'_3 y)}_{u'_z}{}'_y = x(f''_{32}\cdot x + f''_{33}xz) + x\cdot f'_3\cdot 1 =$$
$$= x(x(f''_{32} + f''_{33}z)) + x\cdot f'_3 = x^2(f''_{32} + f''_{33}z) + x\cdot f'_3$$

$\boxed{N\ 3622}$

$z = x^2 - (y-1)^2 = x^2 - y^2 + 2y - 1$

екстремуми – ?

1. $\begin{cases} U'_x = 2x = 0 \\ U'_y = -2y + 2 = 0 \end{cases}$ $\begin{cases} x = 0 \\ y = 1 \end{cases}$

$M(0;1)$ – стаціонарна точка

2. Знаход. всеможливі другі часткові похідні:

$U''_{x^2} = (2x)'_x = 2$

$U''_{y^2} = (-2y+2)'_y = -2$

$U''_{xy} = U''_{yx} = (2x)'_y = 0$

$A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$A_1 = 2$

$A_2 = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 < 0$

В т. M ф-я не має екстремуму

$\boxed{N\ 3627}$

$U = x^4 + y^4 - x^2 - 2xy - y^2$

1. $\begin{cases} U'_x = 4x^3 - 2x - 2y = 0 \\ U'_y = 4y^3 - 2x - 2y = 0 \end{cases}$ $\ominus$ $4(x^3 - y^3) = 0$

$x^3 = y^3$

$x = y$

$4x^3 - 4x = 0$

$4x(x^2 - 1) = 0$

$x = 0 \quad x = 1 \quad x = -1$

$y = 0 \quad y = 1 \quad y = -1$

$M_1(0;0)$ і $M_2(1;1)$; $M_3(-1;-1)$ – стаціон. точки

2. Знаход. всеможл. другі часткові похідні

$U''_{x^2} = 12x^2 - 2$ $\qquad U''_{x^2}(M_1) = U''_{x^2}(M_2) = 10$

$U''_{y^2} = 12y^2 - 2$ $\qquad U''_{y^2}(M_1) = U''_{y^2}(M_2) = 10$

$U''_{xy} = U''_{yx} = -2$

• Для точок $M_2(1;1)$ та $M_3(-1;1)$

$A = \begin{pmatrix} 10 & -2 \\ -2 & 10 \end{pmatrix}$ $\qquad A_1 = 10 > 0$

$A_2 = \begin{vmatrix} 10 & -2 \\ -2 & 10 \end{vmatrix} = 100 - 4 = 96 > 0$

Отже, т. $M_2(1;1)$ та $M_3(-1;-1)$ є <u>точками</u> <u>мінімуму</u>

$U(M_2) = U(M_3) = 1+1-1-2-1 = -2$

$M_1(0;0)$ $\qquad U''_{x^2}(M_1) = -2$ $\qquad U''_{y^2}(M_1) = -2$

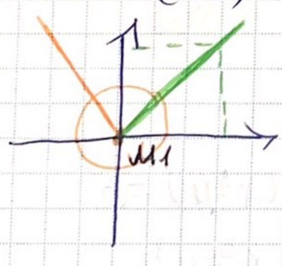
$A = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}$ $\qquad A_1 = -2 < 0$

$A_2 = \begin{vmatrix} -2 & -2 \\ -2 & -2 \end{vmatrix} = \underline{4-4=0}$

<u>Не маємо відповіді, бо</u> $\underline{A_2 = 0}$

• Для т. $M_1(0;0)$ провод. додаткові дослідж.

$\Delta U = U(\Delta x, \Delta y) - \underbrace{U(0;0)}_{0} = U(\Delta x, \Delta y)$



$\Delta x = \Delta y = h$

$\Delta U = U(h,h) = h^4 + h^4 - h^2 - 2h^2 - h^2 =$

$= 2h(h^2 - 2) < 0$, якщо $h < \sqrt{2}$

$\Delta y = -\Delta x = h$

$\Delta U = U(h;-h) = h^4 + h^4 \cancel{-h^2} \cancel{+2h^2} \cancel{-h^2} = 2h^4 > 0$ завжди

В т. <u>$M_1(0;0)$</u> ф-я <u>не має</u> <u>екстремумів.</u>

$\boxed{№3621}$

$U = x^2 + (y-1)^2 = x^2 + y^2 - 2y + 1$

екстремуми - ?

1. $\begin{cases} U'_x = 2x = 0 \\ U'_y = 2y - 2 = 0 \end{cases}$ $\begin{cases} x = 0 \\ y = 1 \end{cases}$

$M(0;1)$ - стац. точка

2. Знаход. всеможлив. другі часткові похідні

$U''_{x^2} = (2x)'_x = 2$

$U''_{y^2} = (2y-2)'_y = 2$

$U''_{xy} = U''_{yx} = (2x)'_y = 0$

$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad A_1 = 2$

$A_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 - 0 = 4 > 0$

$A_2 > 0$, то точка M є точкою мінім.

$U_{min} = U(M) = U(0;1) = 0 + 1 - 2 \cdot 1 + 1 = \boxed{0}$