Astrophysical Objects

General relativity

Based on: An introduction to modern Astrophysics chapter 17

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General Relativity

Gravity plays a fundamental role in sculpting the universe on the largest scale. Newton's law of universal gravitation,

$$F = G\frac{Mm}{r^2},$$

remained a cornerstone of astronomers' understanding of planetary motions until the beginning of the twentieth century.

Its application had **explained the motions of the known planets** and had accurately predicted the existence and position of the planet Neptune in 1846.

However the **problem** with Newtonian gravitation was the inexplicably **large rate of shift in the orientation of Mercury's orbit.**

General Relativity

The gravitational influences of the other planets cause the major axis of Mercury's elliptical orbit to slowly swing around the Sun in a counterclockwise direction relative to the fixed stars. -> **perihelion precession**

The angular position at which perihelion occurs shifts at a rate of $574''$ per century.

However, Newton's law of gravity predicted only $38''$ per century of this shift. -> inconsistency

This led some physicists to suggest that Newton's law should be modified from an exact inverse-square law.

Others thought that an unseen planet, nicknamed Vulcan, might occupy an orbit inside Mercury's.

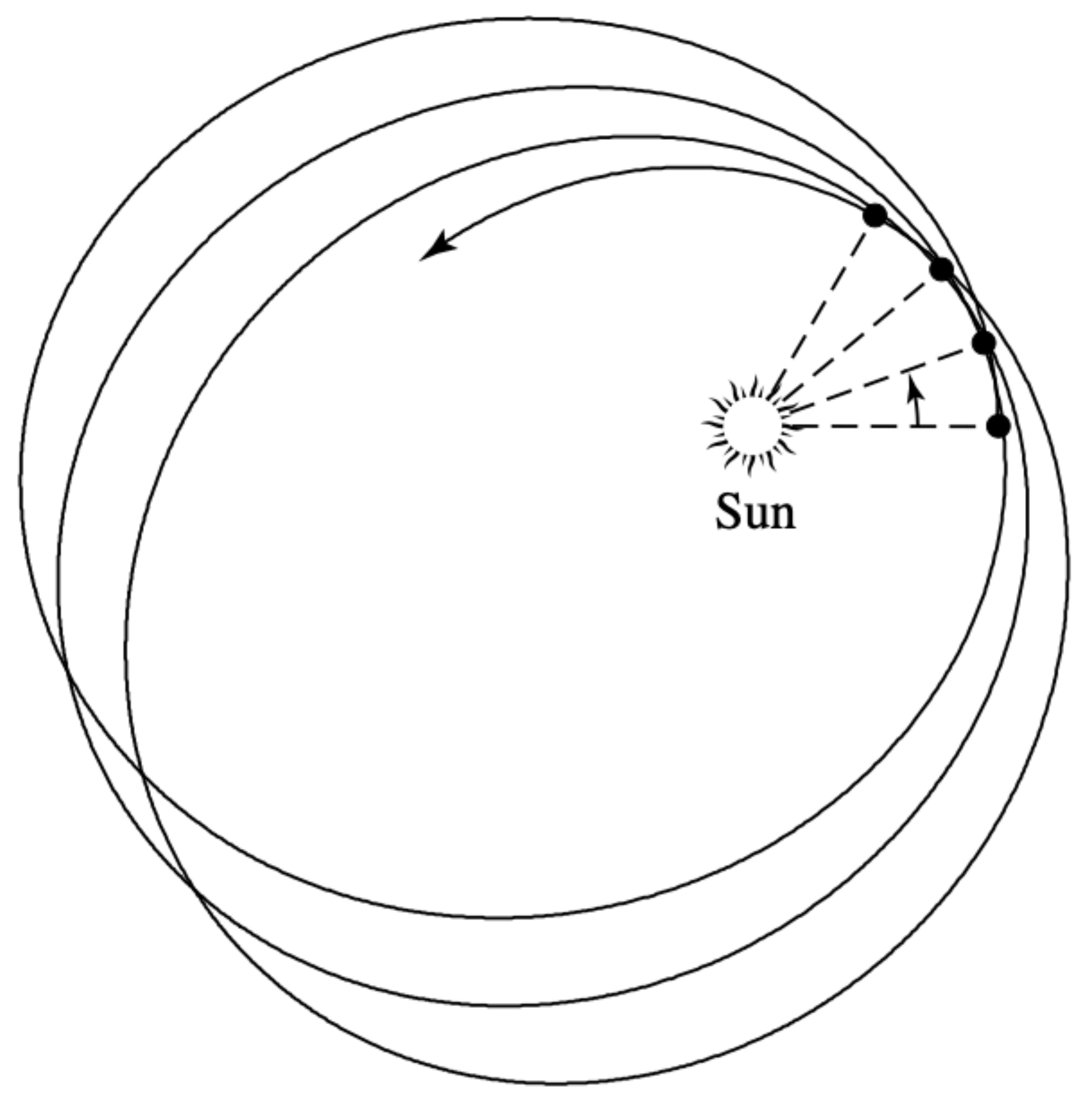

FIGURE 1 The perihelion shift of Mercury's orbit. Both the eccentricity of the orbit and the amount of shift in the location of perihelion in successive orbits have been exaggerated to better show the effect.

The Curvature of spacetime

Between the years 1907 and 1915, Albert Einstein developed a new theory of gravity, his **general theory of relativity**.

- In addition to resolving the mystery of Mercury's orbit,
- it predicted many new phenomena that were later confirmed by experiment.

The general theory of relativity is fundamentally a geometric description of how distances (intervals) in spacetime are measured in the presence of mass. For the moment, the effects on space and time will be considered separately, although you should always keep in mind that relativity deals with a unified spacetime. Near an object, both space and time must be described in a new way.

The Curvature of spacetime

Distances between points in the space surrounding a massive object are altered in a way that can be interpreted as **space becoming *curved*** through a fourth spatial dimension perpendicular to *all* of the usual three spatial directions.

The fact that **mass has an effect on the surrounding space** is the first essential element of general relativity.

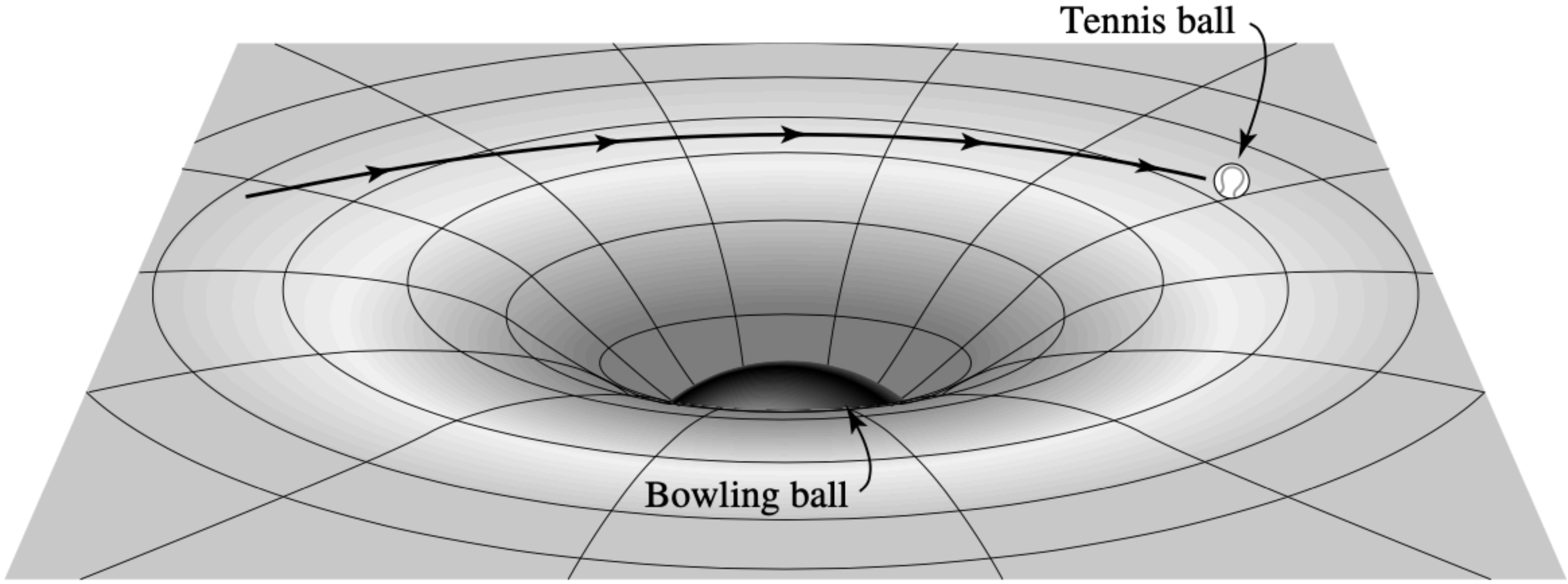

FIGURE 2 Rubber sheet analogy for curved space around the Sun. It is assumed that the rubber sheet is much larger than the area of curvature, so that the edges of the sheet have no effect on the curvature produced by the central mass.

The Curvature of spacetime

The passage of a ray of light near the Sun -> the bend of the photon's trajectory is small because the photon's speed carries it quickly through the curved space.

Can this effect be observed?

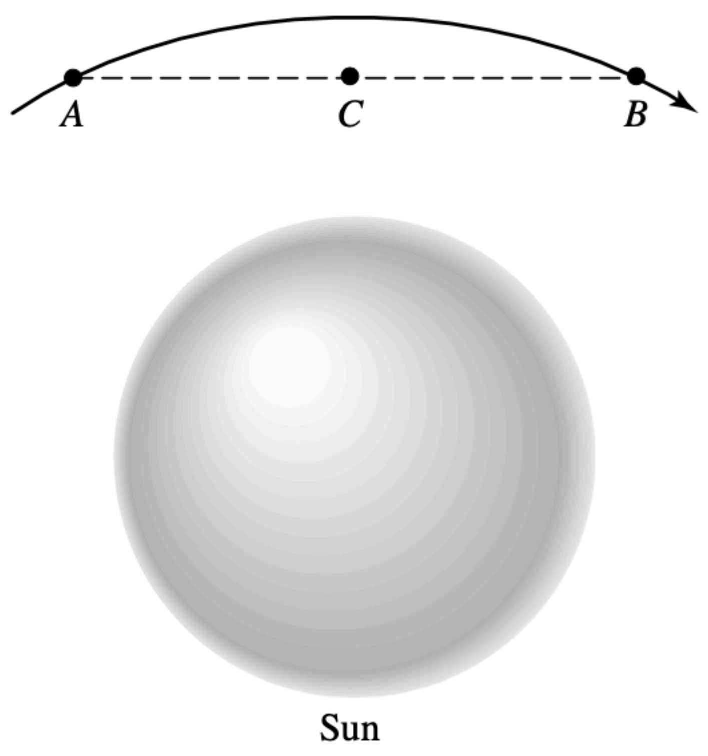

FIGURE 3 A photon's path around the Sun is shown by the solid line. The bend in the photon's trajectory is greatly exaggerated.

The Curvature of spacetime

The passage of a ray of light near the Sun -> the bend of the photon’s trajectory is small because the photon’s speed carries it quickly through the curved space.

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- Prediction: light bending 1.75 arcseconds for light that grazes the Sun.

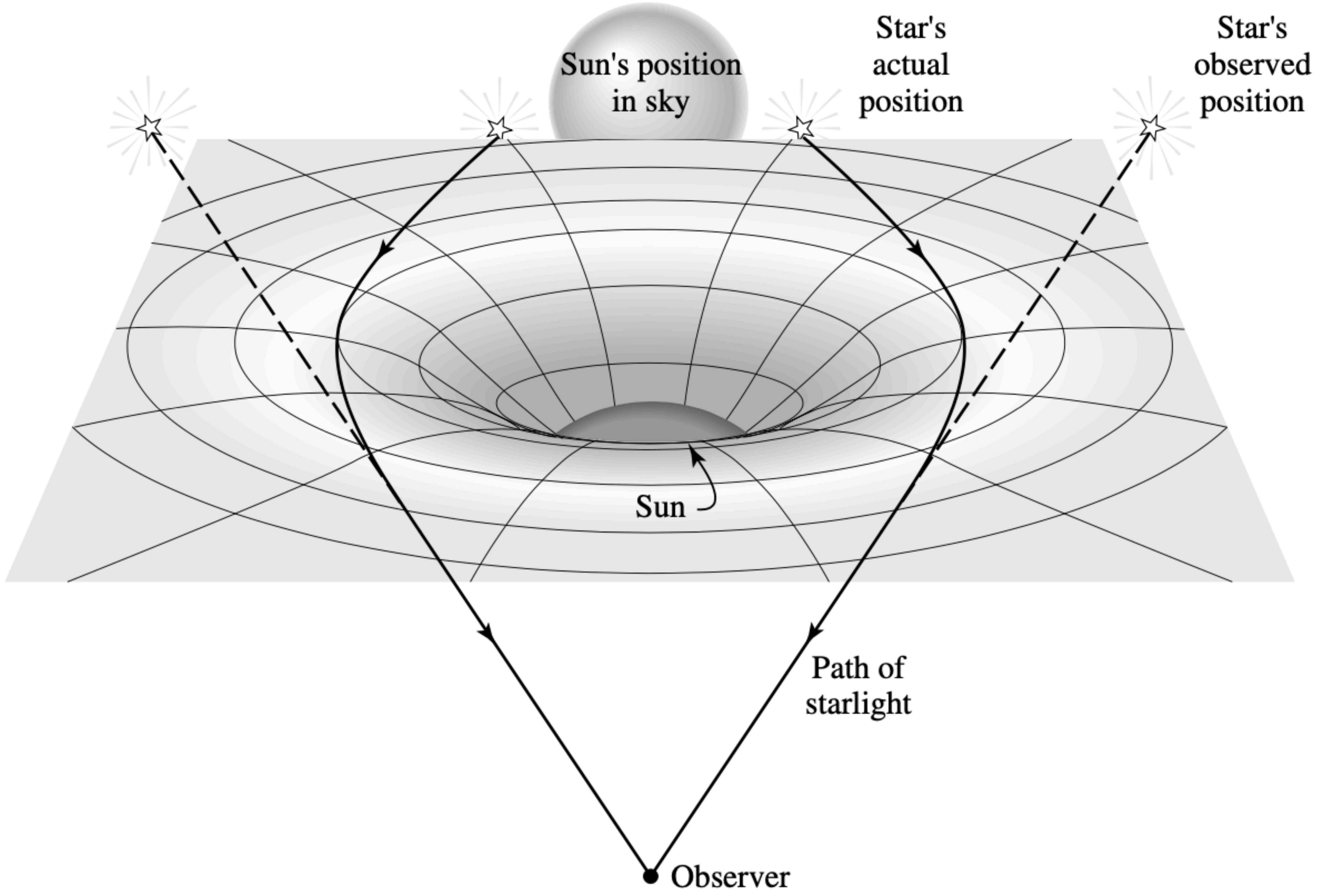


FIGURE 5 Bending of starlight measured during a solar eclipse.

The Curvature of spacetime

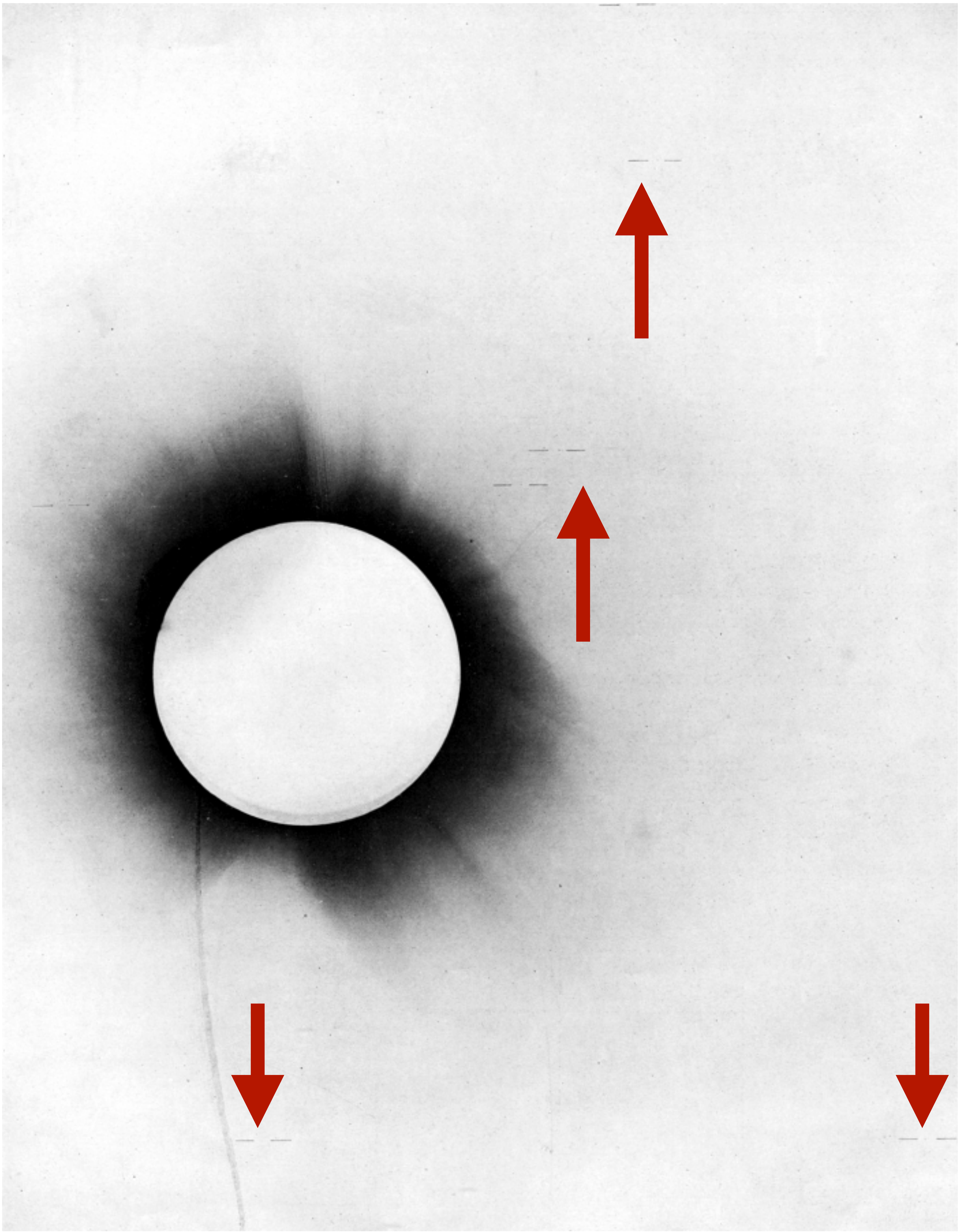
The passage of a ray of light near the Sun -> the bend of the photon's trajectory is small because the photon's speed carries it quickly through the curved space.

Can this effect be observed?

- Prediction: light bending 1.75 arcseconds for light that grazes the Sun.
- Observations: change in position of stars as they passed near the Sun, confirmed in 1919 during a solar eclipse

In general relativity, **gravity is the result of objects moving through curved spacetime, and everything that passes through, even massless particles such as photons, is affected.**

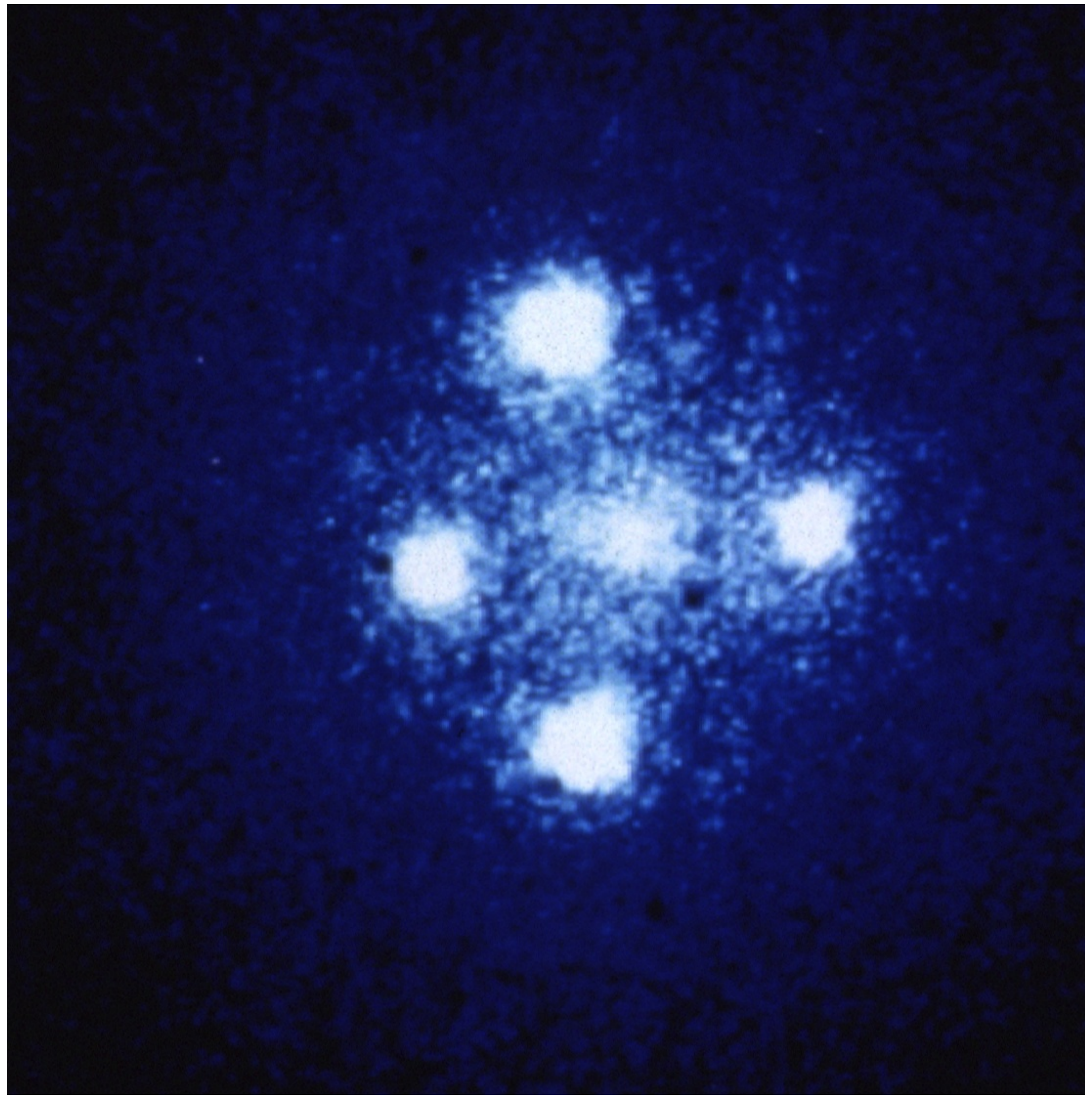
Image of the Solar eclipse in 1919 - shows the position of the stars



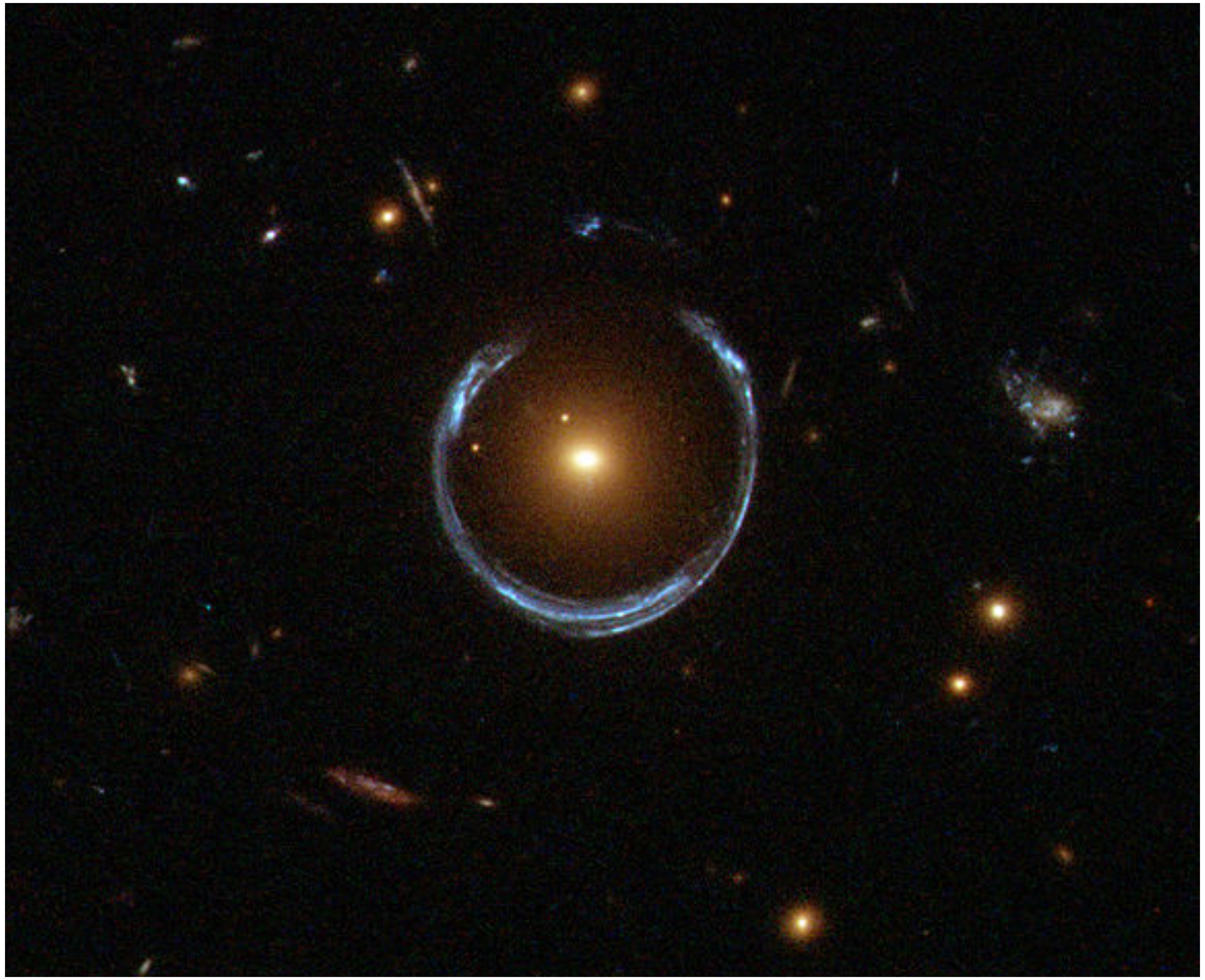
The Curvature of spacetime

More modern observations of the bending of light:
gravitational lensing

Multiple images like this are called Einstein crosses.



Rings like this are now known as Einstein Rings



The Curvature of spacetime

More modern observations of the bending of light:
gravitational lensing

https://www.eso.org/public/videos/eso1522b/

The Curvature of spacetime

In addition, the curvature of space involves a concomitant slowing down of time, so clocks placed along the dashed path would actually *run more slowly*. This is the final essential feature of general relativity: ***Time runs more slowly in curved spacetime.***

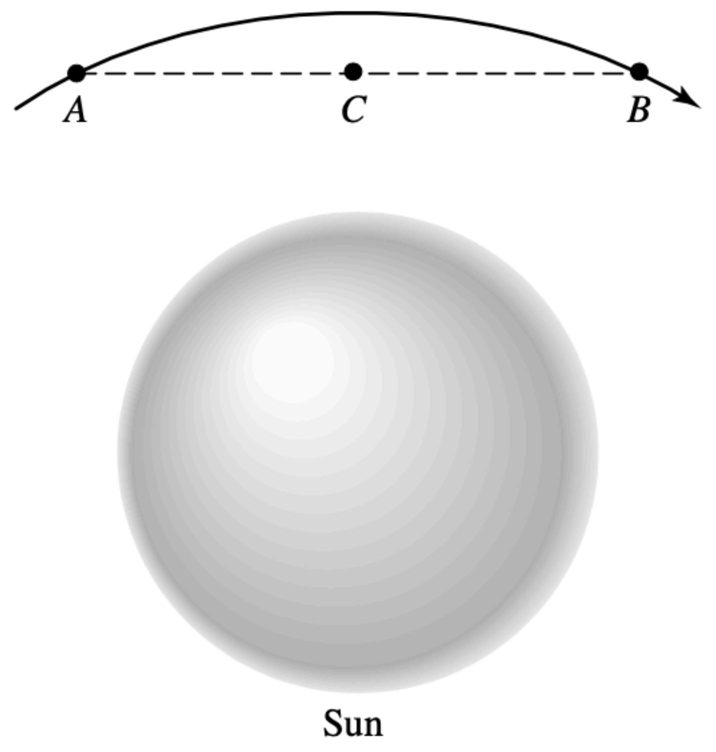


FIGURE 3 A photon's path around the Sun is shown by the solid line. The bend in the photon's trajectory is greatly exaggerated.

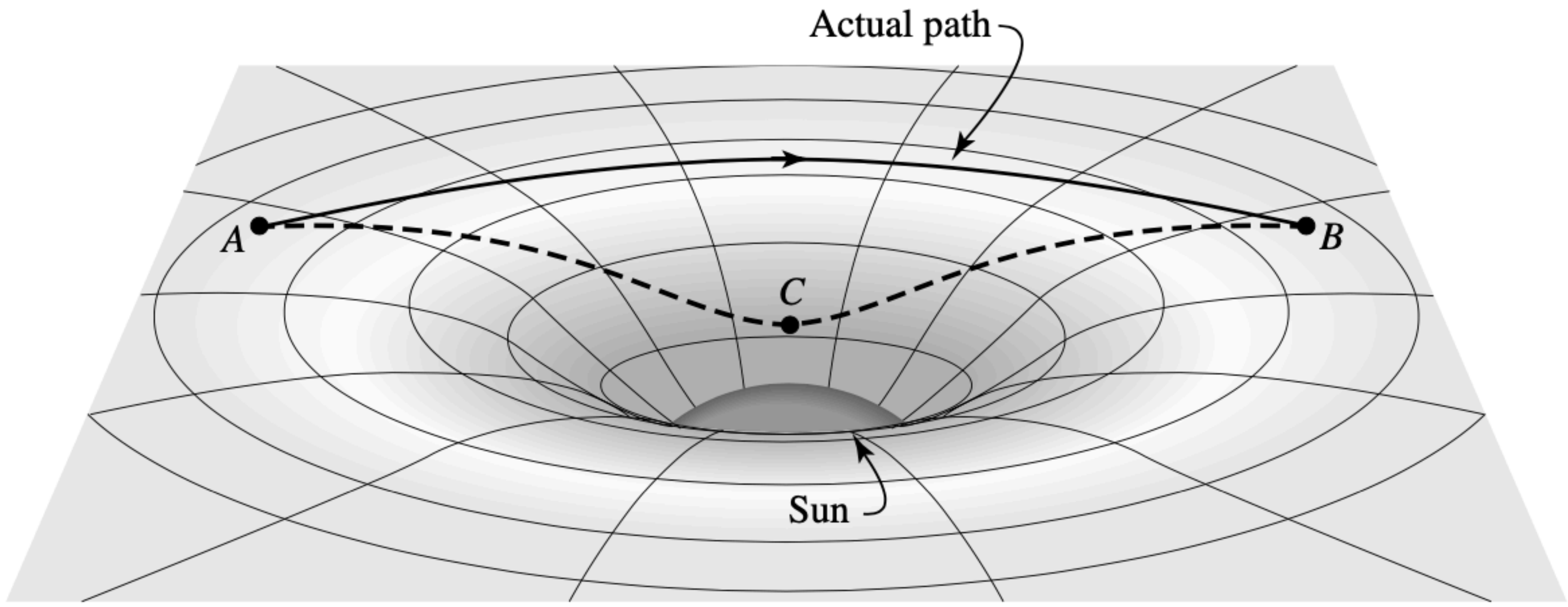


FIGURE 4 Comparison of two photon paths through curved space between points A and B. The projection of the path ACB onto the plane is the straight line depicted in Fig. 3.

The principle of equivalence

One of the postulates of **special relativity** states that the laws of physics are the same in all **inertial reference frames.**

Accelerating frames of reference are not inertial frames.

The Principle of Equivalence: All local, freely falling, nonrotating reference frames are fully equivalent for the performance of all physical experiments. -> **local inertial reference frames.**

Note that special relativity is incorporated into the principle of equivalence. For example, measurements made from two local inertial frames in relative motion are related by the **Lorentz transformations** using the ***instantaneous* value of the relative velocity** between the two frames. Thus general relativity is an extension of the theory of special relativity.

$$\boxed{\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma\left(x - vt\right) \\ y' &= y \\ z' &= z \end{aligned}}$$

The bending of light

Two simple thought experiments:

For the first experiment, imagine a laboratory suspended above the ground by a cable [see Fig. 9(a)].

Let a photon of light leave a horizontal flashlight at the same instant the cable holding the lab is severed [Fig. 9(b)].

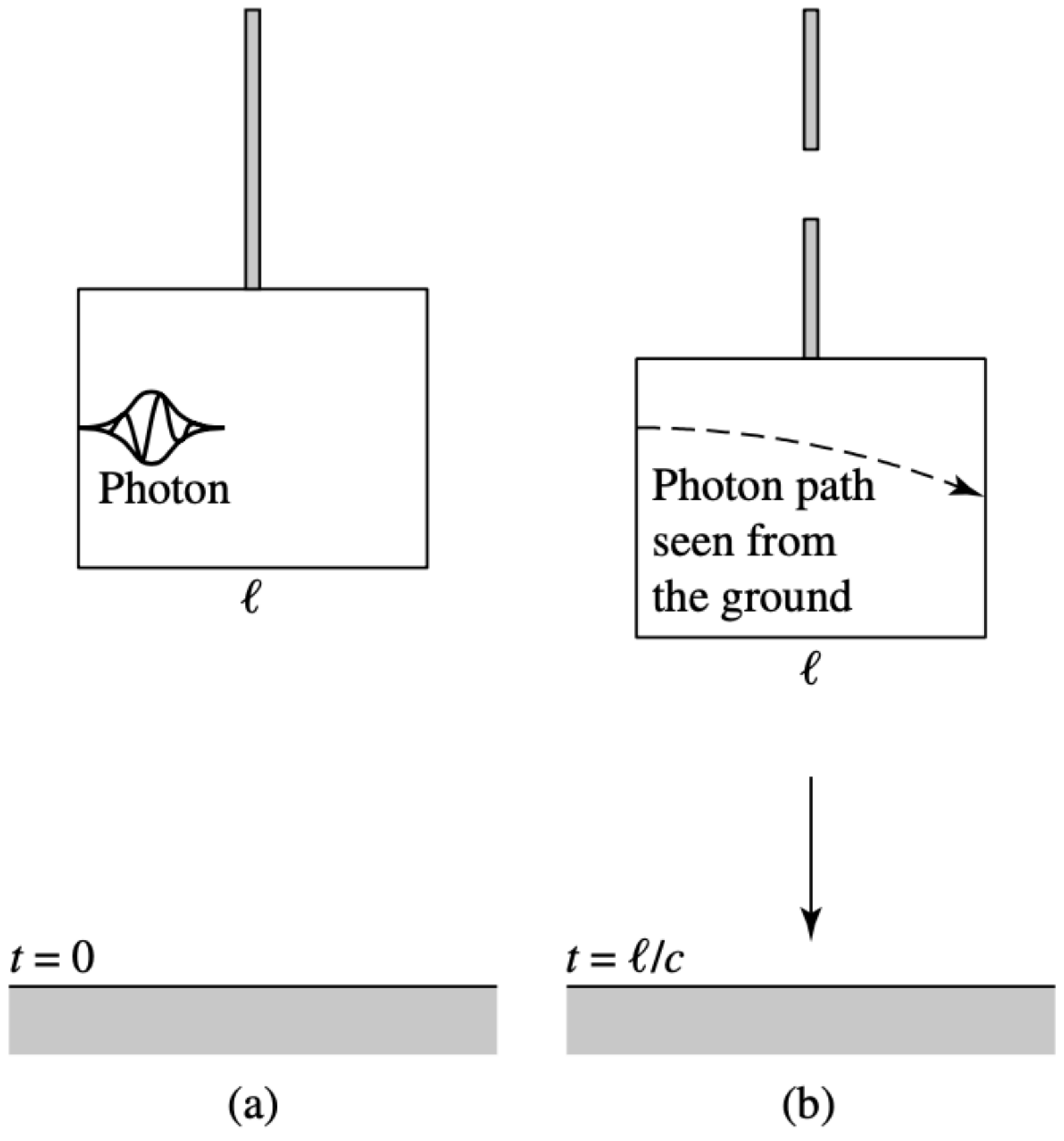


FIGURE 9 The equivalence principle for a horizontally traveling photon. The photon (a) leaves the left wall at $t = 0$, and (b) arrives at the right wall at $t = \ell/c$.

Gravity has been abolished from this freely falling lab, so it is a local inertial reference frame.

According to the equivalence principle, an **observer falling with the lab** will measure the **light's path** across the room as a **straight horizontal line,** in agreement with all of the laws of physics.

But another observer on the ground sees a lab that is falling under the influence of gravity. Because the photon maintains a constant height above the lab's floor, the **ground observer must measure a photon that falls with the lab**, following a **curved path**. The curved path taken by the photon is the quickest route possible through the curved spacetime surrounding Earth.

The bending of light

The angle of deflection, ϕ, of the photon is very slight.

Although the photon does not follow a circular path, we will use the ***best-fitting circle*** of radius r_c to the actual path measured by the ground observer.

The center of the best-fitting circle is at point O, and the arc of the circle subtends an angle ϕ (exaggerated in the figure) between the radii OA and OB. If the width of the lab is l, then the photon crosses the lab in time t = l/c. (The difference between the length of the arc and the width of the lab is negligible.)

In this amount of time, the lab falls a distance

$$d = \frac{1}{2}gt^2$$

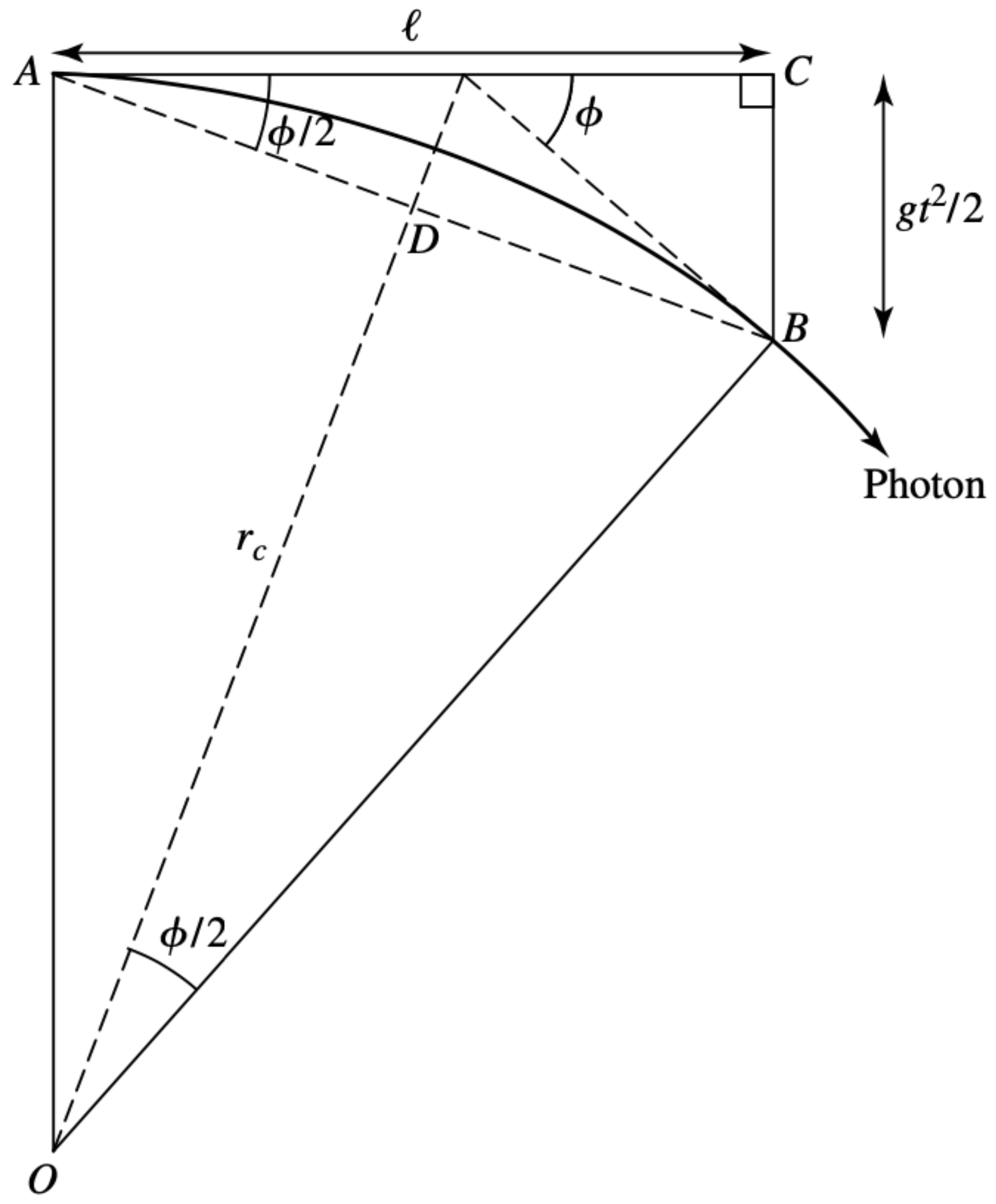


FIGURE 10 Geometry for the radius of curvature, r_c, and angular deflection, ϕ.

The bending of light

Because triangles ABC and OBD are similar (each containing a right angle and another angle $\phi/2$),

$$\overline{BC}/\overline{AC} = \overline{BD}/\overline{OD}$$

$$\left(\frac{1}{2}gt^2\right) \Big/ \ell = \left[\frac{\ell}{2\cos(\phi/2)}\right] \Big/ \overline{OD}.$$

In fact, ϕ is so small that we can set $\cos(\phi/2) \simeq 1$ and the distance OD $\simeq r_c$. Then, using

$t = l/c$ and $g = 9.8\ \mathrm{m\ s^{-2}}$ for the acceleration of gravity near the surface of Earth, we find

$$r_c = \frac{c^2}{g} = 9.17 \times 10^{15}\ \mathrm{m},$$

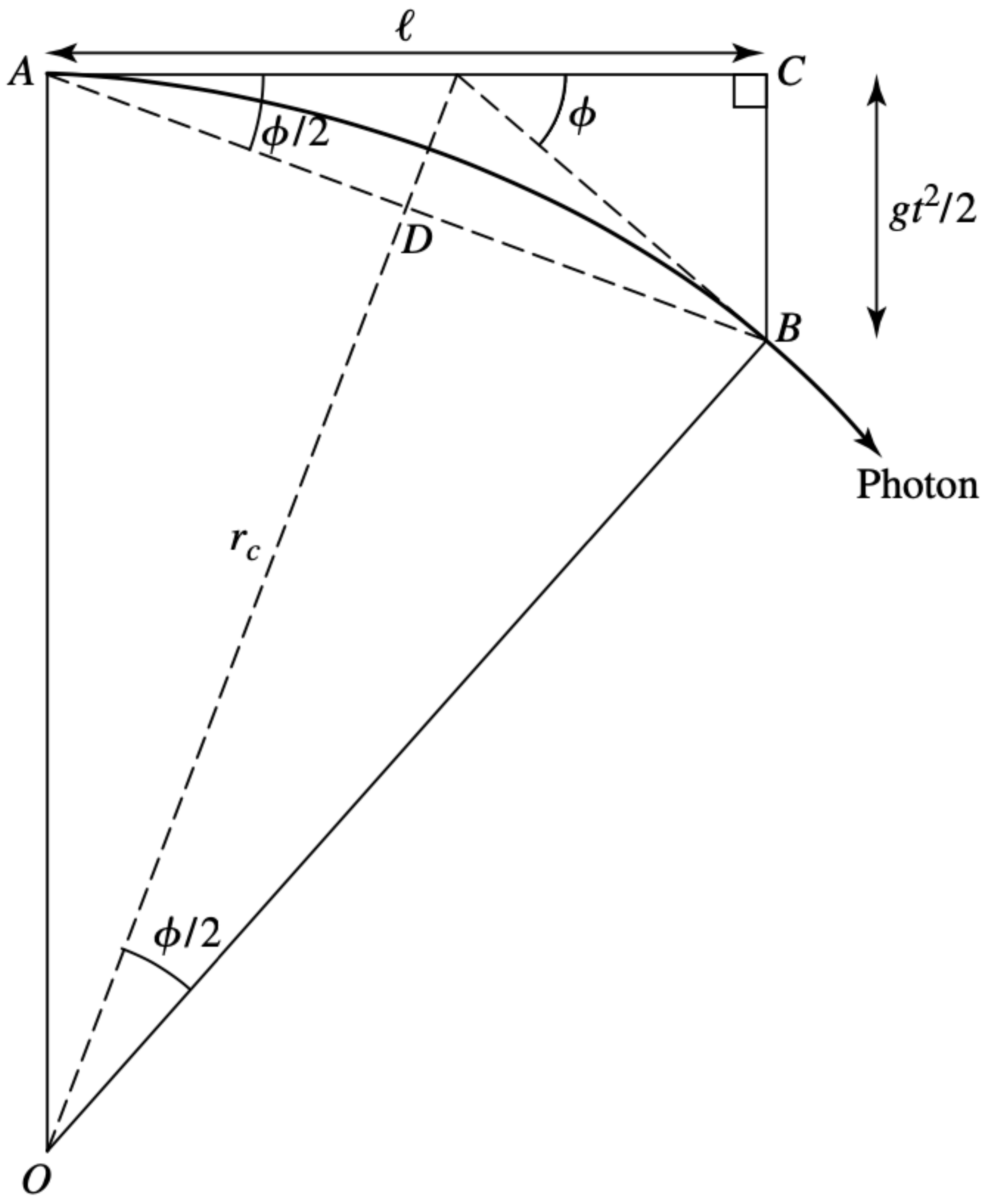


FIGURE 10 Geometry for the radius of curvature, r_c, and angular deflection, ϕ.

for the **radius of curvature of the photon’s path**, which is nearly a light-year!

The bending of light

Of course, the angular deflection φ depends on the width l of the lab. For example, if l = 10 m, then

$$\phi = \frac{\ell}{r_c} = 1.09 \times 10^{-15} \text{ rad},$$

or only 2.25×10^{-10} arcsecond.

The large radius of the photon's path indicates that **spacetime near Earth is only slightly curved.**

Nonetheless, the curvature is great enough to produce the **circular orbits of satellites,** which move slowly (compared to the speed of light) through the curved spacetime.

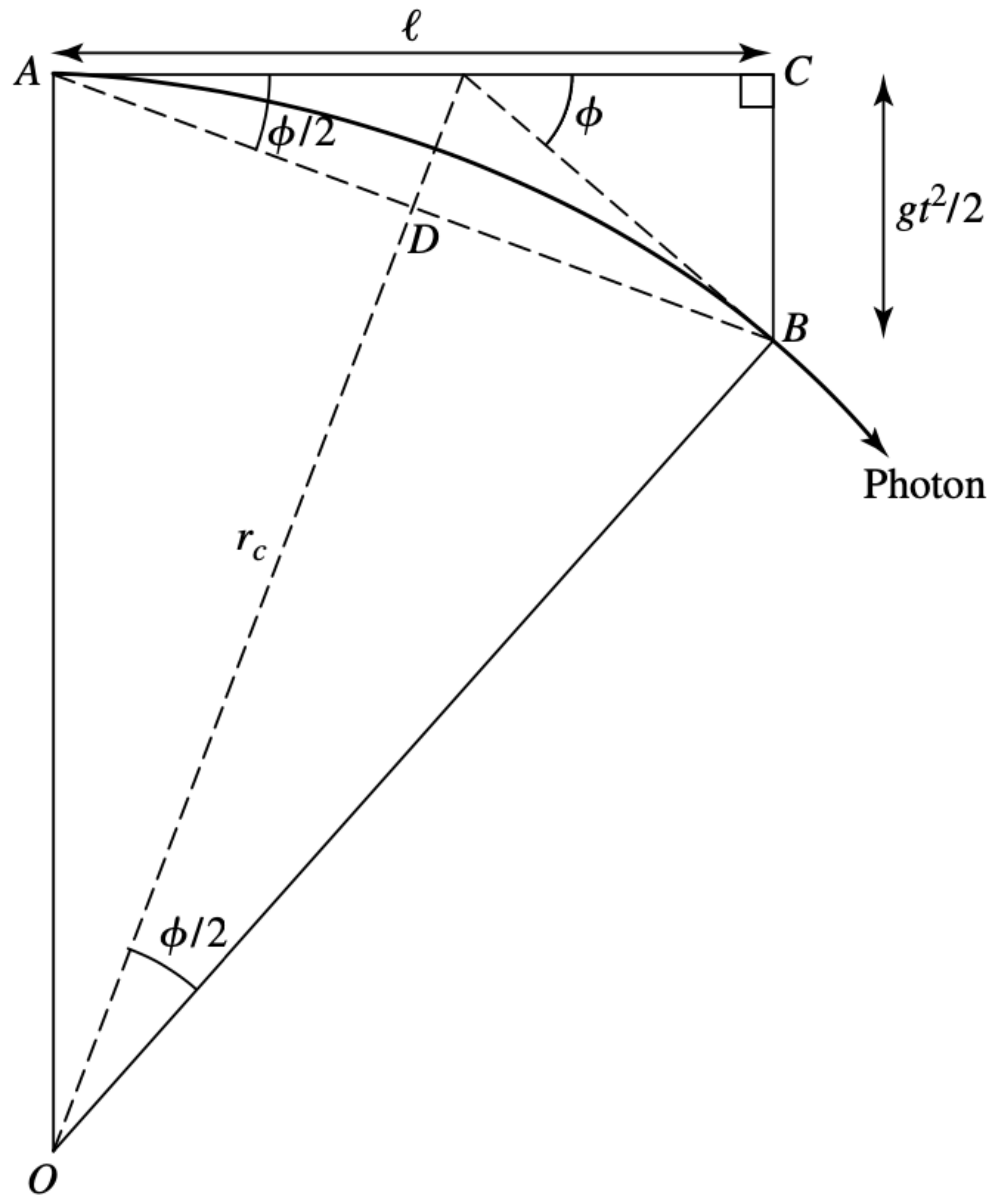


FIGURE 10 Geometry for the radius of curvature, r_c, and angular deflection, ϕ.

Gravitational redshift

Gravitational redshift is not the same as regular redshift due to the motion of the light source!

Our second thought experiment also begins with the laboratory suspended above the ground by a cable.

This time, **monochromatic light of frequency ν_0** leaves a vertical flashlight on the floor at the same instant the cable holding the lab is severed.

The freely falling lab is again a **local inertial frame** where gravity has been abolished, and so the equivalence principle requires that a **frequency meter in the lab's ceiling record the *same* frequency, ν_0,** for the light that it receives.

But an **observer on the ground** sees a lab that is falling under the influence of gravity. As shown in Fig. 11, if the light has traveled upward a height h toward the meter in time t = h/c, then the **meter has gained a downward speed** toward the light of v = gt = gh/c since the cable was released.

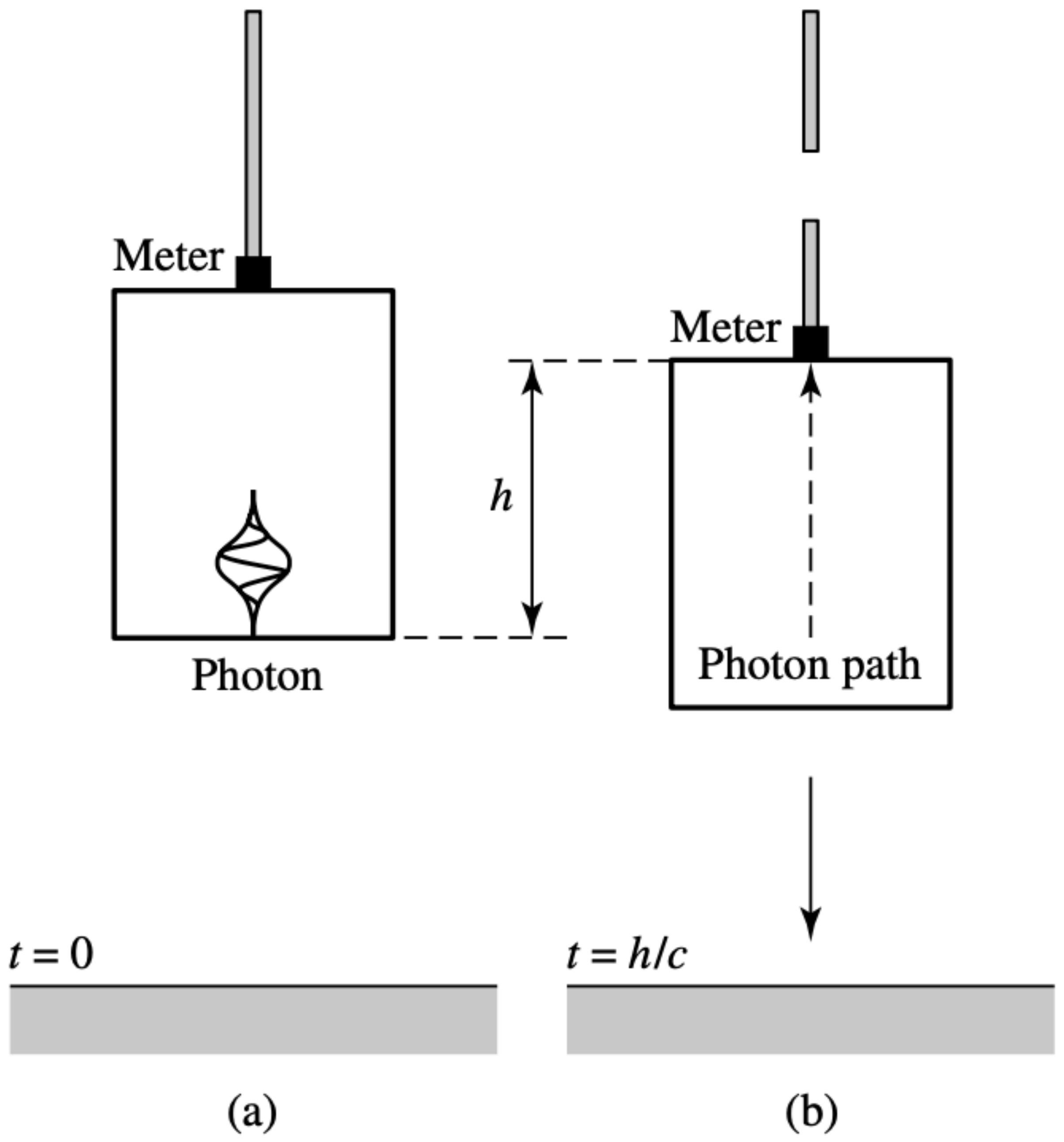


FIGURE 11 Equivalence principle for a vertically traveling light. The photon (a) leaves the floor at $t = 0$, and (b) arrives at the ceiling at $t = h/c$.

Gravitational redshift

Accordingly, we would expect that from the point of view of the **ground observer**, the meter should have measured a **blueshifted frequency** greater than ν_0. For the slow free-fall speeds involved here, this expected *increase* in frequency is

$$\frac{\Delta\nu}{\nu_0} = \frac{v}{c} = \frac{gh}{c^2}.$$

But in fact, the meter recorded *no change* in frequency. Therefore there must be another effect of the light's upward journey through the curved spacetime around Earth that exactly compensates for this blueshift. This is a **gravitational redshift** that tends to *decrease* the frequency of the light as it travels upward a distance h, given by

$$\frac{\Delta\nu}{\nu_0} = -\frac{v}{c} = -\frac{gh}{c^2}.$$

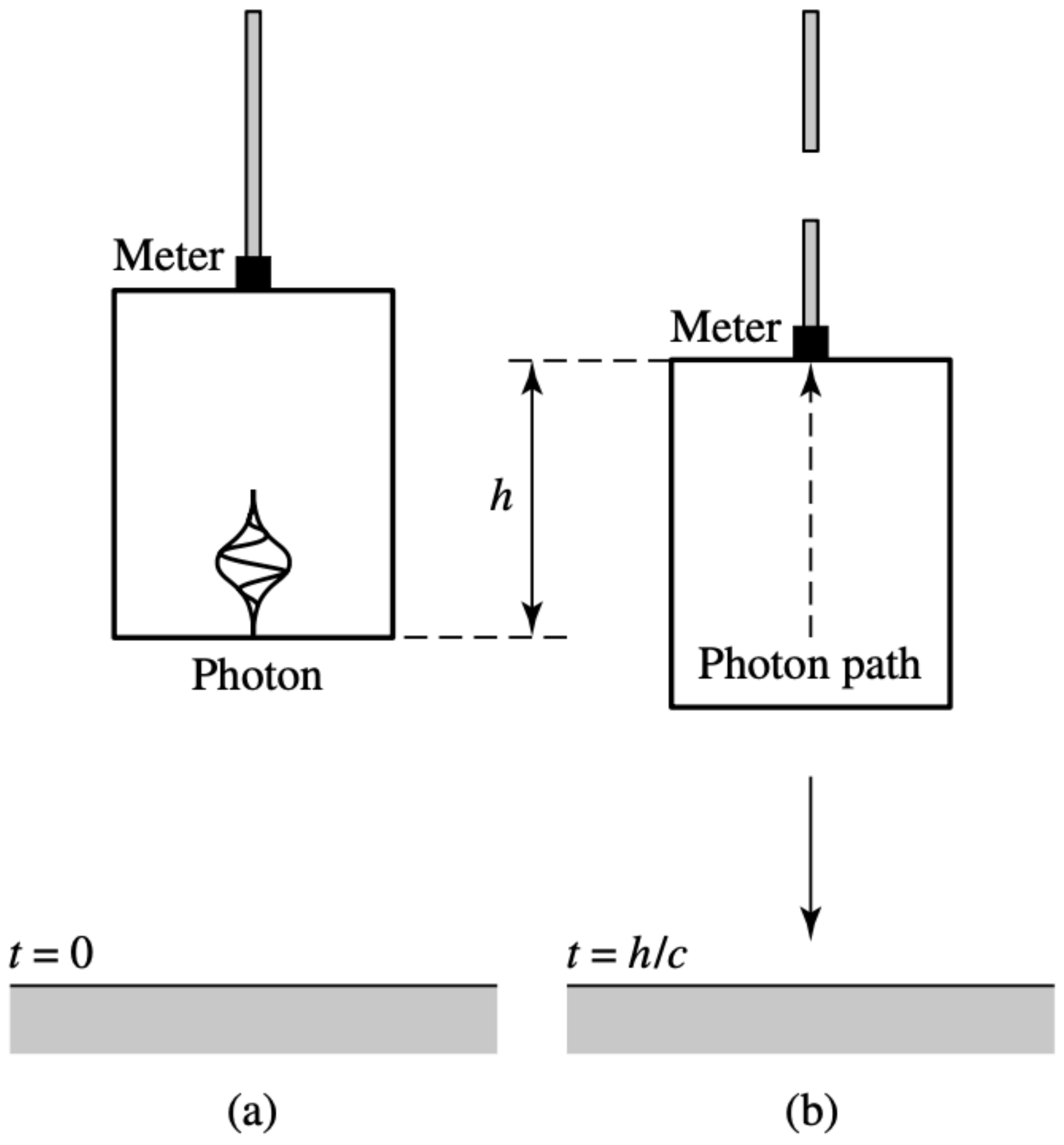


FIGURE 11 Equivalence principle for a vertically traveling light. The photon (a) leaves the floor at $t = 0$, and (b) arrives at the ceiling at $t = h/c$.

Gravitational redshift

An outside observer, not in free-fall inside the lab, would measure only this gravitational redshift. If the light were traveling downward, a corresponding blueshift would be measured. It is left as an exercise to show that this formula remains valid even if the light is traveling at an angle to the vertical, as long as h is taken to be the *vertical* distance covered by the light.

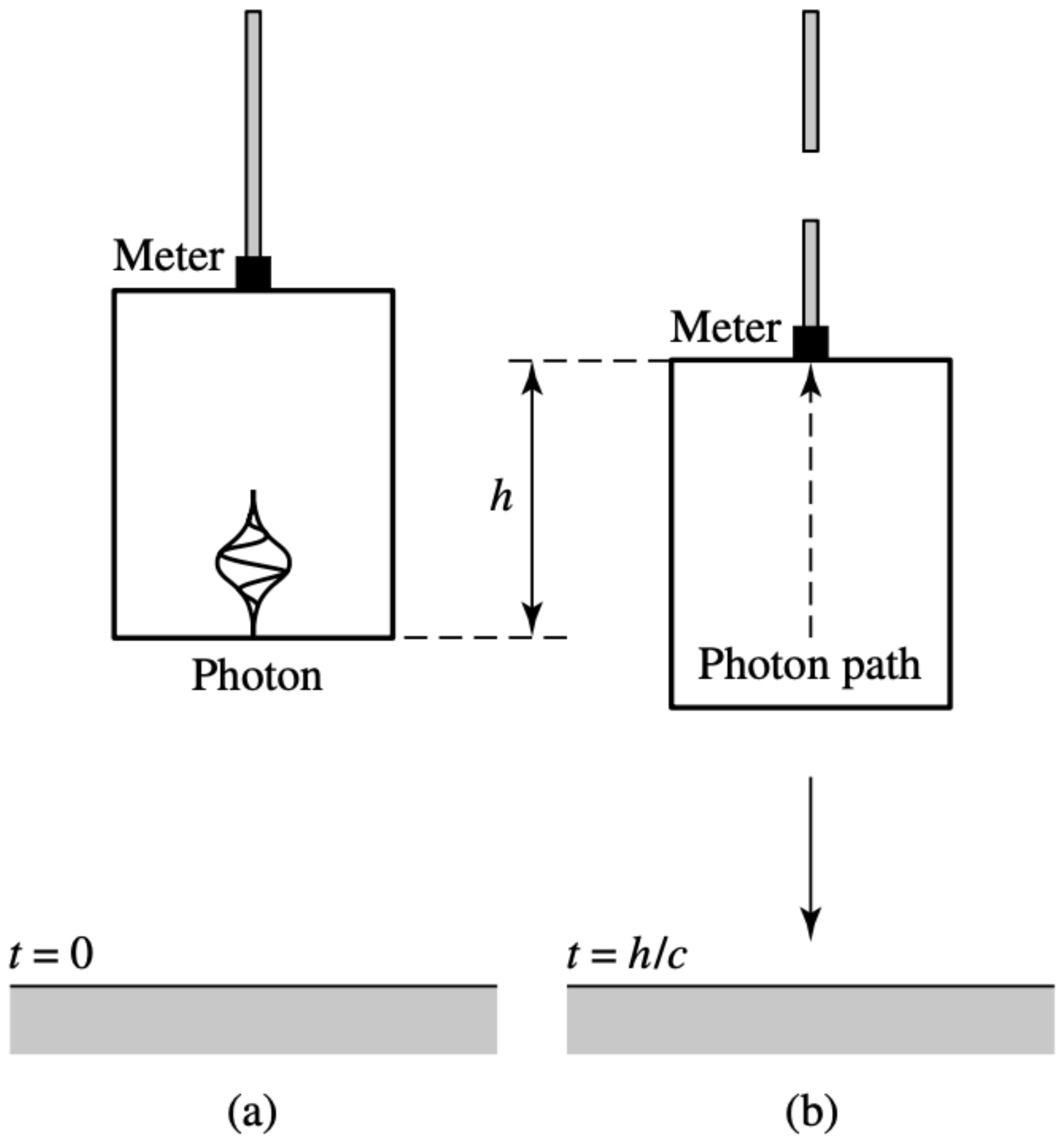


FIGURE 11 Equivalence principle for a vertically traveling light. The photon (a) leaves the floor at $t = 0$, and (b) arrives at the ceiling at $t = h/c$.

Gravitational redshift

Example experiment: In 1960, a test of the gravitational redshift formula was carried out.

A gamma ray was emitted by an unstable isotope of iron, $^{57}_{26}Fe$, at the bottom of a tower 22.6 m tall, and received at the top of the tower.

Using this value for h, the expected decrease in frequency of the gamma ray due to the gravitational redshift is

$$\frac{\Delta\nu}{\nu_0} = -\frac{gh}{c^2} = -2.46 \times 10^{-15},$$

in excellent agreement with the experimental result of $\frac{\Delta\nu}{\nu} = -(2.57 \pm 0.26) \times 10^{-15}$. More precise experiments carried out since that time have obtained agreement to within 0.007%.

The experiment was performed with both upward- and downward-traveling gamma rays, providing tests of both the gravitational redshift and blueshift.

Gravitational redshift

An approximate expression for the total gravitational redshift for a beam of light that escapes out to infinity

$$\frac{\Delta\nu}{\nu_0} = -\frac{v}{c} = -\frac{gh}{c^2}.$$

can be calculated by integrating from an initial position r_0 to infinity, using $g = GM/r^2$ (Newtonian gravity) and setting h equal to the differential radial element, dr for a spherical mass, M, located at the origin.

$$\frac{\Delta\nu}{\nu_0} = -\frac{gh}{c^2} = -2.46 \times 10^{-15},$$

Some care must be taken when carrying out the integration, because was derived using a *local* inertial reference frame. By integrating, we are really adding up the redshifts obtained for a chain of *different* frames.

The radial coordinate r can be used to measure distances for these frames only if spacetime is nearly flat. In

$$\int_{\nu_0}^{\nu_\infty} \frac{d\nu}{\nu} \simeq -\int_{r_0}^{\infty} \frac{GM}{r^2c^2}\,dr,$$

this case, the "stretching" of distances is not too severe, and we can integrate

Gravitational redshift

The result is

$$\ln\left(\frac{\nu_\infty}{\nu_0}\right) \simeq -\frac{GM}{r_0c^2},$$

which is valid **when gravity is weak** (r_0/r_c = GM/$r_0c^2 \ll 1$). This can be rewritten as

$$\frac{\nu_\infty}{\nu_0} \simeq e^{-GM/r_0c^2}.$$

Because the exponent is ≪1, we use $e^{-x} \simeq 1-x$ to get

$$\frac{\nu_\infty}{\nu_0} \simeq 1 - \frac{GM}{r_0c^2}.$$

This approximation shows the first-order correction to the frequency of the photon.

Gravitational redshift

The exact result for the **gravitational redshift, valid even for a strong gravitational field**, is

$$\boxed{\frac{\nu_\infty}{\nu_0} = \left(1 - \frac{2GM}{r_0 c^2}\right)^{1/2} .}$$

When gravity is weak and the exponent is $\ll 1$, we use $(1-x)^{1/2} \simeq 1-x/2$ for the approximation.

The gravitational redshift can be incorporated into the redshift parameter, giving

$$\begin{aligned} z &= \frac{\lambda_\infty - \lambda_0}{\lambda_0} = \frac{\nu_0}{\nu_\infty} - 1 \\ &= \left(1 - \frac{2GM}{r_0 c^2}\right)^{-1/2} - 1 \\ &\simeq \frac{GM}{r_0 c^2}, \end{aligned}$$

valid only for a weak gravitational field.

Gravitational redshift

To understand the origin of the gravitational redshift, imagine a clock that is constructed to tick once with each vibration of a monochromatic light wave. The time between ticks is then equal to the period of the oscillation of the wave, $\Delta t = 1/\nu$. Then according to $\boxed{\frac{\nu_\infty}{\nu_0} = \left(1 - \frac{2GM}{r_0 c^2}\right)^{1/2}.}$, as seen from an infinite distance, the **gravitational redshift implies that the clock at r_0 will be observed to run more slowly than an identical clock at r = ∞**. If an amount of time Δt_0 passes at position r_0 outside a spherical mass, M, then the time Δt_∞ at r = ∞ is

$$\boxed{\frac{\Delta t_0}{\Delta t_\infty} = \frac{\nu_\infty}{\nu_0} = \left(1 - \frac{2GM}{r_0 c^2}\right)^{1/2}}$$

For a weak field,

$$\frac{\Delta t_0}{\Delta t_\infty} \simeq 1 - \frac{GM}{r_0 c^2}.$$

Gravitational redshift

We must conclude that ***time passes more slowly as the surrounding spacetime becomes more curved***, an effect called **gravitational time dilation**. The gravitational redshift is therefore a consequence of time running at a slower rate near a massive object.

In other words, suppose two perfect, identical clocks are initially standing side by side, equally distant from a spherical mass. They are synchronized, and then one is slowly lowered below the other and then raised back to its original level. All observers will agree that when the clocks are again side by side, the clock that was lowered will be running behind the other because time in its vicinity passed more slowly while it was deeper in the mass's gravitational field.

Any examples for this?

Movies?

Gravitational redshift

We must conclude that ***time passes more slowly as the surrounding spacetime becomes more curved***, an effect called **gravitational time dilation**. The gravitational redshift is therefore a consequence of time running at a slower rate near a massive object.

EHT image of the supermassive blackhole in M87

Planet "Miller" orbiting the black hole "Gargantua" in the Movie "Interstellar"

Gravitational redshift

Example: The **white dwarf** Sirius B has a radius of R = 5.5×10^6 m and a mass of M = 2.1×10^{30} kg. The radius of curvature of the path of a horizontally traveling light beam near the surface of Sirius B is given by

$$r_c = \frac{c^2}{g} = \frac{R^2c^2}{GM} = 1.9 \times 10^{10} \text{ m}.$$

The fact that $GM/Rc^2 = R/rc \ll 1$ indicates that **the curvature of spacetime is not severe**. Even at the surface of a white dwarf, gravity is considered relatively weak in terms of its effect on the curvature of spacetime.

The gravitational redshift suffered by a photon emitted at the star's surface is

$$z \simeq \frac{GM}{Rc^2} = 2.8 \times 10^{-4}.$$

This is in excellent agreement with the measured gravitational redshift for Sirius B of $(3.0 \pm 0.5) \times 10^{-4}$.

Gravitational redshift

To compare the **rate at which time passes at the surface of Sirius B** with the rate at a great distance, suppose that exactly one hour is measured by a distant clock. The time recorded by a clock at the surface of Sirius B would be *less* than one hour by an amount found using:

$$\Delta t_\infty - \Delta t_0 = \Delta t_\infty \left(1 - \frac{\Delta t_0}{\Delta t_\infty}\right) \simeq (3600\ \mathrm{s}) \left(\frac{GM}{Rc^2}\right) = 1.0\ \mathrm{s}.$$

The clock at the surface of Sirius B **runs more slowly by about one second per hour** compared to an identical clock far out in space.

Experimental results confirm the curvature of spacetime.

Spacetime

We now consider the united concepts of space and time as expressed in *spacetime*, with four coordinates (x, y, z, t) specifying each *event*. ***Einstein's*** deduction of ***field equations*** for **calculating the geometry of spacetime produced by a given distribution of mass and energy**. His equations have the form

$$\boxed{\mathcal{G} = -\frac{8\pi G}{c^4}\mathcal{T}.}$$

On the right is the ***stress–energy tensor, T***, which evaluates the **effect of a given distribution of mass and energy on the curvature of spacetime**, as described mathematically by the **Einstein tensor, *G*** (for Gravity), on the left.

The appearance of Newton's gravitational constant, G, and the speed of light symbolizes the extension of relativity theory to include gravity.

We will only describe the curvature of spacetime around a spherical object of mass M and radius R, then demonstrate how an object moves through the curved spacetime it encounters.

Note that $E_{rest} = mc^2$ implies that **both mass and energy contribute to the curvature of spacetime.**

Worldlines and light cones

Three examples of some paths traced out in spacetime. In these ***spacetime diagrams*,** time is represented on the vertical axis, while space is depicted by the horizontal x–y plane. The third spatial dimension, z, cannot be shown, so this figure deals only with motion that occurs in a plane.

The **path followed by an object as it moves through spacetime** is called its **worldline**.

The **worldline of a freely falling object in response to the local curvature of spacetime ->** the spatial components of such a worldline describe the trajectory of a planet orbiting the Sun, or a photon attempting to escape from a black hole.

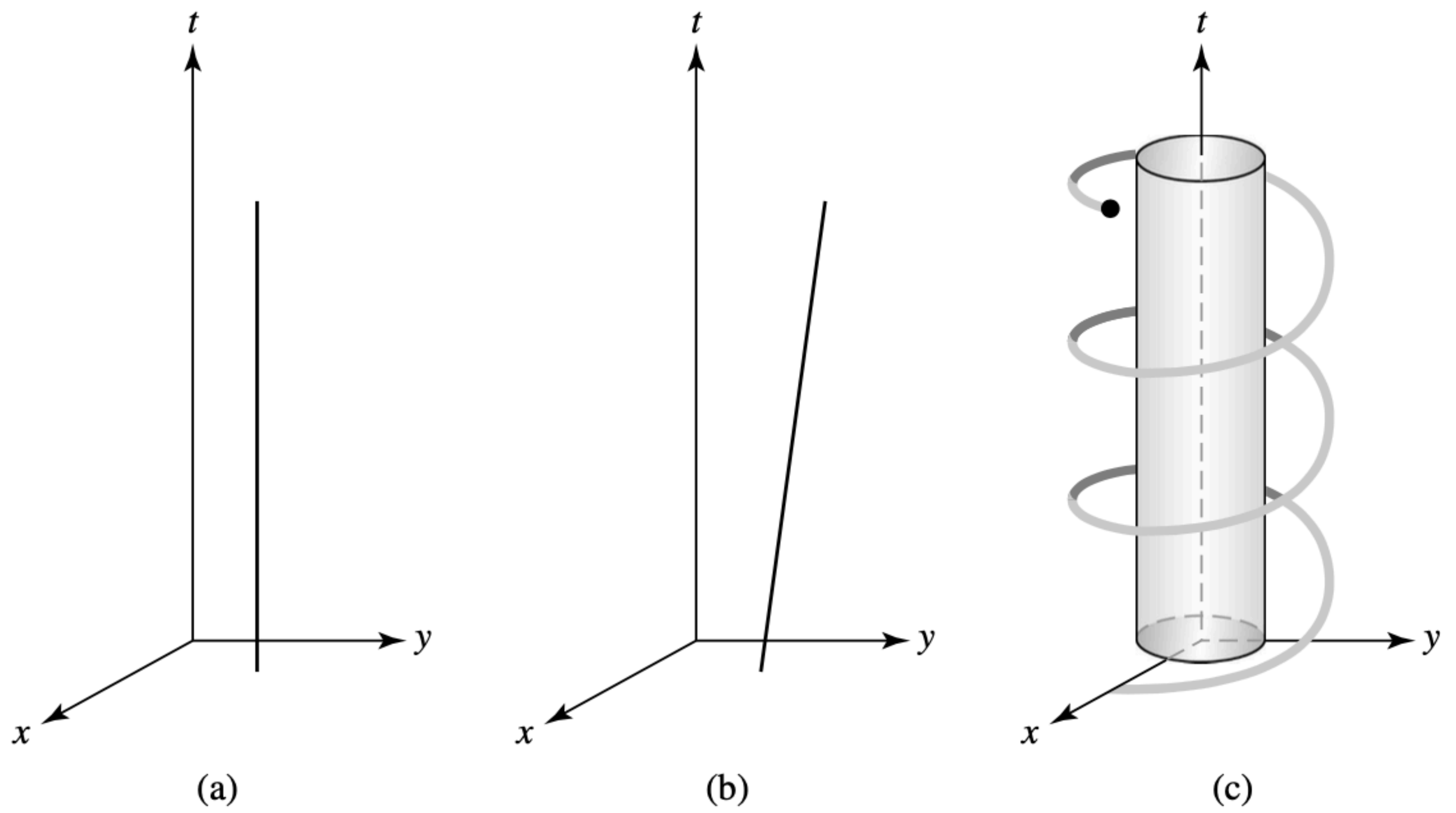


FIGURE 12 Worldlines for (a) a man at rest, (b) a woman running with constant velocity, and (c) a satellite orbiting Earth.

Worldlines and light cones

The worldlines of photons in flat spacetime point the way to an understanding of the geometry of spacetime.

Suppose a lightbulb is set off at the origin at time t = 0; call this event A.

As shown in the Fig, the worldlines of photons traveling in the x–y plane form a **light cone** that represents a widening series of horizontal circular slices through the expanding spherical wavefront of light. The graph's axes are scaled so that **the straight worldlines of light rays make 45° angles with the time axis.**

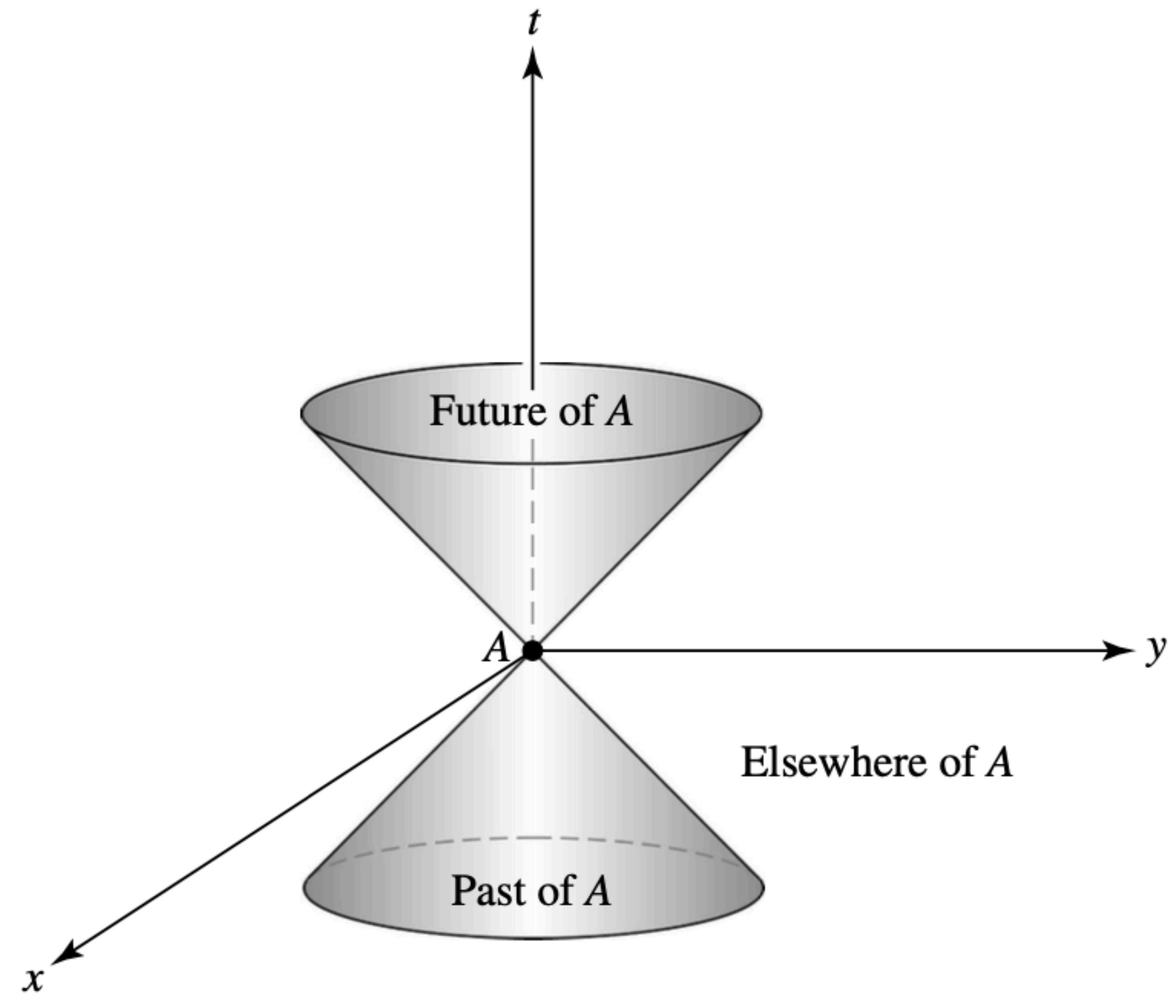


FIGURE 13 Light cones generated by horizontally traveling photons leaving the origin at time $t = 0$.

Worldlines and light cones

An object with mass initially at event A must travel slower than light, so the angle between its worldline and the time axis must be less than 45°. Therefore the region inside the light cone represents the possible ***future* of event A**. It consists of all of the events that can possibly be reached by a traveler initially at event A—and therefore all of the events that the traveler could ever influence in a causal way.

Extending the diverging photon worldlines back through the origin generates a lower light cone. Within this lower light cone is the **possible *past*** of event A, the collection of all events from which a traveler could have arrived just as the bulb flashed. In other words, the possible past consists of the locations in space and time of every event that could possibly have caused the lightbulb to go off.

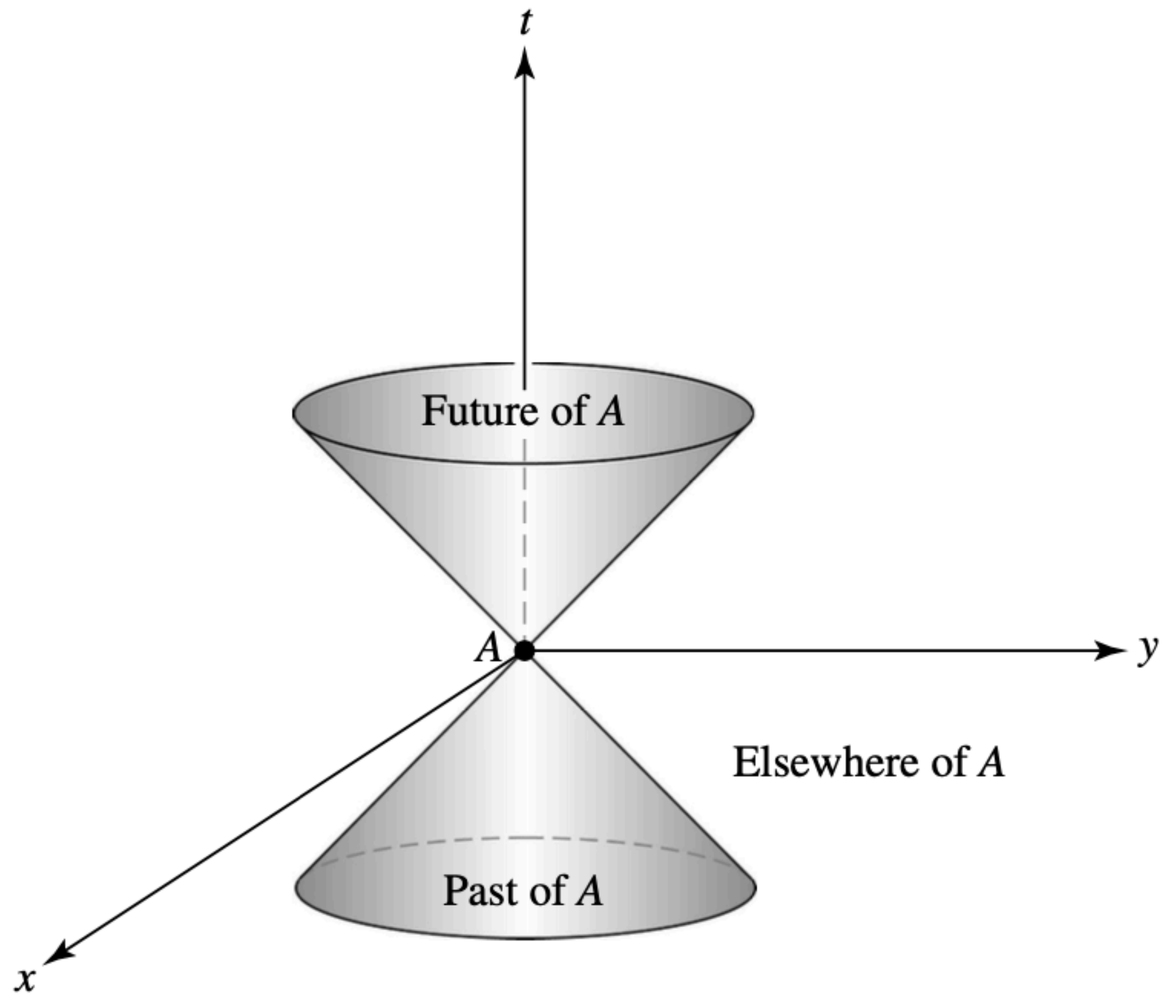


FIGURE 13 Light cones generated by horizontally traveling photons leaving the origin at time $t = 0$.

Worldlines and light cones

Outside the future and past light cones is an unknowable ***elsewhere*****, that part of spacetime of which a traveler at event A can have no knowledge** and over which he or she can have no influence. This means that vast regions of spacetime are hidden from us.

In principle, every event in spacetime has a pair of light cones extending from it. The light cone divides spacetime into that event's future, past, and elsewhere. For any event in the past to have possibly influenced you, that event must lie within your past light cone, just as any event that you can ever possibly affect must lie within your future light cone. Your entire future worldline, must therefore lie within your future light cone at every instant. **Light cones act as spacetime horizons, separating the knowable from the unknowable.**

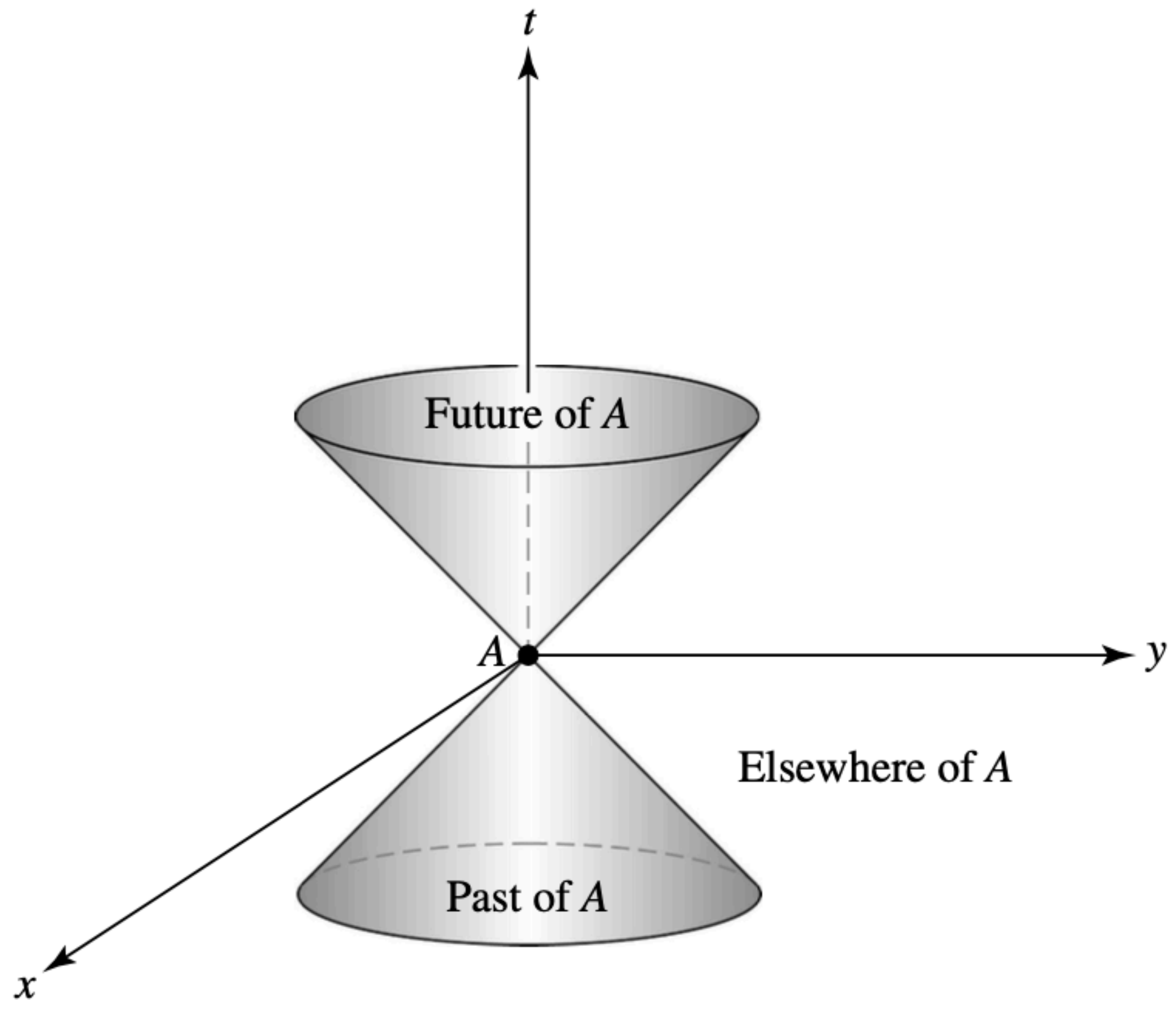


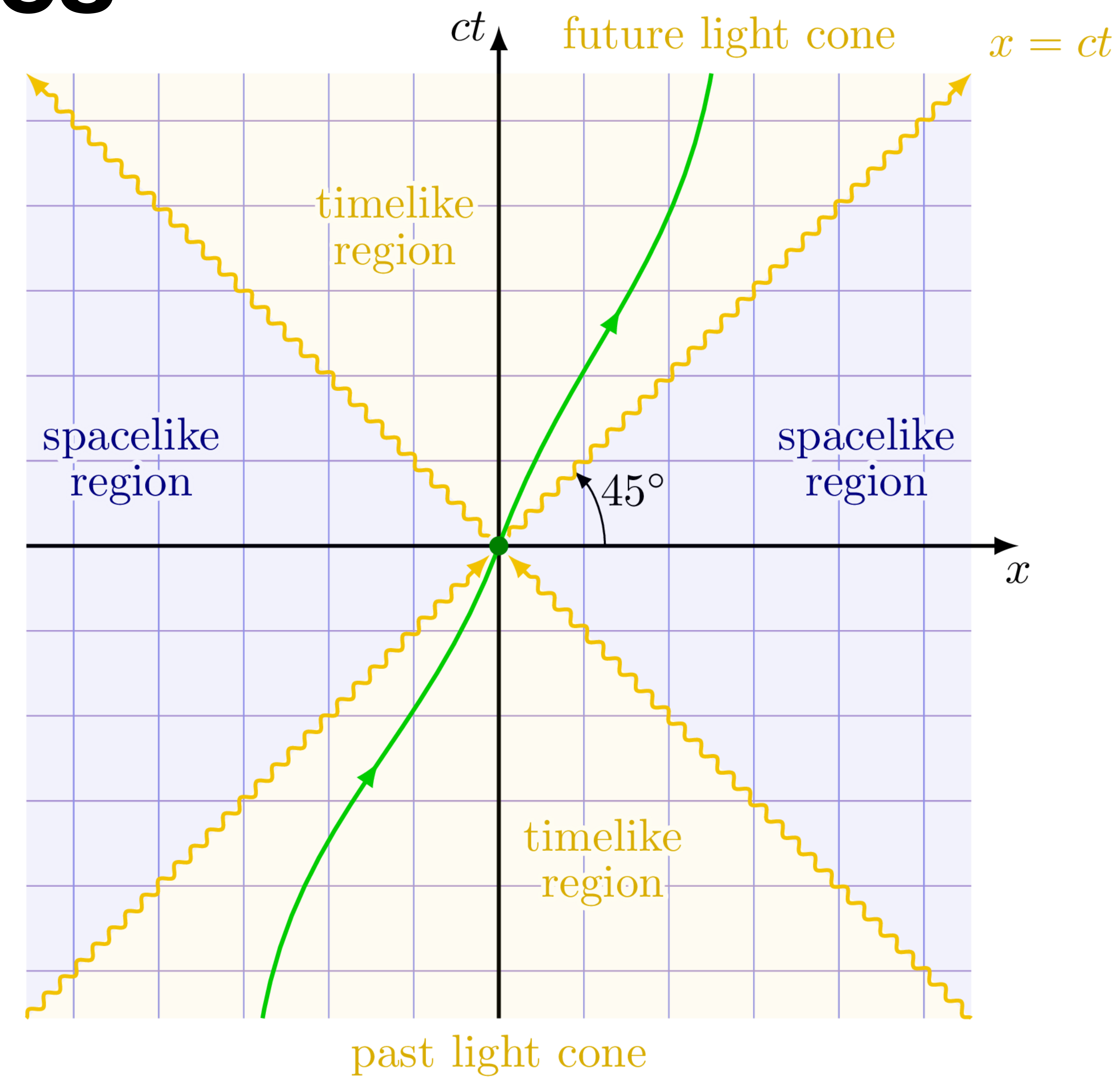
FIGURE 13 Light cones generated by horizontally traveling photons leaving the origin at time $t = 0$.

Worldlines and light cones

Spacetime diagrams are also called **Minkowski diagrams**.

The “elsewhere” region is also referred to as space like regions of the diagram, where there is no absolute time order.

In the time like region there is an absolute time order, which means that there is an absolute past and future for events.



https://tikz.net/relativity_minkowski_diagram/#Coordinate_transformations

Spacetime intervals

Measuring the progress of an object as it moves along its worldline involves defining a “distance” for spacetime. Consider the familiar case of purely spatial distances. If two points have Cartesian coordinates:

(x_1, y_1, z_1) and (x_2, y_2, z_2),

then the distance Δl measured along the straight line between the two points in flat space is defined by

$$(\Delta \ell)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

The analogous measure of “distance” in spacetime is called the **spacetime interval** (or simply *interval* for short). Let two events A and B have spacetime coordinates

(x_A, y_A, z_A, t_A) and (x_B, y_B, z_B, t_B)

measured by an observer in an inertial reference frame, S. Then the interval Δs measured along the straight worldline between the two events in flat spacetime is defined by

$$(\Delta s)^2 = [c(t_B - t_A)]^2 - (x_B - x_A)^2 - (y_B - y_A)^2 - (z_B - z_A)^2.$$

In words, $(\text{interval})^2$ = (distance traveled by light in time $|t_B - t_A|)^2$ – (distance between events A and B)^2.

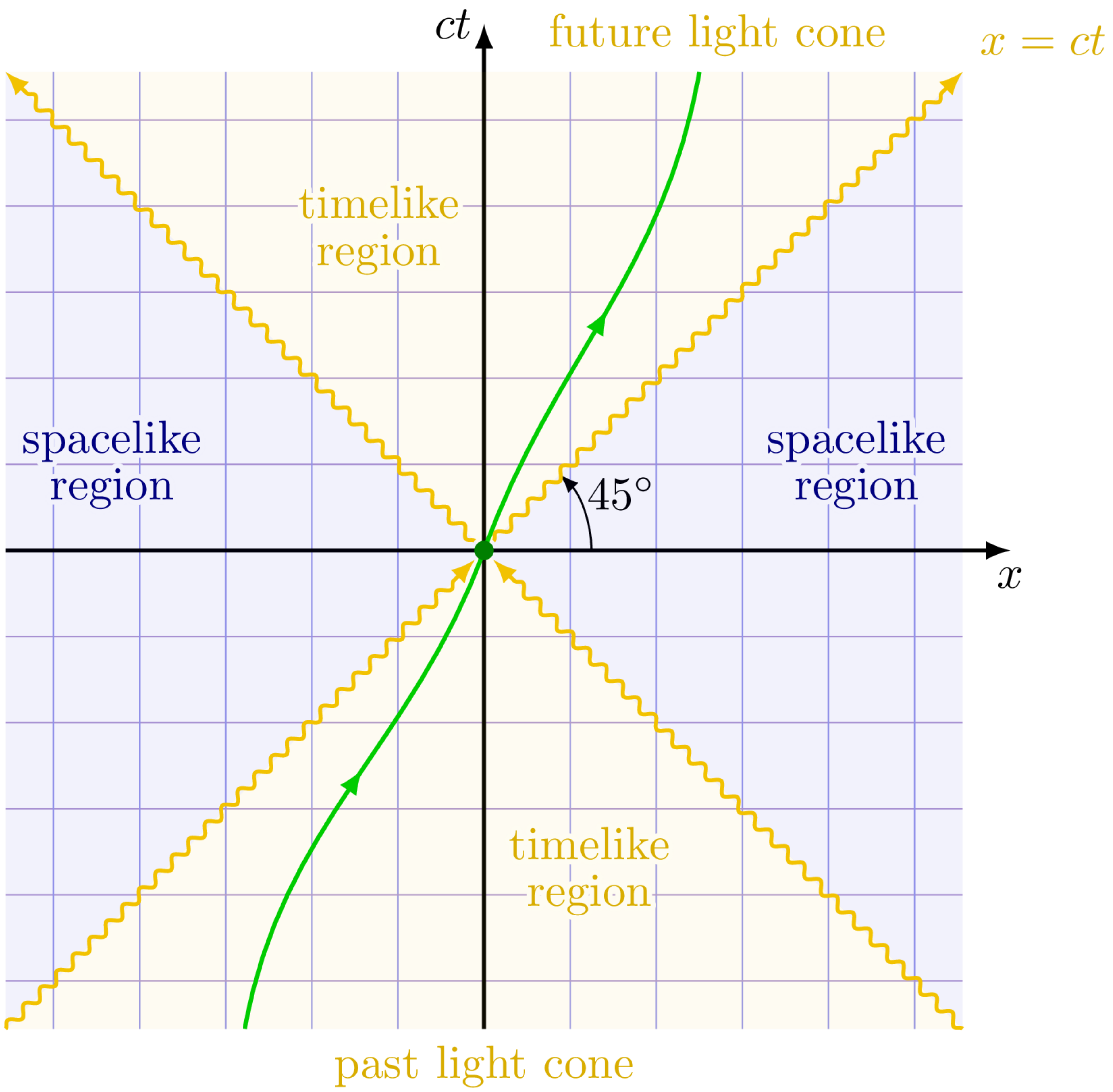
Spacetime intervals

This definition of the interval is very useful because $(\Delta s)^2$ **is *invariant* under a Lorentz transformation.** An observer in another inertial reference frame, S′, will measure the same value for the interval between events A and B; that is, $\Delta s = \Delta s'$.

Note that $(\Delta s)^2$ may be positive, negative, or zero.

The sign tells us whether light has enough time to travel between the two events. If $(\Delta s)^2 > 0$, then the **interval is *timelike*** and light has more than enough time to travel between events A and B.

An inertial reference frame S can therefore be chosen that moves along the straight worldline connecting events A and B so that the two events happen at the same location in S (at the origin, for example). Because the two events occur at the same place in S, the time measured between the two events is $\Delta s/c$.



Spacetime intervals

By definition, the time between two events that occur at the same location is the **proper time**, $\Delta\tau$, where

$$\boxed{\Delta\tau \equiv \frac{\Delta s}{c}}$$

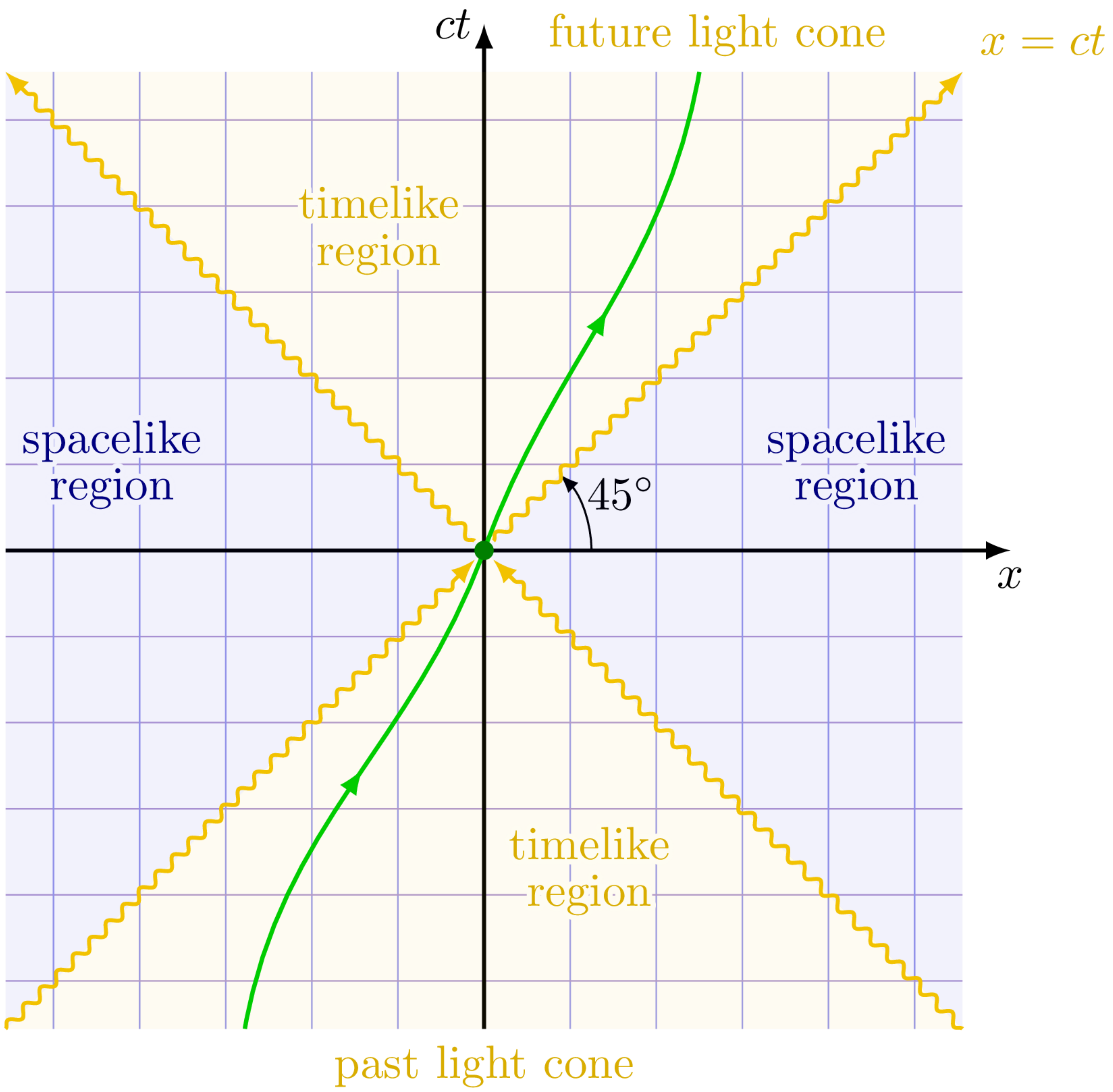
The proper time is just the elapsed time recorded by a watch moving along the worldline from A to B. An observer in any inertial reference frame can use the interval to calculate the proper time between two events that are separated by a timelike interval.

Spacetime intervals

If $(\Delta s)^2 = 0$, then the interval is ***lightlike*** or *null*. In this case, light has exactly enough time to travel between events A and B. Only light can make the journey from one event to the other, and the proper time measured along a null interval is zero.

Finally, if $(\Delta s)^2 < 0$, then the interval is ***spacelike***; light does not have enough time to travel between events A and B. No observer could travel between the two events because speeds greater than c would be required.

The lack of absolute simultaneity in this situation, however, means that there are inertial reference frames in which the two events occur in the opposite temporal order, or even at the *same time*.



Spacetime intervals

By definition, the distance measured between two events A and B in a reference frame for which they occur simultaneously ($t_A = t_B$) is the **proper distance** separating them,

$$\boxed{\Delta \mathcal{L} = \sqrt{-(\Delta s)^2}.}$$

If a straight rod were connected between the locations of the two events, this would be the ***rest length*** of the rod. An observer in any inertial reference frame can use this to calculate the proper distance between two events that are separated by a spacelike interval.

The interval is clearly related to the light cones. Let event A be a flashbulb set off at the origin at time t = 0. The surfaces of the light cones, where the photons are at any time t, are the locations of all events B that are connected to A by a null interval.

The events within the future and past light cones are connected to A by a timelike interval, and the events that occur elsewhere are connected to A by a spacelike interval.

Flat spacetime

Returning to three-dimensional space for a moment, it is obvious that a path connecting two points in space doesn't have to be straight. Two points can be connected by infinitely many curved lines.

To measure the distance along a curved path, P, from one point to the other, we use a **differential distance formula** called a **metric**,

$$(d\ell)^2 = (dx)^2 + (dy)^2 + (dz)^2.$$

Then dl may be integrated along the path P (a *line integral*) to calculate the total distance between the two points,

$$\Delta\ell = \int_1^2 \sqrt{(d\ell)^2} = \int_1^2 \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \quad \text{(along } \mathcal{P}\text{)}.$$

The distance between two points thus depends on the path connecting them. Of course, the *shortest* distance between two points in flat space is measured along a straight line. In fact, we can *define* the **"straightest possible line" between two points as the path for which Δl is a *minimum*.**

Flat spacetime

Similarly, a worldline between two events in spacetime is not required to be straight; the two events can be connected by infinitely many curved worldlines. To measure the interval along a curved worldline, W, connecting two events in spacetime with no mass present, we use the **metric for flat spacetime**,

$$\boxed{(ds)^2 = (c\,dt)^2 - (d\ell)^2 = (c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2.}$$

Then ds is integrated to determine the total interval along the worldline *W*,

$$\Delta s = \int_A^B \sqrt{(ds)^2} = \int_A^B \sqrt{(c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2} \quad (\text{along } \mathcal{W}).$$

The interval is still related to the proper time measured along the worldline by Eq. $\Delta\tau \equiv \dfrac{\Delta s}{c}$

The interval measured along any timelike worldline divided by the speed of light is always the proper time measured by a watch moving along that worldline. The proper time is zero along a null worldline and is undefined for a spacelike worldline.

Flat spacetime

In flat spacetime, the interval measured along a straight timelike worldline between two events is a *maximum*. Any other worldline between the same two events will not be straight and will have a smaller interval.

For a massless particle such as a photon, all worldlines have a null interval (so $\int \sqrt{(ds)^2} = 0$).

The maximal character of the interval of a straight worldline in flat spacetime is easily demonstrated. Figure 15 is a spacetime diagram showing two events, A and B, that occur at times t_A and t_B . The events are observed from an inertial reference frame, S, that moves from A to B, chosen such that the two events occur at the origin of S.

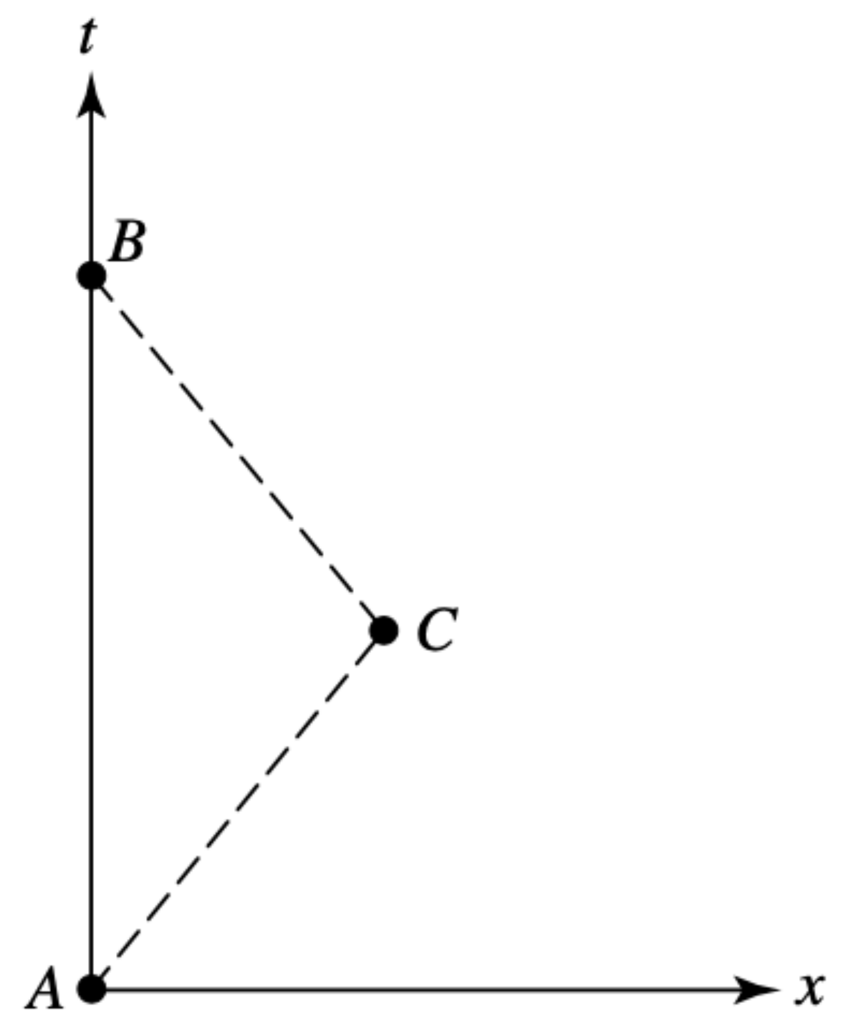


FIGURE 15 Worldlines connecting events A and B.

Flat spacetime

The interval measured along the straight worldline connecting A and B is

$$\Delta s(A \to B) = \int_A^B \sqrt{(ds)^2}$$

$$= \int_A^B \sqrt{(c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}$$

$$= \int_{t_A}^{t_B} c\,dt = c(t_B - t_A).$$

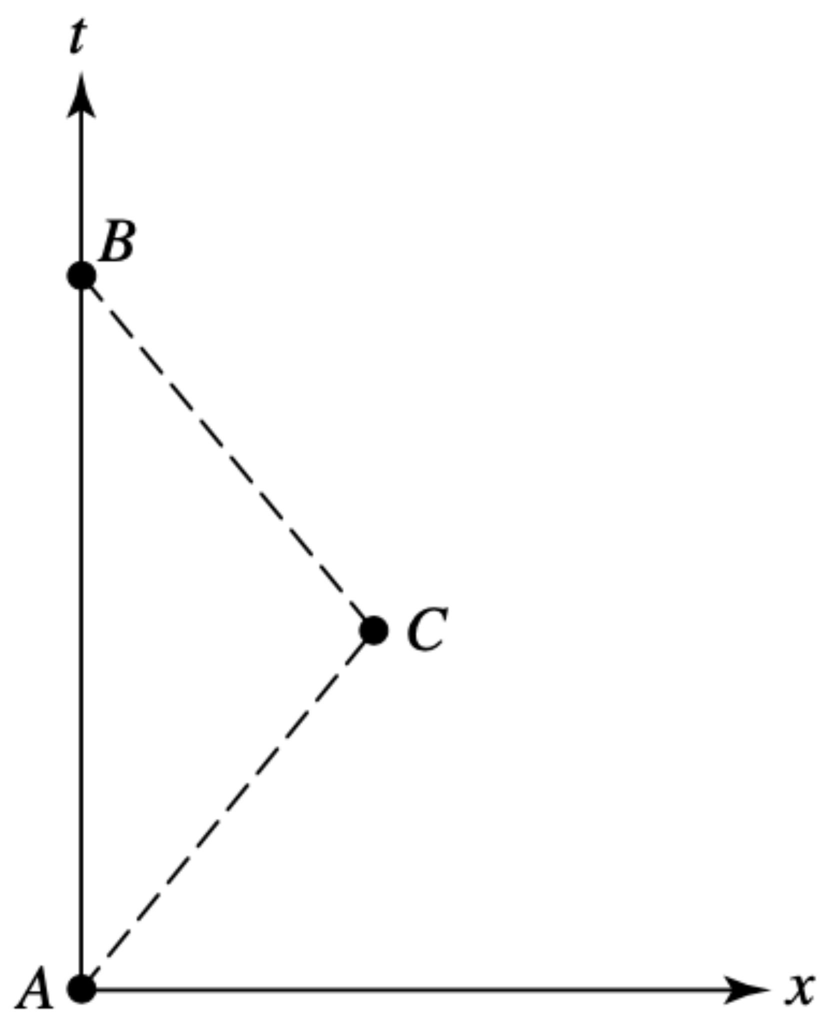


FIGURE 15 Worldlines connecting events A and B.

Flat spacetime

Now consider the interval measured along another worldline connecting A and B that includes event C, which occurs at (x,y,z,t) = (x_C,0,0,t_C). In this case,

$$\Delta s(A \to C \to B) = \int_A^C \sqrt{(ds)^2} + \int_C^B \sqrt{(ds)^2}$$

$$= \int_A^C \sqrt{(c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}$$

$$+ \int_C^B \sqrt{(c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}.$$

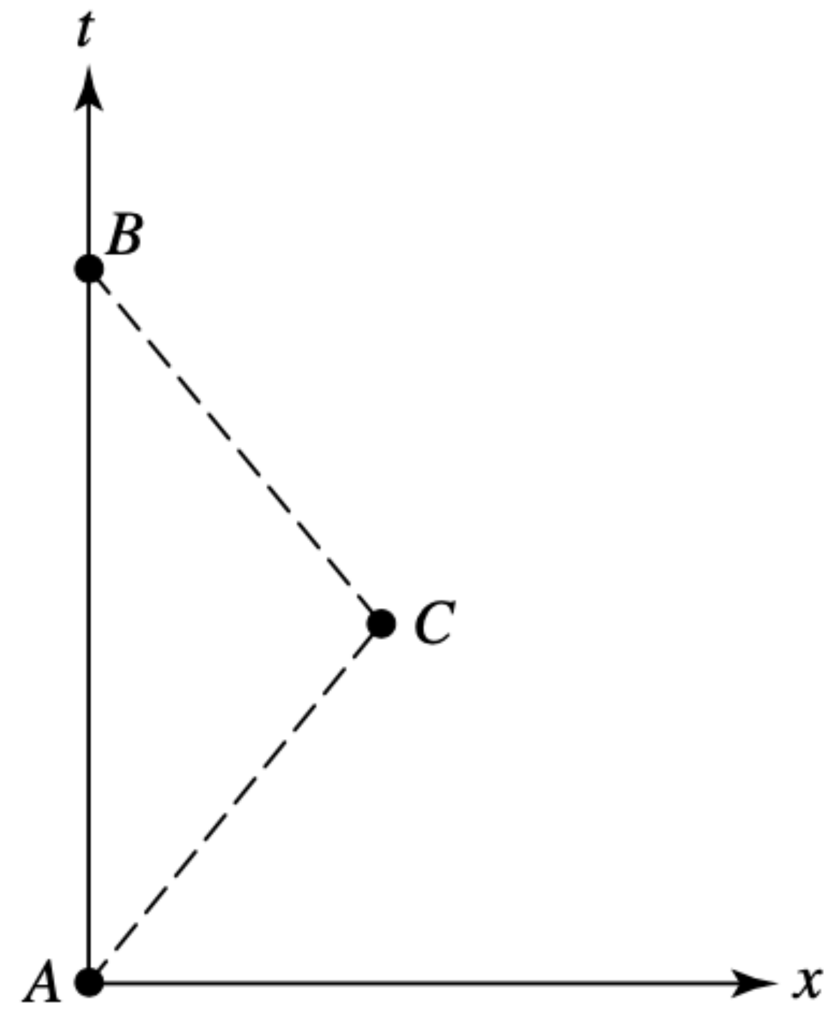


FIGURE 15 Worldlines connecting events A and B.

Using dx/dt = v_{AC} for the constant velocity along worldline A → C in the first integral, and dx/dt = v_{CB} for the constant velocity along C → B in the second integral, leads to

$$\Delta s(A \to C \to B) = (t_C - t_A)\sqrt{c^2 - v_{AC}^2} + (t_B - t_C)\sqrt{c^2 - v_{CB}^2}$$

$$< c(t_C - t_A) + c(t_B - t_C)$$

$$< \Delta s(A \to B).$$

Thus the straight worldline has the longer interval.

Curved spacetime

In a spacetime that is curved by the presence of mass, the situation is slightly more complicated. Even the "straightest possible worldline" will be curved. These straightest possible worldlines are called **geodesics**. In flat spacetime a geodesic is a straight worldline.

In curved spacetime, a timelike geodesic between two events has either a *maximum* or a *minimum* interval. In other words, the value of Δs along a timelike geodesic is an *extremum*, either a maximum or a minimum, when compared with the intervals of nearby worldlines between the same two events.

In the situations we will encounter here, the intervals of timelike geodesics will be maxima.

A massless particle such as a photon follows a *null geodesic*, with $\int \sqrt{(ds)^2} = 0$.

Curved spacetime

The paths followed by freely falling objects through spacetime are geodesics.

Based on the three fundamental features of general relativity:

• Mass acts on spacetime, telling it how to curve.

• Spacetime in turn acts on mass, telling it how to move.

• Any freely falling particle (including a photon) follows the straightest possible worldline, a geodesic, through spacetime. For a massive particle, the geodesic has a *maximum* or a *minimum* interval, while for light, the geodesic has a *null* interval.

Curved spacetime

These components of the theory will allow us to describe the **curvature of spacetime around a massive spherical object** and to **determine how another object will move in response,** whether it is a satellite orbiting Earth or a photon orbiting a black hole.

For situations with spherical symmetry, it will be more convenient to use the spherical coordinates (r,θ,ϕ) instead of Cartesian coordinates. The metric between two nearby points in flat space is then

$$(d\ell)^2 = (dr)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2,$$

and the corresponding expression for the flat spacetime metric is

$$\boxed{(ds)^2 = (c\,dt)^2 - (dr)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2.}$$

Curved spacetime

Of course, spacetime will not be flat in the vicinity of a massive object. The specific situation is the **motion of a particle through the curved spacetime produced by a massive sphere**. It could be a planet, a star, or a black hole.

The first task is to calculate how this massive object acts on spacetime. This requires a **description of the metric for this curved spacetime** that will replace the equation for a flat spacetime.

Before presenting this metric, we must emphasize that the variables r, θ, ϕ, and t that appear in the expression for the metric are the *coordinates* used by an observer at rest a great (≃ infinite) distance from the origin.

Now we place a sphere of **mass M and radius R** (which will be called a "planet") **at the origin** of our coordinate system.

The origin (which is inside the sphere) should not be used as a point of reference, and so we will avoid defining r as "the distance from the origin." Instead, imagine a series of nested concentric spheres centered at the origin. The surface area of a sphere can be measured without approaching the origin, so the coordinate r will be defined by the surface of that sphere having an area $4\pi r^2$. These coordinates can be used with the metric for curved spacetime to measure distances in space and the passage of time near this massive sphere.

As an object moves through this curved spacetime, its **coordinate speed** is just the rate at which its spatial coordinates change.

Curved spacetime

At a large distance ($r \simeq \infty$) from the planet, spacetime is essentially flat, and the gravitational time dilation of a photon received from the planet is given by Eq.

$$\frac{\nu_\infty}{\nu_0} = \left(1 - \frac{2GM}{r_0 c^2}\right)^{1/2}.$$

From this, it might be expected that $\sqrt{1 - 2GM/rc^2}$ would play a role in the metric for the spacetime surrounding the planet. Furthermore, recall that **the stretching of space and the slowing down of time contribute equally to delaying a light beam's passage through curved spacetime.** This provides a hint that the same factor will be involved in the metric's radial term. The angular terms are the same as those for flat spacetime.

These effects are indeed present in the metric that describes the curved spacetime surrounding a spherical mass, M. In 1916, just two months after Einstein published his general theory of relativity, the German astronomer Karl Schwarzschild (1873–1916) solved Einstein's field equations to obtain what is now called the **Schwarzschild metric**:

$$\boxed{\begin{aligned}(ds)^2 = &\left(c\,dt\sqrt{1 - 2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1 - 2GM/rc^2}}\right)^2 \\ &- (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2.\end{aligned}}$$

Curved spacetime

It is important to realize that the Schwarzschild metric is the spherically symmetric ***vacuum solution* of Einstein's field equations.** That is, it is valid only in the **empty space *outside* the object**. The mathematical form of the metric is different in the object's interior, which is occupied by matter.

The Schwarzschild metric contains all of the effects considered in the last section. **The "curvature of space" resides in the radial term.** The radial distance measured simultaneously (dt = 0) between two nearby points on the same radial line (dθ = dϕ = 0) is just the **proper distance,**

$$d\mathcal{L} = \sqrt{-(ds)^2} = \frac{dr}{\sqrt{1 - 2GM/rc^2}}.$$

Thus the spatial distance dL between two points on the same radial line is *greater* than the coordinate difference dr. This is precisely what is represented by the stretched grid lines in the rubber sheet analogy.

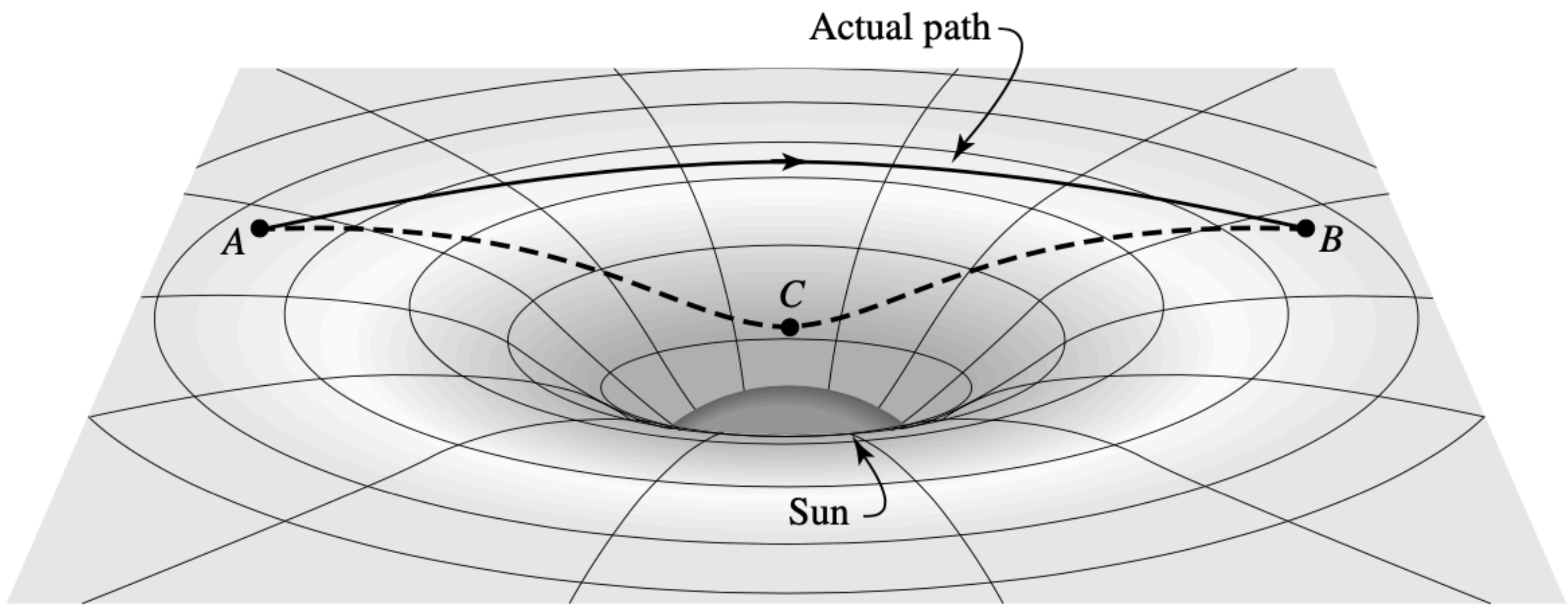


FIGURE 4 Comparison of two photon paths through curved space between points A and B. The projection of the path ACB onto the plane is the straight line depicted in Fig. 3.

Curved spacetime

The factor of $1/\sqrt{1 - 2GM/rc^2}$ must be included.

An example is when planning a hike up a steep trail. The additional information provided by the map's elevation contour lines must be included in any calculation of the actual hiking distance, which is always greater than the difference in map coordinates.

The Schwarzschild metric also incorporates time dilation and the gravitational redshift (two aspects of the same effect). If a clock is at rest at the radial coordinate r , then the **proper time dτ** it records is related to the time dt that elapses at an infinite distance by

$$d\tau = \frac{ds}{c} = dt\sqrt{1 - \frac{2GM}{rc^2}},$$

Since d τ < d t , this shows that time passes more slowly closer to the planet.

The orbit of a satellite

Using general relativity to find the motion of a satellite about the planet. All we need is the rule that it will follow the straightest possible worldline, the worldline with an *extremal* interval.

According to Newton, the motion of a satellite in a circular orbit around Earth is found by simply equating the centripetal and gravitational accelerations. That is,

$$\frac{v^2}{r} = \frac{GM}{r^2},$$

where v is the **orbital speed.** This immediately results in

$$v = \sqrt{\frac{GM}{r}}.$$

It is assumed that the satellite's mass m is small enough that its effect on the surrounding spacetime is negligible.

The orbit of a satellite

Einstein and Newton must agree in the limiting case of weak gravity. We should get the same result using the Schwarzschild metric to find the straightest possible worldline for the satellite's circular orbit.

Calculating the worldline with the maximum interval between two fixed events -> the orbit of the satellite would emerge along with the laws of the conservation of energy, momentum, and angular momentum because they are built into Einstein's field equations.

However, we will use a simpler strategy and assume from the beginning that the satellite travels above Earth's equator ($\theta = 90^\circ$) in a circular orbit with a specified angular speed $\omega = v/r$. Inserting these choices, along with $dr = 0$, $d\theta = 0$, and $d\phi = \omega\, dt$, into **the Schwarzschild metric gives**

$$(ds)^2 = \left[\left(c\sqrt{1 - 2GM/rc^2}\,\right)^2 - r^2\omega^2\right] dt^2 = \left(c^2 - \frac{2GM}{r} - r^2\omega^2\right) dt^2.$$

Integrating, the spacetime interval for one orbit is just

$$\Delta s = \int_0^{2\pi/\omega} \sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\, dt.$$

The orbit of a satellite

When finding the value of r for which the interval is an extremum, we must be certain that the **endpoints of the satellite's worldline remain fixed.** That is, the satellite's orbit must always begin and end at the same position, r_0, for all of the worldlines.

To accommodate orbits of different radii, consider the "orbit" shown in Fig. 17. We start the satellite at r_0 and then move it (at nearly the speed of light) radially outward to the radius r of its actual orbit. At the end of the orbit, the satellite returns just as rapidly to its starting point at r_0.

The quick radial excursions at the beginning and the end of the orbit can be made with negligible contribution to the integral for the spacetime interval. (At almost the speed of light, the contribution is nearly null.) The net effect is a purely circular motion.

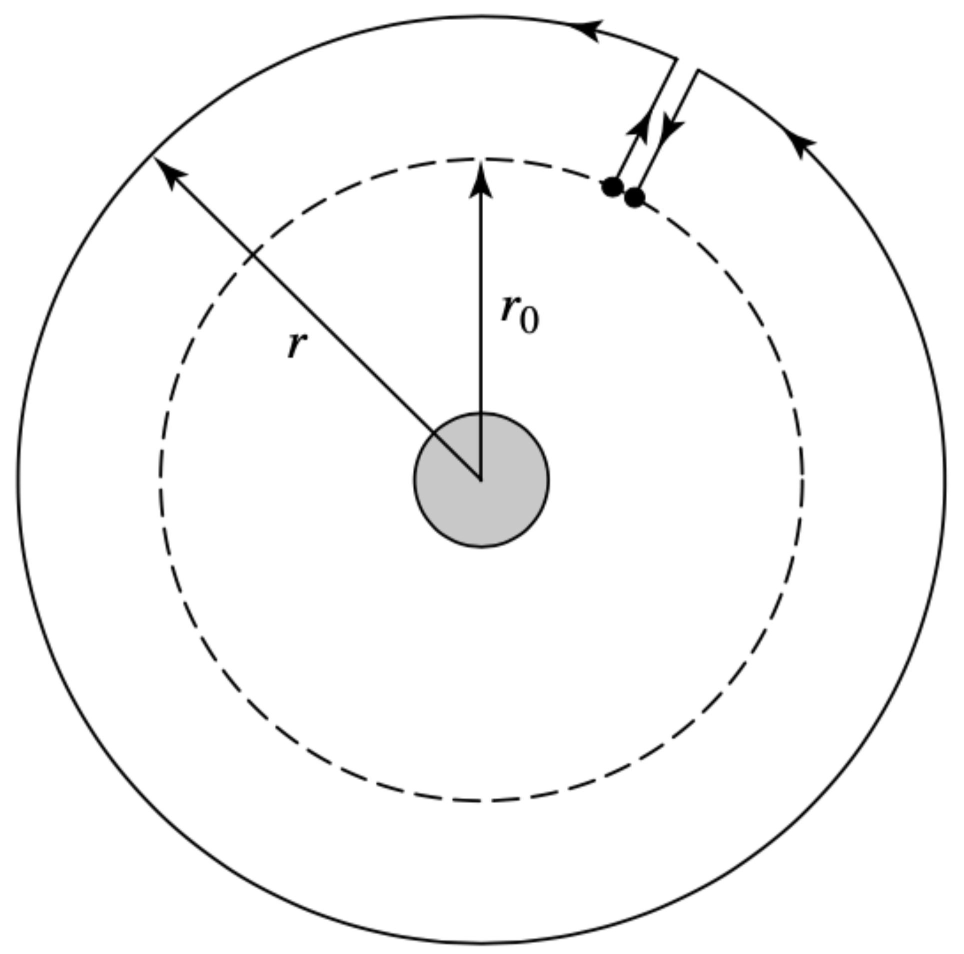


FIGURE 17 The "orbit" of a satellite, showing the radial motions used to keep the endpoints of the satellite's worldline fixed. The net effect is a circular orbit.

The orbit of a satellite

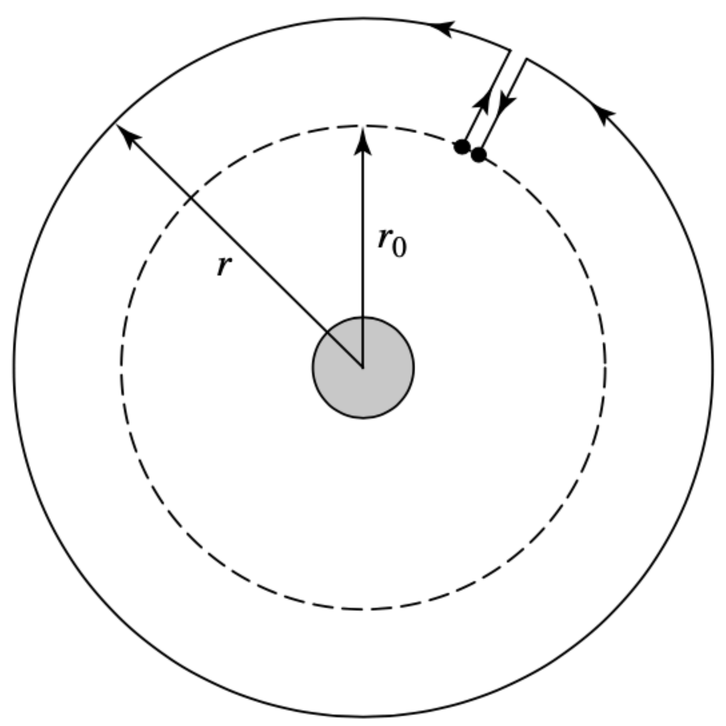


FIGURE 17 The "orbit" of a satellite, showing the radial motions used to keep the endpoints of the satellite's worldline fixed. The net effect is a circular orbit.

$$\Delta s = \int_0^{2\pi/\omega} \sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\, dt.$$

In , the limits of integration are constant and the only variable is r. The value of the radial coordinate r for the orbit actually followed by the satellite must be the one for which Δs is an extremum. This value may be found by taking the derivative of Δs with respect to r and setting it equal to zero:

$$\frac{d}{dr}(\Delta s) = \frac{d}{dr}\left(\int_0^{2\pi/\omega} \sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\, dt\right) = 0.$$

The derivative may be taken inside the integral to obtain

$$\frac{d}{dr}\sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2} = 0,$$

The orbit of a satellite

implying $$\frac{2GM}{r^2} - 2r\omega^2 = 0.$$

Rearranging $$v = r\omega = \sqrt{\frac{GM}{r}}$$

is the coordinate speed of the satellite for a circular orbit.

Figure 18 illustrates how this **straightest possible worldline through curved spacetime** is projected onto the orbital plane, resulting in the satellite's circular orbit around Earth.

This result is valid even for the very large spacetime curvature encountered around a black hole.

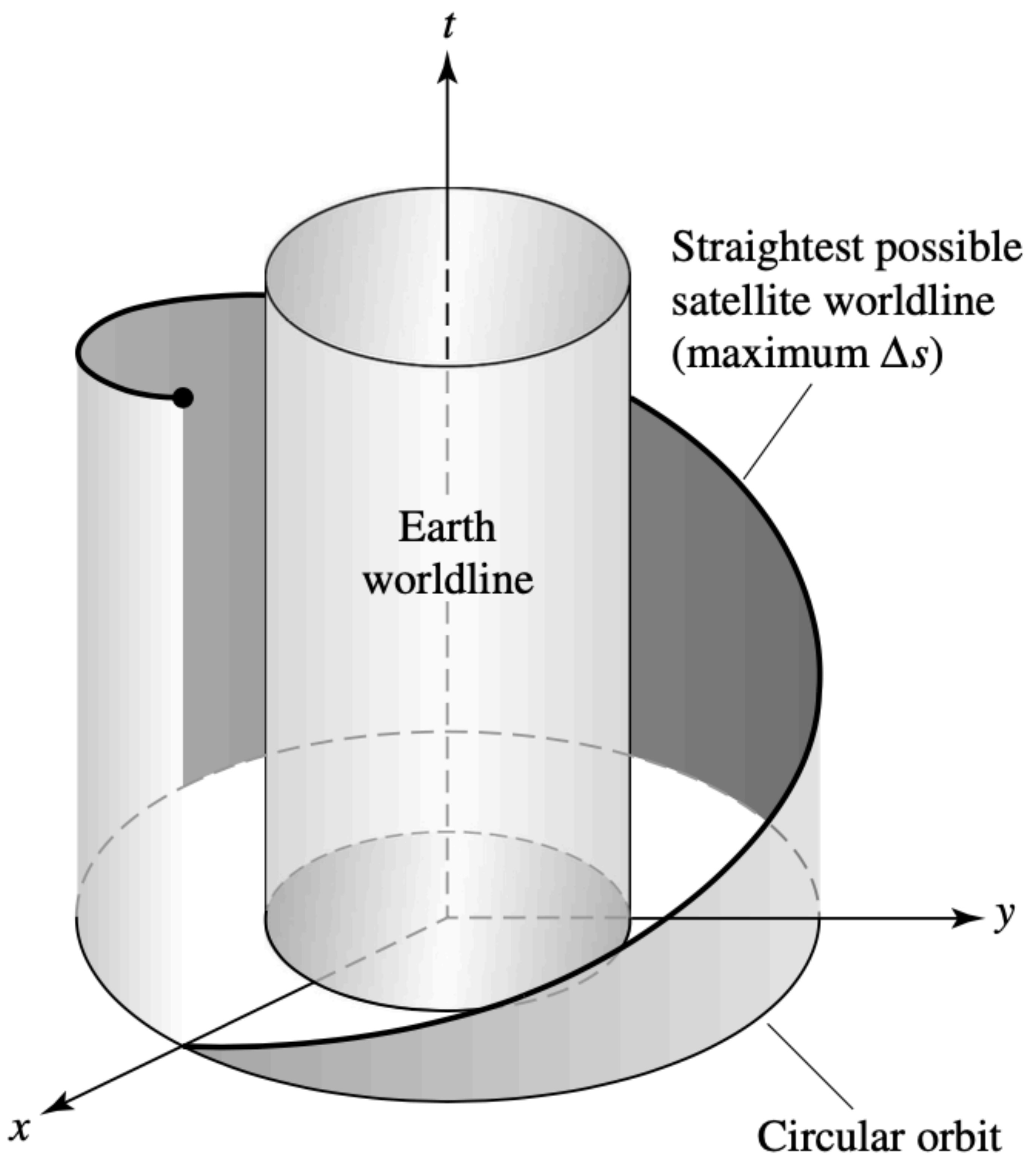


FIGURE 18 The straightest possible worldline through curved spacetime and its projection onto the orbital plane of the satellite.